

# A Tight Lower Bound for EDGE-DISJOINT PATHS on Planar DAGs\*

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## Abstract

Given a graph  $G$  and a set  $\mathcal{T} = \{(s_i, t_i) : 1 \leq i \leq k\}$  of  $k$  pairs, the VERTEX-DISJOINT PATHS (resp. EDGE-DISJOINT PATHS) problems asks to determine whether there exist pairwise vertex-disjoint (resp. edge-disjoint) paths  $P_1, P_2, \dots, P_k$  in  $G$  such that  $P_i$  connects  $s_i$  to  $t_i$  for each  $1 \leq i \leq k$ . Unlike their undirected counterparts which are FPT (parameterized by  $k$ ) from Graph Minor theory, both the edge-disjoint and vertex-disjoint versions in directed graphs were shown by Fortune et al. (TCS '80) to be NP-hard for  $k = 2$ . This strong hardness for DISJOINT PATHS on general directed graphs led to the study of parameterized complexity on special graph classes, e.g., when the underlying undirected graph is planar. For VERTEX-DISJOINT PATHS on planar directed graphs, Schrijver (SICOMP '94) designed an  $n^{O(k)}$  time algorithm which was later improved upon by Cygan et al. (FOCS '13) who designed an FPT algorithm running in  $2^{2^{O(k^2)}} \cdot n^{O(1)}$  time. To the best of our knowledge, the parameterized complexity of EDGE-DISJOINT PATHS on planar<sup>1</sup> directed graphs is unknown.

We resolve this gap by showing that EDGE-DISJOINT PATHS is W[1]-hard parameterized by the number  $k$  of terminal pairs, even when the input graph is a planar directed acyclic graph (DAG). This answers a question of Slivkins (ESA '03, SIDMA '10). Moreover, under the Exponential Time Hypothesis (ETH), we show that there is no  $f(k) \cdot n^{o(k)}$  algorithm for EDGE-DISJOINT PATHS on planar DAGs, where  $k$  is the number of terminal pairs,  $n$  is the number of vertices and  $f$  is any computable function. Our hardness holds even if both the maximum in-degree and maximum out-degree of the graph are at most 2.

We now place our result in the context of previously known algorithms and hardness for EDGE-DISJOINT PATHS on special classes of directed graphs:

- **Implications for EDGE-DISJOINT PATHS on DAGs:** Our result shows that the  $n^{O(k)}$  algorithm of Fortune et al. (TCS '80) for EDGE-DISJOINT PATHS on DAGs is asymptotically tight, even if we add an extra restriction of planarity. The previous best lower bound (also under ETH) for EDGE-DISJOINT PATHS on DAGs was  $f(k) \cdot n^{o(k/\log k)}$  by Amiri et al. (MFCS '16, IPL '19) which improved upon the  $f(k) \cdot n^{o(\sqrt{k})}$  lower bound implicit in Slivkins (ESA '03, SIDMA '10).
- **Implications for EDGE-DISJOINT PATHS on planar directed graphs:** As a special case of our result, we obtain that EDGE-DISJOINT PATHS on planar directed graphs is W[1]-hard parameterized by the number  $k$  of terminal pairs. This answers a question of Cygan et al. (FOCS '13) and Schrijver (pp. 417-444, Building Bridges II, '19), and completes the landscape (see Table 2) of the parameterized complexity status of edge and vertex versions of the DISJOINT PATHS problem on planar directed and planar undirected graphs.

## 1 Introduction

The DISJOINT PATHS problem is one of the most fundamental problems in graph theory: given a graph and a set of  $k$  terminal pairs, the question is to determine whether there exists a collection of  $k$  pairwise

\*A preliminary version of this paper appeared in CIAC 2021.

<sup>1</sup>A directed graph is planar if its underlying undirected graph is planar.

disjoint paths where each path connects one of the given terminal pairs. There are four natural variants of this problem depending on whether we consider undirected or directed graphs and the edge-disjoint or vertex-disjoint requirement. In undirected graphs, the edge-disjoint version is reducible to the vertex-disjoint version in polynomial time by considering the line graph. In directed graphs, the edge-disjoint version and vertex-disjoint version are known to be equivalent in terms of designing exact algorithms. Besides its theoretical importance, the DISJOINT PATHS problem has found applications in VLSI design, routing, etc. The interested reader is referred to the surveys [20] and [42, Chapter 9] for more details.

The case when the number of terminal pairs  $k$  are bounded is of special interest: given a graph with  $n$  vertices and  $k$  terminal pairs the goal is to try to design either FPT algorithms, i.e., algorithms whose running time is  $f(k) \cdot n^{O(1)}$  for some computable function  $f$ , or XP algorithms, i.e., algorithms whose running time is  $n^{g(k)}$  for some computable function  $g$ . We now discuss some of the known results on exact<sup>2</sup> algorithms for different variants of the DISJOINT PATHS problem before stating our result.

### Prior work on exact algorithms for DISJOINT PATHS on undirected graphs:

The NP-hardness for EDGE-DISJOINT PATHS and VERTEX-DISJOINT PATHS on undirected graphs was shown by Even et al. [16]. Solving the VERTEX-DISJOINT PATHS problem on undirected graphs is an important subroutine in checking whether a fixed graph  $H$  is a minor of a graph  $G$ . Hence, a core algorithmic result of the seminal work of Robertson and Seymour was their FPT algorithm [40] for VERTEX-DISJOINT PATHS (and hence also EDGE-DISJOINT PATHS) on general undirected graphs which runs in  $O(g(k) \cdot n^3)$  time for some function  $g$ . The cubic dependence on the input size was improved to quadratic by Kawarabayashi et al. [28] who designed an algorithm running in  $O(h(k) \cdot n^2)$  time for some function  $h$ . Both the functions  $g$  and  $h$  are quite large (at least quintuple exponential as per [2]). This naturally led to the search for faster FPT algorithms on planar graphs: Adler et al. [2] designed an algorithm for VERTEX-DISJOINT PATHS on planar graphs which runs in  $2^{2^{O(k^2)}} \cdot n^{O(1)}$  time. Very recently, this was improved to a single-exponential time FPT algorithm which runs in  $2^{O(k^2)} \cdot n^{O(1)}$  time by Lokshtanov et al. [32].

There are two more variants of the DISJOINT PATHS problem: the *half-integral* version where each vertex/edge can belong to at most two paths, and the *parity* version where the length of each path is required to respect a given parity (even or odd) condition. FPT algorithms are known for each of the following versions of VERTEX-DISJOINT PATHS on general undirected graphs: the half-integral version [24, 31], the half-integral version with parity [25] and finally just the parity version (without half-integral) [27].

### Prior work on exact algorithms for DISJOINT PATHS on directed graphs:

Unlike undirected graphs where both EDGE-DISJOINT PATHS and VERTEX-DISJOINT PATHS are FPT parameterized by  $k$ , the DISJOINT PATHS problem becomes significantly harder for directed graphs: Fortune et al. [19] showed that both EDGE-DISJOINT PATHS and VERTEX-DISJOINT PATHS on general directed graphs are NP-hard even for  $k = 2$ . For general directed graphs, Giannopoulou et al. [21] recently designed an XP algorithm for the half-integral version of DISJOINT PATHS: here the goal is to either find a set of  $k$  paths  $P_1, P_2, \dots, P_k$  such that  $P_i$  is an  $s_i \rightsquigarrow t_i$  path for each  $i \in [k]$  and each vertex in the graph appears in at most two of the paths, or conclude that the given instance has no solution with pairwise disjoint paths. This algorithm improves upon an older XP algorithm of Kawarabayashi et al. [26] for the quarter-integral case in general digraphs.

The DISJOINT PATHS problem has also been extensively studied on special subclasses of digraphs:

- **DISJOINT PATHS on DAGs:** It is easy to show that VERTEX-DISJOINT PATHS and EDGE-DISJOINT PATHS are equivalent on the class of directed acyclic graphs (DAGs). Fortune et al. [19]

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<sup>2</sup>This paper focuses on exact algorithms for the DISJOINT PATHS problem so we do not discuss here the results regarding (in)approximability.

designed an  $n^{O(k)}$  algorithm for EDGE-DISJOINT PATHS on DAGs. Slivkins [44] showed W[1]-hardness for EDGE-DISJOINT PATHS on DAGs and a  $f(k) \cdot n^{o(\sqrt{k})}$  lower bound (for any computable function  $f$ ) under the Exponential Time Hypothesis [22, 23] (ETH) follows from that reduction. Amiri et al. [3]<sup>3</sup> improved the lower bound to  $f(k) \cdot n^{o(k/\log k)}$  thus showing that the algorithm of Fortune et al. [19] is almost-tight.

- **DISJOINT PATHS on directed planar graphs:** Schrijver [41] designed an  $n^{O(k)}$  algorithm for VERTEX-DISJOINT PATHS on directed planar graphs. This was improved upon by Cygan et al. [12] who designed an FPT algorithm running in  $2^{2^{O(k^2)}} \cdot n^{O(1)}$  time. As pointed out by Cygan et al. [12], their FPT algorithm for VERTEX-DISJOINT PATHS on directed planar graphs does not work for the EDGE-DISJOINT PATHS problem. The status of parameterized complexity (parameterized by  $k$ ) of EDGE-DISJOINT PATHS on directed planar graphs remained an open question. Table 1 gives a summary of known results for exact algorithms for DISJOINT PATHS on (subclasses of) directed graphs.

Graph class	Problem type	Algorithm	Lower Bound
General graphs	Vertex-disjoint = edge-disjoint	????	NP-hard for $k = 2$
DAGs	Vertex-disjoint = edge-disjoint	$n^{O(k)}$ [19]	$f(k) \cdot n^{o(\sqrt{k})}$ [44] $f(k) \cdot n^{o(k/\log k)}$ [3] $f(k) \cdot n^{o(k)}$ [this paper]
Planar graphs	Vertex-disjoint	$n^{O(k)}$ [41] $2^{2^{O(k^2)}} \cdot n^{O(1)}$ [12]	????
	Edge-disjoint	????	$f(k) \cdot n^{o(k)}$ [this paper]
Planar DAGs	Vertex-disjoint	$2^{2^{O(k^2)}} \cdot n^{O(1)}$ [12]	????
	Edge-disjoint	$n^{O(k)}$ [19]	$f(k) \cdot n^{o(k)}$ [this paper]

**Table 1:** The landscape of parameterized complexity results for DISJOINT PATHS on directed graphs. All lower bounds are under the Exponential Time Hypothesis (ETH). To the best of our knowledge, the entries marked with ???? have no known non-trivial results.

### Our result:

We resolve this open question by showing a slightly stronger result: the EDGE-DISJOINT PATHS problem is W[1]-hard parameterized by  $k$  when the input graph is a planar DAG whose max in-degree and max out-degree are both at most 2. First we define the EDGE-DISJOINT PATHS problem formally below, and then state our result:

#### EDGE-DISJOINT PATHS

*Input:* A directed graph  $G = (V, E)$ , and a set  $\mathcal{T} \subseteq V \times V$  of  $k$  terminal pairs given by  $\{(s_i, t_i) : 1 \leq i \leq k\}$ .

*Question:* Do there exist  $k$  pairwise edge-disjoint paths  $P_1, P_2, \dots, P_k$  such that  $P_i$  is an  $s_i \rightsquigarrow t_i$  path for each  $1 \leq i \leq k$ ?

*Parameter:*  $k$

**Theorem 1.1.** *The EDGE-DISJOINT PATHS problem on planar DAGs is W[1]-hard parameterized by the number  $k$  of terminal pairs. Moreover, under ETH, the EDGE-DISJOINT PATHS problem on planar DAGs cannot be solved  $f(k) \cdot n^{o(k)}$  time where  $f$  is any computable function,  $n$  is the number of vertices and  $k$  is the number of terminal pairs. The hardness holds even if both the maximum in-degree and maximum out-degree of the graph are at most 2.*

<sup>3</sup>We note that [3] considers a more general version than DISJOINT PATHS which allows congestion

Recall that the Exponential Time Hypothesis (ETH) states that  $n$ -variable  $m$ -clause 3-SAT cannot be solved in  $2^{o(n)} \cdot (n+m)^{O(1)}$  time [22, 23]. Prior to our result, only the NP-completeness of EDGE-DISJOINT PATHS on planar DAGs was known [45]. The reduction used in [Theorem 1.1](#) is heavily inspired by some known reductions: in particular, the planar DAG structure ([Figure 2](#)) is from [6, 7] and the splitting operation ([Figure 3](#) and [Definition 2.4](#)) is from [4, 5]. We view the simplicity of our reduction as evidence of success of the (now) established methodology of showing W[1]-hardness (and ETH-based hardness) for planar graph problems using GRID-TILING and its variants.

**Placing [Theorem 1.1](#) in the context of prior work:**

[Theorem 1.1](#) answers a question of Slivkins [44] regarding the parameterized complexity of EDGE-DISJOINT PATHS on planar DAGs. As a special case of [Theorem 1.1](#), one obtains that EDGE-DISJOINT PATHS on planar directed graphs is W[1]-hard parameterized by the number  $k$  of terminal pairs: this answers a question of Cygan et al. [12] and Schrijver [43]. The W[1]-hardness result of [Theorem 1.1](#) completes the landscape (see [Table 2](#)) of parameterized complexity of edge-disjoint and vertex-disjoint versions of the DISJOINT PATHS problem on planar directed and planar undirected graphs. [Theorem 1.1](#) also shows that the  $n^{O(k)}$  algorithm of Fortune et al. [19] for EDGE-DISJOINT PATHS on DAGs is asymptotically optimal, even if we add an extra restriction of planarity to the mix. [Theorem 1.1](#) adds another problem (EDGE-DISJOINT PATHS on DAGs) to the relatively small list of problems for which it is provably known that the planar version has the same asymptotic complexity as the problem on general graphs: the only such other problems we are aware of are [5, 7, 38]. This is in contrast to the fact that for several problems [1, 14, 17, 18, 29, 30, 33, 34, 36, 38, 39], the planar version is easier by (roughly) a square root factor in the exponent as compared to general graphs, and there are lower bounds indicating that this improvement is essentially the best possible [35].

Graph class	Problem type	Parameterized Complexity parameterized by $k$
Planar undirected	Vertex-disjoint	FPT [2, 28, 32, 40]
	Edge-disjoint	
Planar directed	Vertex-disjoint	FPT [12]
	Edge-disjoint	W[1]-hard [ <a href="#">this paper</a> ]

**Table 2:** The landscape of parameterized complexity results for the four different versions (edge-disjoint vs vertex-disjoint & directed vs undirected) of DISJOINT PATHS on planar graphs.

**Organization of the paper:**

In [Section 2.1](#) we describe the construction of the instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS. The two directions of the reduction are shown in [Section 2.2](#) and [Section 2.3](#) respectively. Finally, [Section 2.4](#) contains the proof of [Theorem 1.1](#). We conclude with some open questions in [Section 3](#).

**Notation:**

All graphs considered in this paper are directed and do not have self-loops or multiple edges. We use (mostly) standard graph theory notation [15]. The set  $\{1, 2, 3, \dots, M\}$  is denoted by  $[M]$  for each  $M \in \mathbb{N}$ . A directed edge (resp. path) from  $s$  to  $t$  is denoted by  $s \rightarrow t$  (resp.  $s \rightsquigarrow t$ ). We use the **non-standard** notation (to avoid having to consider different cases in our proofs):  $s \rightsquigarrow s$  **does not** represent a self-loop but rather is to be viewed as “*just staying put*” at the vertex  $s$ . If  $A, B \subseteq V(G)$  then we say that there is an  $A \rightsquigarrow B$  path if and only if there exists two vertices  $a \in A, b \in B$  such that there is an  $a \rightsquigarrow b$  path. For  $A \subseteq V(G)$  we define  $N_G^+(A) = \{x \notin A : \exists y \in A \text{ such that } (y, x) \in E(G)\}$  and  $N_G^-(A) = \{x \notin A : \exists y \in A \text{ such that } (x, y) \in E(G)\}$ . For  $A \subseteq V(G)$  we define  $G[A]$  to be the graph induced on the vertex set  $A$ , i.e.,  $G[A] := (A, E_A)$  where  $E_A := E(G) \cap (A \times A)$ .

## 2 W[1]-hardness of EDGE-DISJOINT PATHS on Planar DAGs

To obtain W[1]-hardness for EDGE-DISJOINT PATHS on planar DAGs, we reduce from the GRID-TILING- $\leq$  problem [37] which is defined below:

<b>GRID-TILING-<math>\leq</math></b>		
<i>Input:</i> Integers $k, N$ , and a collection $\mathcal{S}$ of $k^2$ sets given by $\{S_{x,y} \subseteq [N] \times [N] : 1 \leq x, y \leq k\}$ .		
<i>Question:</i> For each $1 \leq x, y \leq k$ does there exist a pair $\gamma_{x,y} \in S_{x,y}$ such that		
<ul style="list-style-type: none"> <li>• if <math>\gamma_{x,y} = (a, b)</math> and <math>\gamma_{x+1,y} = (a', b')</math> then <math>b \leq b'</math>, and</li> <li>• if <math>\gamma_{x,y} = (a, b)</math> and <math>\gamma_{x,y+1} = (a', b')</math> then <math>a \leq a'</math></li> </ul>		

$S_{1,3}$	(1,1) (1,3) <b>(4,2)</b>	$S_{2,3}$	(1,5) <b>(5,2)</b> (3,5)	$S_{3,3}$	(1,1) <b>(4,5)</b> (3,3)
$S_{1,2}$	<b>(2,1)</b> (4,1)	$S_{2,2}$	(1,3) <b>(4,2)</b>	$S_{3,2}$	<b>(4,4)</b> (3,2)
$S_{1,1}$	(3,1) <b>(1,2)</b> (3,3)	$S_{2,1}$	(1,1) <b>(2,3)</b>	$S_{3,1}$	<b>(4,3)</b> (3,5)

**Figure 1:** An instance of GRID-TILING- $\leq$  with  $k = 3, N = 5$  and a solution highlighted in red. Note that in a solution, the second coordinates in a row are non-decreasing as we go from left to right and the first coordinates in a column are non-decreasing as we go from bottom to top.

Figure 1 gives an illustration of an instance of GRID-TILING- $\leq$  along with a solution. It is known [13, Theorem 14.30] that GRID-TILING- $\leq$  is W[1]-hard parameterized by  $k$ , and under the Exponential Time Hypothesis (ETH) has no  $f(k) \cdot N^{o(k)}$  algorithm for any computable function  $f$ . We will exploit this result by reducing an instance  $(k, N, \mathcal{S})$  of GRID-TILING- $\leq$  in  $\text{poly}(N, k)$  time to an instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS such that  $G_2$  is a planar DAG, number of vertices in  $G_2$  is  $|V(G_2)| = O(N^2 k^2)$  and number of terminal pairs is  $|\mathcal{T}| = 2k$ .

**Remark 2.1.** Our definition of GRID-TILING- $\leq$  above is slightly different than the one given in [13, Theorem 14.30]: there the constraints are first coordinate of  $\gamma_{x,y}$  is  $\leq$  first coordinate of  $\gamma_{x+1,y}$  and second coordinate of  $\gamma_{x,y}$  is  $\leq$  second coordinate of  $\gamma_{x,y+1}$ . By rotating the axis by  $90^\circ$ , i.e., swapping the indices, our version of GRID-TILING- $\leq$  is equivalent to that from [13, Theorem 14.30].

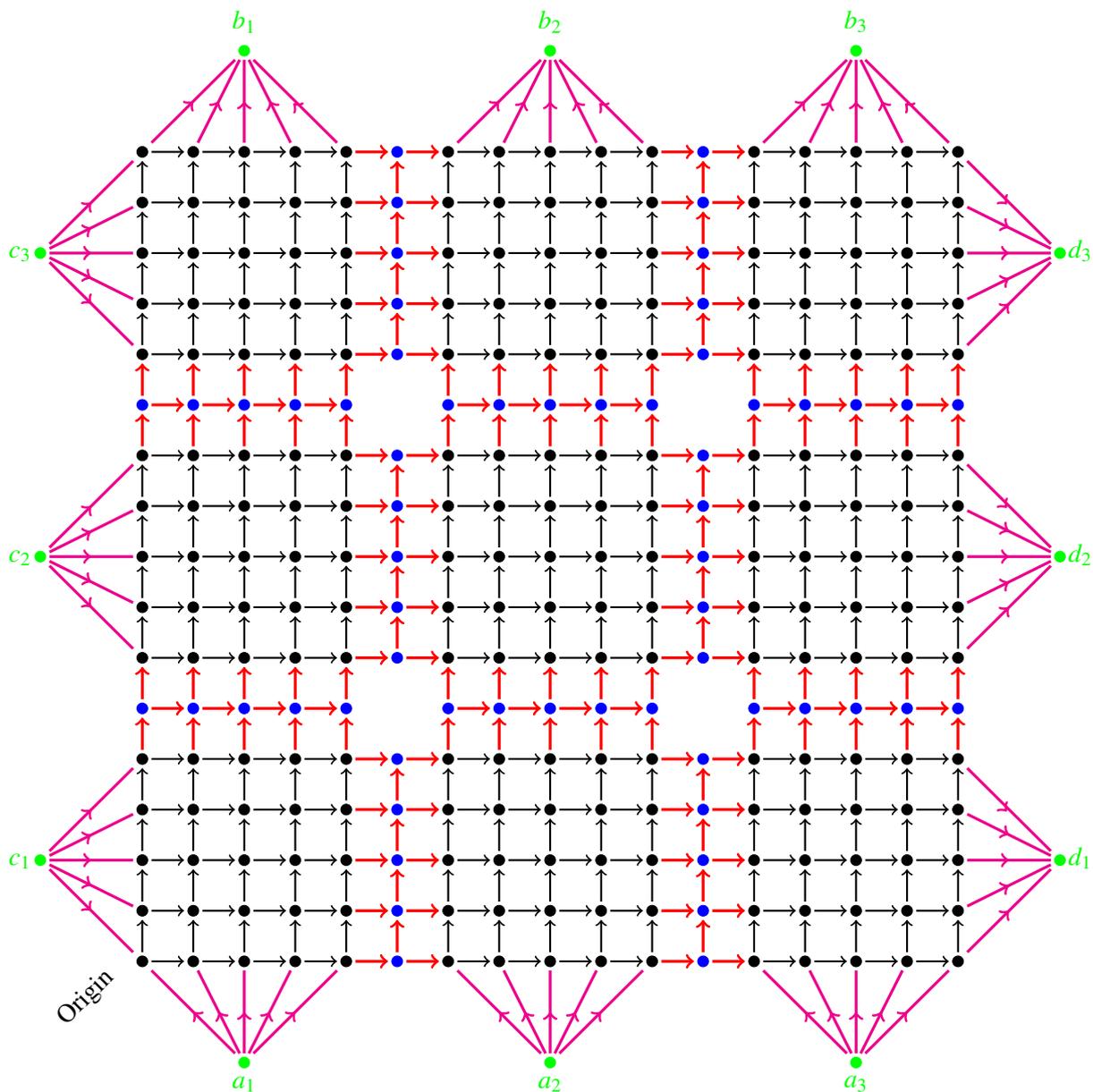
### 2.1 Construction of the instance $(G_2, \mathcal{T})$ of EDGE-DISJOINT PATHS

Consider an instance  $(N, k, \mathcal{S})$  of GRID-TILING- $\leq$ . We now build an instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS as follows: first in Section 2.1.1 we describe the construction of an intermediate graph  $G_1$  (Figure 2). The splitting operation is defined in Section 2.1.2, and the graph  $G_2$  is obtained from  $G_1$  by splitting each (black) grid vertex.

#### 2.1.1 Construction of the graph $G_1$

Given integers  $k$  and  $N$ , we build a directed graph  $G_1$  as follows (refer to Figure 2):

1. **Origin:** The origin is marked at the bottom left corner of Figure 2. This is defined just so we can view the naming of the vertices as per the usual  $X - Y$  coordinate system: increasing horizontally towards the right, and vertically towards the top.
2. **Grid (black) vertices and edges:** For each  $1 \leq i, j \leq k$  we introduce a (directed)  $N \times N$  grid  $G_{i,j}$  where the column numbers increase from 1 to  $N$  as we go from left to right, and the row numbers increase from 1 to  $N$  as we go from bottom to top. For each  $1 \leq q, \ell \leq N$  the unique vertex which



**Figure 2:** The graph  $G_1$  constructed for the input  $k = 3$  and  $N = 5$  via the construction described in Section 2.1.1. The final graph  $G_2$  for the EDGE-DISJOINT PATHS instance is obtained from  $G_1$  by the splitting operation (Definition 2.4) as described in Section 2.1.2.

is the intersection of the  $q^{\text{th}}$  column and  $\ell^{\text{th}}$  row of  $G_{i,j}$  is denoted by  $\mathbf{w}_{i,j}^{q,\ell}$ . The vertex set and edge set of  $G_{i,j}$  is defined formally as:

$$\begin{aligned} \bullet V(G_{i,j}) &= \{\mathbf{w}_{i,j}^{q,\ell} : 1 \leq q, \ell \leq N\} \\ \bullet E(G_{i,j}) &= \left( \bigcup_{(q,\ell) \in [N] \times [N-1]} \mathbf{w}_{i,j}^{q,\ell} \rightarrow \mathbf{w}_{i,j}^{q,\ell+1} \right) \cup \left( \bigcup_{(q,\ell) \in [N-1] \times [N]} \mathbf{w}_{i,j}^{q,\ell} \rightarrow \mathbf{w}_{i,j}^{q+1,\ell} \right) \end{aligned}$$

All vertices and edges of  $G_{i,j}$  are shown in Figure 2 using black color. Note that each horizontal edge of the grid  $G_{i,j}$  is oriented to the right, and each vertical edge is oriented towards the top. We will later (Definition 2.4) modify the grid  $G_{i,j}$  to represent the set  $S_{i,j}$ .

For each  $1 \leq i, j \leq k$  we define the set of boundary vertices of the grid  $G_{i,j}$  as follows:

$$\begin{aligned} \text{Left}(G_{i,j}) &:= \{\mathbf{w}_{i,j}^{1,\ell} : \ell \in [N]\} ; \text{Right}(G_{i,j}) := \{\mathbf{w}_{i,j}^{N,\ell} : \ell \in [N]\} \\ \text{Top}(G_{i,j}) &:= \{\mathbf{w}_{i,j}^{\ell,N} : \ell \in [N]\} ; \text{Bottom}(G_{i,j}) := \{\mathbf{w}_{i,j}^{\ell,1} : \ell \in [N]\} \end{aligned} \quad (1)$$

3. **Arranging the  $k^2$  different  $N \times N$  grids  $\{G_{i,j}\}_{1 \leq i, j \leq k}$  into a large  $k \times k$  grid:** We place the grids  $G_{i,j}$  into a big  $k \times k$  grid of grids left to right according to growing  $i$  and from bottom to top according to growing  $j$  (see the naming of the sets in Figure 1 in blue color). In particular, the grid  $G_{1,1}$  is at bottom left corner of the construction, the grid  $G_{k,k}$  at the top right corner, and so on.
4. **Blue vertices and red edges for horizontal connections:** For each  $(i, j) \in [k-1] \times [k]$  we add a set of vertices  $H_{i,j}^{i+1,j} := \{\mathbf{h}_{i,j}^{i+1,j}(\ell) : \ell \in [N]\}$  shown in Figure 2 using blue color. We also add the following three sets of edges (shown in Figure 2 using red color):
  - a directed path of  $N-1$  edges given by  $\text{Path}(H_{i,j}^{i+1,j}) := \{\mathbf{h}_{i,j}^{i+1,j}(\ell) \rightarrow \mathbf{h}_{i,j}^{i+1,j}(\ell+1) : \ell \in [N-1]\}$
  - a directed perfect matching from  $\text{Right}(G_{i,j})$  to  $H_{i,j}^{i+1,j}$  given by  $\text{Matching}(G_{i,j}, H_{i,j}^{i+1,j}) := \{\mathbf{w}_{i,j}^{N,\ell} \rightarrow \mathbf{h}_{i,j}^{i+1,j}(\ell) : \ell \in [N]\}$
  - a directed perfect matching from  $H_{i,j}^{i+1,j}$  to  $\text{Left}(G_{i+1,j})$  given by  $\text{Matching}(H_{i,j}^{i+1,j}, G_{i+1,j}) := \{\mathbf{h}_{i,j}^{i+1,j}(\ell) \rightarrow \mathbf{w}_{i+1,j}^{1,\ell} : \ell \in [N]\}$
5. **Blue vertices and red edges for vertical connections:** For each  $(i, j) \in [k] \times [k-1]$  we add a set of vertices  $V_{i,j}^{i,j+1} := \{\mathbf{v}_{i,j}^{i,j+1}(\ell) : \ell \in [N]\}$  shown in Figure 2 using blue color. We also add the following three sets of edges (shown in Figure 2 using red color):
  - a directed path of  $N-1$  edges given by  $\text{Path}(V_{i,j}^{i,j+1}) := \{\mathbf{v}_{i,j}^{i,j+1}(\ell) \rightarrow \mathbf{v}_{i,j}^{i,j+1}(\ell+1) : \ell \in [N-1]\}$
  - a directed perfect matching from  $\text{Top}(G_{i,j})$  to  $V_{i,j}^{i,j+1}$  given by  $\text{Matching}(G_{i,j}, V_{i,j}^{i,j+1}) := \{\mathbf{w}_{i,j}^{\ell,N} \rightarrow \mathbf{v}_{i,j}^{i,j+1}(\ell) : \ell \in [N]\}$
  - a directed perfect matching from  $V_{i,j}^{i,j+1}$  to  $\text{Bottom}(G_{i,j+1})$  given by  $\text{Matching}(V_{i,j}^{i,j+1}, G_{i,j+1}) := \{\mathbf{v}_{i,j}^{i,j+1}(\ell) \rightarrow \mathbf{w}_{i,j+1}^{\ell,1} : \ell \in [N]\}$
6. **Green (terminal) vertices and magenta edges:** For each  $i \in [k]$  we add the following four sets of (terminal) vertices (shown in Figure 2 using green color)

$$\begin{aligned} A &:= \{a_i : i \in [k]\} \quad ; \quad B := \{b_i : i \in [k]\} \\ C &:= \{c_i : i \in [k]\} \quad ; \quad D := \{d_i : i \in [k]\} \end{aligned} \quad (2)$$

For each  $i \in [k]$  we add the edges (shown in Figure 2 using magenta color)

$$\text{Source}(A) := \{a_i \rightarrow \mathbf{w}_{i,1}^{\ell,1} : \ell \in [N]\} ; \text{Sink}(B) := \{\mathbf{w}_{i,N}^{\ell,N} \rightarrow b_i : \ell \in [N]\} \quad (3)$$

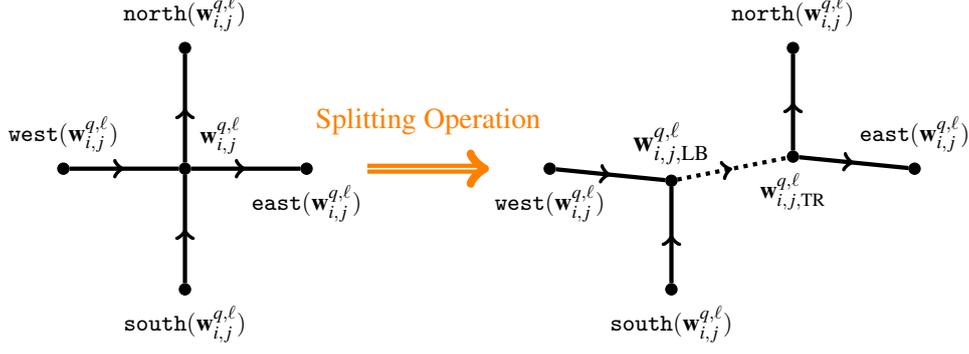
For each  $j \in [k]$  we add the edges (shown in Figure 2 using magenta color)

$$\text{Source}(C) := \{c_j \rightarrow \mathbf{w}_{1,j}^{1,\ell} : \ell \in [N]\} ; \text{Sink}(D) := \{\mathbf{w}_{N,j}^{N,\ell} \rightarrow d_j : \ell \in [N]\} \quad (4)$$

This completes the construction of the graph  $G_1$  (see Figure 2).

**Claim 2.2.**  $G_1$  is a planar DAG

*Proof.* Figure 2 gives a planar embedding of  $G_1$ . It is easy to verify from the construction of  $G_1$  described at the start of Section 2.1.1 (see also Figure 2) that  $G_1$  is a DAG.  $\square$



**Figure 3:** The splitting operation for the vertex  $w_{i,j}^{q,\ell}$  when  $(q,\ell) \notin S_{i,j}$ . The idea behind this splitting is if we want edge-disjoint paths then we can go **either** left-to-right or bottom-to-top but not in **both** directions. On the other hand, if  $(q,\ell) \in S_{i,j}$  then the picture on the right-hand side (after the splitting operation) would look exactly like that on the left-hand side.

### 2.1.2 Obtaining the graph $G_2$ from $G_1$ via the splitting operation

Observe (see Figure 2) that every (black) grid vertex in  $G_1$  has in-degree two and out-degree two. Moreover, the two in-neighbors and two out-neighbors do not appear alternately. For each (black) grid vertex  $z \in G_1$  we set up the notation:

**Definition 2.3. (four neighbors of each grid vertex in  $G_1$ )** For each (black) grid vertex  $z \in G_1$  we define the following four vertices

- $\text{west}(z)$  is the vertex to the left of  $z$  (as seen by the reader) which has an edge incoming into  $z$
- $\text{south}(z)$  is the vertex below  $z$  (as seen by the reader) which has an edge incoming into  $z$
- $\text{east}(z)$  is the vertex to the right of  $z$  (as seen by the reader) which has an edge outgoing from  $z$
- $\text{north}(z)$  is the vertex above  $z$  (as seen by the reader) which has an edge outgoing from  $z$

We now define the splitting operation which allows us to obtain the graph  $G_2$  from the graph  $G_1$  constructed in Section 2.1.1.

**Definition 2.4. (splitting operation)** For each  $i, j \in [k]$  and each  $q, \ell \in [N]$

- If  $(q,\ell) \notin S_{i,j}$ , then we **split** the vertex  $w_{i,j}^{q,\ell}$  into two **distinct** vertices  $w_{i,j}^{q,\ell, LB}$  and  $w_{i,j}^{q,\ell, TR}$  and add the edge  $w_{i,j}^{q,\ell, LB} \rightarrow w_{i,j}^{q,\ell, TR}$  (denoted by the dotted edge in Figure 3). The 4 edges (see Definition 2.3) incident on  $w_{i,j}^{q,\ell}$  are now changed as follows (see Figure 3):
  - Replace the edge  $\text{west}(w_{i,j}^{q,\ell}) \rightarrow w_{i,j}^{q,\ell}$  by the edge  $\text{west}(w_{i,j}^{q,\ell}) \rightarrow w_{i,j}^{q,\ell, LB}$
  - Replace the edge  $\text{south}(w_{i,j}^{q,\ell}) \rightarrow w_{i,j}^{q,\ell}$  by the edge  $\text{south}(w_{i,j}^{q,\ell}) \rightarrow w_{i,j}^{q,\ell, LB}$
  - Replace the edge  $w_{i,j}^{q,\ell} \rightarrow \text{east}(w_{i,j}^{q,\ell})$  by the edge  $w_{i,j}^{q,\ell, TR} \rightarrow \text{east}(w_{i,j}^{q,\ell})$
  - Replace the edge  $w_{i,j}^{q,\ell} \rightarrow \text{north}(w_{i,j}^{q,\ell})$  by the edge  $w_{i,j}^{q,\ell, TR} \rightarrow \text{north}(w_{i,j}^{q,\ell})$
- Otherwise, if  $(q,\ell) \in S_{i,j}$  then the vertex  $w_{i,j}^{q,\ell}$  is **not split**, and we define  $w_{i,j}^{q,\ell, LB} = w_{i,j}^{q,\ell} = w_{i,j}^{q,\ell, TR}$ . Note that the four edges (Definition 2.3) incident on  $w_{i,j}^{q,\ell}$  are unchanged.

**Remark 2.5.** To avoid case distinctions in the forthcoming proof of correctness of the reduction, we will use the following non-standard notation: the edge  $s \rightsquigarrow s$  **does not** represent a self-loop but rather is to be viewed as “*just staying put*” at the vertex  $s$ . Note that this does not affect edge-disjointness.

We are now ready to define the graph  $G_2$  and the set  $\mathcal{T}$  of terminal pairs:

**Definition 2.6.** The graph  $G_2$  is obtained by applying the splitting operation (Definition 2.4) to each (black) grid vertex of  $G_1$ , i.e., the set of vertices given by  $\bigcup_{1 \leq i, j \leq k} V(G_{i,j})$ . The set of terminal pairs is  $\mathcal{T} := \{(a_i, b_i) : i \in [k]\} \cup \{(c_j, d_j) : j \in [k]\}$

Note that in  $G_2$  we have

- All vertices in  $G_2$  except  $A \cup C$  have out-degree at most 2
- All vertices in  $G_2$  except  $B \cup D$  have in-degree at most 2

We will later show (see last paragraph in the proof of [Theorem 1.1](#)) how to edit  $G_2$  such that each vertex has both in-degree and out-degree at most 2. The next claim shows that  $G_2$  is also both planar and acyclic (like  $G_1$ ).

**Claim 2.7.**  $G_2$  is a planar DAG

*Proof.* In [Claim 2.2](#), we have shown that  $G_1$  is a planar DAG. By [Definition 2.6](#),  $G_2$  is obtained from  $G_1$  by applying the splitting operation ([Definition 2.4](#)) on every (black) grid vertex, i.e., every vertex from the set  $\bigcup_{1 \leq i, j \leq k} V(G_{i,j})$ .

By [Definition 2.3](#), every vertex of  $G_1$  that is split has exactly two in-neighbors and two out-neighbors in  $G_1$ . Hence, it is easy to see ([Figure 3](#)) that the splitting operation ([Definition 2.4](#)) does not destroy planarity when we construct  $G_2$  from  $G_1$ . Since  $G_1$  is a DAG, replacing each split (black) grid vertex  $w$  in  $G_1$  by  $w_{LB}$  followed by  $w_{TR}$  in the topological order of  $G_1$  gives a topological order for  $G_2$ . Hence,  $G_2$  is a planar DAG.  $\square$

We now set up notation for the grids in  $G_2$ :

**Definition 2.8.** For each  $i, j \in [k]$ , we define  $G_{i,j}^{\text{split}}$  to be the graph obtained by applying the splitting operation ([Definition 2.4](#)) to each vertex of  $G_{i,j}$ . For each  $i, j \in [k]$  and each  $q, \ell \in [N]$  we define  $\text{split}(w_{i,j}^{q,\ell}) := \{w_{i,j,LB}^{q,\ell}, w_{i,j,TR}^{q,\ell}\}$ .

## 2.2 Solution for EDGE-DISJOINT PATHS $\Rightarrow$ Solution for GRID-TILING- $\leq$

In this section, we show that if the instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS has a solution then the instance  $(k, N, \mathcal{S})$  of GRID-TILING- $\leq$  also has a solution.

Suppose that the instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS has a solution, i.e., there is a collection of  $2k$  pairwise edge-disjoint paths  $\{P_1, P_2, \dots, P_k, Q_1, Q_2, \dots, Q_k\}$  in  $G_2$  such that

$$\begin{aligned} P_i &\text{ is an } a_i \rightsquigarrow b_i \text{ path } \forall i \in [k] \\ Q_j &\text{ is an } c_j \rightsquigarrow d_j \text{ path } \forall j \in [k] \end{aligned} \tag{5}$$

To streamline the arguments of this section, we define the following subsets of vertices of  $G_2$ :

**Definition 2.9. (horizontal & vertical levels)**

For each  $j \in [k]$ , we define the following set of vertices:

$$\text{HORIZONTAL}(j) = \{c_j, d_j\} \cup \left( \bigcup_{i=1}^k V(G_{i,j}^{\text{split}}) \right) \cup \left( \bigcup_{i=1}^{k-1} H_{i,j}^{i+1,j} \right)$$

For each  $i \in [k]$ , we define the following set of vertices:

$$\text{VERTICAL}(i) = \{a_i, b_i\} \cup \left( \bigcup_{j=1}^k V(G_{i,j}^{\text{split}}) \right) \cup \left( \bigcup_{j=1}^{k-1} V_{i,j}^{i,j+1} \right)$$

From [Definition 2.9](#), it is easy to verify that  $\text{VERTICAL}(i) \cap \text{VERTICAL}(i') = \emptyset = \text{HORIZONTAL}(i) \cap \text{HORIZONTAL}(i')$  for every  $1 \leq i \neq i' \leq k$ .

**Definition 2.10. (boundary vertices in  $G_2$ )** For each  $1 \leq i, j \leq k$  we define the set of boundary vertices of the grid  $G_{i,j}^{\text{split}}$  in the graph  $G_2$  as follows:

$$\begin{aligned} \text{Left}(G_{i,j}^{\text{split}}) &:= \{w_{i,j,LB}^{1,\ell} : \ell \in [N]\}; \text{Right}(G_{i,j}^{\text{split}}) := \{w_{i,j,TR}^{N,\ell} : \ell \in [N]\} \\ \text{Top}(G_{i,j}^{\text{split}}) &:= \{w_{i,j,TR}^{\ell,N} : \ell \in [N]\}; \text{Bottom}(G_{i,j}^{\text{split}}) := \{w_{i,j,LB}^{\ell,1} : \ell \in [N]\} \end{aligned} \tag{6}$$

**Lemma 2.11.** For each  $i \in [k]$  the path  $P_i$  satisfies the following two structural properties:

- every edge of the path  $P_i$  has both end-points in  $\text{VERTICAL}(i)$
- $P_i$  contains an  $\text{Bottom}(G_{i,j}^{\text{split}}) \rightsquigarrow \text{Top}(G_{i,j}^{\text{split}})$  path for each  $j \in [k]$ .

*Proof.* For this proof, define  $H_{0,j}^{1,j} := \{c_j\}$  and  $H_{k,j}^{k+1,j} := \{d_j\}$  for each  $j \in [k]$ .

Fix any  $i^* \in [k]$ . Note that  $P_{i^*}$  is an  $a_{i^*} \rightsquigarrow b_{i^*}$  path and hence starts and ends at a vertex in  $\text{VERTICAL}(i^*)$ . We now prove the first part of lemma by showing two claims which state that  $P_{i^*}$  cannot contain any vertex of  $N_{G_2}^+(\text{VERTICAL}(i^*))$  and  $N_{G_2}^-(\text{VERTICAL}(i^*))$  respectively.

**Claim 2.12.**  $P_{i^*}$  does not contain any vertex of  $N_{G_2}^+(\text{VERTICAL}(i^*))$ .

*Proof.* The structure of  $G_2$  implies that

- $N_{G_2}^+(\text{VERTICAL}(i)) = \bigcup_{j=1}^k H_{i,j}^{i+1,j}$  for each  $i \in [k]$
- $N_{G_2}^+(\bigcup_{j=1}^k H_{i,j}^{i+1,j}) \subseteq \text{VERTICAL}(i+1)$  for each  $0 \leq i \leq k-1$
- $N_{G_2}^+(\bigcup_{j=1}^k H_{k,j}^{k+1,j}) = \emptyset$  since each vertex of  $D$  is a sink in  $G_2$

Hence, if  $P_{i^*}$  contains a vertex from  $N_{G_2}^+(\text{VERTICAL}(i^*))$  then it cannot ever return back to  $\text{VERTICAL}(i^*)$  which contradicts the fact that the last vertex of  $P_{i^*}$  is  $b_{i^*} \in \text{VERTICAL}(i^*)$ .  $\square$

**Claim 2.13.**  $P_{i^*}$  does not contain any vertex of  $N_{G_2}^-(\text{VERTICAL}(i^*))$ .

*Proof.* The structure of  $G_2$  implies that

- $N_{G_2}^-(\text{VERTICAL}(i)) = \bigcup_{j=1}^k H_{i-1,j}^{i,j}$  for each  $i \in [k]$
- $N_{G_2}^-(\bigcup_{j=1}^k H_{i-1,j}^{i,j}) \subseteq \text{VERTICAL}(i)$  for each  $1 \leq i \leq k$
- $N_{G_2}^-(\bigcup_{j=1}^k H_{0,j}^{1,j}) = \emptyset$  since each vertex of  $C$  is a source in  $G_2$

Hence, if  $P_{i^*}$  contains a vertex from  $N_{G_2}^-(\text{VERTICAL}(i^*))$  then  $P_{i^*}$  cannot have started at a vertex of  $\text{VERTICAL}(i^*)$  which contradicts the fact that the first vertex of  $P_{i^*}$  is  $a_{i^*} \in \text{VERTICAL}(i^*)$ .  $\square$

This concludes the proof of the first part of the lemma. We now show the second part of the lemma. We define  $V_{i^*,0}^{i^*,1} := \{a_{i^*}\}$  and  $V_{i^*,k}^{i^*,k+1} := \{b_{i^*}\}$ . The structure of  $G_2$  implies that

- $N_{G_2[\text{VERTICAL}(i^*)]}^+(G_{i^*,j}^{\text{split}}) = V_{i^*,j}^{i^*,j+1}$  and  $N_{G_2[\text{VERTICAL}(i^*)]}^-(G_{i^*,j}^{\text{split}}) = V_{i^*,j-1}^{i^*,j}$  for each  $j \in [k]$
- $N_{G_2[\text{VERTICAL}(i^*)]}^+(V_{i^*,j}^{i^*,j+1}) = \text{Bottom}(G_{i^*,j+1}^{\text{split}})$  for each  $0 \leq j \leq k-1$
- $N_{G_2[\text{VERTICAL}(i^*)]}^-(V_{i^*,j}^{i^*,j+1}) = \text{Top}(G_{i^*,j}^{\text{split}})$  for each  $1 \leq j \leq k$

These three relations, combined with the first part of the lemma which states that  $P_{i^*}$  lies within  $G_2[\text{VERTICAL}(i^*)]$ , implies that  $P_{i^*}$  contains an  $\text{Bottom}(G_{i^*,j}^{\text{split}}) \rightsquigarrow \text{Top}(G_{i^*,j}^{\text{split}})$  path for each  $j \in [k]$ . This concludes the proof of [Lemma 2.11](#).  $\square$

The proof of the next lemma is very similar to that of [Lemma 2.11](#), and we skip repeating the details.

**Lemma 2.14.** For each  $j \in [k]$  the path  $Q_j$  satisfies the following two structural properties:

- every edge of the path  $Q_j$  has both end-points in  $\text{HORIZONTAL}(j)$
- $Q_j$  contains an  $\text{Left}(G_{i,j}^{\text{split}}) \rightsquigarrow \text{Right}(G_{i,j}^{\text{split}})$  path for each  $i \in [k]$

**Lemma 2.15.** For any  $(i, j) \in [k] \times [k]$ , let  $P', Q'$  be any  $\text{Bottom}(G_{i,j}^{\text{split}}) \rightsquigarrow \text{Top}(G_{i,j}^{\text{split}})$ ,  $\text{Left}(G_{i,j}^{\text{split}}) \rightsquigarrow \text{Right}(G_{i,j}^{\text{split}})$  paths in  $G_2$  respectively. If  $P'$  and  $Q'$  are edge-disjoint then there exists  $(\mu, \delta) \in S_{i,j}$  such that the vertex  $\mathbf{w}_{i,j,\text{LB}}^{\mu,\delta} = \mathbf{w}_{i,j}^{\mu,\delta} = \mathbf{w}_{i,j,\text{TR}}^{\mu,\delta}$  belongs to both  $P'$  and  $Q'$

*Proof.* Let  $P'', Q''$  be the paths obtained from  $P', Q'$  by contracting all the dotted edges on  $P', Q'$  respectively. By the construction of  $G_2$  ([Definition 2.6](#)) and the splitting operation ([Definition 2.4](#)), it follows that  $P'', Q''$  are  $\text{Bottom}(G_{i,j}) \rightsquigarrow \text{Top}(G_{i,j})$ ,  $\text{Left}(G_{i,j}) \rightsquigarrow \text{Right}(G_{i,j})$  paths in  $G_1$  respectively. Hence, there exist  $x_1, x_2 \in [N]$  such that  $P''$  is a  $\mathbf{w}_{i,j}^{x_1,1} \rightarrow \mathbf{w}_{i,j}^{x_2,N}$  path and  $y_1, y_2 \in [N]$  such that  $Q''$  is a  $\mathbf{w}_{i,j}^{1,y_1} \rightarrow \mathbf{w}_{i,j}^{N,y_2}$  path. We now show that  $P''$  and  $Q''$  must intersect in  $G_1$

**Claim 2.16.**  $P''$  and  $Q''$  have a common vertex in  $G_1$

*Proof.* For each  $x \in [N]$  such that  $x_1 \leq x \leq x_2$  define  $P''(x) = \{y \in [N] : \mathbf{w}_{i,j}^{x,y} \in P''\}$ . For each  $x \in [N]$  such that  $x_1 \leq x \leq x_2$  define  $Q''(x) = \{y \in [N] : \mathbf{w}_{i,j}^{x,y} \in Q''\}$ . We will prove the claim by showing that there exists  $x^*, y^* \in [N]$  such that  $y^* \in (P''(x^*) \cap Q''(x^*))$ . By the orientation of the edges in  $G_{i,j}$ , it follows that

$$\begin{aligned} \max P''(z) &= \min P''(z+1) \text{ and } \max Q''(z) = \min Q''(z+1) \quad \forall x_1 \leq z < x_2 \\ \text{If } 1 \leq u \leq z \leq N \text{ then } \max P''(u) &\leq \min P''(z) \text{ and } \max Q''(u) \leq \min Q''(z) \end{aligned} \quad (7)$$

By definition of  $Q''$ , we have  $y_1 \in Q''(1)$  and hence  $y \geq y_1 \geq 1$  for each  $y \in Q''(x_1)$ . If  $(P''(x_1) \cap Q''(x_1)) \neq \emptyset$  then we are done. Otherwise, we have that  $\min Q''(x_1) > \max P''(x_1)$  since  $1 \in P''(x_1)$ . Now if  $(P''(x_1+1) \cap Q''(x_1+1)) \neq \emptyset$  then we are done. Otherwise, we have  $\min Q''(x_1+1) > \max P''(x_1+1)$  since  $\min Q''(x_1+1) = \max Q''(x_1)$ . Continuing this way, we must find an  $x^* \in \mathbb{N}$  such that  $x_1 \leq x^* \leq x_2$  and  $(P''(x^*) \cap Q''(x^*)) \neq \emptyset$ : this is because  $N \in P''(x_2)$  and hence  $\min Q''(x_2) \leq N = \max P''(x_2)$ . Since  $(P''(x^*) \cap Q''(x^*)) \neq \emptyset$  let  $y^* \in (P''(x^*) \cap Q''(x^*))$ , i.e., the vertex  $\mathbf{w}_{i,j}^{x^*,y^*}$  belongs to both  $P''$  and  $Q''$ .  $\square$

By [Claim 2.16](#), the paths  $P'', Q''$  have a common vertex in  $G_1$ . Let this vertex be  $\mathbf{w}_{i,j}^{\mu,\delta}$ . Viewing the paths  $P'', Q''$  in  $G_2$ , i.e., “un-contracting” the dotted edges ([Definition 2.4](#)), it follows that both  $P'$  and  $Q'$  share the dotted edge  $\mathbf{w}_{i,j}^{\mu,\delta, \text{LB}} \rightarrow \mathbf{w}_{i,j}^{\mu,\delta, \text{TR}}$ . Since  $P'$  and  $Q'$  are given to be edge-disjoint, this implies that the edge  $\mathbf{w}_{i,j}^{\mu,\delta, \text{LB}} \rightarrow \mathbf{w}_{i,j}^{\mu,\delta, \text{TR}}$  cannot exist in  $G_2$ , i.e.,  $(\mu, \delta) \in S_{i,j}$  and the vertex  $\mathbf{w}_{i,j}^{\mu,\delta} = \mathbf{w}_{i,j}^{\mu,\delta} = \mathbf{w}_{i,j, \text{TR}}^{\mu,\delta}$  belongs to both  $P'$  and  $Q'$  (recall [Definition 2.4](#)). This concludes the proof of [Lemma 2.15](#).  $\square$

**Lemma 2.17.** The instance  $(k, N, \mathcal{S})$  of GRID-TILING- $\leq$  has a solution.

*Proof.* Fix any  $(i, j) \in [k] \times [k]$ . By [Lemma 2.11](#),  $P_i$  contains an  $\text{Bottom}(G_{i,j}^{\text{split}}) \rightsquigarrow \text{Top}(G_{i,j}^{\text{split}})$  path say  $P_{i,j}$ . By [Lemma 2.14](#),  $Q_j$  contains an  $\text{Left}(G_{i,j}^{\text{split}}) \rightsquigarrow \text{Right}(G_{i,j}^{\text{split}})$  path say  $Q_{i,j}$ . Since  $P_i$  and  $Q_j$  are edge-disjoint ([Equation 5](#)), it follows that the paths  $P_{i,j}$  and  $Q_{i,j}$  are also edge-disjoint. Applying [Lemma 2.15](#) to the paths  $P_{i,j}$  and  $Q_{i,j}$  we get that there exists  $(\mu_{i,j}, \delta_{i,j}) \in [N] \times [N]$  such that  $(\mu_{i,j}, \delta_{i,j}) \in S_{i,j}$  and the vertex  $\mathbf{w}_{i,j, \text{LB}}^{\mu_{i,j}, \delta_{i,j}} = \mathbf{w}_{i,j}^{\mu_{i,j}, \delta_{i,j}} = \mathbf{w}_{i,j, \text{TR}}^{\mu_{i,j}, \delta_{i,j}}$  belongs to  $P_{i,j}$  (and hence also to  $P_i$ ) and  $Q_{i,j}$  (and hence also to  $Q_j$ ).

We now claim that the values  $\{(\mu_{i,j}, \delta_{i,j}) : (i, j) \in [k] \times [k]\}$  form a solution for the instance  $(k, N, \mathcal{S})$  of GRID-TILING- $\leq$ . In the last paragraph, we have already shown that  $(\mu_{i,j}, \delta_{i,j}) \in S_{i,j}$  for each  $(i, j) \in [k] \times [k]$ . For each  $(i, j) \in [k-1] \times [k]$  both the vertices  $\mathbf{w}_{i,j, \text{LB}}^{\mu_{i,j}, \delta_{i,j}} = \mathbf{w}_{i,j, \text{TR}}^{\mu_{i,j}, \delta_{i,j}}$  and  $\mathbf{w}_{i+1, j, \text{LB}}^{\mu_{i+1, j}, \delta_{i+1, j}} = \mathbf{w}_{i+1, j, \text{TR}}^{\mu_{i+1, j}, \delta_{i+1, j}}$  belong to the path  $Q_j$  which is contained in  $G_2[\text{HORIZONTAL}(j)]$  ([Lemma 2.14](#)). Hence, by the orientation of the edges in  $G_2$ , it follows that  $\delta_{i,j} \leq \delta_{i+1, j}$ . Similarly, it can be shown that  $\mu_{i,j} \leq \mu_{i, j+1}$  for each  $(i, j) \in [k] \times [k-1]$ .  $\square$

### 2.3 Solution for GRID-TILING- $\leq \Rightarrow$ Solution for EDGE-DISJOINT PATHS

In this section, we show that if the instance  $(k, N, \mathcal{S})$  of GRID-TILING- $\leq$  has a solution then the instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS also has a solution.

Suppose that the instance  $(k, N, \mathcal{S})$  of GRID-TILING- $\leq$  has a solution given by the pairs  $\{(\alpha_{i,j}, \beta_{i,j}) : i, j \in [k]\}$ . Hence, we have

$$\begin{aligned} (\alpha_{i,j}, \beta_{i,j}) &\in S_{i,j} \quad \text{for each } (i, j) \in [k] \times [k] \\ \alpha_{i,j} &\leq \alpha_{i, j+1} \quad \text{for each } (i, j) \in [k] \times [k-1] \\ \beta_{i,j} &\leq \beta_{i+1, j} \quad \text{for each } (i, j) \in [k-1] \times [k] \end{aligned} \quad (8)$$

**Definition 2.18. (row-paths and column-paths in  $G_2$ )** For each  $(i, j) \in [k] \times [k]$  and  $\ell \in [N]$  we define

- $\text{RowPath}_\ell(G_{i,j}^{\text{split}})$  to be the  $\mathbf{w}_{i,j, \text{LB}}^{1, \ell} \rightsquigarrow \mathbf{w}_{i,j, \text{TR}}^{N, \ell}$  path in  $G_2[G_{i,j}^{\text{split}}]$  consisting of the following edges (in order): for each  $r \in [N-1]$ 
  - $\mathbf{w}_{i,j, \text{LB}}^{r, \ell} \rightarrow \mathbf{w}_{i,j, \text{TR}}^{r, \ell}$  and  $\mathbf{w}_{i,j, \text{TR}}^{r, \ell} \rightarrow \mathbf{w}_{i,j, \text{LB}}^{r+1, \ell}$
- followed finally by the edge  $\mathbf{w}_{i,j, \text{LB}}^{N, \ell} \rightarrow \mathbf{w}_{i,j, \text{TR}}^{N, \ell}$

- $\text{ColumnPath}_\ell(G_{i,j}^{\text{split}})$  to be the  $\mathbf{w}_{i,j,\text{LB}}^{\ell,1} \rightsquigarrow \mathbf{w}_{i,j,\text{TR}}^{\ell,N}$  path in  $G_2$  consisting of the following edges (in order): for each  $r \in [N-1]$ 
  - $\mathbf{w}_{i,j,\text{LB}}^{\ell,r} \rightarrow \mathbf{w}_{i,j,\text{TR}}^{\ell,r}$  and  $\mathbf{w}_{i,j,\text{TR}}^{\ell,r} \rightarrow \mathbf{w}_{i,j,\text{LB}}^{\ell,r+1}$
followed finally by the edge  $\mathbf{w}_{i,j,\text{LB}}^{\ell,N} \rightarrow \mathbf{w}_{i,j,\text{TR}}^{\ell,N}$

Using the special types of paths from [Definition 2.18](#), we can now show the following lemma:

**Lemma 2.19.** *The instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS has a solution.*

*Proof.* We build a collection of  $2k$  paths  $\mathcal{P} := \{R_1, R_2, \dots, R_k, T_1, T_2, \dots, T_k\}$  and show that it forms a solution for the instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS. First, we describe this collection of paths below:

**- Description of the set of paths  $\{R_1, R_2, \dots, R_k\}$  :**

For each  $i \in [k]$ , we build the path  $R_i$  as follows:

- Start with the edge  $a_i \rightarrow \mathbf{w}_{i,1,\text{LB}}^{\alpha_{i,1},1}$
- For each  $j \in [k-1]$  use the  $\mathbf{w}_{i,j,\text{LB}}^{\alpha_{i,j},1} \rightsquigarrow \mathbf{w}_{i,j+1,\text{LB}}^{\alpha_{i,j+1},1}$  path obtained by concatenating
  - the  $\mathbf{w}_{i,j,\text{LB}}^{\alpha_{i,j},1} \rightsquigarrow \mathbf{w}_{i,j,\text{TR}}^{\alpha_{i,j},N}$  path  $\text{ColumnPath}_{\alpha_{i,j}}(G_{i,j}^{\text{split}})$  from [Definition 2.18](#)
  - the  $\mathbf{w}_{i,j,\text{TR}}^{\alpha_{i,j},N} \rightsquigarrow \mathbf{w}_{i,j+1,\text{LB}}^{\alpha_{i,j+1},1}$  path  $\mathbf{w}_{i,j,\text{TR}}^{\alpha_{i,j},N} \rightarrow \mathbf{v}_{i,j}^{\alpha_{i,j},1}(\alpha_{i,j}) \rightarrow \dots \rightarrow \mathbf{v}_{i,j}^{\alpha_{i,j},1}(\alpha_{i,j+1}) \rightarrow \mathbf{w}_{i,j+1,\text{LB}}^{\alpha_{i,j+1},1}$  which exists since [Equation 8](#) implies  $\alpha_{i,j} \leq \alpha_{i,j+1}$ .
- Now, we have reached the vertex  $\mathbf{w}_{i,k,\text{LB}}^{\alpha_{i,k},1}$ . Use the  $\mathbf{w}_{i,k,\text{LB}}^{\alpha_{i,k},1} \rightsquigarrow \mathbf{w}_{i,k,\text{TR}}^{\alpha_{i,k},N}$  path  $\text{ColumnPath}_{\alpha_{i,k}}(G_{i,k}^{\text{split}})$  from [Definition 2.18](#) to reach the vertex  $\mathbf{w}_{i,k,\text{TR}}^{\alpha_{i,k},N}$ .
- Finally, use the edge  $\mathbf{w}_{i,k,\text{TR}}^{\alpha_{i,k},N} \rightarrow b_i$  to reach  $b_i$ .

**- Description of the set of paths  $\{T_1, T_2, \dots, T_k\}$  :**

For each  $j \in [k]$ , we build the path  $T_j$  as follows:

- Start with the edge  $c_j \rightarrow \mathbf{w}_{1,j,\text{LB}}^{1,\beta_{1,j}}$
- For each  $i \in [k-1]$  use the  $\mathbf{w}_{i,j,\text{LB}}^{1,\beta_{i,j}} \rightsquigarrow \mathbf{w}_{i+1,j,\text{LB}}^{1,\beta_{i+1,j}}$  path obtained by concatenating
  - the  $\mathbf{w}_{i,j,\text{LB}}^{1,\beta_{i,j}} \rightsquigarrow \mathbf{w}_{i,j,\text{TR}}^{N,\beta_{i,j}}$  path  $\text{RowPath}_{\beta_{i,j}}(G_{i,j}^{\text{split}})$  from [Definition 2.18](#)
  - the  $\mathbf{w}_{i,j,\text{TR}}^{N,\beta_{i,j}} \rightsquigarrow \mathbf{w}_{i+1,j,\text{LB}}^{1,\beta_{i+1,j}}$  path  $\mathbf{w}_{i,j,\text{TR}}^{N,\beta_{i,j}} \rightarrow \mathbf{h}_{i,j}^{i+1,j}(\beta_{i,j}) \rightarrow \dots \rightarrow \mathbf{h}_{i,j}^{i+1,j}(\beta_{i+1,j}) \rightarrow \mathbf{w}_{i+1,j,\text{LB}}^{1,\beta_{i+1,j}}$  which exists since [Equation 8](#) implies  $\beta_{i,j} \leq \beta_{i+1,j}$ .
- Now, we have reached the vertex  $\mathbf{w}_{k,j,\text{LB}}^{1,\beta_{k,j}}$ . Use the  $\mathbf{w}_{k,j,\text{LB}}^{1,\beta_{k,j}} \rightsquigarrow \mathbf{w}_{k,j,\text{TR}}^{N,\beta_{k,j}}$  path  $\text{RowPath}_{\beta_{k,j}}(G_{k,j}^{\text{split}})$  from [Definition 2.18](#) to reach the vertex  $\mathbf{w}_{k,j,\text{TR}}^{N,\beta_{k,j}}$ .
- Finally, use the edge  $\mathbf{w}_{k,j,\text{TR}}^{N,\beta_{k,j}} \rightarrow d_j$  to reach  $d_j$ .

By [Definition 2.9](#), it follows that every edge of the path  $R_i$  has both endpoints in  $\text{VERTICAL}(i)$  for every  $i \in [k]$ . Since  $\text{VERTICAL}(i) \cap \text{VERTICAL}(i') = \emptyset$  for every  $1 \leq i \neq i' \leq k$ , it follows that the collection of paths  $\{R_1, R_2, \dots, R_k\}$  are pairwise edge-disjoint.

By [Definition 2.9](#), it follows that every edge of the path  $T_j$  has both endpoints in  $\text{HORIZONTAL}(j)$  for every  $j \in [k]$ . Since  $\text{HORIZONTAL}(j) \cap \text{HORIZONTAL}(j') = \emptyset$  for every  $1 \leq j \neq j' \leq k$ , it follows that the collection of paths  $\{T_1, T_2, \dots, T_k\}$  are pairwise edge-disjoint.

Fix any  $(i, j) \in [k] \times [k]$ . We now conclude the proof of this lemma by showing that  $R_i$  and  $T_j$  are edge-disjoint. By the construction of  $G_2$  ([Figure 2](#) and [Figure 3](#)) and definitions of the paths  $R_i$  and  $T_j$ , it follows that the only common edge between  $R_i$  and  $T_j$  could be  $\mathbf{w}_{i,j,\text{LB}}^{\alpha_{i,j},\beta_{i,j}} \rightarrow \mathbf{w}_{i,j,\text{TR}}^{\alpha_{i,j},\beta_{i,j}}$ . By [Equation 8](#), we have that  $(\alpha_{i,j}, \beta_{i,j}) \in S_{i,j}$ . Hence, by the splitting operation ([Definition 2.4](#)), we have that  $\mathbf{w}_{i,j,\text{LB}}^{\alpha_{i,j},\beta_{i,j}} = \mathbf{w}_{i,j}^{\alpha_{i,j},\beta_{i,j}} = \mathbf{w}_{i,j,\text{TR}}^{\alpha_{i,j},\beta_{i,j}}$ , i.e., the only possible common edge  $\mathbf{w}_{i,j,\text{LB}}^{\alpha_{i,j},\beta_{i,j}} \rightarrow \mathbf{w}_{i,j,\text{TR}}^{\alpha_{i,j},\beta_{i,j}}$  between  $R_i$  and  $T_j$  is not an edge in  $G_2$ . Hence,  $R_i$  and  $T_j$  are edge-disjoint.  $\square$

## 2.4 Proof of Theorem 1.1

Finally we are ready to prove our main theorem (Theorem 1.1) which is restated below:

**Theorem 1.1.** *The EDGE-DISJOINT PATHS problem on planar DAGs is W[1]-hard parameterized by the number  $k$  of terminal pairs. Moreover, under ETH, the EDGE-DISJOINT PATHS problem on planar DAGs cannot be solved  $f(k) \cdot n^{o(k)}$  time where  $f$  is any computable function,  $n$  is the number of vertices and  $k$  is the number of terminal pairs. The hardness holds even if both the maximum in-degree and maximum out-degree of the graph are at most 2.*

*Proof.* Given an instance  $(k, N, \mathcal{S})$  of GRID-TILING- $\leq$ , we use the construction from Section 2.1 to build an instance  $(G_2, \mathcal{T})$  of EDGE-DISJOINT PATHS such that  $G_2$  is a planar DAG (Claim 2.7). It is easy to see that  $n = |V(G_2)| = O(N^2 k^2)$  and  $G_2$  can be constructed in  $\text{poly}(N, k)$  time.

It is known [13, Theorem 14.30] that GRID-TILING- $\leq$  is W[1]-hard parameterized by  $k$ , and under ETH cannot be solved in  $f(k) \cdot N^{o(k)}$  time for any computable function  $f$ . Combining the two directions from Section 2.2 and Section 2.3, we get a parameterized reduction from GRID-TILING- $\leq$  to an instance of EDGE-DISJOINT PATHS which is a planar DAG and has  $|\mathcal{T}| = 2k$  terminal pairs. Hence, it follows that EDGE-DISJOINT PATHS on planar DAGs is W[1]-hard parameterized by number  $k$  of terminal pairs, and under ETH cannot be solved in  $f(k) \cdot n^{o(k)}$  time for any computable function  $f$ .

Finally we show how to edit  $G_2$ , without affecting the correctness of the reduction, so that both the max out-degree and max in-degree are at most 2. We present the argument for reducing the out-degree: the argument for reducing the in-degree is analogous. Note that the only vertices in  $G_2$  with out-degree  $> 2$  are  $A \cup C$ . For each  $c_j \in C$  we replace the directed star whose edges are from  $c_j$  to each vertex of  $\text{Left}(G_{1,j})$  with a directed binary tree whose root is  $c_j$ , leaves are the set of vertices  $\text{Left}(G_{1,j})$  and each edge is directed away from the root. It is easy to see that in this directed binary tree the set of paths from  $c_j$  to the different leaves (i.e., vertices of  $\text{Left}(G_{1,j})$ ) are pairwise edge-disjoint, and we have only increased the number of vertices by  $O(k)$  while maintaining both planarity and (directed) acyclicity. We do a similar transformation for each  $a_i \in A$ . It is easy to see that this editing adds  $O(k^2)$  new vertices and takes  $\text{poly}(k)$  time, and therefore it is still true that  $n = |V(G_2)| = O(N^2 k^2)$  and  $G_2$  can be constructed in  $\text{poly}(N, k)$  time.  $\square$

## 3 Conclusion & Open Questions

In this paper we have shown that EDGE-DISJOINT PATHS on planar DAGs is W[1]-hard parameterized by  $k$ , and has no  $f(k) \cdot n^{o(k)}$  algorithm under the Exponential Time Hypothesis (ETH) for any computable function  $f$ . The hardness holds even if both the maximum in-degree and maximum out-degree of the graph are at most 2. Our result answers a question of Slivkins [44] regarding the parameterized complexity of EDGE-DISJOINT PATHS on planar DAGS, and a question of Cygan et al. [12] and Schrijver [43] regarding the parameterized complexity of EDGE-DISJOINT PATHS on planar directed graphs.

We now propose some open questions related to the complexity of the DISJOINT PATHS problem:

- What is the *correct* parameterized complexity of EDGE-DISJOINT PATHS on planar graphs parameterized by  $k$ ? Can we design an XP algorithm, or is the problem NP-hard even for  $k = O(1)$  like the general version? Note that to prove the latter result, one would need to have directed cycles involved in the reduction since there is  $n^{O(k)}$  algorithm of Fortune et al. [19] for EDGE-DISJOINT PATHS on DAGs.
- Is the half-integral version<sup>4</sup> of EDGE-DISJOINT PATHS FPT on directed planar graphs or DAGs? It is easy to see that our W[1]-hardness reduction does not work for this problem.
- Given our W[1]-hardness result, can we obtain FPT (in)approximability results for the EDGE-DISJOINT PATHS problem on planar DAGs? To the best of our knowledge, there are no known (non-trivial) FPT (in)approximability results for any variants of the DISJOINT PATHS problem. This question might be worth considering even for those versions of the DISJOINT PATHS problem

<sup>4</sup>Each edge can belong to at most two of the paths

which are known to be FPT since the running times are astronomical (except maybe [32]). Some of the recent work [8, 9, 10, 11] on polynomial time (in)approximability of the DISJOINT PATHS problem might be relevant.

## Acknowledgements

We thank the anonymous reviewers of CIAC 2021 for their helpful comments. In particular, one of the reviewers suggested the strengthening of [Theorem 1.1](#) for the case when the input graph has both in-degree and out-degree at most 2.

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