Detection of Topology Changes in Dynamical System: An Information Theoretic Approach

Pierre-Alain Toupance, Bastien Chopard, Laurent Lefèvre

Detection of topology changes in dynamical system: an information theoretic approach

TOUPANCE Pierre-Alain[1,2], CHOPARD Bastien[1], LEFÈVRE Laurent[2]

[1] University of Geneva, Geneva, Switzerland
[2] Univ. Grenoble Alpes, Grenoble INP, LCIS, Valence, France

Abstract. In this paper, we show that the theory of information offers some tools to detect changes in the interaction topology of a dynamical system defined on a graph. As an illustrative example, the system we consider is a probabilistic voter model defined on a scale-free network. We show that, using time-delayed mutual-information, the interaction topology of an unknown graph can be reconstructed to some level. We apply this approach on a sliding time window to detect possible changes in the interaction topology over time.

Keywords: dynamical systems, complex network, probabilistic models, delayed mutual information, interaction topology reconstruction, online topology identification, voter model.

1 Introduction

The knowledge of the interconnection topology of a complex network is important in order to have results on structural observability and controllability, as discussed in [1,2].

In this paper we consider the concept of causality as a way to obtain the interaction topology among the variables of complex dynamical systems. For probabilistic models, the theory of information proposed by Shannon in 1948 ([3]) offers tools to define causality, for example the transfer entropy introduced by Schreiber [4], as a measure of directed (time-asymmetric) information transfer between joint processes.

In [5] we use the time-delayed mutual- and multi-information, defined in the section 2.2, to analyze the most influential components of a complex system with no a priori knowledge of the interconnection topology. This approach is non-intrusive in the sense that it may be performed by a simple sampling of the system state, even if the underlying dynamics is unknown. We proved – on the example of the so-called voter model– that the nodes (voters) may be ranked according to their influence (the impact of their opinion on the average opinion of the entire group) by monitoring the time-delayed multi-information. This ranking closely relates to controllability/observability Grammians singular values, as defined in classical system theory (see [6]). Furthermore, by sampling the state of the dynamical system, we showed [7] that time-delayed mutual- and multi-information can be used to reconstruct the interaction topology.

In this paper, the problem of a change of the interaction topology during time is investigated. The goal is to detect structural modifications in a dynamical system defined on a graph, by simply observing its state variables. To this end, we compute the delayed mutual-information on sliding time-windows and study whether this quantity can alert us of a change in the structure of the system.

The paper is organized as follows: section 2 introduces the voter model which will be used throughout the paper as an example. The metrics from information theory that we will use are recalled. An overview of results previously obtained to measure the relative influence of the agents is presented. Then we present how we used our approach to reconstruct the interconnection topology of a complex system. Our main new contributions are presented in section 4 where we discuss how the delayed mutual-information can be measured in a time sliding-window and how this leads to the identification of dynamical topology changes.

2 Dynamical system and mutual information

2.1 Voter Model

As an illustration of our approach we consider a voter model as a representative dynamical system on a graph. Various versions of voter models have been studied. For example Castellano et al. [8] have defined a q-voter model in which an agent votes like its neighbors if the opinion is unanimous; otherwise the vote is random. This model has been used by Nycska et al. [9]. Our model is closer to those used by Mobilia et al. [10], Masuda [11] or Galam [12]: it is a model where the vote of an agent depends on the average vote of its neighbors. The version we consider here is a time synchronous agent-based model defined on a graph of arbitrary topology, whether directed or not.

Our model can be described in the following way. Each node i of the network represents an agent whose opinion is either $s_i = 0$ or $s_i = 1$. The dynamics is specified by assuming that each agent i looks at every other agents in its neighborhood, and counts the fraction ρ_i of those neighbors which are in state $+1$. In case an agent is linked to itself, it belongs to its own neighborhood. A function f is specified such that $0 \leq f(\rho_i) \leq 1$ gives the probability for agent i to be in state $+1$ at the next iteration. For instance, if f would be chosen as $f(\rho) = \rho$, an agent for which all neighbors are in state $+1$ would turn into state $+1$ with certainty. The update is performed synchronously over all n agents.

Formally, the dynamics of our voter model can be express as

$$s_i(t+1) = \begin{cases} 1 & \text{with probability } f(\rho_i(t)) \\ 0 & \text{with probability } 1 - f(\rho_i(t)) \end{cases} \tag{1}$$

where $s_i(t) \in \{0, 1\}$ is the state of agent i at iteration t, and

$$\rho_i(t) = \frac{1}{|N_i|} \sum_{j \in N_i} s_j(t) \tag{2}$$

The set N_i is the set of agents j that are neighbors of agent i, as specified by the network topology.

The global density of all n agents with opinion 1 is

$$\rho(t) = \frac{1}{n}\sum_{i=1}^{n} s_i(t) \tag{3}$$

In the present case, we consider a voter model in which agent can vote differently than the majority of their neighbors. According to the total probability formula, the probability p_i that agent i votes +1 is

$$\begin{aligned} p_i(t+1) &= (1-\epsilon)p_{V_i}(t)) + \epsilon(1 - p_{V_i}(t)) \\ &= (1-2\epsilon)p_{V_i}(t) + \epsilon \end{aligned}$$

where ϵ is the probability to take a decision different from that of the neighborhood and $p_{V_i}(t)$ is the probability that the majority of neighbors of agent i votes 1 at time t. Thus, we defined $f(\rho)$ as

$$f(\rho) = (1-\epsilon)\rho + \epsilon(1-\rho) = (1-2\epsilon)\rho + \epsilon \tag{4}$$

From now on, the quantity ϵ will be called the noise. We limit the noise in the range $0 \leq \epsilon \leq 1/2$. The upper value $\epsilon = 1/2$ corresponds to a blind vote, i.e a probability 1/2 for each outcome.

To illustrate the behavior of this model, we consider a random scale-free graph G [13] which is considered as some instance of a social network [14]. In a scale-free network, a small number of particular nodes have many connections. These nodes, often referred to as hubs, are the leaders of the social network. Most other nodes have very few connections. The majority of voters are in this situation. The scale free graph structure is based on communities built around a leader, as discussed for instance in Wu et al. [15]. We use the algorithm of Bollobás and Riordan [16] to generate the random scale free graphs throughout this paper.

2.2 Delayed mutual- and multi-information

Let us consider a set of random variables $X_i(t)$ associated with each agent i, taking their values in a set A. For instance, $X_i(t) = s_i(t)$ would be the opinion of agent i at iteration t.

To measure the influence of an agents i on j, we define the τ-delayed mutual information $w_{i,j}$ as

$$\begin{aligned} w_{i,j}(t,\tau) &= I(X_i(t), X_j(t+\tau)) && (5) \\ &= \sum_{(x,y)\in A^2} p_{xy} \log\left(\frac{p_{xy}}{p_x p_y}\right) && (6) \end{aligned}$$

with

$$p_{xy} = \mathbb{P}(X_i(t) = x, X_j(t+\tau) = y)$$

$$p_x = \mathbb{P}(X_i(t) = x) \text{ and } p_y = \mathbb{P}(X_j(t+\tau) = y)$$

We also defined the τ-delayed multi-information w_i to measure the influence of one agent i on all the others

$$w_i(t,\tau) = I(X_i(t), Y_i(t+\tau)) \tag{7}$$

$$Y_i(t+\tau) = \sum_{k \neq i} X_k(t+\tau) \tag{8}$$

2.3 Controllability and information theory

In our recent paper [5] entitled "Controllability of the Voter Model: an information theoretic approach", we define the influence of an agent in two different ways. The so-called *intrusive* approach consists in forcing (or controlling) the opinion of an agent and to measure the impact on the global density of opinions 1 in the system. More specifically we average ρ as given by eq. (3) over a large number N of independent realizations (ensemble average). This gives a quantity $\langle\rho(t)\rangle_i$, where subscript i indicates which agent has been forced to 1. For large enough t, $\langle\rho\rangle_i$ no longer depends on t and provides a measure of the influence of agent i on the system.

A second way to define the influence of agent i is to use the delayed multi-information introduced in eq. (7). The quantity $w_i(t,\tau)$ provides a non-intrusive measure (no forcing is required) of the influence of agent i. Here the time delay τ is taken as the diameter of the network, so that the influence of an agent can propagate to all the vertices of the graph.

We showed in [5] that the intrusive and non-intrusive measurements are very similar, as illustrated in Fig. 1. The gray scale representation for the nodes shows the intensities of the multi-information $w_i(t,\tau)$ or the influence $\langle\rho\rangle_i$ of the corresponding agent i. The multi-information gives also indication about the controllability of the system as it clearly identifies the agents that are best to control the system when their vote is forced.

3 Topology of the system

3.1 1-delayed mutual information and adjacency matrix

After these first results about the controllability, we were interested in the topology of the system. The aim was to use information theory, to reconstruct the graph of interaction, assuming it was not known beforehand. In the following sections we present the results we have obtained in [7].

In Figure 2, we can see the values of the $1-$delayed mutual information, $w_{i,j}(1)$, between one agent i and any other agent j in the system. These values were calculated by sampling the system when it has reached its steady state. The peaks that we observe in $w_{i,j}(1)$ for some values of $j \in \{1,\ldots,50\}$ suggest that node i is a direct neighbor of this node j, as there is a causal effect after one time step.

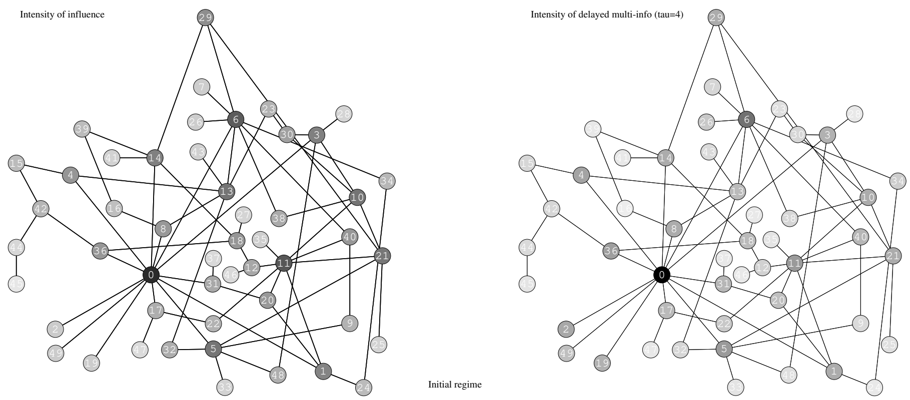

Fig. 1. Scale free graph colored as a function of the values of the influence (left) and the τ-delayed multi-information (right), for $\tau = 4$. In this case, the multi-information is computed in the transient initial regime.

In order to infer the edges of the interaction graph and build its adjacency matrix M, we used the 1-delayed mutual-information in the following way. For each agent i, we fixed a threshold T_i on the value of $w_{i,j}(1)$ to decide whether or not i is a neighbor of j. The value of T_i was determined empirically as

$$T_i = \mu_i + a_i \sigma_i$$

where μ_i is the mean value of the 1-delayed mutual-information between agent i and the other agents, and σ_i its standard deviation. We chose two different values for the coefficient a_i, to reflect the different nature of the agents (influential agents have more neighbors). We propose the following values for a_i:

$$a_i = \begin{cases} 0.2 & \text{if } w_i(t,\tau) > \alpha + \frac{1}{2}\beta \text{ (then agent } i \text{ is considered as very influential)} \\ 0.7 & \text{otherwise} \end{cases} \tag{9}$$

where $w_i(t, \tau)$ is agent i's $\tau-$delayed multi-information at time t, as defined in eq. (7). The value of t is chosen to be in the initial regime and τ is taken large enough to capture the influence over the rest of the system. The values α and β are respectively the average and the standard deviation of $w_i(t, \tau)$ over i.

The elements of the adjacency matrix, $M = (m_{ij})_{1 \leq i,j \leq n}$, are computed as

$$m_{ij} = \begin{cases} 1 & \text{if } w_{ij}(1) > T_i \text{ or } w_{j,i}(1) > T_j \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

When $w_{ij}(1) > T_i$ or $w_{ji}(1) > T_j$ then it is assumed that agents i and j are neighbors and interact symmetrically. One could of course also defined a criterion for non-symmetrical graphs.

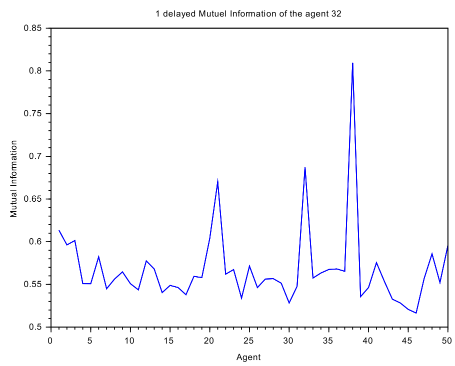

Fig. 2. 1-delayed mutual information of agent 32 with every other agent.

The values of a_i have been chosen in order to minimize the error rate r between the reconstructed matrix M and the actual graph adjacency matrix A. We have considered several scale-free graphs, by testing all values of a from 0 to 1 with a step of 0.1. The error rate is defined as $r = \frac{\Delta(M,A)}{n^2}$, where $\Delta(M, A)$ is the number of different values between M and A and n^2 the number of elements in M, n being the number of nodes.

The left panel of Fig. 3 shows the graph G_1 we obtain with an error rate $r = 1.3\%$. To have a better result, we computed the 1-delayed mutual information when the system is in its initial regime, as shown in the right side of Fig. 3 (graph G_2). There, the error rate dropped down to $r = 0.24\%$. This suggests that the results are better in a transient mode than in the steady state. Such a transient regime can be created artificially by disrupting the system temporarily by increasing the noise, when calculating the mutual information. This method has been tested by randomly generating 20 scale-free graphs. The average error rate was found to be $r = 0.9\%$ with a standard deviation 0.0026.

3.2 Comparison of system behavior between the original and reconstructed graph

To compare the vote dynamics of the original system associated with graph G, with the one associated with graph G_1 (obtained with the 1-delayed mutual information computed in the steady state, see Fig. 3), we look at the evolution of the fraction of voters in state 1, starting with the same initial state and making use of the same noise history (i.e. same seed for the random generator). In the left side of Fig. 4, we can see that the behaviors of G and $G1$ are similar whereas the evolution of the fraction of vote 1 looks in general very different for another arbitrarily chosen graph, such as it is illustrated in the right side of Fig. 4. This suggests that our method of reconstructing the topology in the steady state

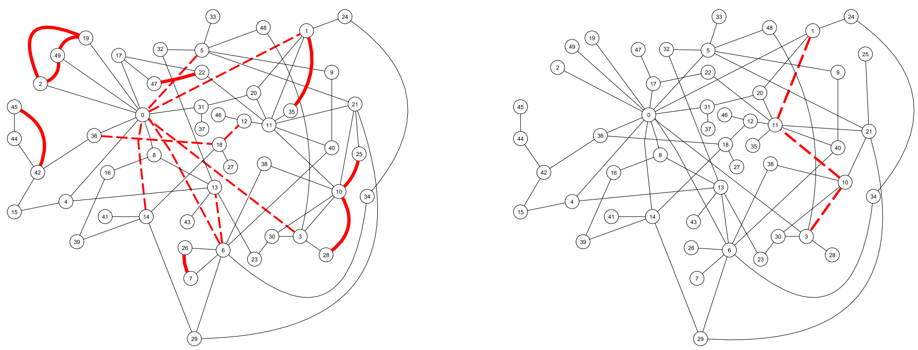

Fig. 3. (Left:) Graph G_1 built with the 1-delayed mutual information calculated when the system is in steady state. (Right:) Graph G_2 built with the 1-delayed mutual information computed when the system is in the initial transient regime. Red lines are the errors we got on the edges during reconstructions : the dashed line for the extra links and thick line for the missing links

selects the most important edges from the point of view of their influence on the dynamical evolution.

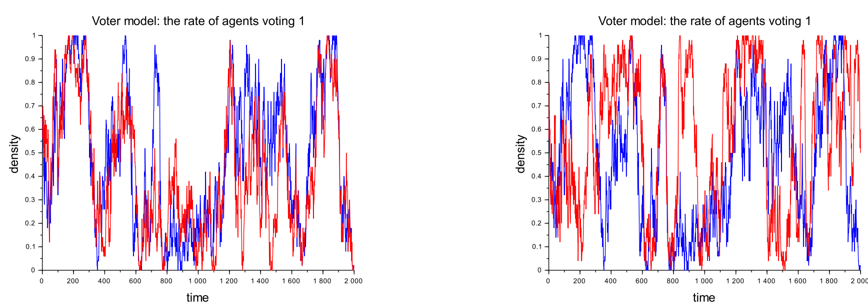

Fig. 4. Left: Plot of the time evolution of the density of opinion 1 with noise $\epsilon = 0.01$ for graph G (blue curve) and for the reconstructed graph G_1 (red curve). Right: the same quantity as produced by G (blue curve) and another, non-related graph (red curve).

4 Delayed mutual-information to detect topology changes

In this section, we assume that the topology of the dynamical system changes over time. Our goal is the real time detection of such changes through online analysis of the 1-delayed mutual information $w_{ij}(t,\tau)$ between the two corresponding vertices i and j.

Therefore, we compute $w_{ij}(t,\tau)$ at time $t = t_0$ by sampling $s_i(t-\tau)$ and $s_j(t)$ on a sliding window $t \in [t_0 - \Delta - 1, t_0]$ with width Δt. During this time interval, we record the pairs $(s_i(t-\tau), s_j(t)$ of the states of vertices i and j with a time delay $\tau = 1$. This leads to the following quantities:

$$N^{00}_{i,j}(t_0) = \sum_{k=0}^{\Delta t} \bar{s_i}(t_0 - k - 1) \times \bar{s_j}(t_0 - k) \qquad N^{01}_{i,j}(t_0) = \sum_{k=0}^{\Delta t} \bar{s_i}(t_0 - k - 1) \times s_j(t_0 - k)$$

$$N^{10}_{i,j}(t_0) = \sum_{k=0}^{\Delta t} s_i(t_0 - k - 1) \times \bar{s_j}(t_0 - k) \qquad N^{11}_{i,j}(t_0) = \sum_{k=0}^{\Delta t} s_i(t_0 - k - 1) \times s_j(t_0 - k)$$

with $\bar{s_i}(t) = 1 - s_i(t)$.

According to eq. (6) the estimation of the mutual information on the time interval $[t_0 - \Delta - 1, t_0]$ is given by

$$\begin{aligned} w_{i,j} = & \frac{N^{00}_{i,j}}{\Delta t} \log\Big(\frac{(\Delta t) \times N^{00}_{i,j}}{(N^{00}_{i,j} + N^{01}_{i,j})(N^{00}_{i,j} + N^{10}_{i,j})}\Big) + \frac{N^{01}_{i,j}}{\Delta t} \log\Big(\frac{(\Delta t) \times N^{01}_{i,j}}{(N^{01}_{i,j} + N^{00}_{i,j})(N^{01}_{i,j} + N^{11}_{i,j})}\Big) \\ & + \frac{N^{10}_{i,j}}{\Delta t} \log\Big(\frac{(\Delta t) \times N^{10}_{i,j}}{(N^{10}_{i,j} + N^{11}_{i,j})(N^{10}_{i,j} + N^{00}_{i,j})}\Big) + \frac{N^{11}_{i,j}}{\Delta t} \log\Big(\frac{(\Delta t) \times N^{11}_{i,j}}{(N^{11}_{i,j} + N^{10}_{i,j})(N^{11}_{i,j} + N^{01}_{i,j})}\Big) \end{aligned}$$

We now consider a graph of size $n = 500$ whose topology is modified over time in a prescribed way. The 1-delayed mutual information was computed using a sliding-window with $\Delta = 300$. Fig. 5 shows the time evolution of $w_{ij}(t)$ between two selected vertices of low degree on one side and between two selected hubs (vertices of high degree) on the other side. Between these pairs of vertices an edge was alternatively added and removed every 2000 time steps. In the case of low degree vertices, we see that the value of w_{ij} informs us of this change of topology. On the other hand, when the changes occur between the two hubs, $w_{ij}(t)$ does not detect them. However, in this case, it was found that the corresponding edge does not have a great influence on the system dynamics, using a similar analysis as the one reported in section 3.2.

If the link is modified between a vertex i of low degree and a hub j, w_{ij} hardly detects this change, as seen in Fig. 6 (right). This is expected as a terminal node does not influence a hub. But this change can be detected by measuring the opposite delayed mutual-information, w_{ji}, (see Fig 6, left panel), reflecting the fact that the hub does influence a neighboring vertex.

As mentioned before, the 1-delayed mutual information computed on a sliding window seems to be a good metric to detect when the link between two nodes appears or disappears. This is confirmed by simulations in which the graph topology changes randomly over time. At each iteration, a change can occur with probability p. A change corresponds to choosing randomly a pair of nodes and adding an edge between them (if none is present) or removing the existing one. For example, with a scale free graph with $n = 500$ nodes and a probability of change $p = 0.01$, 54 changes were observed over 6000 iterations. In this simulation all changes were detected because no changes between two hubs occured. For

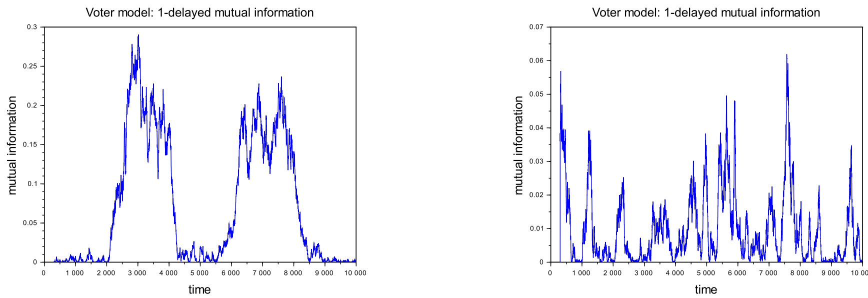

Fig. 5. Plots of the 1-delayed mutual information between two vertices of low degree (left) and between two hubs (right). The parameters of the simulation are $\epsilon = 0.01$, $n = 500$, $\Delta t = 300$.

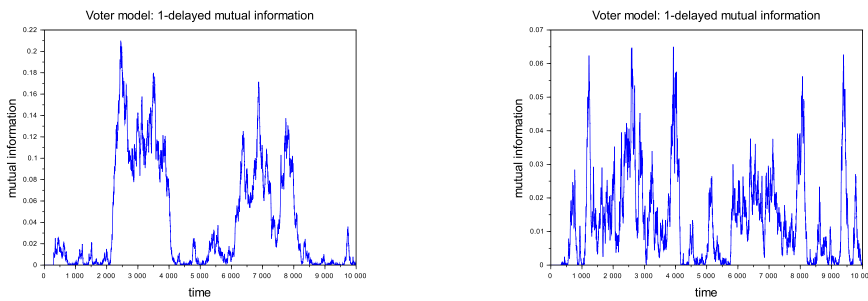

Fig. 6. Left: plot of the 1-delayed mutual information between a hub and a low degree vertex. Right: plot of the 1-delayed mutual information between the low degree vertex and the hub. The parameters of the simulation are $\epsilon = 0.01$, $n = 500$, $\Delta t = 300$.

each pair of nodes i and j, we computed $w_{i,j}$ and $w_{j,i}$, the $1-$delayed mutual information on a sliding window of size 100. When these two values exceed and remain above a threshold, we assume that a new edge has appeared. And when these values become lower than this same threshold, we assume that an edge has disappeared. With this method, the average time to detect changes is 160 iteration, with a standard deviation of 110.

5 Conclusions

In this paper, we have described a way to detect topology changes in a dynamical system on a graph, such as the voter model. Detecting structural changes is important as it can provide an early warning of tipping points. We expect that our approach can be applied to many other complex systems. The 1-delayed mutual information computed on a sliding-window was used to identify the possible changes of connectivity. We saw that this quantity allows us to detect whether an edge is added or removed between two vertices of low degree, or between a hub and a vertex of low degree. Between two hubs this method is not effective, but the actual presence or absence of a link between them does not affect much the global behavior of the system.

Delayed mutual- and multi-information can also be used to determine communities in a graph, as discussed in [7]. In a forthcoming study we will investigate, using the approach developed here, how a change of community can be detected in a dynamic system, thus indicating a possible loss of controllability.

References

1. Liu, Y.Y., Slotine, J.J., Barabasi, A.L.: Controllability of complex networks. Nature **473** (2011) 167
2. Liu, Y.Y., Slotine, J.J., Barabási, A.L.: Observability of complex systems. Proceedings of the National Academy of Sciences **110**(7) (2013) 2460–2465
3. Shannon, C.E.: A mathematical theory of communication. Bell system technical journal **27**(3) (1948) 379–423
4. Schreiber, T.: Measuring information transfer. Physical review letters **85**(2) (2000) 461
5. Toupance, P.A., Lefèvre, L., Chopard, B.: Controllability of the voter model: an information theoretic approach. J. Comp. Science, in revision; arXiv:1911.06540 (2019)
6. Antoulas, A.C.: Approximation of large-scale dynamical systems. Volume 6. Siam (2005)
7. Toupance, P.A., Chopard, B., Lefèvre, L.: Identification of complex network topologies through delayed mutual information. In: 21st IFAC World Congress, accepted; arXiv:2003.08792, Berlin, Germany (2020)
8. Castellano, C., Muñoz, M.A., Pastor-Satorras, R.: Nonlinear q-voter model. Physical Review E **80**(4) (2009) 041129
9. Nyczka, P., Sznajd-Weron, K.: Anticonformity or independence??insights from statistical physics. Journal of Statistical Physics **151**(1-2) (2013) 174–202
10. Mobilia, M., Petersen, A., Redner, S.: On the role of zealotry in the voter model. Journal of Statistical Mechanics: Theory and Experiment **2007**(08) (2007) P08029
11. Masuda, N.: Opinion control in complex networks. New Journal of Physics **17**(3) (2015) 033031
12. Galam, S., Chopard, B., Masselot, A., Droz, M.: Competing species dynamics: Qualitative advantage versus geography. Eur. Phys. J. B **4** (1998) 529–531
13. Barabási, A.L., Albert, R., Jeong, H.: Scale-free characteristics of random networks: the topology of the world-wide web. Physica A: statistical mechanics and its applications **281**(1-4) (2000) 69–77
14. Newman, M.E., Watts, D.J., Strogatz, S.H.: Random graph models of social networks. Proceedings of the National Academy of Sciences **99**(suppl 1) (2002) 2566–2572
15. Wu, A.Y., Garland, M., Han, J.: Mining scale-free networks using geodesic clustering. In: Proceedings of the tenth ACM SIGKDD international conference on Knowledge discovery and data mining, ACM (2004) 719–724
16. Bollobás, B., Riordan, O.M.: Mathematical results on scale-free random graphs. Handbook of graphs and networks: from the genome to the internet (2003) 1–34