

Exploring Scale-Measures of Data Sets

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Abstract Measurement is a fundamental building block of numerous scientific models and their creation. This is in particular true for data driven science. Due to the high complexity and size of modern data sets, the necessity for the development of understandable and efficient scaling methods is at hand. A profound theory for scaling data is scale-measures, as developed in the field of formal concept analysis. Recent developments indicate that the set of all scale-measures for a given data set constitutes a lattice and does hence allow efficient exploring algorithms. In this work we study the properties of said lattice and propose a novel scale-measure exploration algorithm that is based on the well-known and proven attribute exploration approach. Our results motivate multiple applications in scale recommendation, most prominently (semi-)automatic scaling.

keywords: FCA, Conceptual Measures, Data Scaling, Measurements, Formal Concept, Lattice

1 Introduction

An inevitable step of any data-based knowledge discovery process is *measurement* [24] and the associated (explicit or implicit) *scaling* of the data [27]. The latter is particularly constrained by the underlying mathematical formulation of the data representation, e.g., real-valued vector spaces or weighted graphs, the requirements of the data procedures, e.g., the presence of a distance function, and, more recently, the need for human understanding of the results. Considering the scaling of data as part of the analysis itself, in particular formalizing it and thus making it controllable, is a salient feature of formal concept analysis (FCA) [7]. This field of research has spawned a variety of specialized scaling methods, such as logical scaling [25], and in the form of *scale-measures* links the scaling process with the study of *continuous mappings* between *closure systems*.

Recent results by the authors [13] revealed that the set of all scale-measures for a given data set constitutes a lattice. Furthermore, it was shown that any scale-measure can be expressed in simple propositional terms using disjunction, conjunction and negation. Among other things, the previous results allow a computational transition between different scale-measures, which we may call *scale-measure navigation*, as well as their *interpretability* by humans.

Despite these advances, the question of how to identify appropriate and meaningful scale-measures for a given data set with respect to a human data analyst

and how to express that meaningfulness in the first place remains unanswered. In this paper, we propose an answer to this question by adapting the well-known *attribute exploration algorithm* from FCA to present a method for exploring scale measures. Very similar to the original algorithm does *scale-measure exploration* inquire a (human) scaling expert for how to aggregate, separate, omit, or introduce data set features. Our efforts do finally result in a (semi-)automatic scaling framework which may be applied to large and complex data sets.

In detail, after recalling scale-measure basics in Section 3 we apply theoretical results for ideals in closure systems to the lattice of all scale-measures. From this we derive notions for the relevance of scale-measures as well as the mentioned novel exploration method in Section 4, which is supported by a detailed example. Finally, in Section 4.2, we outline the (semi-)automatic scaling framework and conclude in Section 6 after revisiting related work about scaling in Section 5.

2 Scales and Measurement

FCA Recap Formalizing and understanding the process of *measurement* is, in particular in data science, an ongoing discussion, for which we refer the reader to *Representational Theory of Measurement* [20, 29] as well as *Numerical Relational Structure* [24], and *algebraic (measurement) structures* [26, p. 253].

Formal concept analysis (FCA) [7, 31] is well equipped to handle and comprehend data scaling tasks. In FCA the basic data structure is the *formal contexts* as seen in the example Figure 1 (top), i.e., a triple (G, M, I) with non-empty and finite set G (called *objects*), non-empty and finite set M (called *attributes*) and a binary relation $I \subseteq G \times M$ (called *incidence*). We say $(g, m) \in I$ is equivalent to “ g has attribute m ”. We call $\mathbb{S} = (H, N, J)$ an *induced sub-context* of \mathbb{K} , iff $H \subseteq G, N \subseteq M$ and $I_{\mathbb{S}} = I \cap (H \times N)$, and write $\mathbb{S} \leq \mathbb{K}$. We find two operators $\cdot' : \mathcal{P}(G) \rightarrow \mathcal{P}(M), A \mapsto A' = \{m \in M \mid \forall a \in A : (a, m) \in I\}$, and $\cdot' : \mathcal{P}(M) \rightarrow \mathcal{P}(G), B \mapsto B' = \{g \in G \mid \forall b \in B : (g, b) \in I\}$, called *derivations*. Pairs $(A, B) \in \mathcal{P}(G) \times \mathcal{P}(M)$ with $A' = B$ and $A = B'$, are called *formal concepts*, where A is called *extent* and B *intent*. Consecutive application leads to two *closure spaces* $\text{Ext}(\mathbb{K}) := (G, '')$ and $\text{Int}(\mathbb{K}) := (M, '')$. Both closure systems are represented in the (*concept*) *lattice* $\mathfrak{B}(\mathbb{K}) = (\mathcal{B}(\mathbb{K}), \subseteq)$, where $\mathcal{B}(\mathbb{K}) := \{(A, B) \in \mathcal{P}(G) \times \mathcal{P}(M) \mid A' = B \wedge B' = A\}$ is the set of concepts of \mathbb{K} and the order relation is $(A, B) \leq (C, D) \Leftrightarrow A \subseteq C$.

2.1 Scales-Measures

A fundamental approach to comprehensible scaling, in particular for nominal and ordinal data as studied in this work, is the following.

Definition 1 (Scale-Measure (cf. Definition 91, [7])). *Let $\mathbb{K} = (G, M, I)$ and $\mathbb{S} = (G_{\mathbb{S}}, M_{\mathbb{S}}, I_{\mathbb{S}})$ be a formal contexts. The map $\sigma : G \rightarrow G_{\mathbb{S}}$ is called an \mathbb{S} -measure of \mathbb{K} into the scale \mathbb{S} iff the preimage $\sigma^{-1}(A) := \{g \in G \mid \sigma(g) \in A\}$ of every extent $A \in \text{Ext}(\mathbb{S})$ is an extent of \mathbb{K} .*

	has limbs (L)	breast feeds (BF)	needs chlorophyll (Ch)	needs water to live (W)	lives on land (LL)	lives in water (LW)	can move (M)	monocotyledon (MC)	dicotyledon (DC)
dog	x	x		x	x		x		
fish				x		x	x		
leech				x		x			
corn			x		x			x	
bream	x			x		x	x		
water weeds			x	x		x		x	
bean			x	x	x				x
frog	x			x	x	x	x		
reed			x	x	x	x		x	

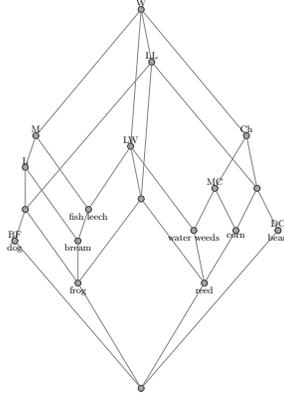


Figure 1. This Figure shows the *Living Beings and Water* context in the top. Its concept lattice is displayed at the bottom and contains nineteen concepts.

This definition resembles the idea of *continuity between closure spaces* (G_1, c_1) and (G_2, c_2) . We say that the map $f : G_1 \rightarrow G_2$ is *continuous* if and only if for all $A \in \mathcal{P}(G_2)$ we have $c_1(f^{-1}(A)) \subseteq f^{-1}(c_2(A))$. This property is equivalent to the requirement in Definition 1 that the preimage of closed sets is closed.

In the light of the definition above we understand σ as an interpretation of the objects from \mathbb{K} in \mathbb{S} . Therefore we view the set $\sigma^{-1}(\text{Ext}(\mathbb{S})) := \bigcup_{A \in \text{Ext}(\mathbb{S})} \sigma^{-1}(A)$ as the set of extents that is *reflected* by the scale context \mathbb{S} .

We present in Figure 2 the scale-context for some scale-measure and its concept lattice, derived from our *running example* context *Living Beings and Water* \mathbb{K}_W , cf. Figure 1. This scaling is based on the original object set G , however, the attribute set is comprised of nine, partially new, elements, which may reflect specie taxons. We observe in this example that the concept lattice of the scale-measure context reflects twelve out of the nineteen concepts from $\mathfrak{B}(\mathbb{K}_W)$.

In our work [13] we derived a *scale-hierarchy* on the set of scale-measures, i.e., $\mathfrak{S}(\mathbb{K}) := \{(\sigma, \mathbb{S}) \mid \sigma \text{ is a } \mathbb{S}\text{-measure of } \mathbb{K}\}$, from a natural order of scales introduced by Ganter and Wille [7, Definition 92]). We say for two scale-measures $(\sigma, \mathbb{S}), (\psi, \mathbb{T})$ that (σ, \mathbb{S}) is finer than (ψ, \mathbb{T}) , iff $\psi^{-1}(\text{Ext}(\mathbb{T})) \subseteq \sigma^{-1}(\text{Ext}(\mathbb{S}))$, from which also follows a natural equivalence relation \sim .

Definition 2 (Scale-Hierarchy (cf. Definition 7, [13])). For a formal context \mathbb{K} we call $\underline{\mathfrak{S}}(\mathbb{K}) = (\mathfrak{S}(\mathbb{K})/\sim, \leq)$ the scale-hierarchy \mathbb{K} .

	W	LW	plants	animals	land plants	water plants	land animal	water animal	mammal
dog	x			x			x		x
fish	x	x		x				x	
leech	x		x						
corn	x				x				
bream	x	x		x				x	
water weeds	x	x	x			x			
bean	x		x		x				
frog	x	x		x			x	x	
reed	x	x	x		x	x			

plants := Ch
 animals := M
 land plants := LL \wedge plant
 water plants := LW \wedge plant
 land animal := LL \wedge animal
 water animal := LW \wedge animal
 mammal := animal \wedge BF

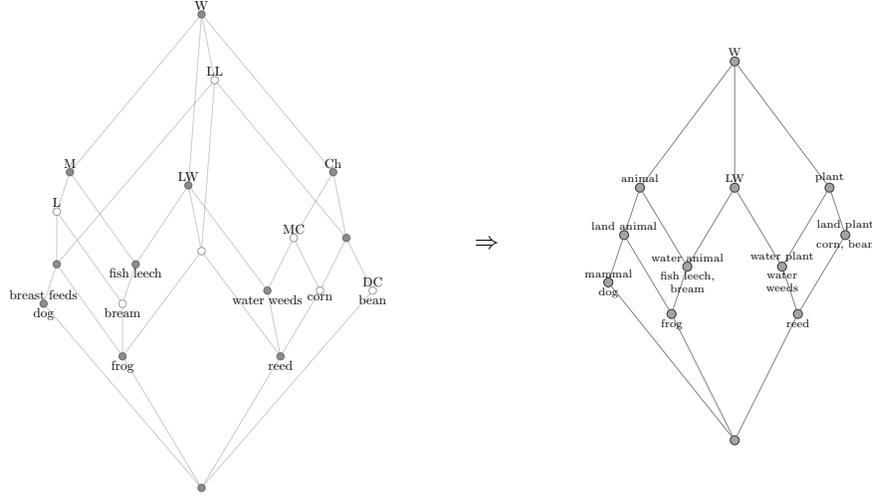


Figure 2. A scale context (top), its concept lattice (bottom right) for which id_G is a scale-measure of the context in Figure 1. The reflected extents by the scale $\sigma^{-1}(\text{Ext}(\mathbb{S}))$ of the scale-measure are indicated in gray in the contexts concept lattice (bottom left).

Also in [13], we have shown that the scale-hierarchy of a context \mathbb{K} is lattice ordered and isomorphic to the set of all sub-closure systems of $\text{Ext}(\mathbb{K})$, i.e., $\{Q \subseteq \text{Ext}(\mathbb{K}) \mid Q \text{ is a Closure System on } G\}$ that is ordered by set inclusion \subseteq . To show this, we defined a *canonical representation* of scale-measures, using the so called *canonical scale* $\mathbb{K}_{\mathcal{A}} := (G, \mathcal{A}, \in)$ for $\mathcal{A} \subseteq \text{Ext}(\mathbb{K})$ with $\text{Ext}(\mathbb{K}_{\mathcal{A}}) = \mathcal{A}$.

Proposition 1 (Canonical Representation (cf. Proposition 10, [13])).
 Let $\mathbb{K} = (G, M, I)$ be a formal context with scale-measure $(\mathbb{S}, \sigma) \in \mathfrak{S}(\mathbb{K})$, then $(\sigma, \mathbb{S}) \sim (\text{id}, \mathbb{K}_{\sigma^{-1}(\text{Ext}(\mathbb{S}))})$.

We argued in [13] that the canonical representation eludes human explanation to some degree. To remedied this issue by means of logical scaling [25] which led to to scales with logical attributes $M_{\mathbb{S}} \subseteq \mathcal{L}(M, \{\wedge, \vee, \neg\})$ ([13, Problem 1]).

Proposition 2 (Conjunctive Normalform (cf. Proposition 23, [13])).
 Let \mathbb{K} be a context, $(\sigma, \mathbb{S}) \in \mathfrak{S}(\mathbb{K})$. Then the scale-measure $(\psi, \mathbb{T}) \in \mathfrak{S}(\mathbb{K})$ given by

$$\psi = \text{id}_G \quad \text{and} \quad \mathbb{T} = \bigwedge_{A \in \sigma^{-1}(\text{Ext}(\mathbb{S}))} (G, \{\phi = \wedge A^I\}, I_{\phi})$$

is equivalent to (σ, \mathbb{S}) and is called conjunctive normalform of (σ, \mathbb{S}) .

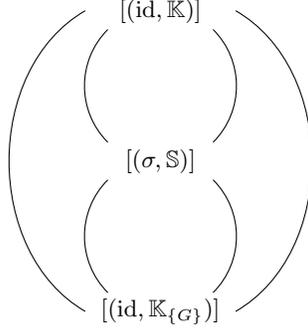


Figure 3. Scale-hierarchy of \mathbb{K} with indicated scale-measures.

3 Ideals in the Lattice of Closure Systems

The goal for the rest of this work is to identify *outstanding* and particularly interesting data scalings. This quest leads to the natural question for a structural understanding of the scale-hierarchy and its elements. In order to do this we rely on the isomorphism [13, Proposition 11] between a context's scale-hierarchy $\underline{\mathfrak{S}}(\mathbb{K})$ and the lattice of all sub-closure systems of the extent set, as explained in the last section. The latter forms an order ideal in the lattice of all closure systems \mathfrak{F}_G on a set G , to which we refer by $\downarrow_{\mathfrak{F}_G} \text{Ext}(\mathbb{K})$. This ideal is well studied [1] and we may often omit the index \mathfrak{F}_G to improve the readability.

Equipped with this structure we have to recall a few notions and definitions for a complete lattices (L, \leq) . In the following, we denote by \prec the *cover relation* of \leq . Furthermore, we say L is 1) *lower semi-modular* if and only if $\forall x, y \in L : x \prec x \vee y \implies x \wedge y \prec y$, 2) *join-semidistributive* iff $\forall x, y, z \in L : x \vee y = x \vee z \implies x \vee y = x \vee (y \wedge z)$, 3) *meet-distributive (lower locally distributive, cf [1])* iff L is join-semidistributive and lower semi-modular, 4) *join-pseudocomplemented* iff $x \in L$ the set $\{y \in L \mid y \vee x = \top\}$ has a least, 5) *ranked* iff there is a function $\rho : L \mapsto \mathbb{N}$ with $x \prec y \implies \rho(x) + 1 = \rho(y)$, 6) *atomistic* iff every $x \in L$ can be written as the join of atoms in L . In addition to the just introduced lattice properties, there are properties for elements in L that we consider. An element $x \in L$ is 1) *neutral* iff every triple $\{x, y, z\} \subseteq L$ generates a distributive sublattice of L , 2) *distributive* iff the equalities $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for every $y, z \in L$ hold, 3) *meet irreducible* iff $x \neq \top$ and $\bigwedge_{y \in Y} y$ for $Y \subseteq L$ implies $x \in Y$, 4) *join irreducible* iff $x \neq \perp$ and $\bigvee_{y \in Y} y$ for $Y \subseteq L$ implies $x \in Y$. Throughout the rest of this work, we denote by $\mathcal{M}(L)$ the set of all meet-irreducible elements of L .

We can derive from literature [1, Proposition 19] the following statement.

Corollary 1. For $\mathbb{K} = (G, M, I)$, $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ and $\mathcal{R}, \mathcal{R}' \in \downarrow \text{Ext}(\mathbb{K})$ we find the equivalence: $\mathcal{R}' \prec \mathcal{R} \iff \mathcal{R}' \cup \{A\} = \mathcal{R}$ with A is meet-irreducible in \mathcal{R}

Of special interest in lattices are the meet- and join-irreducibles, since every element of a lattice can be represented as a join or meet of these elements.

Proposition 3. For \mathbb{K} , $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ and $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$ we find the equivalence: \mathcal{R} join-irreducible in $\downarrow \text{Ext}(\mathbb{K}) \iff \exists A \in \text{Ext}(\mathbb{K}) \setminus \{G\} : \mathcal{R} = \{G, A\}$

Proof. \Leftarrow : For $A \in \text{Ext}(\mathbb{K}) \setminus \{G\}$ is $\{A, G\}$ a closure system on G and thereby in $\downarrow \text{Ext}(\mathbb{K})$. Further, the set $\{A, G\}$ is of cardinality two and thereby an atom of $\downarrow \text{Ext}(\mathbb{K})$ and thus join-irreducible. \Rightarrow : By contradiction assume that $\nexists A \in \text{Ext}(\mathbb{K}) \setminus \{G\} : \mathcal{R} = \{G, A\}$, then for every $D \in \mathcal{R} \setminus \{G\}$ is $\{D, G\}$ an atom of $\downarrow \text{Ext}(\mathbb{K})$, hence, $\mathcal{R} = \bigvee_{D \in \mathcal{R} \setminus \{G\}} \{D, G\}$, i.e., not join-irreducible.

Next, we investigate the meet-irreducibles of $\downarrow \text{Ext}(\mathbb{K})$ using a similar approach as done for \mathfrak{F}_G [1] based on propositional logic. We recall, that an (object) implication for some context \mathbb{K} is a pair $(A, B) \in \mathcal{P}(G) \times \mathcal{P}(G)$, shortly denoted by $A \rightarrow B$. We say $A \rightarrow B$ is valid in \mathbb{K} iff $A' \subseteq B'$. The set $\mathcal{F}_{A,B} := \{D \subseteq G : A \not\subseteq B \vee B \subseteq D\}$ contains all *models* of $A \rightarrow B$. Additionally, $\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})} := \mathcal{F}_{A,B} \cap \text{Ext}(\mathbb{K})$ is the set of all extents $D \in \text{Ext}(\mathbb{K})$ that are models of $A \rightarrow B$. The set $\mathcal{F}_{A,B}$ is a closure system [1] and therefore $\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})}$, too. Furthermore, we can deduce that $\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})} \in \downarrow \text{Ext}(\mathbb{K})$.

Lemma 1. For context \mathbb{K} , $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$, $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$ with closure operator $\phi_{\mathcal{R}}$ we find $\mathcal{R} = \bigcap \{\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})} \mid A, B \subseteq G \wedge B \subseteq \phi_{\mathcal{R}}(A)\}$.

Proof. We know that $\mathcal{R} = \bigcap \{\mathcal{F}_{A,B} \mid A, B \subseteq G \wedge B \subseteq \phi_{\mathcal{R}}(A)\}$ [1, Proposition 22]. Since $\mathcal{R} \subseteq \text{Ext}(\mathbb{K})$ it holds that $\mathcal{R} = \bigcap \{\mathcal{F}_{A,B} \mid A, B \subseteq G \wedge B \subseteq \phi_{\mathcal{R}}(A)\} \cap \text{Ext}(\mathbb{K})$ and thus equal to $\bigcap \{\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})} \mid A, B \subseteq G \wedge B \subseteq \phi_{\mathcal{R}}(A)\}$.

Note that for any $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$ the set $\{\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})} \mid A, B \subseteq G \wedge B \subseteq \phi_{\mathcal{R}}(A)\}$ contains only closure systems in $\downarrow \text{Ext}(\mathbb{K})$ and thus possibly meet-irreducible elements of $\downarrow \text{Ext}(\mathbb{K})$.

Proposition 4. For context \mathbb{K} , $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ and $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$, we find *tfac*: 1. \mathcal{R} is meet-irreducible in $\downarrow \text{Ext}(\mathbb{K})$ 2. $\exists A \in \text{Ext}(\mathbb{K}), i \in G$ with $A \prec_{\text{Ext}(\mathbb{K})} (A \cup \{i\})''$ such that $\mathcal{R} = \mathcal{F}_{A,\{i\}}|_{\text{Ext}(\mathbb{K})}$

Proof. [1. \Rightarrow 2.] Due to Lemma 1 we can represent $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$ by the equation $\mathcal{R} = \bigcap \{\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})} \mid A, B \subseteq G \wedge B \subseteq \phi_{\mathcal{R}}(A)\}$. Moreover, since \mathcal{R} is meet-irreducible in $\downarrow \text{Ext}(\mathbb{K})$, we can infer that $\mathcal{R} \in \{\mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})} \mid A, B \subseteq G \wedge B \subseteq \phi_{\mathcal{R}}(A)\}$. In particular there exist $A, B \subseteq G$ with $B \subseteq \phi_{\mathcal{R}}(A)$ such that $\mathcal{R} = \mathcal{F}_{A,B}|_{\text{Ext}(\mathbb{K})}$, and thus $\mathcal{R} = \mathcal{F}_{A'',B}|_{\text{Ext}(\mathbb{K})}$. Hence, we identify A'' by A for the rest of this proof. Using the fact that $\mathcal{F}_{A,\{i\}} \cap \mathcal{F}_{A,\{j\}} = \mathcal{F}_{A,\{i,j\}}$ we can infer that $\mathcal{F}_{A,\{i\}}|_{\text{Ext}(\mathbb{K})} \cap \mathcal{F}_{A,\{i\}}|_{\text{Ext}(\mathbb{K})} = \mathcal{F}_{A,\{i,j\}}|_{\text{Ext}(\mathbb{K})}$. Therefore, there must exist $A, \{i\} \subseteq G$ with $\mathcal{R} = \mathcal{F}_{A,\{i\}}|_{\text{Ext}(\mathbb{K})}$ (*).

In the case that $A = (A \cup \{i\})''$ the set $\mathcal{F}_{A,\{i\}}|_{\text{Ext}(\mathbb{K})} = \text{Ext}(\mathbb{K})$ and \mathcal{R} is thereby not meet-irreducible. Assume that $A \not\prec_{\text{Ext}(\mathbb{K})} (A \cup \{i\})''$, then there is a $D \in \text{Ext}(\mathbb{K})$ with $A \prec_{\text{Ext}(\mathbb{K})} D \subseteq (A \cup \{i\})''$ and $i \notin D$. Hence $A, D \not\subseteq A \rightarrow \{i\}$ (see *) and thus $A, D \notin \mathcal{R}$. Using this, we construct two sets $\mathcal{R} \cup \{A\}$ and $\mathcal{R} \cup \{D\}$. The set $\mathcal{R} \cup \{D\}$ is closed by intersection, since an intersection of D with an element in \mathcal{R} is a model of $A \rightarrow i$, thus $\mathcal{R} \cup \{D\} \in \downarrow \text{Ext}(\mathbb{K})$. The same

holds for $\mathcal{R} \cup \{A\}$ respectively. The intersection of $\mathcal{R} \cup \{A\}$ and $\mathcal{R} \cup \{D\}$ is equal to \mathcal{R} which is thereby not meet-irreducible, a contradiction.

[1. \Leftarrow 2.] Consider a closure system $\hat{\mathcal{F}} \in \downarrow \text{Ext}(\mathbb{K})$ with $\hat{\mathcal{F}}$ covers \mathcal{R} in $\downarrow \text{Ext}(\mathbb{K})$. By Corollary 1, we can represent $\hat{\mathcal{F}} = \mathcal{R} \cup \{D\}$ (*) with $D \notin \mathcal{R}$ and D is meet-irreducible in $\hat{\mathcal{F}}$ (and therefore $D \in \text{Ext}(\mathbb{K})$). Due to $\mathcal{R} \subseteq \hat{\mathcal{F}}$ the set $(A \cup \{i\})''$ is an element of $\hat{\mathcal{F}}$ and thereby the intersection $(A \cup \{i\})'' \cap D \in \hat{\mathcal{F}}$. Since $D \notin \mathcal{R}$, we can deduce that $D \not\models A \rightarrow i$ and therefor $A \subseteq D$ and $i \notin D$. From $A \prec_{\text{Ext}(\mathbb{K})} (A \cup \{i\})''$ we know that $(A \cup \{i\})'' \cap D = A$. Finally, $D \in \hat{\mathcal{F}} \implies A \in \hat{\mathcal{F}}$, and using (*), we can infer that $D = A$. Hence, $\mathcal{R} \cup \{A\}$ is the sole upper neighbour of \mathcal{R} in $\downarrow \text{Ext}(\mathbb{K})$ and thereby \mathcal{R} is meet-irreducible.

Propositions 3 and 4 provide a characterization of irreducible elements in $\downarrow \text{Ext}(\mathbb{K})$ and thereby in the scale-hierarchy of \mathbb{K} . Those may be of particular interest, since any element of $\downarrow \text{Ext}(\mathbb{K})$ is representable by irreducible elements.

Proposition 5. *For context \mathbb{K} , $A, B \in \text{Ext}(\mathbb{K})$ with $A \prec_{\text{Ext}(\mathbb{K})} B$, then if A is meet-irreducible in $\text{Ext}(\mathbb{K})$, follows $\mathcal{F}_{A,B} \upharpoonright_{\text{Ext}(\mathbb{K})}$ is a maximum meet-irreducible element in $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$.*

Proof. For $A \prec_{\text{Ext}(\mathbb{K})} B$, A is the only extent that that is not a model of implication $A \rightarrow B$, since every other superset of A in $\text{Ext}(\mathbb{K})$ is also a superset of B . Hence $\mathcal{F}_{A,B} \upharpoonright_{\text{Ext}(\mathbb{K})}$ is equal to $\text{Ext}(\mathbb{K}) \setminus \{A\}$. The only superset in $\downarrow \text{Ext}(\mathbb{K})$ is $\text{Ext}(\mathbb{K})$, which is not meet-irreducible.

Equipped with this characterization we look into counting the irreducibles.

Proposition 6. *For context \mathbb{K} , the number of meet-irreducible elements in the lattice $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ is equal to $|\prec_{\downarrow \text{Ext}(\mathbb{K})}|$.*

Proof. According to Proposition 4, an element $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$ is meet-irreducible iff it can be represented as $\mathcal{F}_{A,\{i\}} \upharpoonright_{\text{Ext}(\mathbb{K})}$ for some $A \in \text{Ext}(\mathbb{K})$ with $A \prec_{\text{Ext}(\mathbb{K})} (A \cup \{i\})''$. Hence the number of meet-irreducible elements is bound by the number of covering pairs $A \prec_{\text{Ext}(\mathbb{K})} B$ in $\text{Ext}(\mathbb{K})$. It remains to be shown that for \mathcal{R} there is only one pair $(A, B) \in \prec_{\text{Ext}(\mathbb{K})}$ with $B = (A \cup \{i\})''$ for some $i \in B \setminus A$ such that $\mathcal{R} = \mathcal{F}_{A,\{i\}} \upharpoonright_{\text{Ext}(\mathbb{K})}$. Assume there are $(A, B), (C, D) \in \prec_{\text{Ext}(\mathbb{K})}$ with $(A, B) \neq (C, D)$ and $\mathcal{F}_{A,B} \upharpoonright_{\text{Ext}(\mathbb{K})} = \mathcal{F}_{C,D} \upharpoonright_{\text{Ext}(\mathbb{K})}$. First, consider the case $A \neq C$. Without loss of generality let $A \not\subseteq C$, then we have $C \models A \rightarrow B$, but $C \not\models C \rightarrow D$. Therefore $C \in \mathcal{F}_{A,B} \upharpoonright_{\text{Ext}(\mathbb{K})}$ but $C \notin \mathcal{F}_{C,D} \upharpoonright_{\text{Ext}(\mathbb{K})}$. In the second case, $A = C$, we have $B \neq D$ and thus $B \not\models C \rightarrow D$, but $B \models A \rightarrow B$. This implies that $B \in \mathcal{F}_{A,B} \upharpoonright_{\text{Ext}(\mathbb{K})}$ but $B \notin \mathcal{F}_{C,D} \upharpoonright_{\text{Ext}(\mathbb{K})}$. Thus, $\mathcal{F}_{A,B} \upharpoonright_{\text{Ext}(\mathbb{K})} \neq \mathcal{F}_{C,D} \upharpoonright_{\text{Ext}(\mathbb{K})}$.

Next, we turn ourselves to other lattice properties of $\downarrow \text{Ext}(\mathbb{K})$ and its elements.

Lemma 2 (Join Complement). *For \mathbb{K} , $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ and $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$, the set $\hat{\mathcal{R}} = \bigvee_{A \in \mathcal{M}(\text{Ext}(\mathbb{K})) \setminus \mathcal{M}(\mathcal{R})} \{A, G\}$ is the inclusion minimum closure-system for which $\mathcal{R} \vee \hat{\mathcal{R}} = \text{Ext}(\mathbb{K})$.*

Proof. A set $\mathcal{A} \subseteq \text{Ext}(\mathbb{K})$ is a generator of $\text{Ext}(\mathbb{K})$ iff all meet-irreducible elements of $\text{Ext}(\mathbb{K})$ are in \mathcal{A} . Hence, for every $\mathcal{D} \in \downarrow \text{Ext}(\mathbb{K})$ with $\mathcal{R} \vee \mathcal{D} = \text{Ext}(\mathbb{K})$, we have \mathcal{D} is a superset of $\mathcal{M}(\text{Ext}(\mathbb{K})) \setminus \mathcal{M}(\mathcal{R})$ and thus of \hat{R} , since \hat{R} it is the closure of $\mathcal{M}(\text{Ext}(\mathbb{K})) \setminus \mathcal{M}(\mathcal{R})$ in $\downarrow \text{Ext}(\mathbb{K})$.

All the above result in the following statement about $\downarrow \text{Ext}(\mathbb{K})$:

Proposition 7. *For context \mathbb{K} , the lattice $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$:*

- | | |
|------------------------------------|---------------------------------------|
| <i>i) is join-semidistributive</i> | <i>iv) is join-pseudocomplemented</i> |
| <i>ii) is lower semi-modular</i> | <i>v) is ranked</i> |
| <i>iii) is meet-distributive</i> | <i>vi) is atomistic</i> |

Proof. i) According to [1, Corollary 30] \mathfrak{F}_G is join-semidistributive and therefore $\downarrow \text{Ext}(\mathbb{K})$ too, since the meet and join operations of \mathfrak{F}_G are closed in $\downarrow \text{Ext}(\mathbb{K})$. ii) Analogue to i). iii) Follows from i) and ii) (cf. Definition 15 (5) [1]). iv) The join-complement of any $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$ is given by $\hat{\mathcal{R}}$ according to Lemma 2. v) The lattice \mathfrak{F}_G is ranked by the cardinality function (cf. [1, Corollary 30]). Since $\downarrow \text{Ext}(\mathbb{K})$ is an order ideal in \mathfrak{F}_G , it is ranked by the same function. vi) Follows directly from the characterization of join-irreducibles in Proposition 3.

This result can be employed for the recommendation of scale-measures, in particular with respect to Libkins decomposition theorem [19, Theorem 1]. This would allow for a divide-and-conquer procedure within the scale-hierarchy, based on the fact: for context \mathbb{K} the lattice $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ is decomposable into the direct product of two lattices $\downarrow \text{Ext}(\mathbb{K}) \sim L_1 \times L_2$ iff $L_1 = (n]$, $L_2 = (\bar{n}]$ and n is neutral in $\downarrow \text{Ext}(\mathbb{K})$. Here \bar{n} indicates the complement of n with respect to $\downarrow \text{Ext}(\mathbb{K})$, which can be computed using Lemma 2. That this approach is reasonable can be drawn from the fact that $\downarrow \text{Ext}(\mathbb{K})$ fullfils all requirements of Lemma 2 and Theorem 1 from Libkin's work [18, 19] by considering Proposition 7.

In the rest of this section we investigate distributive and neutral elements in $\downarrow \text{Ext}(\mathbb{K})$ more deeply. For this, let $\psi, \phi \in \Phi(L)$, i.e., the set of all closure operators on lattice L . We say that $\phi \leq_{\Phi} \psi$ iff for all $x \in L$: $\phi(x) \leq_{\Phi} \psi(x)$.

Lemma 3. *For context \mathbb{K} , $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ and $\Phi(\text{Ext}(\mathbb{K}))$, we find that the map $i : \downarrow \text{Ext}(\mathbb{K}) \mapsto \Phi(\text{Ext}(\mathbb{K}))$ with $i(\mathcal{A}) \rightarrow \phi_{\mathcal{A}}|_{\text{Ext}(\mathbb{K})}$ is a dual-isomorphism.*

Proof. For $\mathcal{A}, \mathcal{D} \in \downarrow \text{Ext}(\mathbb{K})$ with $A \in \mathcal{A}$, $A \notin \mathcal{D}$ is $i(\mathcal{A})(A) = A$ but $i(\mathcal{D})(A) \neq A$. Thus $i(\mathcal{A}) \neq i(\mathcal{D})$ and i injective. For $\phi \in \Phi(\text{Ext}(\mathbb{K}))$ is $\phi[\text{Ext}(\mathbb{K})] \subseteq \text{Ext}(\mathbb{K})$ a closure system with $G \in \phi[\text{Ext}(\mathbb{K})]$ and therefore $\phi[\text{Ext}(\mathbb{K})] \in \downarrow \text{Ext}(\mathbb{K})$ with $i(\phi[\text{Ext}(\mathbb{K})]) = \phi$. Hence i is bijective. For $\mathcal{A}, \mathcal{D} \in \downarrow \text{Ext}(\mathbb{K})$ with $\mathcal{A} \prec_{\downarrow \text{Ext}(\mathbb{K})} \mathcal{D}$ is $\mathcal{A} \cup \{D\} = \mathcal{D}$ for D meet-irreducible in \mathcal{D} (Corollary 1). Thus for all $A \in \text{Ext}(\mathbb{K})$ is $i(\mathcal{A})(A) = i(\mathcal{D})(A)$ except for the pre-images of D , i.e., $i(\mathcal{D})^{-1}(D)$. For $A \in i(\mathcal{D})^{-1}(D)$ is $i(\mathcal{D})(A) = D \subseteq i(\mathcal{A})(A)$ and thus $i(\mathcal{D}) \leq i(\mathcal{A})$, as required.

Corollary 2. *For a context \mathbb{K} , $\downarrow \text{Ext}(\mathbb{K}) \subseteq \mathfrak{F}_G$ and $\mathcal{R} \in \downarrow \text{Ext}(\mathbb{K})$ tfae: i) \mathcal{R} is distributive ii) \mathcal{R} is neutral iii) For $A, B, C \in \mathcal{R}$ with $C = A \wedge B$ and A, B incomparable in $\text{Ext}(\mathbb{K})$, we have $A \in \mathcal{R} \vee B \in \mathcal{R} \vee C \in \mathcal{R}$ implies $A, B, C \in \mathcal{R}$.*

Proof. Using Lemma 3, $i) \Leftrightarrow ii)$ due to Thm. 2 [23] and $i) \Leftrightarrow iii)$ due to Thm. 1 [23].

An additional accompanying property is that the set of neutral elements of $\downarrow \text{Ext}(\mathbb{K})$ is a complete lattice [22]. Thus, the iterative procedure that results from Corollary 2, iii) yields a closure operator on $\downarrow \text{Ext}(\mathbb{K})$ to compute the neutral elements. To nourish our understanding of the neutral elements take the following example: in the lattice \mathfrak{F}_G are only the top and bottom elements neutral [1, Proposition 33 (5)]. In contrast, take the chain $\mathcal{C} \subseteq \mathfrak{F}_G$ with $G \in \mathcal{C}$, for which $\downarrow \text{Ext}(\mathbb{K}_{\mathcal{C}})$ is a distributive lattice, hence, every element is neutral.

4 Recommending Conceptual Scale-Measures

Our theoretical findings unveils several possibilities to recommend scale-measures. First, there are meet- and join-irreducible elements of the scale-hierarchy (Propositions 3 and 4). These elements are a minimum representation from which every other scale-measure can be retrieved. However, the number of meet- and join-irreducible elements is in the size of the concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$ (Proposition 3) and thereby potentially exponential large. Hence, it is necessary to narrow down the set of join-irreducible scale-measures, for example, by constraining the selection to irreducible elements in $\underline{\mathfrak{B}}(\mathbb{K})$ or by applying conceptual importance measure.

Other scale-measures of interest can be depicted based on their structural placement in the scale-hierarchy, i.e., element-wise modularity, distributivity, or neutrality. A further advantage of latter two selection methods is that they allow a decomposition of the scale-hierarchy using divide-and-conquer strategies. The existence of such neutral elements, however, cannot be guaranteed, as it can be observed in \mathfrak{F}_G . When a starting scale-measure (σ, \mathbb{S}) is selected, an obvious choice is to recommend the join-complemented scale-measure (Proposition 7), i.e., the minimum scale-measure such that the join with (σ, \mathbb{S}) yields $\text{Ext}(\mathbb{K})$. The said join-complemented scale-measure can then be used as additional information or be the starting point for a thorough search.

In general, whenever multiple scale-measures of interest $\{(\sigma_j, \mathbb{S}_j)\}_{j \in J}$ are selected, we are able to combine all those by the apposition of scale-measures ([13, Proposition 19]) to combine their conceptual views on the data set.

4.1 Exploration

For the task of efficiently determining a scale-measure, based on human preferences, we propose the following approach. Motivated by the representation of meet-irreducible elements in the scale-hierarchy through object implications of the context (Proposition 4), we employ the dual of the *attribute exploration* algorithm ([8]) by Ganter. We modified said algorithm toward exploring scale-measures and present its pseudo-code in Algorithm 1. In this depiction we highlighted our modifications with respect to the original exploration algorithm (Algorithm 19, [9]) with darker print. This algorithm semi-automatically computes a scale context \mathbb{S} and its canonical base. In each iteration of the inner loop of our

Algorithm 1: Scale-measure Exploration: A modified Exploration with Background Knowledge

Input : Context $\mathbb{K} = (G, M, I)$
Output: $(id_G, \mathbb{S}) \in \mathfrak{S}(\mathbb{K})$ and optionally $\mathcal{L}_{\mathbb{S}}$
Init Scale $\mathbb{S} = (G, \emptyset, \in)$
Init $A = \emptyset, \mathcal{L}_{\mathbb{S}} = \text{CanonicalBase}(\mathbb{K})$ (or $\mathcal{L}_{\mathbb{S}} = \{\}$ for larger contexts)
while $A \neq G$ **do**
 while $A \neq A^{I_{\mathbb{S}}}$ **do**
 if *Can $A^{I_{\mathbb{K}}} \setminus (A)^{I_{\mathbb{S}}I_{\mathbb{K}}}$ for objects having $(A)^{I_{\mathbb{S}}I_{\mathbb{K}}}$ be neglected?* **then**
 $\mathcal{L}_{\mathbb{S}} = \mathcal{L}_{\mathbb{S}} \cup \{A \rightarrow A^{I_{\mathbb{S}}I_{\mathbb{S}}}\}$
 Exit While
 else
 Enter $B \subseteq A^{I_{\mathbb{K}}} \setminus (A)^{I_{\mathbb{S}}I_{\mathbb{K}}}$ that should be considered
 Add attribute $B^{I_{\mathbb{K}}}$ to \mathbb{S}
 $A = \text{Next_Closure}(A, G, \mathcal{L}_{\mathbb{S}})$
return : (id_G, \mathbb{S}) and optionally \mathcal{L}

exploring algorithm the query that is stated to the *scaling expert* is if an object implication $A \implies B$ is true in the closure system of preferences. If the implication holds, it is added to the implicational base of \mathbb{S} and the algorithm continues with the next implication query. Otherwise a counter example in the form of a closed set $C \in \text{Ext}(\mathbb{K})$ with $A \subseteq C$ but $B \not\subseteq C$. This closed set is then added as attribute to the scale context \mathbb{S} with the incidences given by \in . If $C \notin \text{Ext}(\mathbb{K})$ the scale \mathbb{S} would contradict the scale-measure property (Proposition 20, [13]).

The object implicational theory $L_{\mathbb{S}}$ is initialized to the object canonical base of \mathbb{K} , which is an instance of according to attribute exploration with background knowledge [8]. This initialization can be neglected for larger contexts, however it may reduce the number of queries. The algorithm terminates when the implication premise of the query is equal to G . The returned scale-measure is in canonical form, i.e., the canonical representation $(id_G, (G, \text{Ext}(\mathbb{S}), \in))$ (Proposition 1). The motivation behind attribute exploration queries is to determine if an implication holds in the unknown representational context of the learning domain. In contrast, the exploration of scale-measures determines if a given $\text{Ext}(\mathbb{K})$ can be coarsened by implications $A \implies B$, resulting in a smaller and thus more human comprehensible concept lattice $\underline{\mathfrak{B}}(\mathbb{S})$, adjusted to the preferences (or view) of the scaling expert.

Querying object implications may be less intuitive compared to attribute implications, hence, we suggest to rather not test for $A \implies A^{I_{\mathbb{S}}}$ for $A \subseteq G$ but to test if the difference of the intents $A^{I_{\mathbb{K}}}$ and $(A^{I_{\mathbb{S}}})'$ in \mathbb{K} , is of relevance to the scaling expert. Finally, as a post-processing, one may apply the *conjunctive normalform* [13, Proposition 23] of scale-measures to further increase the human-comprehension. Yet, deriving other human-comprehensible representations of scale-measures is deemed future work.

4.2 (Semi-)Automatic Large Data Set Scaling

To demonstrate the applicability of the presented exploring algorithm, we have implemented it in the `conexp-clj` ([11]) software suite for formal concept analysis. For this, we apply the scale-measure exploration Algorithm 1 on our running example \mathbb{K}_W , see Figure 1. In Figure 4 (left) we depicted the evaluation steps of algorithm, the first two columns represent the object implication that is queried, the third column contains the query translated in terms of attributes. For example, in row two the implication $\{\} \implies \{\mathbf{D}, \mathbf{FL}, \mathbf{Br}, \mathbf{F}\}$ is true in the so far generated scale \mathbb{S} and is queried if it should hold. All objects of the implication do have at least the attributes *can move* and *needs water to live*, as indicated in the third column (left). In the same column (right) we find attributes from $(1)^{I_{\mathbb{K}}} \setminus (2)^{I_{\mathbb{K}}} \subseteq M_W$ that can be considered by the scaling expert to narrow the object implication, i.e., to shrinken the size of the conclusion. The by us envisioned answer of the scaling expert is given in column four, the attribute *lives on land*. Thus, the object counter example is then the attribute-derivation the union $\{\mathbf{M}, \mathbf{W}, \mathbf{LL}\}^{I_W} = \{\mathbf{D}, \mathbf{F}\}$. In our example of the scale-measure exploration the algorithm terminates after the scaling expert provided in total nine counter examples and four accepts. The output is a scale context in canonical representation with twelve concepts as depicted in Figure 4 (right).

The just demonstrated application of the scale-measure exploration can be supported in every step by conceptual importance measures [16]. Furthermore, these measures can also be used to automate the exploration algorithm by randomly selecting the counterexample from the top-k of the list of outstanding concepts with respect to one or more of said conceptual measures. We show illustrate this idea for the spices planer data set [12, 13, 21] and depict the resulting scale-measure in Figure 5. This data set is comprised of 56 dishes (objects) and 37 spices (attributes), resulting in the context $\mathbb{K}_{\text{Spices}}$. The dishes are picked from multiple categories, such as vegetables, meats, or fish dishes. The incidence $I_{\mathbb{K}_{\text{Spices}}}$ indicates that a spice m is necessary to cook dish g . The concept lattice of $\mathbb{K}_{\text{Spices}}$ has 421 concepts and is therefore too large for a meaningful human comprehension. Thus, using our automatic approach for scale-measure recommendation, we are able to generate a small-scaled view of readable size.

For this example of automatic scale-measure exploration, we considered the importance measure *separation index* [15, 16] on the set of objects. We consider the maximum number of concepts that are human readable to be thirty and therefore we restricted the number of counter examples to be computed accordingly. We depicted the concept lattice of the resulting scale-measure in Figure 5 using the conjunctive normalform. To improve the readability, we only annotated meet-irreducible attribute concepts in the lattice diagram and omitted redundant attribute conjunctions, e.g., for $\text{Anis} \wedge \text{Vanilla} \wedge \text{Cinnamon} \wedge \text{Pastry}$ we annotate $\dots \wedge \text{Pastry}$, since $\text{Anis} \wedge \text{Vanilla} \wedge \text{Cinnamon}$ is already given by an upper neighbor. The so given scale-measure concept lattice seems empirically more human readable and displays extensive information with respect to the original data set $\mathbb{K}_{\text{Spices}}$ and the employed importance measure.

(Object) Premise (1)	(Object) Conclusion (2)	Attribute Question		CE?	Edit
		(2)'	(1)' \ (2)'		
{}	{D, FL, Co, Br, WW, Be, F, R}	{W}:	add more detail using {L, BF, Ch, LL, LW, M, MC, DC}?	{M}	Add {D, FL, Br, F} to M_G
	{D, FL, Br, F}	{W, M}:	add more detail using {L, BF, Ch, LL, LW, MC, DC}?	{LL}	Add {D, F} to M_G
	{D, F}	{L, W, LL, M}:	add more detail using {BF, Ch, LW, MC, DC}?	{BF}	Add {} to M_G
	{D}	{L, BF, W, LL, M}:	add more detail using {Ch, LW, MC, DC}?	{LW}	Add {} to M_G
	{}				
{R}	{D, FL, C, Br, WW, Be, F, R}	{W}:	add more detail using {Ch, LL, LW, MC}?	{Ch}	Add {C, WW, Be, R} to M_G
	{C, WW, Be, R}	{W, Ch}:	add more detail using {LL, LW, MC}?	{LL}	Add {C, Be, R} to M_G
	{C, Be, R}	{W, Ch, LL}:	add more detail using {LW, MC}?	{LW}	Add {R} to M_G
	{R}				
{F}	{F, D}	{L, W, LL, M}:	add more detail using {LW}?	{LW}	Add {F} to M_G
	{F}				
{F, R}	{D, FL, Co, Br, WW, Be, F, R}	{W}:	add more detail using {LW, LL}?	{LW}	Add {FL, Br, WW, F, R} to M_G
	{FL, Br, WW, F, R}	{W}:	add more detail using {LW, LL}?	no	Add (1) \implies (2) to L_G
	{F, Br}	{W, LW, M}:	add more detail using {L}?	no	Add (1) \implies (2) to L_G
	{FL, Br, F}				
	{FL, Br, F}				
	{WW, R}				
	{FL, Br, WW, F, R}				
	{D}				
	{D, F}				
	{D, FL, Br, F}				
	{Co, R}	{W, Ch, LL}:	add more detail using {Mc}?	no	Add (1) \implies (2) to L_G
	{Be, R}	{W, Ch, LL}:	add more detail using {Dc}?	no	Add (1) \implies (2) to L_G
	{Co, Be, R}				
	{Co, WW, Be, R}				DONE

		{D, FL, Br, F}	{}	{Co, WW, Be, R}	{Co, Be, R}	{FL, Br, WW, F, R}	{D, F}	{D}	{F}	{R}
dog (D)		x					x	x		
fish leech (FL)		x				x				
corn (Co)				x	x					
bream (Br)		x				x				
water weeds (WW)				x		x				
bean (Be)				x	x					
frog (F)		x				x	x		x	
reed (R)				x	x	x				x

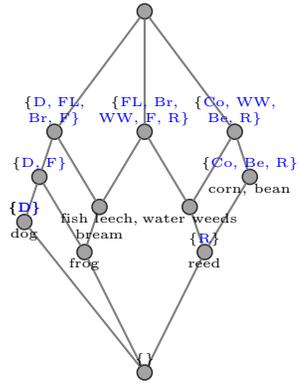


Figure 4. Scale-measure exploration results (left) for the *Living Beings and Water* context, the resulting context (bottom right) and its concept lattice (top right). The employed object order is: $Be > Co > D > WW > FL > Br > F > R$

5 Related Work

Measurement is an important field of study in many (scientific) disciplines that involve the collection and analysis of data. According to Stevens [27] there are four feature categories that can be measured, i.e., *nominal*, *ordinal*, *interval* and *ratio* features. Although there are multiple extensions and re-categorizations of the original four categories, e.g., most recently Chrisman introduced ten [2], for the purpose of our work the original four suffice. Each of these categories describe which operations are supported per feature category. In the realm of formal concept analysis we work often with *nominal* and *ordinal* features, supporting value comparisons by = and <, >. Hence grades of detail/membership cannot be expressed. A framework to describe and analyze the measurement for Boolean data sets has been introduced in [10] and [6], called *scale-measures*. It characterizes the measurement based on object clusters that are formed according to common feature (attribute) value combinations. An accompanied notion of dependency has been studied [30], which led to attribute selection based measurements of boolean data. The formalism includes a notion of consistency enabling the determination of different views and abstractions, called *scales*, to the data set. This approach is comparable to *OLAP* [3] for databases, but on a conceptual level. Similar to the feature dependency study is an approach for selecting relevant attributes in contexts based on a mix of lattice structural features and entropy maximization [14]. All discussed abstractions reduce the complexity of the data, making it easier to understand by humans.

Despite the in this work demonstrated expressiveness of the scale-measure framework, it is so far insufficiently studied in the literature. In particular algorithmical and practical calculation approaches are missing. Comparable and popular machine learning approaches, such as feature compressed techniques, e.g., *Latent Semantic Analysis* [4, 5], have the disadvantage that the newly compressed features are not interpretable by means of the original data and are not guaranteed to be consistent with said original data. The methods presented in this paper do not have these disadvantages, as they are based on meaningful and interpretable features with respect to the original features using propositional expressions. In particular preserving consistency, as we did, is not a given, which was explicitly investigated in the realm scaling many-valued formal contexts [25] and implicitly studied for generalized attributes [17].

Earlier approaches to use scale contexts for complexity reduction in data used constructs such as $(G_N \subseteq \mathcal{P}(N), N, \exists)$ for a formal context $\mathbb{K} = (G, M, I)$ with $N \subseteq M$ and the restriction that at least all intents of \mathbb{K} restricted to N are also intent in the scale [28]. Hence, the size of the scale context concept lattice depends directly on the size of the concept lattice of \mathbb{K} . This is particularly infeasible if the number of intents is exponential, leading to incomprehensible scale lattices. This is in contrast to the notion of scale-measures, which cover at most the extents of the original context, and can thereby display selected and interesting object dependencies of scalable size.

6 Conclusion

With this work we have shed light on the hierarchy of scale-measures. By applying multiple results from lattice theory, especially concerning ideals, to said hierarchy, we were able to give a more thorough structural description of $\downarrow_{\mathfrak{F}_G} \text{Ext}(\mathbb{K})$. Our main theoretical result is Proposition 7, which in turn leads to our practical applications. In particular, based on this deeper understanding we were able to present an algorithm for exploring the scale-hierarchy of a contextual data set \mathbb{K} . Equipped with this algorithm a data scaling expert may explore the lattice of scale-measures for a given data set with respect to her preferences and the requirements of the data analysis task. The practical evaluation and optimization of this algorithm is a promising goal for future investigations. Even more important, however, is the implementation and further development of the automatic scaling framework, as outlined in Section 4.2. This opens the door to empirical scale recommendation studies and a novel approach for data preprocessing.

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