# A Multistage View on 2-Satisfiability

#### Till Fluschnik ©

Technische Universität Berlin, Faculty IV, Algorithmics and Computational Complexity, Germany till.fluschnik@tu-berlin.de

#### Abstract

We study q-SAT in the multistage model, focusing on the linear-time solvable 2-SAT. Herein, given a sequence of q-CNF fomulas and a non-negative integer d, the question is whether there is a sequence of satisfying truth assignments such that for every two consecutive truth assignments, the number of variables whose values changed is at most d. We prove that MULTISTAGE 2-SAT is NP-hard even in quite restricted cases. Moreover, we present parameterized algorithms (including kernelization) for MULTISTAGE 2-SAT and prove them to be asymptotically optimal.

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# 1 Introduction

q-Satisfiability (q-Sat) is one of the most basic and best studied decision problems in computer science: It asks whether a given boolean formula in conjunctive normal form, where each clause consists of at most q literals, is satisfiable. q-Sat is NP-complete for  $q \geq 3$ , while 2-Satisfiability (2-Sat) is linear-time solvable [1]. The recently introduced multistage model [17, 24] takes a sequence of instances of some decision problem (e.g., modeling one instance that evolved over time), and asks whether there is a sequence of solutions to them such that, roughly speaking, any two consecutive solutions do not differ too much. We introduce q-Sat in the multistage model, defined as follows.<sup>1</sup>

Multistage q-SAT (MqSAT)

**Input**: A set X of variables, a sequence  $\Phi = (\phi_1, \dots, \phi_\tau), \tau \in \mathbb{N}$ , of q-CNF formulas over literals over X, and an integer  $d \in \mathbb{N}_0$ .

**Question**: Are there  $\tau$  truth assignments  $f_1, \ldots, f_{\tau} \colon X \to \{\bot, \top\}$  such that

- (i) for each  $i \in \{1, ..., \tau\}$ ,  $f_i$  is a satisfying truth assignment for  $\phi_i$ , and
- (ii) for each  $i \in \{1, ..., \tau 1\}$ , it holds that  $|\{x \in X \mid f_i(x) \neq f_{i+1}(x)\}| \leq d$ ?

Constraint (ii) of MqSAT can also be understood as that the Hamming distance of two consecutive truth assignments interpreted as n-dimensional vectors over  $\{\bot, \top\}$  is at most d, or when considering the sets of variables set true, then the symmetric difference of two consecutive sets is at most d.

In this work, we focus on M2SAT yet relate most of our results to MqSAT. We study M2SAT in terms of classic computational complexity and parameterized algorithmics [13].

<sup>&</sup>lt;sup>1</sup> We identify false and true with  $\perp$  and  $\top$ , respectively.

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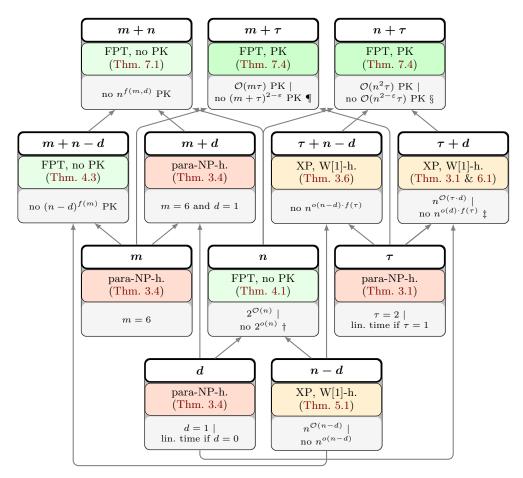


Figure 1 Our results for MULTISTAGE 2-SAT. Each box gives, regarding to a parameterization (top layer), our parameterized classification (middle layer) with additional details on the corresponding result (bottom layer). Arrows indicate the parameter hierarchy: An arrow from parameter  $p_1$  to  $p_2$  indicates that  $p_1 \leq p_2$ . "PK" and "no PK" stand for "polynomial problem kernel" and "no polynomial problem kernel unless NP ⊆ coNP / poly", respectively. †: unless the ETH breaks (Thm. 3.4). ‡: unless the ETH breaks (Thm. 3.1). ¶: unless NP ⊆ coNP / poly (Thm. 7.6) §: unless NP ⊆ coNP / poly (Thm. 3.1).

**Motivation.** In theory as well as in practice, it is common to model problems as q-SAT- or even 2-SAT-instances. Once being modeled, established solvers specialized on q-SAT are employed. In some cases, a sequence of problem instances (e.g., modeling a problem instance that changes over time) is to solve such that any two consecutive solutions are similar in some way (e.g., when costs are inferred for setup changes). Hence, when following the previously described approach, each problem instance is first modeled as a q-SAT instance such that a sequence of q-SAT-instances remains to be solved. Comparably to the single-stage setting, understanding the multistage setting could give raise to a general approach for solving different (multistage) problems. With MqSAT we introduce the first problem that models the described setup. Note that, though a lot of variants of q-SAT exist, MqSAT is one of the very few variants that deal with a sequence of q-SAT-instances [34].

Our Contributions. Our results for Multistage 2-SAT are summarized in Figure 1. We prove Multistage 2-SAT to be NP-hard, even in fairly restricted cases: (i) if d = 1 and

the maximum number m of clauses in any stage is six, or (ii) if there are only two stages. These results are tight in the sense that M2SAT is linear-time solvable when d=0 or  $\tau=1$ . While NP-hardness for d=1 implies that there is no  $(n+m+\tau)^{f(d)}$ -time algorithm for any function f unless P=NP, where n denotes the number of variables, we prove that when parameterized by the dual parameter n-d (the minimum number of variables not changing between any two consecutive layers), M2SAT is W[1]-hard and solvable in  $\mathcal{O}^*(n^{\mathcal{O}(n-d)})$  time. We prove this algorithm to be tight in the sense that, unless the Exponential Time Hypothesis (ETH) breaks, there is no  $\mathcal{O}^*(n^{o(n-d)})$ -time algorithm. Further, we prove that M2SAT is solvable in  $\mathcal{O}^*(2^{\mathcal{O}(n)})$  time but not in  $\mathcal{O}^*(2^{o(n)})$  time unless the ETH breaks. Likewise, we prove that M2SAT is solvable in  $\mathcal{O}^*(n^{\mathcal{O}(\tau \cdot d)})$  time but not in  $\mathcal{O}^*(n^{o(d) \cdot f(\tau)})$  time for any function f unless the ETH breaks. As to efficient and effective data reduction, we prove M2SAT to admit problem kernelizations of size  $\mathcal{O}(m \cdot \tau)$  and  $\mathcal{O}(n^2\tau)$ , but none of size  $(n+m)^{\mathcal{O}(1)}$ ,  $\mathcal{O}((n+m+\tau)^{2-\varepsilon})$ , or  $\mathcal{O}(n^{2-\varepsilon}\tau)$ ,  $\varepsilon>0$ , unless  $NP\subseteq coNP/poly$ .

**Related Work.** q-SAT is one of the most famous decision problems with a central role in NP-completeness theory [12, 30], for the (Strong) Exponential Time Hypothesis [28, 29], and in the early theory on kernelization lower bounds [6, 23], for instance. In contrast to q-SAT with  $q \geq 3$ , 2-SAT is proven to be polynomial- [31], even linear-time [1] solvable. Several applications of 2-SAT are known (see, e.g., [11, 18, 25, 33]). In the multistage model, various problems from different fields were studied, e.g. graph theory [2, 3, 10, 21, 22, 24], facility location [17], knapsack [5], or committee elections [8]. Also variations to the multistage model were studied, e.g. with a global budget [26], an online-version [4], or using different distance measures for consecutive stages [8, 22].

#### 2 Preliminaries

We denote by  $\mathbb{N}$  and  $\mathbb{N}_0$  the natural numbers excluding and including zero, respectively. Frequently, we will tacitly make use of the fact that for every  $n \in \mathbb{N}$ ,  $0 \le k \le n$ , it holds true that  $1 + \sum_{i=1}^k \binom{n}{i} = \sum_{i=0}^k \binom{n}{i} \le 1 + n^k \le 2n^k$ .

Satisfiability. Let X denote a set of variables. A literal is a variable that is either positive or negated (we denote the negation of x by  $\neg x$ ). A clause is a disjunction over literals. A formula  $\phi$  is in conjunctive normal form (CNF) if it is of the form  $\bigwedge_i C_i$ , where  $C_i$  is a clause. A formula  $\phi$  is in q-CNF if it is in CNF and each clause consists of at most q literals. An truth assignment  $f: X \to \{\bot, \top\}$  is satisfying for  $\phi$  (or satisfies  $\phi$ ) if each clause is satisfied, which is the case if at least one literal in the clause is evaluated to true (a positive variable assigned true, or a negated variable assigned false). For  $a, b \in \{\bot, \top\}$ , let  $a \oplus b := \bot$  if a = b, and  $a \oplus b := \top$  otherwise. For  $X' \subset X$ , an truth assignment  $f': X' \to \{\bot, \top\}$  is called partial. We say that we simplify a formula  $\phi$  given a partial truth assignment f' (we denote the simplified formula by  $\phi[f']$ ) if each variable  $x \in X'$  is replaced by f'(x), and then each clause containing an evaluated-to-true literal is deleted.

**Parameterized Algorithmics.** A parameterized problem L is a set of instances  $(x, p) \in \Sigma^* \times \mathbb{N}_0$ , where  $\Sigma$  is a finite alphabet and p is referred to as the parameter. A parameterized problem L is (i) fixed-parameter tractable (in FPT) if each instance (x, p) can be decided

<sup>&</sup>lt;sup>2</sup> The  $\mathcal{O}^*$ -notation suppresses factors polynomial in the input size.

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for L in  $f(p) \cdot |x|^{\mathcal{O}(1)}$  time, and (ii) in XP if each instance (x,p) can be decided for L in  $|x|^{g(p)}$  time, where f,g are computable functions only depending on p. If L is W[1]-hard, it is presumably not in FPT. A problem bikernelization for a parameterized problem L to a parameterized problem L' takes any instance (x,p) of L and maps it in polynomial time to an equivalent instance (x',p') of L' (the so-called problem bikernel) such that  $|x'|+p' \leq f(p)$  for some computable function f. A problem kernelization is a problem bikernelization where L = L'. If f is a polynomial, the problem (bi)kernelization is said to be polynomial. A parametric transformation from a parameterized problem L to a parameterized problem L' maps any instance (x,p) of L in  $f(p) \cdot |x|^{\mathcal{O}(1)}$  time to an equivalent instance (x',p') of L' such that  $p' \leq g(p)$  for some functions f,g each only depending on p. If there is a parametric transformation from L to L' with L being W[1]-hard, then L' is W[1]-hard. If  $f(p) \in p^{\mathcal{O}(1)}$  and  $g(p) \in \mathcal{O}(p)$ , then we have a linear parametric transformation [27]. If there is a linear parametric transformation from a problem L to problem L' with L' admitting a problem kernelization of size h(p'), then L admits a problem bikernelization of size h(p').

**Preprocessing on** MULTISTAGE 2-SAT. Due to the following data reduction, we can safely assume each stage to admit a satisfying truth assignment.

▶ Reduction Rule 1. If a stage exists with no satisfying truth assignment, then return no. Reduction Rule 1 is correct and applicable in linear time.

# 3 From Easy to Hard: NP- and W-hardness

MULTISTAGE 2-SAT is linear-time solvable if the input consists of only one stage, or if all or none variables are allowed to change its truth assignment between two consecutive stages.

▶ Observation 1. Multistage 2-SAT is linear-time solvable if (i)  $\tau = 1$ , (ii) d = 0, or (iii) d = n.

**Proof.** Let  $(X, \Phi = (\phi_1, \dots, \phi_{\tau}), d)$  be an instance of M2SAT. Case (i):  $\tau = 1$ . Polynomial-time many-one reduction to 2-SAT with instance  $(X, \phi_1)$ . Case (ii): d = 0. Polynomial-time many-one reduction to 2-SAT with instance  $(X, \phi')$ , wheret  $\phi' = \bigwedge_{i=1}^{\tau} \phi_i$ , Case (iii): d = n. Solve each of the  $\tau$  instances  $(X, \phi_1), \dots, (X, \phi_{\tau})$  of 2-SAT individually (Turing reduction).

We will prove that the cases (i) and (ii) in Observation 1 are tight: MULTISTAGE 2-SAT becomes NP-hard if  $\tau \geq 2$  (Sections 3.1 & 3.3) or d=1 (Section 3.2). For the case (iii) in Observation 1 the picture looks different: we prove MULTISTAGE 2-SAT to be polynomial-time solvable if  $n-d \in \mathcal{O}(1)$  (Section 5).

#### 3.1 From One to Two Stages

In this section, we prove that MULTISTAGE 2-SAT becomes NP-hard if  $\tau \geq 2$ . In fact, we prove the following.

- ▶ Theorem 3.1. Multistage 2-SAT is NP-hard, even for two stages, where the variables appear all negated in one and all positive in the other stage. Moreover, Multistage 2-SAT
  - (i) is W[1]-hard when parameterized by d even if  $\tau = 2$ ,
- (ii) admits no  $n^{o(d)\cdot f(\tau)}$ -time algorithm for any function f unless the ETH breaks, and
- (iii) admits no problem kernelization of size  $\mathcal{O}(n^{2-\varepsilon} \cdot f(\tau))$  for any  $\varepsilon > 0$  and function f, unless  $NP \subseteq coNP / poly$ .

We will reduce from the following NP-hard problem:

Weighted 2-SAT

**Input**: A set of variables X, a 2-CNF  $\phi$  over X, and an integer k.

**Question**: Is there satisfying truth assignment for  $\phi$  with at most k variables set true?

When parameterized by the number k of set-to-true variables, WEIGHTED 2-SAT is W[1]-complete [15, 20]. Moreover, WEIGHTED 2-SAT admits no  $n^{o(k)}$ -time algorithm unless the ETH breaks [9] and no problem bikernelization of size  $\mathcal{O}(n^{2-\varepsilon})$ ,  $\varepsilon > 0$ , unless NP  $\subseteq$  coNP/poly [14].

- ▶ Construction 1. Let  $(X, \phi, k)$  be an instance of WEIGHTED 2-SAT, where  $\phi = \bigwedge_{i=1}^{m} C_i$ . Construct  $\Phi = (\phi_1, \phi_2)$ , where  $\phi_2 := \phi$  and  $\phi_1 := \bigwedge_{x \in X} (\neg x)$  consists of n size-one clauses, where each variable appears negated in one clause. Finally, set d := k.
- ▶ Lemma 3.2. Let  $\mathcal{I} = (X, \phi, k)$  be an instance of WEIGHTED 2-SAT, and let  $\mathcal{I}' = (X, \Phi, d)$  be an instance of MULTISTAGE 2-SAT obtained from  $\mathcal{I}$  using Construction 2. Then,  $\mathcal{I}$  is a yes-instance if and only if  $\mathcal{I}'$  is a yes-instance.
- **Proof.** ( $\Rightarrow$ ) Let f be a satisfying truth assignment for  $\mathcal{I}$ . We claim that  $f_1: X \to \{\bot\}$ ,  $x \mapsto \bot$ , and  $f_2 := f$  is a solution to  $\mathcal{I}'$ . Note that  $f_1$  and  $f_2$  satisfy  $\phi_1$  and  $\phi_2$ , respectively. Moreover,  $|\{x \in X \mid f_1(x) \neq f_2(x)\}| = |\{x \in X \mid f_2(x) = \top\}| \leq d = k$ .
- $(\Leftarrow)$  Let  $(f_1, f_2)$  be a solution to  $\mathcal{I}'$ . Note that  $f_1 \colon X \to \{\bot\}$ . Since d = k, there are at most k variables set to true by  $f_2$ . Hence,  $f_2$  is a satisfying truth assignment for  $\mathcal{I}$  with at most k variables set to true, and thus  $\mathcal{I}$  is yes-instance.
- **Proof of Theorem 3.1.** Construction 1 forms a polynomial-time many-one reduction to an instance with two stages with d=k. Note that WEIGHTED 2-SAT remains NP-hard if all literals are positive (e.g., via a reduction from VERTEX COVER). Hence, M2SAT is NP-hard, even if the variables appear all negated in one and all positive in the other stage. Moreover, unless the ETH breaks, M2SAT admits no  $n^{o(d)\cdot f(\tau)}$ -time algorithm for any function f since no  $n^{o(k)}$ -time algorithm exists for WEIGHTED 2-SAT [9]. As Construction 1 also forms a parametric transformation, M2SAT is W[1]-hard when parameterized by d even if  $\tau = 2$ . Moreover, Construction 1 forms a linear parametric transformation from WEIGHTED 2-SAT parameterized by |X| to M2SAT parameterized by  $n \cdot f(\tau)$  for any function f. Hence, M2SAT admits no problem kernel of size  $\mathcal{O}(n^{2-\varepsilon} \cdot f(\tau))$  for any  $\varepsilon > 0$  and function f, unless NP  $\subseteq$  coNP / poly.
- ▶ Remark 3.3. Theorem 3.1(iii) can be generalized to Multistage q-SAT: Instead from Weighted 2-SAT, we reduce (in an analogous way) from Weighted q-SAT which admits no problem bikernelization of size  $\mathcal{O}(n^{q-\varepsilon})$ ,  $\varepsilon > 0$ , unless NP  $\subseteq$  coNP / poly [14]. Thus, unless NP  $\subseteq$  coNP / poly, Multistage q-SAT admits no problem kernel of size  $\mathcal{O}(n^{q-\varepsilon} \cdot f(\tau))$  for any  $\varepsilon > 0$  and function f.

#### 3.2 From Zero to One Allowed Change

In this section, we prove that MULTISTAGE 2-SAT becomes NP-hard if d=1 and the maximum number m of clauses in any stage is six. In fact, we prove the following.

▶ Theorem 3.4. Multistage 2-SAT is NP-hard, even if the number of clauses in each stage is at most six and d=1. Moreover, unless the ETH breaks, Multistage 2-SAT admits no  $\mathcal{O}^*(2^{o(n)})$ -time algorithm.

▶ Construction 2. Let  $(X, \phi)$  be an instance of 3-SAT, where  $\phi = \bigwedge_{i=1}^m C_i$  and each clause consists of exactly three literals. Let  $\ell_i^j$ ,  $j \in \{1, 2, 3\}$ , denote the literals in  $C_i$  for each  $i \in \{1, \ldots, m\}$ . Construct instance  $(X', \Phi, d)$  of M2SAT as follows. First, construct  $X' := X \cup B$ , where  $B := \{b_1, b_2, b_3\}$ . Let

$$\phi_B := (b_1 \vee b_2) \wedge (b_1 \vee b_3) \wedge (b_2 \vee b_3), \text{ and}$$
$$\phi_{\neg B} := (\neg b_1 \vee \neg b_2) \wedge (\neg b_1 \vee \neg b_3) \wedge (\neg b_2 \vee \neg b_3).$$

Next, construct  $\Phi := (\phi_i, \dots, \phi_{2m})$  as follows. For each  $i \in \{1, \dots, m\}$ , construct

$$\phi_{2i-1} := \phi_{\neg B},$$
 and  $\phi_{2i} := (\ell_1^i \lor b_1) \land (\ell_2^i \lor b_2) \land (\ell_3^i \lor b_3) \land \phi_B$ 

Finally, set d := 1.

▶ **Observation 2.** In every solution to an instance obtained from Construction 2, in each odd stage exactly two  $b_j$  are set to false and in each even stage exactly two  $b_j$  are set to true.

 $\Diamond$ 

**Proof.** Clearly, in every satisfying truth assignment for  $\phi_{2i-1} = \phi_{\neg B}$ ,  $i \in \{1, ..., m\}$ , at least two  $b_j$  are set to false. In every satisfying truth assignment for  $\phi_{2i}$ , to satisfy  $\phi_B$ , at least two variables from B must be set to true. As d=1, exactly one of B being set to false in a satisfying truth assignment  $\phi_{2i-1}$  can be set to true in a satisfying truth assignment for  $\phi_{2i}$ , which implies that already one of B must be set to true in  $\phi_{2i-1}$ .

▶ Lemma 3.5. Let  $\mathcal{I} = (X, \phi)$  be an instance of 3-SAT, and let  $\mathcal{I}' = (X', \Phi, d)$  be an instance of MULTISTAGE 2-SAT obtained from  $\mathcal{I}$  using Construction 2. Then,  $\mathcal{I}$  is a yes-instance if and only if  $\mathcal{I}$ ' is a yes-instance.

**Proof.** ( $\Rightarrow$ ) Let  $f: X \to \{\bot, \top\}$  be a satisfying truth assignment for  $\phi$ . We construct truth assignments  $f_1, \ldots, f_\tau \colon X' \to \{\bot, \top\}$  as follows. Let  $f_i(x) = f(x)$  for all  $i \in \{1, \ldots, \tau\}$  and all  $x \in X$ . Next, for each  $i \in \{1, \ldots, m\}$ ,  $f_{2i}$  assigns exactly two variables from B to true such that  $\phi_{2i}$  is satisfied. This is possible since at least one clause from  $\phi_{2i}$  is already set to true by one true literal. It remains to show that for each  $i \in \{1, \ldots, m\}$ , there is an truth assignment of  $f_{2i-1}$  to the variables from B such that exactly two are set to false (in which case  $\phi_{2i-1}$  is satisfied), and  $|\{b \in B \mid f_{2i-2}(b) \neq f_{2i-1}(b)\}| \leq 1$  (if i = 1, interpret  $f_{2i-2} = f_{2i}$ ) and  $|\{b \in B \mid f_{2i-1}(b) \neq f_{2i}(b)\}| \leq 1$ . Since |B| = 3, there is a  $j \in \{1, \ldots, 3\}$  such that  $f_{2i-2}(b_j) = f_{2i}(b_j) = \top$ . Set  $f_{2i-1}(b_j) = \top$  and  $f_{2i-1}(b_\ell) = \bot$  for  $\ell \in \{1, \ldots, 3\} \setminus \{j\}$ . Observe that  $|\{b \in B \setminus \{b_j\} \mid f_{2i-2}(b) \neq f_{2i-1}(b)\}| = |\{b \in B \setminus \{b_j\} \mid f_{2i-1}(b) \neq f_{2i}(b)\}| = 1$ , what we needed to show.

( $\Leftarrow$ ) By Observation 2, between every two consecutive stages, exactly one variable in B changes its true-false value. Hence, each variable from X is assigned the same value in each stage, i.e.,  $f_i(x) = f_j(x)$  for every  $x \in X$  and every  $i, j \in \{1, ..., \tau\}$ . Let  $f: X \to \{\bot, \top\}$  be the truth assignment of the variables in X with  $f(x) := f_1(x)$  for all  $x \in X$ . Since in every even stage, by Observation 2, exactly one variable from B is set to false, at least one literal must be set to true. It follows that each clause in the 3-SAT instance is satisfied by f, that is, f is a satisfying truth assignment for  $\mathcal{I}$ . Thus,  $\mathcal{I}$  is a yes-instance.

**Proof of Theorem 3.4.** Construction 2 forms a polynomial-time many-one reduction to an instance with d=1, m=6, and n=|X|+3. Hence, M2SAT is NP-hard, even if d=1 and m=6, and, unless the ETH breaks, admits no  $\mathcal{O}^*(2^{o(n)})$ -time algorithm since no  $\mathcal{O}^*(2^{o(|X|)})$ -time algorithm exists for 3-SAT [9].

# 3.3 From All to All But k Allowed Changes

In this section, we prove that MULTISTAGE 2-SAT is W[1]-hard when parameterized by the lower bound n-d on the number of unchanged variables between any two consecutive stages.

▶ Theorem 3.6. MULTISTAGE 2-SAT is W[1]-hard when parameterized by n-d even if  $\tau=2$ , and, unless the ETH breaks, admits no  $\mathcal{O}^*(n^{o(n-d)\cdot f(\tau)})$ -time algorithm for any function f.

We reduce from the following NP-hard problem:

MULTICOLORED INDEPENDENT SET (MIS)

**Input**: An undirected, k-partite graph  $G = (V^1, \dots, V^k, E)$ .

**Question**: Is there an independent set S such that  $|S \cap V^i| = 1$  for all  $i \in \{1, ..., k\}$ ?

MIS is W[1]-hard with respect to k [19] and unless the ETH breaks, there is no  $f(k) \cdot n^{o(k)}$ -time algorithm [32].

▶ Construction 3. Let  $\mathcal{I} = (G = (V^1, \dots, V^k, E))$  be an instance of MIS and let  $V \coloneqq V^1 \uplus \dots \uplus V^k$ ,  $n \coloneqq |V|$ , and  $V^i = \{v^i_1, \dots, v^i_{|V_i|}\}$  for all  $i \in \{1, \dots, k\}$ . We construct an instance  $\mathcal{I}' = (X, (\phi_1, \phi_2), d)$  with  $d \coloneqq n - k$  as follows. Let  $X \coloneqq X^1 \cup \dots \cup X^k$  with  $X^i = \{x^i_i \mid v^i_i \in V_i\}$  for all  $i \in \{1, \dots, k\}$ . Let for all  $i \in \{1, \dots, k\}$ 

$$\phi_i^* \coloneqq \bigwedge_{j,j' \in \{1,\dots,|V^i|\},\, j \neq j'} (\neg x_j^i \vee \neg x_{j'}^i), \qquad \text{and let} \qquad \phi_E \coloneqq \bigwedge_{\{v_j^i,v_{j'}^{i'}\} \in E} (\neg x_j^i \vee \neg x_{j'}^{i'}).$$

Let

$$\phi_1 \coloneqq \bigwedge_{x \in X} (x)$$
 and  $\phi_2 \coloneqq \phi_E \wedge \bigwedge_{i \in \{1, \dots, k\}} \phi_i^*.$ 

This finishes the construction.

▶ Lemma 3.7. Let  $\mathcal{I} = (G = (V^1, ..., V^k, E))$  be an instance of MIS, and let  $\mathcal{I}' = (X, (\phi_1, \phi_2), d)$  be an instance of MULTISTAGE 2-SAT obtained from  $\mathcal{I}$  using Construction 3. Then,  $\mathcal{I}$  is a yes-instance if and only if  $\mathcal{I}'$  is a yes-instance.

 $\Diamond$ 

**Proof.**  $(\Rightarrow)$  Let  $S = \{v_{j_1}^1, \ldots, v_{j_k}^k\} \subseteq V$  be an independent set with  $S \cap V^i = \{v_{j_i}^i\}$  for all  $i \in \{1, \ldots, k\}$ . Let  $X_S \coloneqq \{x_{j_1}^1, \ldots, x_{j_k}^k\}$  be the variables in X corresponding to the vertices in S. Let  $f_1 \colon X \to \{\bot, \top\}, x \mapsto \top$  and  $f_2 \colon X \to \{\bot, \top\}$  be defined as

$$f_2(x) = \begin{cases} \top, & \text{if } x \in X_S, \\ \bot, & \text{otherwise.} \end{cases}$$

Clearly,  $f_1$  satisfies  $\phi_1$ . Further, observe that for each  $r \in \{1, ..., k\}$ ,  $f_2$  satisfies  $\phi_r^*$  since all variables from  $X^i$  except for  $x_{j_i}^i$  is set to  $\bot$ . Since S is an independent set, and only variables corresponding to vertices from S are set to true by  $f_2$ ,  $f_2$  satisfies  $\phi_E$ . It follows that  $f_2$  satisfies  $\phi_2$ , and hence,  $f = (f_1, f_2)$  is a solution for  $\mathcal{I}'$ .

 $(\Leftarrow)$  Let  $f = (f_1, f_2)$  be a solution to  $\mathcal{I}'$ . Let  $S := \{v_j^i \in V \mid f_2(v_j^i) = \top\}$ . We claim that S is an independent set in G with  $|S \cap V^i| = 1$  for all  $i \in \{1, \ldots, k\}$ .

First, observe that S is an independent set in G: Suppose not, then there are  $v_j^i, v_{j'}^{i'} \in S$  such that  $\{v_j^i, v_{j'}^{i'}\} \in E$ . By construction,  $f_2(x_j^i) = f_2(x_{j'}^{i'}) = \top$ . Since  $\phi_E$  contains the clause  $(\neg x_j^i \lor \neg x_{j'}^{i'})$ ,  $f_2$  does not satisfy  $\phi_E$  (and, thus,  $\phi_2$ ), contradicting the fact that f is a solution. It follows that S is an independent set in G.

It remains to show that  $|S \cap V^i| = 1$  for all  $i \in \{1, ..., k\}$ . Observe that for all  $i \in \{1, ..., k\}$ ,  $|\{x \in X^i \mid f_2(x) = \top\}| \le 1$ . Suppose not, that is, there is some  $i \in \{1, ..., k\}$  such that there are  $x, y \in X^i$  with  $x \ne y$  and  $f_2(x) = f_2(y) = \top$ . By construction,  $\phi_i^*$  contains the clause  $(\neg x \vee \neg y)$ , which is evaluated to false under  $f_2$ . This is a contradiction to the fact that f is a solution.

Observe that for all  $i \in \{1, \ldots, k\}$ ,  $|\{x \in X^i \mid f_2(x) = \top\}| > 0$ : By construction of  $\phi_1$ , we know that  $f_1(x) = \top$  for all  $x \in X$ . Since d = n - k, there are at least k vertices set to true by  $f_2$ . If for some  $i \in \{1, \ldots, k\}$  we have that  $|\{x \in X^i \mid f_2(x) = \top\}| = 0$ , then, by the pigeonhole principle, there is an  $i' \in \{1, \ldots, k\}$  with  $i \neq i'$  such that  $|\{x \in X^i \mid f_2(x) = \top\}| \geq 2$ , which yields a contradiction as discussed. Thus,  $|S \cap V^i| = 1$  for all  $i \in \{1, \ldots, k\}$ . It follows that S is a solution to  $\mathcal{I}$ .

**Proof of Theorem 3.6.** Construction 3 runs in polynomial time and outputs an equivalent instance (Lemma 3.7) with two stages and d = n - k. As Construction 1 also forms a parametric transformation, M2SAT is W[1]-hard when parameterized by n - d even if  $\tau = 2$ . Moreover, unless the ETH breaks, M2SAT admits no  $n^{o(n-d)\cdot f(\tau)}$ -time algorithm for any function f since no  $n^{o(k)}$ -time algorithm exists for MIS.

# Fixed-Parameter Tractability Regarding the Number of Variables and m+n-d

In this section, we prove that MULTISTAGE 2-SAT is fixed-parameter tractable regarding the number of variables (Section 4.1) and regarding the parameter m + n - d, the maximum number of clauses over all input formulas and the minimum number of variables not changing between any two consecutive stages (Section 4.2).

# 4.1 Fixed-Parameter Tractability Regarding the Number of Variables

We prove that MULTISTAGE 2-SAT is fixed-parameter tractable regarding the number of variables.

▶ Theorem 4.1. MULTISTAGE 2-SAT is solvable in  $\mathcal{O}(\min\{2^n n^d, 4^n\} \cdot \tau \cdot (n+m))$  time.

**Proof.** Let  $\mathcal{I}=(X,\phi,d)$  be an instance of M2SAT with  $\Phi=(\phi_1,\ldots,\phi_{\tau})$ . Construct the digraph D with vertex set  $V=V^1\uplus\cdots\uplus V^\tau\uplus \{s,t\}$  and arc set A as follows. Add two designated vertices s and t to V. For each  $i\in\{1,\ldots,\tau\}$ , for every truth assignment f satisfying  $\phi_i$ , add a vertex  $v_f^i$  to  $V^i$ . Note that there are at most  $2^n$  truth assignments, where we can test for each truth assignment whether it is satisfying in  $\mathcal{O}(n+m)$  time. Add the arc (s,v) for all  $v\in V^1$  and the arc (v,t) for all  $v\in V^\tau$ . Moreover, for each  $i\in\{1,\ldots,\tau-1\}$ , add the arc  $(v_f^i,v_g^{i+1})$  if and only if  $|\{x\in X\mid f(x)\neq g(x)\}|\leq d$ . This finishes the construction of D. Note that  $|V^i|\leq 2^n$ , and each vertex (except for s) has outdegree at most  $\sum_{j=1}^d\binom{n}{j}\leq n^d$ . Hence,  $|A|\in\mathcal{O}(\min\{2^nn^d,4^n\}\tau)$ .

It is not difficult to see that D admits an s-t path if and only if  $\mathcal{I}$  is a yes-instance (see, e.g., [8, 21, 22]). Checking whether D admits an s-t path can be done in  $\mathcal{O}(|V| + |A|)$ .

▶ Remark 4.2. Theorem 4.1 is asymptotically optimal regarding n unless the ETH breaks (Theorem 3.4). Moreover, Theorem 4.1 is easily adaptable to MULTISTAGE q-SAT with  $q \ge 3$  as, for every  $q \ge 3$ , the number of truth assignments is  $2^n$  and each is verifiable in linear time.

# 4.2 Fixed-Parameter Tractability Regarding m + n - d

We prove that Multistage 2-SAT is fixed-parameter tractable regarding the parameter m + n - d.

▶ Theorem 4.3. MULTISTAGE 2-SAT is solvable in  $\mathcal{O}(4^{2(m+n-d)}\tau(n+m))$  time.

To prove Theorem 4.3, we will show that either Theorem 4.1 applies with  $n \le 2(m+n-d)$  or the following.

▶ Lemma 4.4. MULTISTAGE 2-SAT solvable in  $\mathcal{O}(\tau(n+m))$  time if 2m < d.

**Proof.** Let  $\mathcal{I}=(X,\phi,d)$  be an instance of M2SAT with  $\Phi=(\phi_1,\ldots,\phi_{\tau})$  on n variables and each formula contains at most m clauses. Due to Reduction Rule 1, we can safely assume that each formula of  $\Phi$  admits a satisfying truth assignment. Let  $X_i\subseteq X$  be the set of variables appearing as literals in  $\phi_i$  for each  $i\in\{1,\ldots,\tau\}$ . Note that  $|X_i|\leq 2m$  for each  $i\in\{1,\ldots,\tau\}$ . Compute in linear time a satisfying truth assignment  $f_1\colon X\to\{\bot,\top\}$  for  $\phi_1$ . Compute for each  $i\in\{2,\ldots,\tau\}$  in linear time a satisfying truth assignment  $f_i'\colon X_i\to\{\bot,\top\}$  for  $\phi_i$ . Next, iteratively for  $i=2,\ldots,\tau$ , set for all  $x\in X$ 

$$f_i(x) = \begin{cases} f_i'(x), & \text{if } x \in X_i, \\ f_{i-1}(x), & \text{if } x \in X \setminus X_i. \end{cases}$$

Clearly, truth assignment  $f_i$  satisfies  $\phi_i$ . Moreover, for all  $i \in \{2, ..., \tau\}$  it holds that  $|\{x \in X \mid f_{i-1}(x) \neq f_i(x)\}| \leq |X_i| \leq 2m < d$ , and hence  $(f_1, ..., f_\tau)$  is a solution to  $\mathcal{I}$ .

**Proof of Theorem 4.3.** Let  $\mathcal{I}=(X,\phi,d)$  be an instance of M2SAT with  $\Phi=(\phi_1,\ldots,\phi_\tau)$  on n variables and each formula contains at most m clauses. We distinguish how 2(m+n-d) relates to 2n-d.

Case 1:  $2(m+n-d) \ge 2n-d$ . Since  $d \le n$ , it follows that  $2(m+n-d) \ge n$ . Due to Theorem 4.1, we can solve  $\mathcal{I}$  in  $\mathcal{O}(\min\{2^n n^d, 4^n\}\tau(n+m)) \subseteq \mathcal{O}(4^{2(m+n-d)}\tau(n+m))$  time.

Case 2: 2(m+n-d) < 2n-d. We have that

$$2(m+n-d) < 2n-d \iff 2m < d.$$

Due to Lemma 4.4, we can solve  $\mathcal{I}$  in  $\mathcal{O}(\tau(n+m))$  time.

▶ Remark 4.5. Theorem 4.3 can be adapted for MULTISTAGE q-SAT for every  $q \geq 3$ , where Lemma 4.4 is restated for qm < d and we check for a satisfying truth assignment for each stage in  $\mathcal{O}^*(2^{qm})$  time. To adapt the proof of Theorem 4.3, we then relate q(m+n-d) with qn-(q-1)d and either employ the adapted Theorem 4.1 (see Remark 4.2), or the adapted Lemma 4.4.

# 5 XP Regarding the Number of Consecutive Non-Changes

We prove that MULTISTAGE 2-SAT is in XP when parameterized by the lower bound n-d on non-changes between consecutive stages, the parameter "dual" to d.

▶ Theorem 5.1. MULTISTAGE 2-SAT is solvable in  $\mathcal{O}(n^{4(n-d)+1} \cdot 2^{4(n-d)}\tau(n+m))$  time.

Let  $\mathcal{I} = (X, \Phi = (\phi_1, \dots, \phi_{\tau}), d)$  be a fixed yet arbitrary instance with n variables. Two partial truth assignments  $f_Y \colon Y \to \{\bot, \top\}$  and  $f_Z \colon Z \to \{\bot, \top\}$  with  $Y, Z \subseteq X$  are called *compatible* if for all  $x \in Y \cap Z$  it holds that  $f_Y(x) = f_Z(x)$ . For two compatible assignments  $f_Y, f_Z$ , we denote by

$$f_Y \cup f_Z := \begin{cases} f_Y(x), & x \in Y, \\ f_Z(x), & x \in Z \setminus Y. \end{cases}$$

With a similar idea as in the proof of Theorem 4.1, we will construct a directed graph with terminals s and t such that there is an s-t path in G if and only if  $\mathcal{I}$  is a yes-instance.

▶ Construction 4. Given  $\mathcal{I}$ , we construct a graph G = (V, E) with vertex set

$$V := V^{1 \to 3} \cup V^{2 \to 3} \cup \dots \cup V^{\tau - 2 \to \tau} \cup \{s, t\},\$$

where for each  $Y,Z\in \binom{X}{n-d}$ , we have that  $(f_Y,f_Z)\in V^{i\to i+2}$  if and only if  $f_Y,f_Z$  are compatible and each of  $\phi_i[f_Y]$ ,  $\phi_{i+1}[f_Y\cup f_Z]$ , and  $\phi_{i+2}[f_Z]$  is satisfiable, and the following arcs: (i) (s,v) for all  $v\in V^{1\to 3}$ , (ii) (v,t) for all  $v\in V^{\tau-2\to\tau}$ , and (iii)  $((f_Y,f_Z),(f_{Y'},f_{Z'}))\in V^{i\to i+2}\times V^{j\to j+2}$  if j=i+1 and  $f_Z=f_{Y'}$  (implying that Z=Y').

▶ Lemma 5.2. Construction 4 computes a graph of size  $\mathcal{O}(n^{4(n-d)+1} \cdot 2^{4(n-d)}\tau)$  and can be done in  $\mathcal{O}(n^{4(n-d)+1} \cdot 2^{4(n-d)}\tau(n+m))$  time.

**Proof.** To construct a set  $V^{i\to i+2}$ , we compute each tuple  $(f_Y, f_Y)$  in  $\mathcal{O}(n^{2(n-d)} \cdot 2^{2(n-d)})$  time, and check whether they are compatible in  $\mathcal{O}(n+m)$  time, and whether each of  $\phi_i[f_Y]$ ,  $\phi_{i+1}[f_Y \cup f_Z]$ , and  $\phi_{i+2}[f_Z]$  is satisfiable, each in  $\mathcal{O}(n+m)$  time. Since for any  $f_Y, f_Z$  we can check whether  $f_Y = f_Z$  in  $\mathcal{O}(n)$  time, we add the  $\mathcal{O}(n^{4(n-d)} \cdot 2^{4(n-d)})$  many arcs from  $V^{i\to i+2}$  to  $V^{i+1\to i+3}$  in  $\mathcal{O}(n^{4(n-d)+1} \cdot 2^{4(n-d)})$  time. In total, G can be constructed in  $\mathcal{O}(n^{4(n-d)+1} \cdot 2^{4(n-d)}\tau(n+m))$  time.

- ▶ Lemma 5.3. Let  $\mathcal{I}$  be an instance of MULTISTAGE 2-SAT and let G be the graph obtained from applying Construction 4 to  $\mathcal{I}$ . Then,  $\mathcal{I}$  is a yes-instance if and only if G admits an s-t paths.
- **Proof.** ( $\Rightarrow$ ) Let  $f = (f_1, \ldots, f_{\tau})$  be a solution to  $\mathcal{I}$ . For each  $i \in \{1, \ldots, \tau 1\}$ , since  $|\{x \in X \mid f_i(x) \neq f_{i+1}(x)\}| \leq d$ , there is a set  $Y_i \subseteq \{x \in X \mid f_i(x) = f_{i+1}(x)\}$  with  $|Y_i| = n d$ . Observe that  $v_i^f \coloneqq (f_i|_{Y_i}, f_{i+1}|_{Y_{i+1}}) \in V^{i \to i+2}$ : Clearly  $\phi_i[f_i|_{Y_i}]$  and  $\phi_i[f_{i+2}|_{Y_{i+1}}]$  satisfiable. Note that  $f_i|_{Y_i}, f_{i+1}|_{Y_{i+1}}$  are compatible since  $Y_i \cap Y_{i+1} \subseteq \{x \in X \mid f_i(x) = f_{i+1}(x) = f_{i+2}(x)\}$ . Moreover,  $\phi_{i+1}[f_i|_{Y_i} \cup f_{i+1}|_{Y_{i+1}}]$  is satisfiable since  $f_{i+1}(x) = f_i|_{Y_i} \cup f_{i+1}|_{Y_{i+1}}(x)$  for all  $x \in Y_i \cup Y_{i+1}$ . It follows that there is an s-t path in G with the arc sequence  $((s, v_1^f), (v_1^f, v_2^f), \ldots, (v_{\tau-1}^f, t))$ .
- ( $\Leftarrow$ ) Let P be an s-t path in G. By construction of G, P contains s, t, and from each  $V^{i \to i+2}$  exactly one vertex. Moreover, if arc  $((f_X, f_Y), (f_{X'}, f_{Y'}))$  is contained in P, then  $f_Y = f_{X'}$ . Let  $(s, ((f_{Y_i}, f_{Y_{i+1}}))_{i=1}^{\tau-2}, t)$  be the sequence of vertices in P. We know that there exists an  $f_1' \colon X \setminus Y_1 \to \{\bot, \top\}$  that satisfies  $\phi_1[f_{Y_1}]$ , and hence  $f_1 \coloneqq f_1' \cup f_{Y_1}$  satisfies  $\phi_1$ . Moreover, we know that for all  $i \in \{2, \ldots, \tau-1\}$ , there exists  $f_i' \colon X \setminus (Y_{i-1} \cup Y_i) \to \{\bot, \top\}$  that satisfies  $\phi_i[f_{Y_{i-1}} \cup f_{Y_i}]$ , and hence  $f_i \coloneqq f_i' \cup f_{Y_{i-1}} \cup f_{Y_i}$  satisfies  $\phi_i$ . Finally, we know that there exists an  $f_\tau' \colon X \setminus Y_{\tau-1} \to \{\bot, \top\}$  that satisfies  $\phi_\tau[f_{Y_{\tau-1}}]$ , and hence  $f_\tau \coloneqq f_\tau' \cup f_{Y_{\tau-1}}$  satisfies  $\phi_\tau$ .

It remains to show that  $|\{x \in X \mid f_i(x) \neq f_{i+1}(x)\}| \leq d$  for all  $i \in \{1, \dots, \tau - 1\}$ . Note that  $f_i(x) = f_{i+1}(x)$  for all  $x \in Y_i$ , and since  $|Y_i| = n - d$ , the claim follows.

**Algorithm 1** XP-algorithm on input instance  $(X, \phi, d)$ .

```
ı foreach X' \subseteq X : |X'| \le \tau \cdot d do
                                                                                                        //1+n^{\tau \cdot d} many
                                                                                                             // 2^{|X'|} many
        foreach f_1: X' \to \{\bot, \top\} do
2
              \phi_1^* \leftarrow \mathbf{simplify}(\phi_1, f_1);
3
              foreach g_2, g_3, \ldots, g_{\tau} : g_i \in \mathcal{F}(X') \ \forall i \in \{2, \ldots, \tau\} \ \mathbf{do}
                                                                                                      // 2^{\tau}|X'|^{\tau \cdot d} many
4
                   foreach i \in \{2, \ldots, \tau\} do
5
                       f_i(x) \leftarrow f_{i-1}(x) \oplus g_i(x) \ \forall x \in X'; \quad \phi_i^* \leftarrow \mathbf{simplify}(\phi_i, f_i);
6
                   if (X \setminus X', (\phi_1^*, \dots, \phi_{\tau}^*), 0) is a yes-instance of M2SAT then
                        return yes // decidable in linear time (Observation 1)
9 return no
```

**Proof of Theorem 5.1.** Given an instance  $\mathcal{I} = (X, \Phi = (\phi_1, \dots, \phi_\tau), d)$  of M2SAT, apply Construction 4 in  $\mathcal{O}(n^{4(n-d)+1} \cdot 2^{4(n-d)}\tau(n+m))$  time to obtain graph G with terminals s and t of size  $\mathcal{O}(n^{4(n-d)+1} \cdot 2^{4(n-d)}\tau)$  (Lemma 5.2). Return, in time linear in the size of G, yes if G admits an s-t path, and no otherwise (Lemma 5.3).

▶ Remark 5.4. Theorem 5.1 is asymptotically optimal regarding n-d unless the ETH breaks (Theorem 3.6). Moreover, Theorem 5.1 does not generalize to MULTISTAGE q-SAT for  $q \ge 3$ , as MqSAT is already NP-hard for one stage and hence for any number n-d.

# **6** XP Regarding Number of Stages and Consecutive Changes

In this section, we prove that MULTISTAGE 2-SAT is in XP when parameterized by  $\tau + d$ .

▶ Theorem 6.1. MULTISTAGE 2-SAT is solvable in  $\mathcal{O}(n^{2\tau \cdot d} \cdot 2^{\tau \cdot d+1} \cdot \tau \cdot (n+m))$  time.

Let  $\mathcal{I} = (X, \Phi = (\phi_1, \dots, \phi_{\tau}), d)$  be a fixed yet arbitrary instance with  $\tau \cdot d < n$ , as otherwise Theorem 4.1 applies. On a high level, our Algorithm 1 works as follows:

- (1) Guess  $q \leq \tau \cdot d$  variables  $X' \subseteq X$  that will change over time.
- (2) Guess an initial truth assignment of the variables in X'.
- (3) For each but the first stage, guess the at most  $\min\{q,d\}$  possible variables to change.
- (4) Set the variables to the guessed true or false values, delete clauses which are set to true.
- (5) Return yes if the resulting instance with d=0 is a yes-instance (linear-time checkable).
- (6) If the algorithm never (for all possible guesses) returned yes, then return no.

For any  $X' \subseteq X$ , define the set of all truth assignments to variables of X' with at most  $\min\{|X'|,d\}$  true values by

$$\mathcal{F}(X') := \{ f \colon X' \to \{\bot, \top\} \mid | \{x \in X' \mid f(x) = \top\} | \le \min\{|X'|, d\} \}.$$

With the next two lemmas, we prove that Algorithm 1 is correct and runs in XP-time regarding  $\tau + d$ .

▶ Lemma 6.2. Algorithm 1 returns yes if and only if the input instance is a yes-instance.

**Proof.** ( $\Rightarrow$ ) If Algorithm 1 returns yes, then for some  $X' \subseteq X$ , and some  $f_1, \ldots, f_{\tau}$  that simplified  $\phi_1, \ldots, \phi_{\tau}$  to  $\phi_1^*, \ldots, \phi_{\tau}^*$ , instance  $\mathcal{I}^* := (X \setminus X', (\phi_1^*, \ldots, \phi_{\tau}^*), 0)$  is a yes-instance of M2SAT. Let  $f_1^*, \ldots, f_{\tau}^* \colon X \setminus X' \to \{\bot, \top\}$  be a solution to  $\mathcal{I}^*$ . Let  $h_1, \ldots, h_{\tau}$  be defined

as  $h_i(x) := f_i(x)$  if  $x \in X'$ , and  $h_i(x) := f_i^*(x)$  otherwise, i.e., if  $x \in X \setminus X'$ . We claim that  $(h_1, \ldots, h_{\tau})$  is a solution to  $\mathcal{I}$ . Observe that  $h_i$  satisfies  $\phi_i$  for each  $i \in \{1, \ldots, \tau\}$ . Moreover, for each  $i \in \{1, \ldots, \tau - 1\}$  we have  $|\{x \in X \mid h_i(x) \neq h_{i+1}(x)\}| = |\{x \in X' \mid g_{i+1}(x) = \top\}| \leq d$ .

- ( $\Leftarrow$ ) Let  $h=(h_1,\ldots,h_{\tau})$  be a solution to  $\mathcal{I}$ . Let  $X'\subseteq X$  with  $|X'|\leq \tau\cdot d$  the set of all variables which change at least once over the stages their true-false value. Algorithm 1 guesses X' in line 1. Let  $f=(f_1,\ldots,f_{\tau})$  be such that for each  $i\in\{1,\ldots,\tau\}$ ,  $f_i\colon X'\to\{\bot,\top\}$  is  $h_i$  restricted to the variables in X'. In line 2, Algorithm 1 guesses  $f_1$ . Since h is a solution to  $\mathcal{I}$ , we know that  $|\{x\in X\mid h_i(x)\neq h_{i+1}(x)\}|=|\{x\in X\mid f_i(x)\neq f_{i+1}(x)\}|\leq \min\{|X'|,d\}$  for each  $i\in\{1,\ldots,\tau-1\}$ . It follows that for each  $i\in\{2,\ldots,\tau\}$  there exists a  $g_i\in\mathcal{F}(X')$  such that  $f_i(x)=f_{i-1}(x)\oplus g_i(x)$ . Algorithm 1 guesses  $g_2,\ldots,g_{\tau}$  in line 4, and finds f in line 6. Let  $(\phi_1^*,\ldots,\phi_{\tau}^*)$  be the formulas  $(\phi_1,\ldots,\phi_{\tau})$  simplified according to f, as done by Algorithm 1 in line 3 and line 6. Since h is a solution,  $f'=(f'_1,\ldots,f'_{\tau})$  where for each  $i\in\{1,\ldots,\tau\}$ ,  $f'_i\colon X\setminus X'\to\{\bot,\top\}$  is  $h_i$  restricted to the variables in  $X\setminus X'$ , is a solution to  $(X\setminus X',(\phi_1^*,\ldots,\phi_{\tau}^*),0)$ . Hence,  $(X\setminus X',(\phi_1^*,\ldots,\phi_{\tau}^*),0)$  is a yes-instance, and consequently Algorithm 1 returns yes in line 8.
- ▶ Lemma 6.3. Algorithm 1 runs in  $\mathcal{O}(n^{2\tau \cdot d} \cdot 2^{\tau \cdot d+1}\tau(n+m))$  time.

**Proof.** The running time  $T(\mathcal{I})$  is  $T(\mathcal{I}) \leq (1 + n^{\tau \cdot d}) \cdot T_1(\mathcal{I})$ , where  $T_1(\mathcal{I})$  is the worst-case running time inside the first for-loop (line 2 to line 8). Analogously, we have  $T_1(\mathcal{I}) \leq 2^{\tau \cdot d} \cdot T_2(\mathcal{I})$ , and  $T_2(\mathcal{I}) \in \mathcal{O}(n+m) + (1 + (\tau \cdot d)^d)^{\tau-1} \cdot T_3(\mathcal{I})$ . Now,  $T_3(\mathcal{I}) \in \mathcal{O}(\tau(n+m))$ , as line 6 can be done in  $\mathcal{O}(n+m)$  time with  $(\tau-1)$  executions of this line, and checking the if-condition for line 8 can be done in  $\mathcal{O}(\tau(n+m))$  time. We arrive at

$$T(\mathcal{I}) \in \mathcal{O}((1+n^{\tau \cdot d}) \cdot 2^{\tau \cdot d} \cdot ((n+m) + (1+\tau \cdot d)^{\tau \cdot d} \cdot \tau(n+m)))$$

$$\subseteq \mathcal{O}(n^{2\tau \cdot d} \cdot 2^{\tau \cdot d+1} \cdot \tau(n+m))$$

We are set to prove the main result from this section.

**Proof of Theorem 6.1.** Let  $\mathcal{I} = (X, \Phi = (\phi_1, \dots, \phi_{\tau}), d)$  be an instance of M2SAT with n variables and at most m clauses in each stage's formula. If  $\tau \cdot d \geq n$ , then, by Theorem 4.1, we know that M2SAT is solvable in  $\mathcal{O}(2^{2\tau \cdot d} \cdot \tau(n+m))$  time. Otherwise, if  $\tau \cdot d < n$ , then Algorithm 1 runs in  $\mathcal{O}(n^{2\tau \cdot d} \cdot 2^{\tau \cdot d+1}\tau(n+m))$  time (Lemma 6.3) and correctly decides  $\mathcal{I}$  (Lemma 6.2).

▶ Remark 6.4. Theorem 6.1 is asymptotically optimal regarding d unless the ETH breaks (Theorem 3.1). Moreover, Theorem 6.1 is not adaptable to MULTISTAGE q-SAT with  $q \geq 3$  unless P = NP since MULTISTAGE q-SAT with  $q \geq 3$  is NP-hard even with  $\tau + d \in \mathcal{O}(1)$ .

#### 7 Efficient and Effective Data Reduction

In this section, we study efficient and provably effective data reduction for MULTISTAGE 2-SAT in terms of problem kernelization. We focus on the parameter combinations n+m,  $n+\tau$ , and  $m+\tau$ . We prove that no problem kernelization of size polynomial in n+m exists unless NP  $\subseteq$  coNP / poly (Section 7.1), and that a problem kernelization of size quadratic in  $m+\tau$  and of size cubic in  $n+\tau$  exists (Section 7.2). Finally, we prove that no problem kernel of size truly subquadratic in  $m+\tau$  exists unless NP  $\subseteq$  coNP / poly (Section 7.2.1).

#### 7.1 No Time-Independent Polynomial Problem Kernelization

When parameterized by n + m, efficient and effective data reduction appears unlikely.

▶ **Theorem 7.1.** Unless NP  $\subseteq$  coNP / poly, MULTISTAGE 2-SAT admits no problem kernel of size polynomial in  $n^{f(m,d)}$ , for any function f only depending on m and d.

We will prove Theorem 7.1 via an AND-composition [6, 7], that is, we prove that given t instances of MULTISTAGE 2-SAT, each with d=1 and the same number of variables and stages, we can compute in polynomial time an instance of MULTISTAGE 2-SAT such that all input instances are yes if and only if the output instance is yes, and the number of variables and the maximum number of clauses in one stage does not exceed those from all input instances. Drucker [16] proved that if a parameterized problem admits an AND-composition from an NP-hard problem, then it admits no polynomial problem kernelization, unless NP  $\subseteq$  coNP / poly.

▶ Construction 5. Let  $\mathcal{I}_1, \ldots, \mathcal{I}_t$  be t instances of M2SAT with d=1, m=6, n variables, and  $\tau$  stages, where  $\mathcal{I}_i = (X^i, \Phi^i, d)$  with  $X^i = \{x_1^i, \ldots, x_n^i\}$  and  $\Phi^i = (\phi_1^i, \ldots, \phi_{\tau}^i)$ . Construct the instance  $\mathcal{I} := (X, \Phi, d)$  as follows. Construct the set  $X = \{x_1, \ldots, x_n\}$  of variables, and identify  $x_j$  with  $x_j^i$  for each  $i \in \{1, \ldots, t\}, j \in \{1, \ldots, n\}$ . In a nutshell, we construct the sequence of formulas by chaining up the input instances' formulas, and add n stages between any two consecutive instances each consisting of the always-true formula  $(x_1 \vee \neg x_1)$ —these ensure a reconfiguration of the last truth assignment to the initial truth assignment of the subsequent instance. Formally, construct  $\Phi = (\phi_1, \ldots, \phi_{t \cdot (\tau + n)})$  as follows. For all  $i \in \{1, \ldots, t\}, j \in \{1, \ldots, \tau + n\}$ , set (where  $S(i) := (i-1) \cdot (\tau + n)$ )

$$\phi_{S(i)+j} := \begin{cases} \phi_j^i, & \text{if } 1 \le j \le \tau, \\ (x_1 \lor \neg x_1), & \text{if } \tau + 1 \le j \le \tau + n. \end{cases}$$

Finally, set d = 1.

▶ Lemma 7.2. Let  $\mathcal{I}_1, \ldots, \mathcal{I}_t$  be t instances of MULTISTAGE 2-SAT with d=1, m=6, n variables, and  $\tau$  stages, and let  $\mathcal{I}$  be the instance obtained from Construction 5. Then, each  $\mathcal{I}_i$  is a yes-instance if and only if  $\mathcal{I}$  is a yes-instance.

 $\Diamond$ 

**Proof.** ( $\Leftarrow$ ) Let  $(f_1, \ldots, f_{t(\tau+n)})$  be a solution to  $\mathcal{I}$ . It is not difficult to see that, for each  $i \in \{1, \ldots, t\}$ , the sequence  $(f_{S(i)+1}, \ldots, f_{S(i)+\tau})$  is a solution to  $\mathcal{I}_i$ .

( $\Rightarrow$ ) For each  $i \in \{1, ..., t\}$ , let  $(f_1^i, ..., f_{\tau}^i)$  denote a solution for  $\mathcal{I}_i$ . We construct a solution  $f = (f_1, ..., f_{t(\tau+n)})$  for  $\mathcal{I}$  as follows. For each  $i \in \{1, ..., t\}$  and  $j \in \{1, ..., \tau\}$ , set  $f_{S(i)+j} := f_j^i$ . For each  $i \in \{1, ..., t-1\}$ , we define  $f_{S(i)+\tau+1}, ..., f_{S(i)+\tau+n}$  iteratively as follows. For j = 1, ..., n, let

$$f_{S(i)+\tau+j}(x) := \begin{cases} f_{S(i)+\tau+(j-1)}(x), & \text{if } x \in X \setminus \{x_j\}, \\ f_{S(i+1)+1}(x), & \text{if } x = x_j. \end{cases}$$

Observe that for each  $j \in \{1, ..., n\}$ , it holds true that  $|\{x \in X \mid f_{S(i)+\tau+(j-1)}(x) \neq f_{S(i)+\tau+j}(x)\}| \leq 1$ , and that  $f_{S(i)+\tau+n} = f_{S(i+1)+1}$ .

**Proof of Theorem 7.1.** Construction 5 forms an AND-composition (Lemma 7.2) from an NP-hard special case of M2SAT (Theorem 3.4) to M2SAT when parameterized by n + m, in fact, mapping m and d to a constant. Thus, due to Drucker [16], M2SAT admits no problem kernelization of size polynomial in  $n^{f(m,d)}$  for any function f only depending on m and d.

▶ Remark 7.3. Due to Theorem 4.1, MULTISTAGE 2-SAT yet admits a problem kernel of size  $2^{\mathcal{O}(n)}$ .

# 7.2 Polynomial Problem Kernelizations

We prove problem kernelizations of size polynomial in  $n + \tau$  and  $m + \tau$ .

▶ **Theorem 7.4.** MULTISTAGE 2-SAT admits a linear-time computable problem kernelization of size  $\mathcal{O}(n^2\tau)$  and of size  $\mathcal{O}(m \cdot \tau)$ .

We employ the following two immediate reduction rules (each is clearly correct and applicable in linear time):

- ▶ **Reduction Rule 2.** In each stage, delete all but one appearances of a clause in the formula.
- ▶ Reduction Rule 3. Delete a variable that appears in no stage's formula as a literal.

**Proof of Theorem 7.4.** Observe that there are at most  $N := 2n + \binom{2n}{2} \in \mathcal{O}(n^2)$  many pairwise different clauses. After exhaustively applying Reduction Rule 2, we have  $m \leq N \in \mathcal{O}(n^2)$ . After exhaustively applying Reduction Rule 3, it follows that for each variable, there is at least one clause, and hence,  $n \leq 2 \cdot m \cdot \tau$ .

▶ Remark 7.5. Theorem 7.4 adapts easily to MULTISTAGE q-SAT. Herein, the problem kernel sizes are  $\mathcal{O}(n^q \cdot \tau)$  and  $\mathcal{O}(q \cdot m \cdot \tau)$ .

Subsequently, we prove that a linear kernel appears unlikely.

# 7.2.1 No Subquadratic Problem Kernelization

▶ **Theorem 7.6.** Unless NP  $\subseteq$  coNP / poly, MULTISTAGE 2-SAT admits no problem kernel of size  $\mathcal{O}((m+n+\tau)^{2-\varepsilon})$  for any  $\varepsilon > 0$ .

To prove Theorem 7.6, we show that there is a linear parametric transformation from Vertex Cover parameterized by |V| to Multistage 2-SAT parameterized by  $n + m + \tau$ .

▶ Construction 6. Let  $\mathcal{I} = (G, k)$  with G = (V, E) be an instance of VERTEX COVER. Denote the vertices  $V = \{v_1, \dots, v_n\}$ . We construct the instance  $\mathcal{I}' = (X, \Phi, d)$  of M2SAT with d = k and  $\Phi = (\phi_0, \phi_1, \dots, \phi_n)$  as follows. Let  $X = X_V \cup B$  with  $X_V = \{x_i \mid v_i \in V\}$  and  $B = \{b_1, \dots, b_k\}$ . Let

$$\phi_0 := \bigwedge_{i=1}^n (\neg x_i) \wedge \bigwedge_{j=1}^k (\neg b_j) \quad \text{and}$$

$$\phi_i := \bigwedge_{\{v_i, v_j\} \in E} (x_i \vee x_j) \wedge \begin{cases} \bigwedge_{j=1}^k (b_j) & \text{if } i \bmod 2 = 0, \\ \bigwedge_{j=1}^k (\neg b_j) & \text{if } i \bmod 2 = 1, \end{cases} \quad \forall i \in \{1, \dots, n\}.$$

Note that  $\tau + m + |X| \in \mathcal{O}(n)$ , since each vertex degree is at most n - 1.

▶ Lemma 7.7. Let  $\mathcal{I} = (G, k)$  be an instance of Vertex Cover, and let  $\mathcal{I}' = (X', \Phi', d)$  be the instance of Multistage 2-SAT obtained from  $\mathcal{I}$  using Construction 6. Then,  $\mathcal{I}$  is a yes-instance if and only if  $\mathcal{I}'$  is a yes-instance.

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**Proof.** ( $\Rightarrow$ ) Let  $V' \subseteq V$  be a size-at-most-k vertex cover of G. Let  $X_W := \{x_i \in X_V \mid v_i \in W\}$ . Define  $f_0 \colon X \to \{\bot, \top\}$  such that  $f_0(x) = \bot$  for all  $x \in X$ . Define  $f_1, \ldots, f_n \colon X \to \{\bot, \top\}$  and  $f^* \colon X_V \to \{\bot, \top\}$  as

$$f_i(x) = \begin{cases} f^*(x), & \text{if } x \in X_V, \\ \top, & \text{if } x \in B \text{ and } i \text{ mod } 2 = 0, \\ \bot, & \text{if } x \in B \text{ and } i \text{ mod } 2 = 1, \end{cases} \text{ where } f^*(x) = \begin{cases} \top, & \text{if } x \in X_W, \\ \bot, & \text{if } x \in X_V \setminus X_W. \end{cases}$$

Observe that  $|\{x \in X \mid f_0(x) \neq f_1(x)\}| = |\{x \in X_V \mid f^*(x) = \top\}| = |X_W| = |W| \leq k$ . Moreover, for each  $i \in \{1, \dots, n-1\}$ , we have that  $|\{x \in X \mid f_i(x) \neq f_{i+1}(x)\}| = |B| = k$ . It is not difficult to see that  $f_i$  satisfies  $\phi_i$  for each  $i \in \{0, \dots, \tau\}$ . Hence,  $(f_0, f_1, \dots, f_n)$  is a solution to  $\mathcal{I}'$ .

( $\Leftarrow$ ) Let  $f = (f_0, f_1, \ldots, f_n)$  be a solution to  $\mathcal{I}'$ . By construction of  $\phi_0$ , it must hold that  $f_0(x) = \bot$  for all  $x \in X$ . Moreover, by construction of  $\phi_1$ , we know that  $f_1(x) = \bot$  for all  $x \in B$ , and hence  $X' \coloneqq \{x \in X_V \mid f_0(x) \neq f_1(x)\} = \{x \in X_V \mid f_1(x) = \top\}$  has  $|X'| \leq k$ . Since for each  $i \in \{1, \ldots, n-1\}$ , we have that  $\{x \in X \mid f_i(x) \neq f_{i+1}(x)\} = B$  by construction, we know that for each  $i, j \in \{1, \ldots, n\}$  it holds true that  $f_i(x) = f_j(x)$  for all  $x \in X_V$ . We claim that  $W = \{v_i \in V \mid x_i \in X'\}$  is a size-at-most-k vertex cover of K. We know that  $|K| = |K'| \leq k$ . Suppose towards a contradiction that there is an edge  $\{v_i, v_j\} \in E$  disjoint from K. This implies that K is a size-at-most-K vertex cover of K is a satisfying truth assignment. It follows that K is a size-at-most-K vertex cover of K, and thus, K is a yes-instance.

**Proof of Theorem 7.6.** Construction 6 is a linear parametric transformation (Lemma 7.7) such that  $\tau + m + |X| \in \mathcal{O}(|V|)$ . Since VERTEX COVER admits no problem bikernelization of size  $\mathcal{O}(|V|^{2-\varepsilon})$ ,  $\varepsilon > 0$  [14], the statement follows.

▶ Remark 7.8. Theorem 7.6 can be easily adapted to MULTISTAGE q-SAT when taking q-HITTING SET as source problem [14], ruling out problem kernelizations of size  $\mathcal{O}((n+m+\tau)^{q-\varepsilon})$ ,  $\varepsilon > 0$  (unless NP  $\subseteq$  coNP / poly).

#### 8 Conclusion

While 2-SAT is linear-time solvable, its multistage model Multistage 2-SAT is intractable in even surprisingly restricted cases. This is also reflected by the fact that several of our direct upper bounds are already asymptotically optimal. By our results, the most interesting difference between Multistage 2-SAT and Multistage q-SAT, with  $q \geq 3$ , is that the former is efficiently solvable if the numbers of stages and allowed consecutive changes are constant, which is not the case for the latter (unless P = NP). Finally, our results show that exact solutions are far from practical, waving the path for randomized or heuristic approaches.

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