

Boolean Substructures in Formal Concept Analysis

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Abstract. It is known that a (concept) lattice contains an n -dimensional Boolean suborder if and only if the context contains an n -dimensional contra-nominal scale as subcontext. In this work, we investigate more closely the interplay between the Boolean subcontexts of a given finite context and the Boolean suborders of its concept lattice. To this end, we define mappings from the set of subcontexts of a context to the set of suborders of its concept lattice and vice versa and study their structural properties. In addition, we introduce closed-subcontexts as an extension of closed relations to investigate the set of all sublattices of a given lattice.

Keywords: Formal Concept Analysis · Contranominal Scales · Boolean Contexts · Boolean Lattices · Sublattices · Subcontexts · Closed Relations

1 Introduction

In the field of Formal Concept Analysis (FCA) the basic data structure is a so-called formal context. It consists of a set of objects, a set of attributes, and an incidence relation on those sets representing which object *has* which attribute. Each such context gives rise to concepts which consist of a maximal set of objects that all share the same maximal set of attributes. The concepts, ordered by subset relation, form a complete lattice.

One frequently occurring type of substructure (more precisely: suborder or sub(semi)lattice) of a concept lattice are Boolean algebras. In the formal context, they correspond to subcontexts that are isomorphic to a contranominal scale, i. e., a context of type $(\{1, \dots, k\}, \{1, \dots, k\}, \neq)$. This means in particular the existence of k objects that just differ slightly on k attributes. However, despite of the only slight difference, these Boolean subcontexts are responsible for an exponential growth of the concept lattice [3]. Such Boolean subcontexts occur in real-world data as well as in randomly generated formal contexts [5].

In this paper we investigate the connection between the Boolean substructures in the formal context and in its corresponding concept lattice. Based on closed subrelations of a formal context [14], that provide a method to characterize

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the complete sublattices of the corresponding concept lattice, we introduce *closed-subcontexts* and present a one-to-one correspondence to all sublattices. Through this, we merge the obvious two-step-approach of limiting the lattice to an interval and determining its complete sublattices in one structure. Since this construction is an – almost arbitrary and difficult to handle – mixture of subcontext and subrelation and in addition is not directly specific to the field of Boolean substructures, we investigate the connection between Boolean subcontexts and Boolean sublattices and suborders, respectively, in Section 6 in a direct way without having to manipulate the incidence relation. To this end, we lift two well-known order embeddings [7] to the level of subcontexts and suborders to find the Boolean suborders corresponding to a Boolean subcontext. In addition, we introduce a construction to generate the Boolean subcontext associated to a given Boolean suborder. We combine these methods to investigate to which degree the join and meet operators of the lattice are respected by those maps.

As our work is triggered by complexity issues in data analysis where only finite sets are considered, **all statements in this paper are about finite sets and structures only**, unless explicitly stated otherwise.

As for the structure of this paper, in Section 2 we recall some basic notions and give a brief introduction to the approaches our investigations are based on. Afterwards, in Section 3 we give a short overview of previous works applied to the investigation of substructures of formal contexts and concept lattices. In Section 4 we introduce some notions required for our investigation on Boolean substructures. We introduce closed-subcontexts in Section 5 to determine the set of all Boolean sublattices. Our second approach is presented in Section 6 where we use embeddings of Boolean structures in concept lattices and construct the subcontexts associated to Boolean suborders. In Section 7 we compare both approaches, and discuss the differences and their overlap. We conclude our work and give an outlook in Section 8.

To advanced readers, we recommend proceeding directly to Section 4 and Figure 1 as it illustrates the connections investigated in this work.

2 Recap on FCA and Notations

2.1 Foundations

Following, we recall some basic notions from FCA. For a detailed introduction we refer to [7]. A formal context is triple $\mathbb{K} := (G, M, I)$, where G is the finite *object set*, M the finite *attribute set*, and $I \subseteq G \times M$ a binary *incidence relation*. Instead of writing $(g, m) \in I$ for an object $g \in G$ and an attribute $m \in M$, we also write gIm and say *object g has attribute m* . One kind of formal context is the family of *contranominal scales*, denoted by $\mathbb{N}^c(k) := (\{1, 2, \dots, k\}, \{1, 2, \dots, k\}, \neq)$.

On the power set of the objects and the power set of the attributes there are two operations given: $\cdot' : \mathcal{P}(G) \rightarrow \mathcal{P}(M)$, $A \mapsto A' := \{m \in M \mid \forall g \in A : (g, m) \in I\}$ and $\cdot' : \mathcal{P}(M) \rightarrow \mathcal{P}(G)$, $B \mapsto B' := \{g \in G \mid \forall m \in B : (g, m) \in I\}$. Instead of A' we also write A^I to specify which incidence relation is used for the operation. A *formal concept* $C = (A, B)$ of the context (G, M, I) is a pair consisting of an object subset

$A \subseteq G$, called *extent*, and an attribute subset $B \subseteq M$, called *intent*, that satisfies $A' = B$ and $B' = A$. An object set $O \subseteq G$ is called *minimal object generator* of a concept (A, B) if $O'' = A$ and $P'' \neq A$ for every proper subsets $P \subsetneq O$. Analogous, the *minimal attribute generator* of a concept (A, B) is defined. The set of all minimal object generators (or rather all minimal attribute generators) of (A, B) is denoted by $\min G_{obj}(A, B)$ ($\min G_{att}(A, B)$). The set of all formal concepts $\mathfrak{B}(\mathbb{K})$ together with the order defined by $(A_1, B_1) \leq (A_2, B_2)$ iff $A_1 \subseteq A_2$ for two concepts (A_1, B_1) and (A_2, B_2) determines the *concept lattice* $\underline{\mathfrak{B}}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}), \leq)$. The concept lattice of $\mathbb{N}^c(k)$ is called *Boolean lattice of dimension k* and is denoted by $\underline{\mathfrak{B}}(k) := \underline{\mathfrak{B}}(\mathbb{N}^c(k))$.

There are two tools for basic structural investigations of a formal context $\mathbb{K} = (G, M, I)$ in FCA. An object $g \in G$ is called *clarifiable* if another object $g \neq h \in G$ with $g' = h'$ exists. Furthermore, an object $g \in G$ is called *reducible* if a set of objects $X \subseteq G$ with $g \not\subseteq X$ and $g' = X'$ exists. Otherwise g is called *irreducible*. The same applies to the set of attributes. The concept lattice of a context \mathbb{K} that has no clarifiable/reducible objects and attributes is isomorphic to the lattice of any context that can be constructed by adding reducible or clarifiable objects or attributes to \mathbb{K} . The stepwise elimination of all clarifiable/reducible attributes and objects of a formal context results in a *clarified/reduced* context, the *standard context* of $\underline{\mathfrak{B}}(\mathbb{K})$.

To study particular parts of a formal context the selection of a *subcontext* is useful. A *subcontext* $\mathbb{S} := (H, N, J)$ of a formal context $\mathbb{K} = (G, M, I)$ is a formal context with $H \subseteq G$, $N \subseteq M$ and $J = I \cap (H \times N)$. We write $\mathbb{S} \leq \mathbb{K}$ to describe \mathbb{S} as a subcontext of \mathbb{K} and use the notion $[H, N]$ instead of $(H, N, I \cap (H \times N))$. The set of all subcontexts of a formal context \mathbb{K} is denoted by $\mathcal{S}(\mathbb{K})$.

$1_{\underline{L}}$ and $0_{\underline{L}}$ denote the top and the bottom element of a lattice \underline{L} . The elements covering $0_{\underline{L}}$ are called *atoms* and the elements covered by $1_{\underline{L}}$ *coatoms*. We denote by $At(\underline{L})$ and $CoAt(\underline{L})$, respectively, the set of all atoms and coatoms of \underline{L} . $\underline{S} = (S, \leq)$, a subset $S \subseteq L$ together with the same order relation as \underline{L} , is called *suborder* of \underline{L} . The set of all suborders of \underline{L} is denoted by $\mathcal{SO}(\underline{L})$. If $(a, b \in S \Rightarrow (a \vee b) \in S)$ holds we call \underline{S} *sub- \vee -semilattice* of \underline{L} . If $(a, b \in S \Rightarrow (a \wedge b) \in S)$ holds we call \underline{S} *sub- \wedge -semilattice* of \underline{L} . A \underline{S} that is both, a sub- \vee -semilattice and a sub- \wedge -semilattice, is called *sublattice* of \underline{L} . The set of all sublattices of \underline{L} is denoted by $\mathcal{SL}(\underline{L})$. If $(T \subseteq S \Rightarrow (\bigvee T), (\bigwedge T) \in S)$ holds $\forall T \subseteq S$ we call \underline{S} *complete sublattice* of \underline{L} . The requirement for completeness can be translated into $1_{\underline{L}}$ and $0_{\underline{L}}$ being included in \underline{S} if \underline{L} is a finite lattice.

2.2 Relating Substructures in FCA

Wille [14] presents closed relations to characterize complete sublattices of a concept lattice. A relation $J \subseteq I$ is called *closed relation* of a formal context $\mathbb{K} = (G, M, I)$ if every concept of the context (G, M, J) is a concept of \mathbb{K} as well. Closed relations are linked to the complete sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ [7, chap. 3.3]: The set of all closed subrelations of \mathbb{K} and all complete sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ have a one-to-one correspondence. The bijection $C(\underline{S}) := \bigcup \{A \times B \mid (A, B) \in \underline{S}\}$ maps the set of all complete sublattices to the set of all closed relations. By limiting the lattice to an interval, the described one-to-one correspondence can be found between the complete lattices of the interval and the closed relations of the formal context associated to the interval.

A connection of the concept lattices of a formal context $\mathbb{K} = (G, M, I)$ and its subcontext $\mathbb{S} = [H, N]$ is given by Ganter and Wille [7, Proposition 32] by the two maps $\varphi_1 : \mathfrak{B}[H, N] \rightarrow \mathfrak{B}(G, M, I)$, $(A, B) \mapsto (A'', A')$ and $\varphi_2 : \mathfrak{B}[H, N] \rightarrow \mathfrak{B}(G, M, I)$, $(A, B) \mapsto (B', B'')$. Both maps are *order embeddings*. This means for all $(A_1, B_1), (A_2, B_2) \in \mathfrak{B}[H, N]$ that $(A_1, B_1) \leq (A_2, B_2)$ in $\mathfrak{B}[H, N]$ if and only if $\varphi_i(A_1, B_1) \leq \varphi_i(A_2, B_2)$ in $\mathfrak{B}(G, M, I)$ for both $i \in \{1, 2\}$. Hence, every structure contained in $\mathfrak{B}(\mathbb{S})$ also appears in $\mathfrak{B}(\mathbb{K})$.

3 Related Work

In the field of Formal Concept Analysis, there are several approaches to analyze smaller parts of a formal context or a concept lattice, as well as to investigate the connection between the two data structures. In [2] local changes to a formal context and their effects on the corresponding concept lattice, namely the number of concepts, are explored. Albano [1] studies the impact of contranominal scales in a formal context to the size of the corresponding concept lattice by giving an upper bound for $\mathfrak{B}(k)$ -free lattices. The approach of Wille [14] on the one-to-one correspondence between closed subrelations of a formal context complete sublattices of the associated concept lattice is the basis for the work of Kauer and Krupke [9]. They investigate the problem of constructing the closed subrelation referring to a complete sublattice generated by a given subset of elements while not computing the whole concept lattice. Based on granulation as introduced in [15] the authors of [12] analyze substructures of formal contexts and concept lattices by considering them as granules that provide different levels of accuracy.

Also, many common methods deal with the detection of substructures in the first place. They are based on the selection of structurally meaningful attributes and objects of a formal context. For this purpose, Hanika et al. [8] search for a *relevant* attribute set that reflects the original lattice structure and the distribution of the objects as good as possible. Considering many-valued contexts, Ganter and Kuznetsov [6] select features based on their scaling. Another approach is to generate a meaningful subset by selecting entire concepts directly of the formal context by measuring their individual value for the context and the associated concept lattice. A natural idea is the consideration of extent and intent size of the concepts. Based on this, Kuznetsov [10] proposed a stability measure for formal concepts, measuring the ratio of extent subsets generating the same intent. Another measure, the support, was used by Stumme et al. [13] to generate so-called *iceberg lattices*, which also have a use in the field of mining of frequent association rules.

Besides meaningful reduction, altering the dataset is a standard method in FCA, which is motivated by an attempt to reduce the complexity of the dataset or deal with noise. In this realm, Dias and Vierira investigate the replacement of *similar* objects by a single representative [4]. Approximate frequent itemsets have been investigated to handle noisy data [11], where the authors state an additional threshold for both rows and columns of the dataset.

Since we aim to investigate existing substructures of formal contexts and concept lattices, we turn away from those notions in general.

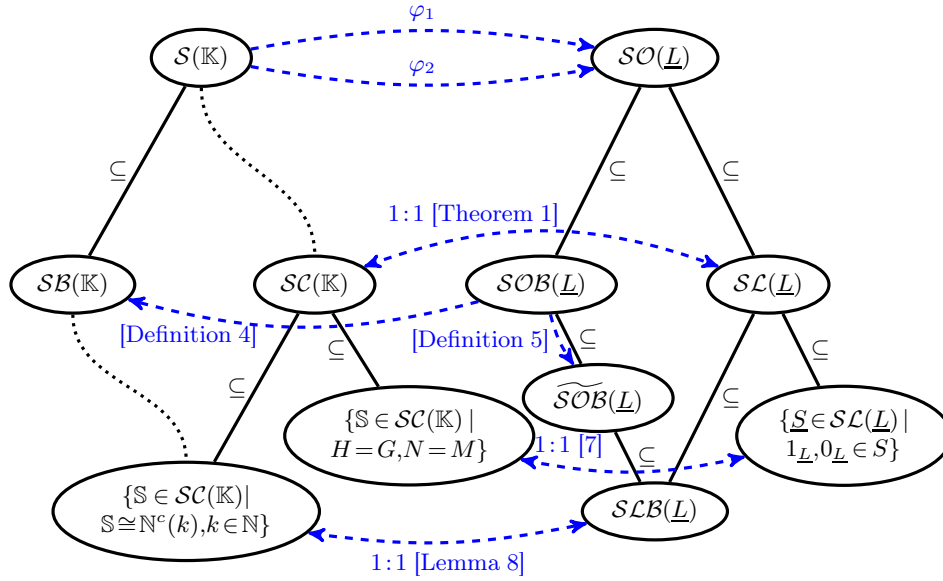


Fig. 1. Connections between the subcontexts of a formal context \mathbb{K} and the suborders of the corresponding concept lattice $\underline{L} := \underline{\mathfrak{B}}(\mathbb{K})$. The set of all subsemilattices of \underline{L} is denoted by $\widetilde{SOB}(\underline{L})$.

4 Boolean Subcontexts and Sublattices

In this work, we investigate Boolean substructures in formal contexts as well as in the corresponding concept lattices. Therefore, as illustrated in Figure 1, we link the different substructures of a formal context with the substructures of the corresponding concept lattice. In this section we introduce the concrete definitions that serve as a foundation to analyze those connections.

Definition 1. Let \mathbb{K} be a formal context, $\mathbb{S} \leq \mathbb{K}$. \mathbb{S} is called Boolean subcontext of dimension k of \mathbb{K} , if $\underline{\mathfrak{B}}(\mathbb{S}) \cong \underline{\mathfrak{B}}(k)$. \mathbb{S} is called reduced if \mathbb{S} is a reduced context. The set of all Boolean subcontexts of dimension k of \mathbb{K} and the set of all reduced Boolean subcontexts of dimension k of \mathbb{K} are denoted by $SB_k(\mathbb{K})$ and $SRB_k(\mathbb{K})$.

Note that a reduced Boolean subcontext of dimension k is isomorphic to the contranominal scale $\mathbb{N}^c(k)$.

Definition 2. Let \underline{L} be a lattice and \underline{S} a suborder of \underline{L} . \underline{S} is called Boolean suborder of dimension k if $\underline{S} \cong \underline{\mathfrak{B}}(k)$. If \underline{S} is a sublattice of \underline{L} , \underline{S} is called Boolean sublattice of dimension k . The set of all Boolean suborders of dimension k of a lattice \underline{L} is denoted by $SOB_k(\underline{L})$. The set of all Boolean sublattices of dimension k of a lattice \underline{L} is denoted by $SLB_k(\underline{L})$.

If all dimensions are considered, the number k is left out in the following.

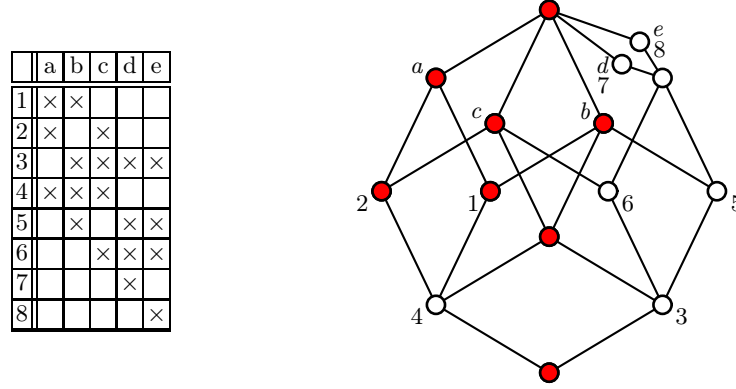


Fig. 2. Example of a formal context $\mathbb{K} = (G, M, I)$ with $G = \{1, 2, \dots, 8\}$ and $M = \{a, b, \dots, e\}$ containing three reduced Boolean subcontexts and its corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K})$.

Note that $\mathcal{SLB}_k(\underline{L})$ is a subset of $\mathcal{SOB}_k(\underline{L})$ and the standard context of a Boolean lattice \underline{L} of dimension k consists of a formal context $\mathbb{K} \cong \mathbb{N}^c(k)$ [7, Proposition 12]. Conversely, a formal context \mathbb{K} consisting of a reduced Boolean subcontext of dimension k and an arbitrary number of additional reducible attributes and objects has a corresponding concept lattice $\underline{\mathfrak{B}}(\mathbb{K}) \cong \underline{\mathfrak{B}}(k)$.

For a better understanding of these structures, we introduce the example given in Figure 2. We will refer back to this illustration throughout the paper.

Example 1. $\mathbb{S} = (\{4, 5, 6\}, \{b, c, d, e\}, J)$ with $J = I \cap (\{4, 5, 6\} \times \{b, c, d, e\})$ is a Boolean subcontext of dimension 3 of the formal context \mathbb{K} given in Figure 2. \mathbb{S} is not reduced, since $d^J = e^J$ holds. However, \mathbb{S} includes the reduced Boolean subcontexts $\mathbb{S}_1 = [\{4, 5, 6\}, \{b, c, d\}]$ and $\mathbb{S}_2 = [\{4, 5, 6\}, \{b, c, e\}]$. The third reduced Boolean subcontext in \mathbb{K} is $\mathbb{S}_3 = [\{1, 2, 3\}, \{a, b, c\}]$. The concept lattice of \mathbb{K} in Figure 2 contains 15 Boolean suborders of dimension 3, two of which are also Boolean sublattices.

5 Closed-Subcontexts

At first, we leave the field of (Boolean) suborders and narrow our focus on (Boolean) sublattices. On the context side, we introduce so-called *closed-subcontexts* and show their one-to-one relationship to the sublattices of the concept lattice.

In [14], Wille introduced closed relations of a context to characterize the complete sublattices of its concept lattice. In finite lattices, complete sublattices differ from (non-complete) sublattices in that they always include the top element and the bottom element of the lattice. We adopt Wille's construction to match with (non necessarily complete) sublattices.

Definition 3. Let $\mathbb{K} = (G, M, I)$ and $\mathbb{S} = (H, N, J)$ be two formal contexts. We call \mathbb{S} closed-subcontext of \mathbb{K} iff $H \subseteq G$, $N \subseteq M$, $J \subseteq I \cap (H \times N)$ and every concept of \mathbb{S} is a concept of \mathbb{K} as well. The set of all closed-subcontexts of \mathbb{K} is denoted by $\mathcal{SC}(\mathbb{K})$.

The sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ have a one-to-one correspondence to closed-subcontexts of \mathbb{K} as follows.

Theorem 1. *Let \mathbb{K} be a formal context and \underline{S} be a sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$. Then*

$$\mathbb{K}_{\underline{S}} := \left(\bigcup_{(A,B) \in \underline{S}} A, \bigcup_{(A,B) \in \underline{S}} B, \bigcup_{(A,B) \in \underline{S}} A \times B \right)$$

is a closed-subcontext of \mathbb{K} . Conversely, for every closed-subcontext \mathbb{S} of \mathbb{K} , $\underline{\mathfrak{B}}(\mathbb{S})$ is a sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$.

Furthermore, the map $f(\underline{S}) := \mathbb{K}_{\underline{S}}$ maps the set of sublattices of $\underline{\mathfrak{B}}(\mathbb{K})$ bijectively onto the set of closed-subcontexts of \mathbb{K} .

Proof. For each formal concept $(A, B) \in \underline{S}$ the formal concept $(A, B) \in \underline{\mathfrak{B}}(\mathbb{K}_{\underline{S}})$ is due to construction a concept in \mathbb{K} . On the other side let $\mathbb{S} = (H, N, J)$ be a closed-subcontext of \mathbb{K} . The concept set of \mathbb{S} is a subset of the concept set of \mathbb{K} and therefore $\underline{\mathfrak{B}}(\mathbb{S})$ is a suborder of $\underline{\mathfrak{B}}(\mathbb{K})$. Let $(A_1, B_1), (A_2, B_2) \in \underline{\mathfrak{B}}(\mathbb{S})$. Let (A_S, B_S) be the infimum of both in \mathbb{S} and (A_K, B_K) the infimum of both in \mathbb{K} . So $A_S = A_1 \cap A_2 = A_K$, which implies $(A_S, B_S) = (A_K, B_K)$ since (A_S, B_S) is by definition a concept in \mathbb{K} . The dual argument shows that \mathbb{S} is closed under suprema. So $\underline{\mathfrak{B}}(\mathbb{S})$ is a sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$. \square

Note that the closed-subsets of a formal context do not form a closure system since the intersection of two closed-subcontexts, in general, is not a closed-subcontext, even though the sublattices of formal concept do so.

In the construction of $\mathbb{K}_{\underline{S}}$, $\bigcup_{(A,B) \in \underline{S}} A$ is the concept extent of the top element of the sublattice and $\bigcup_{(A,B) \in \underline{S}} B$ is the concept intent of its bottom element.

Lemma 1. *Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathbb{S} = (H, N, J)$ a closed-subcontext of \mathbb{K} . Then $H = G$ or $m \in N$ with $m' = H$ exists. And $N = M$ or $g \in H$ with $g' = N$ exists.*

Proof. Due to Definition 3, every concept of \mathbb{S} is a concept of \mathbb{K} as well. In particular, this has to hold for the concepts $(\emptyset'', \emptyset')$ and (H'', H') of \mathbb{S} . \square

We provide next some basic statements about closed-subcontexts. Since the following lemmas are based on the work of Wille [14] and lifted to our approach, the proofs are similar to the ones in [7, Section 3.3].

Lemma 2. *For every set $T \subseteq \underline{\mathfrak{B}}(G, M, I)$ there is a smallest closed-subcontext \mathbb{S} of \mathbb{K} , that contains all $(A \times B)$ for $(A, B) \in T$. $\underline{\mathfrak{B}}(\mathbb{S})$ is the sublattice of $\underline{\mathfrak{B}}(\mathbb{K})$ generated by T .*

Proof. The proof follows the structure of the proof of Proposition 45 in [7]. \square

Lemma 3. *$\mathbb{S} = (H, N, J)$ is a closed-subcontext of the formal context $\mathbb{K} = (G, M, I)$ iff $X^{JJ} \supseteq X^{JI}$ holds for each $X \subseteq H$ and for each $X \subseteq N$.*

Proof. The proof follows the structure of the proof of Proposition 46 in [7]. \square

Lemma 4. *The closed-subcontexts (H, N, J) of (G, M, I) are exactly the subcontexts that satisfy the condition: (C) If $(g, m) \in (H \times N)$ and $(g, m) \in I \setminus J$ then $(h, m) \notin I$ for $h \in H$ with $g^J \subseteq h^J$ and $(g, n) \notin I$ for $n \in N$ with $m^J \subseteq n^J$.*

Proof. The proof follows the structure of the proof of Proposition 47 in [7]. \square

Lemma 5. *Let $\mathbb{K} = (G, M, I)$ be a formal context. A clarified formal context $\mathbb{S} = (H, N, J)$ is a closed-subcontext of \mathbb{K} if and only if $H \subseteq G$, $N \subseteq M$ and $J \subseteq I \cap (H \times N) \subseteq H \times N \setminus (\nearrow^J \cup \swarrow^J)$.*

Proof. The proof follows the structure of the proof of Proposition 49 in [7]. \square

Lemma 6. *Let $\mathbb{K} = (G, M, I)$ be a formal context and (A, B) and (C, D) concepts of \mathbb{K} . Then $(A, B, A \times B)$, $(A, M, I \cap (A \times M))$ and $(G, B, I \cap (G \times B))$ are closed-subcontexts. If $(A, B) \leq (C, D)$ also $(C, B, (A \times B \cup C \times D))$ and $(C, B, I \cap (C \times B))$ are closed-subcontexts. The corresponding concept lattices are given through $\mathfrak{B}(A, B, A \times B) = \{(A, B)\}$, $\mathfrak{B}(A, M, I \cap (A \times M)) = [(A, B)]$, $\mathfrak{B}(G, B, I \cap (G \times B)) = [(A, B)]$, $\mathfrak{B}(C, B, (A \times B \cup C \times D)) = \{(A, B), (C, D)\}$, and $\mathfrak{B}(C, B, I \cap (C \times B)) = [(A, B), (C, D)]$.*

Proof. The proof follows the structure of the proof of Proposition 50 in [7]. \square

Also, the set of the arrow relations of a closed-subcontext \mathbb{S} is a subset of the set of the arrow relations of the original context \mathbb{K} .

Lemma 7. *Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathbb{S} = (H, N, J)$ a closed-subcontext. Then $\nearrow^J \subseteq \nearrow^I$ and $\swarrow^J \subseteq \swarrow^I$ holds.*

Proof. Let $g \in H, m \in N$ and $g \swarrow^J m$. Assumed $g \not\swarrow^I m$. Then there exists $h \in G$ with $g^I \subseteq h^I$ and $(h, m) \notin I$. It follows $g^J \subseteq g^{I \cap (G \times H)} \subseteq h^{I \cap (G \times H)} \Rightarrow h \in h^{I \cap (G \times H)} \subseteq g^{JI} = g^{JJ} \subseteq H \Rightarrow g^J \subseteq h^J$. This is a conflict to $g \swarrow^J m$. \square

Now we transfer our approach to the field of Boolean substructures. To find all Boolean sublattices (of dimension k) in a lattice $\mathfrak{B}(\mathbb{K})$ the closed-subcontexts of \mathbb{K} that are Boolean subcontexts as well have to be found. Hence, Theorem 1 can be restricted in the following way:

Lemma 8. *Let \mathbb{K} be a formal context. $\underline{S} \in \mathcal{SLB}_k(\mathfrak{B}(\mathbb{K}))$ iff $\mathfrak{B}(\mathbb{K}_{\underline{S}}) \cong \mathfrak{B}(k)$ for $\mathbb{K}_{\underline{S}} = (\bigcup_{(A, B) \in \underline{S}} A, \bigcup_{(A, B) \in \underline{S}} B, \bigcup_{(A, B) \in \underline{S}} A \times B)$.*

To directly identify the Boolean closed-subcontexts in a formal context \mathbb{K} , the properties of closed-subcontexts can be utilized. Since every concept in \mathbb{K} is either retained or erased but not altered in a closed-subcontext \mathbb{S} , the Boolean structure of \mathbb{S} has to be preserved from \mathbb{K} . Every Boolean subcontext $\mathbb{T} = (H, N, J) \in \mathcal{SRB}(\mathbb{K})$ provides the Boolean structure. Lifting each concept $(A_{\mathbb{T}}, B_{\mathbb{T}}) \in \mathfrak{B}(\mathbb{T})$ to a concept $(A_{\mathbb{K}}, B_{\mathbb{K}}) \in \mathfrak{B}(\mathbb{K})$ with $A_{\mathbb{T}} \subseteq A_{\mathbb{K}}$ and $B_{\mathbb{T}} \subseteq B_{\mathbb{K}}$, generates an extension of the sets H, N and J that provides a Boolean closed-subcontext $\mathbb{S} = (\tilde{H}, \tilde{N}, \tilde{J}) \in \mathcal{SC}(\mathbb{K})$ as follows: $\tilde{H} := H \cup \bigcup_{(A_{\mathbb{T}}, B_{\mathbb{T}}) \in \mathfrak{B}(\mathbb{T})} A_{\mathbb{K}}$, $\tilde{N} := N \cup \bigcup_{(A_{\mathbb{T}}, B_{\mathbb{T}}) \in \mathfrak{B}(\mathbb{T})} B_{\mathbb{K}}$ and $\tilde{J} := \bigcup_{(A_{\mathbb{T}}, B_{\mathbb{T}}) \in \mathfrak{B}(\mathbb{T})} (A_{\mathbb{K}} \times B_{\mathbb{K}})$. This approach is represented through the dotted lines in Figure 1.

6 Connecting Boolean Suborders and Boolean Subcontexts

In this section we investigate the relationship between Boolean subcontexts and Boolean suborders. For this purpose, we use the embeddings φ_1 and φ_2 and expand them to the set of Boolean subcontexts. Further, we present a construction to get from a Boolean suborder to a corresponding Boolean subcontext. Both approaches are analyzed with focus on the structural information they transfer and their interplay.

6.1 Embeddings of Boolean Substructures

To investigate the connection between Boolean subcontexts \mathbb{S} of a formal context \mathbb{K} and Boolean suborders of $\mathfrak{B}(\mathbb{K})$ we consider embeddings of $\mathfrak{B}(\mathbb{S})$ in $\mathfrak{B}(\mathbb{K})$. Therefore we lift the embeddings φ_1 and φ_2 introduced in Section 2 to the level of subcontexts and suborders:

$$\begin{aligned} \varphi_1 : \mathcal{S}(\mathbb{K}) &\rightarrow \mathcal{SO}(\mathfrak{B}(\mathbb{K})), \mathbb{S} \mapsto (\{\varphi_1(C) \mid C \in \mathfrak{B}(\mathbb{S})\}, \leq) \text{ and} \\ \varphi_2 : \mathcal{S}(\mathbb{K}) &\rightarrow \mathcal{SO}(\mathfrak{B}(\mathbb{K})), \mathbb{S} \mapsto (\{\varphi_2(C) \mid C \in \mathfrak{B}(\mathbb{S})\}, \leq). \end{aligned}$$

From the input (concept or context), it is clear whether the original or the lifted versions of the embeddings φ_1 and φ_2 are used in the following. We will, in particular, study these mappings for Boolean subcontexts. In this case, an additional structural benefit arises: The images of reduced Boolean subcontexts are sub- \vee -semilattice and sub- \wedge -semilattices of the original concept lattice:

Lemma 9. *Let \mathbb{K} be a formal context, $\mathbb{S} = [H, N] \in \mathcal{SRB}_k(\mathbb{K})$. Then $\varphi_1(\mathfrak{B}(\mathbb{S}))$ is a sub- \vee -semilattice of $\mathfrak{B}(\mathbb{K})$ and $\varphi_2(\mathfrak{B}(\mathbb{S}))$ is a sub- \wedge -semilattice of $\mathfrak{B}(\mathbb{K})$.*

Proof. Consider φ_1 : Let $J := I \cap (H \times N)$ and (A, B) and (C, D) be two concepts of $\mathfrak{B}(\mathbb{S})$. Then $\varphi_1(A, B) \vee \varphi_1(C, D) = (A'', A') \vee (C'', C') = ((A'' \cup C'')'', (A' \cap C')) = ((A' \cap C')', (A \cup C)') = ((A \cup C)'', (A \cup C)')$ and in addition $((A \cup C)'', (A \cup C)') = \varphi_1((A \cup C), (B \cap D)) = \varphi_1((A, B) \vee (C, D))$. Since \mathbb{S} is a reduced Boolean context, it includes all possible object combinations as extents so that $E = E^{JJ}$ holds for every $E \subseteq H$. Therefore, in $\mathfrak{B}(\mathbb{S})$ holds $(A, B) \vee (c; D) = ((A \cup C)^{JJ}, B \cap D) = (A \cup C, B \cap D)$. The procedure for φ_2 is analogous. \square

Note that this conclusion does not hold for Boolean reducible subcontexts, e.g., the formal context given in Figure 2 and its subcontext $\mathbb{S} = [\{1237\}, \{abce\}]$.

The images of the two maps of a reduced Boolean context are in general just a sub- \vee -semilattice and a sub- \wedge -semilattice, respectively. Hence, the images of φ_1 and φ_2 have to be identical for $\mathbb{S} \in \mathcal{SRB}_k(\mathbb{K})$ to generate a lattice. This means $\varphi_1(A, B) = (A'', A) = (B', B'') = \varphi_2(A, B)$ has to hold for all $(A, B) \in \mathfrak{B}(\mathbb{S})$.

For every subcontext $\mathbb{S} = (H, N, J) \leq \mathbb{K}$ we can differ between the four cases: Case 1 with $A' = A^J = B$, $B' = B^J = A$, case 2 with $A' = A^J = B$, $A = B^J \subset B'$, case 3 with $B = A^J \subset A'$, $B' = B^J = A$ and case 4 with $B = A^J \subset A'$, $A = B^J \subset B'$. The condition under which $\varphi_1(A, B) = \varphi_2(A, B)$ holds is the following:

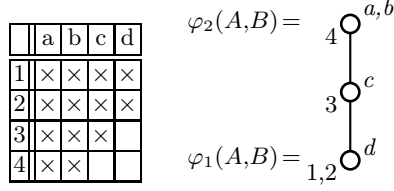


Fig. 3. An example of a formal context \mathbb{K} and its subcontext $\mathbb{S} = [\{1, 2\}, \{a, b\}] = [A, B]$ with $[\varphi_1(A, B), \varphi_2(A, B)] = \underline{\mathfrak{B}}(\mathbb{K})$.

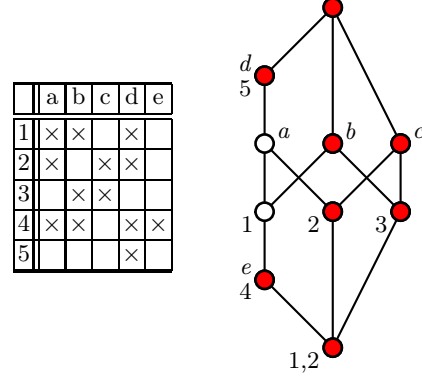


Fig. 4. Example of a formal context \mathbb{K} with $|\mathcal{SRB}_3(\mathbb{K})| = |\mathcal{SOB}_3(\underline{\mathfrak{B}}(\mathbb{K}))| = 4$.

Lemma 10. Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathbb{S} \leq \mathbb{K}$. $\varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S})$ holds if and only if for all $(A, B) \in \underline{\mathfrak{B}}(\mathbb{S})$ $(A' \setminus B) \times (B' \setminus A) \subseteq I$ holds. If case 1, 2 or 3 holds for all $(A, B) \in \underline{\mathfrak{B}}(\mathbb{S})$, then $\varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S})$ holds directly.

Proof. For a concept $(A, B) \in \underline{\mathfrak{B}}(\mathbb{S})$ the identity of both embeddings leads to $\varphi_1(A, B) = \varphi_2(A, B) \Leftrightarrow (A'', A') = (B', B'') = (B', A') \Leftrightarrow (B' \times A') \subseteq I$. This set can be written as $B' \times A' = A \times B \cup (B' \setminus A) \times B \cup A \times (A' \setminus B) \cup (B' \setminus A) \times (A' \setminus B)$. We know $A \times B \subseteq I$ since $(A, B) \in \underline{\mathfrak{B}}(\mathbb{S})$ and $A \times A' \subseteq I$ and $B' \times B \subseteq I$ by definition of the \cdot' operator. The remaining part equals $(A' \setminus B) \times (B' \setminus A)$. In cases 1 to 3 $(A'', A') = (B', B'')$ holds by construction. \square

Proposition 1. Let $\mathbb{K} = (G, M, I)$ be a formal context and $\mathbb{S} = [H, N] \in \mathcal{SB}_k(\mathbb{K})$. If $H = G$ or $N = M$, then $\varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S})$ holds.

However, the relationship between the images of both mappings φ_1 and φ_2 of a specific concept is always (not only in the Boolean case) the same, namely:

Proposition 2. Let \mathbb{K} be a formal context and $\mathbb{S} \leq \mathbb{K}$. Then $\varphi_1(A, B) \leq \varphi_2(A, B)$ for all $(A, B) \in \underline{\mathfrak{B}}(\mathbb{S})$.

In particular, an interval containing exactly the concepts $(C, D) \in \underline{\mathfrak{B}}(\mathbb{K})$ with $A \subseteq C$ and $B \subseteq D$ exists between $\varphi_1(A, B)$ and $\varphi_2(A, B)$ with $\varphi_1(A, B)$ as its bottom element and $\varphi_2(A, B)$ as its top element. In the extreme case, this interval can comprise all of $\underline{\mathfrak{B}}(\mathbb{K})$, as the following example shows.

Example 2. Let \mathbb{K} be the formal context in Figure 3 and $\mathbb{S} = [\{1, 2\}, \{a, b\}] \leq \mathbb{K}$. For the concept $(A, B) = (\{1, 2\}, \{a, b\})$ of \mathbb{S} , $\varphi_1(A, B) = (\{1, 2\}, \{a, b, c, d\})$ and $\varphi_2(A, B) = (\{1, 2, 3, 4\}, \{a, b\})$ hold. These are the bottom and the top element of the whole concept lattice of \mathbb{K} .

This raises the question whether there is a concept lattice where a Boolean suborder exists that can not be obtained by embedding. This is indeed the case also in Figure 2; see, e.g., the Boolean order marked with filled red circles.

An approach to make any Boolean suborder of a (concept) lattice reachable is to expand \mathbb{K} by additional objects and attributes so that every formal concept $C \in \underline{\mathfrak{B}}(\mathbb{K})$ can be generated by one object and by one attribute. For a (concept) lattice \underline{L} this is the case with the context $\mathbb{K} = (L, L, \leq)$. Here $\underline{S} \in \mathcal{SOB}_k(\underline{L})$ is the image of both $\varphi_1(\mathbb{S})$ and $\varphi_2(\mathbb{S})$ for the Boolean subcontext $\mathbb{S} = (S, S, \leq)$.

Since we are interested in the connections between the existence of Boolean subcontexts on the one hand and the existence of Boolean suborders on the other hand, we observe a first relationship between these sets.

Lemma 11. *Let \mathbb{K} be a formal context, $\mathcal{SB}_k(\mathbb{K}) \neq \emptyset$. Then $\mathcal{SOB}_k(\underline{\mathfrak{B}}(\mathbb{K})) \neq \emptyset$.*

Proof. Let $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$. By definition $\underline{\mathfrak{B}}(\mathbb{S}) \cong \mathfrak{B}(k)$. Since $\varphi_1 : \underline{\mathfrak{B}}(\mathbb{S}) \mapsto \underline{\mathfrak{B}}(\mathbb{K})$ is an order embedding $\varphi_1(\underline{\mathfrak{B}}(\mathbb{S}))$ is a Boolean suborder of dimension k in $\underline{\mathfrak{B}}(\mathbb{K})$. \square

In general the images of $\varphi_1(\mathbb{S})$ and $\varphi_2(\mathbb{S})$ are neither lattices nor semilattices. However, we know from Lemma 9 that if \mathbb{S} is a reduced Boolean subcontext and $\varphi_1(\mathfrak{B}(\mathbb{S})) = \varphi_2(\mathfrak{B}(\mathbb{S}))$ holds, there exists a Boolean sublattice \underline{S} of the same dimension in $\underline{\mathfrak{B}}(\mathbb{K})$. We can generalize the previous statement as follows:

Lemma 12. *Let \mathbb{K} be a clarified formal context and $\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{SRB}_k(\mathbb{K})$ with $\mathbb{S}_1 = [H_1, N_1], \mathbb{S}_2 = [H_2, N_2]$ and $\mathbb{S}_1 \neq \mathbb{S}_2$. If $H_1 \neq H_2$, then $\varphi_1(\mathbb{S}_1) \neq \varphi_1(\mathbb{S}_2)$ holds. If $N_1 \neq N_2$, then $\varphi_2(\mathbb{S}_1) \neq \varphi_2(\mathbb{S}_2)$ holds.*

Proof. Since $\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{SRB}_k(\mathbb{K})$, $|H_1| = |H_2|$ holds. If $H_1 \neq H_2$ holds, $g_1 \in H_1$ with $g_1 \notin H_2$ and $g_2 \in H_2$ with $g_2 \notin H_1$ exist. Since \mathbb{S}_1 and \mathbb{S}_2 are reduced and Boolean there is a concept $C_1 = (g_1, g_1'') \in \mathfrak{B}(\mathbb{S}_1)$ and a concept $C_2 = (g_2, g_2'') \in \mathfrak{B}(\mathbb{S}_2)$. Hence \mathbb{K} is clarified, $\varphi_1(C_1) = (g_1'', g_1') \neq (g_2'', g_2') = \varphi_1(C_2)$. If $N_1 \neq N_2$ holds, the analogous procedure can be executed using φ_2 . \square

Based on this statement, we can assume that the total number of reduced Boolean subcontexts of a formal context \mathbb{K} is a lower bound of the total number of Boolean suborders of $\underline{\mathfrak{B}}(\mathbb{K})$:

Conjecture 1. Let \mathbb{K} be a clarified formal context with $|\mathcal{SRB}_k(\mathbb{K})| = n$. Then $|\mathcal{SOB}_k(\underline{\mathfrak{B}}(\mathbb{K}))| \geq n$ holds.

This conjecture can not be proved as straight forward as Lemma 12 since φ_1 and φ_2 can be identical for some $\mathbb{S} \in \mathcal{SRB}_k(\mathbb{K})$. In addition not every Boolean suborder is the image of $\varphi_1(\mathbb{S})$ or $\varphi_2(\mathbb{S})$ for a $\mathbb{S} \in \mathcal{SRB}_k(\mathbb{K})$. Both phenomena occur in the example given in Figure 4, where the marked Boolean suborder is not the image of the embedding by φ_1 or φ_2 of any Boolean subcontext contained in the given formal context, although in this case the number of Boolean subcontexts of dimension 3 and Boolean suborders of dimension 3 is identical.

6.2 Subconcepts associated to Suborders

After investigating mappings of Boolean subcontexts to Boolean suborders, we now analyze the connection between those substructures the other way around. As presented by Albano and Chornomaz [3, Prop. 1] every formal context \mathbb{K} contains a Boolean subcontext $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$ if $\mathfrak{B}(\mathbb{K})$ contains a Boolean suborder $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$. Based on this statement, we introduce a construction to generate a (not necessarily reduced) Boolean subcontext of a formal context based on a Boolean suborder of the corresponding concept lattice.

Definition 4. Let \mathbb{K} be a formal context and $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$. We call $\psi(\underline{S}) := [H, N]$ with $H := \bigcup_{C \in \text{At}(\underline{S})} \min G_{\text{obj}}(C)$ and $N := \bigcup_{C \in \text{CoAt}(\underline{S})} \min G_{\text{att}}(C)$ the subcontext of \mathbb{K} associated to \underline{S} .

Indeed the structure arising from the construction given in Definition 4 is a Boolean subcontext of the same dimension as \underline{S} :

Lemma 13. Let \mathbb{K} be a formal context, $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$ and $\mathbb{S} = [H, N] := \psi(\underline{S})$ the subcontext of \mathbb{K} associated to \underline{S} . Then $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$.

Proof. Let $\text{At}(\underline{S}) = \{A_1, A_2, \dots, A_k\}$ and $\text{CoAt}(\underline{S}) = \{C_1, C_2, \dots, C_k\}$. Due to the Boolean structure of \underline{S} the atoms can be ordered holding the following condition: A_i is a lower bound for the set $\text{CoAt}(\underline{S}) \setminus C_i$ for all $1 \leq i \leq k$ and analogous C_i is an upper bound for the set $\text{At}(\underline{S}) \setminus A_i$ for all $1 \leq i \leq k$. It follows gIm for all $g \in \min_{\text{obj}} G(A_i)$, $m \in N \setminus \min G_{\text{att}}(C_i)$ and $g \not Im$ else. So $\mathbb{S} \cong \mathbb{N}^c(k)$. \square

In the following, we study the interplay of the mapping ψ from suborders to subcontexts with the mappings φ_1 and φ_2 from subcontexts to suborders.

Lemma 14. Let \mathbb{K} be a formal context and $\mathbb{S} = [H, N] \in \mathcal{SRB}_k(\mathbb{K})$. Then $\mathbb{S} = \psi(\varphi_1(\mathbb{S}))$ iff for all $n \in N$ ($n', n'' \in \text{CoAt}(\varphi_1(\mathbb{S}))$) holds and $\mathbb{S} = \psi(\varphi_2(\mathbb{S}))$ holds iff for all $h \in H$ ($h'', h' \in \text{At}(\varphi_2(\mathbb{S}))$) holds.

Proof. Consider φ_1 : Let $\psi(\varphi_1(\mathbb{S})) = [\tilde{H}, \tilde{N}]$, $H = \{h_1, h_2, \dots, h_k\}$ and $N = \{n_1, n_2, \dots, n_k\}$. Due to the construction of φ_1 $\text{At}(\varphi_1(\mathbb{S})) = \{A_1, A_2, \dots, A_k\}$ with $A_i = (h_i'', h_i')$. Since every h_i is a minimal object generator of an atom of $\varphi_1(\mathbb{S})$ $\tilde{H} = H$ holds. Let $\text{CoAt}(\varphi_1(\mathbb{S})) = \{C_1, C_2, \dots, C_k\}$. \tilde{N} consists of the minimal attribute generators of the coatoms of $\varphi_1(\mathbb{S})$. Following, $\tilde{N} = N$ if and only if a renumbering of the coatoms exists so that $C_i = (n_i', n_i'')$ for all $i \in \{1, 2, \dots, k\}$. The procedure for φ_2 is analogous. \square

Example 3. Let \mathbb{K} be the formal context in Figure 4 and $\mathbb{S}_1 = [\{1, 2, 3\}, \{a, b, c\}]$, $\mathbb{S}_2 = [\{2, 3, 4\}, \{a, b, c\}]$, $\mathbb{S}_3 = [\{1, 2, 3\}, \{b, c, d\}]$ and $\mathbb{S}_4 = [\{2, 3, 4\}, \{b, c, d\}]$ its reduced Boolean subcontexts of dimension 3. Then $\mathbb{S}_1 = \psi(\varphi_1(\mathbb{S}_1)) = \psi(\varphi_2(\mathbb{S}_1))$, $\mathbb{S}_2 = \psi(\varphi_2(\mathbb{S}_2))$ and $\mathbb{S}_3 = \psi(\varphi_1(\mathbb{S}_3))$ hold.

Lemma 15. Let \mathbb{K} be a formal context, $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$, $\mathbb{S} := \psi(\underline{S})$. Let $C \in \underline{S} \setminus \{0_{\underline{S}}, 1_{\underline{S}}\}$ with either C not being the supremum (in $\mathfrak{B}(\mathbb{K})$) of a subset of $\text{At}(\underline{S})$ or C not being the infimum (in $\mathfrak{B}(\mathbb{K})$) of a subset of $\text{CoAt}(\underline{S})$. Then (A, B) with $A = \bigcup \{\min G_{\text{obj}}(X) \mid X \in \text{At}(\underline{S}), X \leq C\}$ and $B = \bigcup \{\min G_{\text{att}}(X) \mid X \in \text{CoAt}(\underline{S}), X \geq C\}$ is a concept of \mathbb{S} with $\varphi_1(A, B) \neq \varphi_2(A, B)$.

Proof. According to the construction of \mathbb{S} there is a concept $(A, B) \in \mathfrak{B}(\mathbb{S})$ as stated. If C is not the supremum of a subset of $At(\underline{S})$, especially A does not generate C . Therefore $\varphi_1(A, B) = (A'', A') < C$, due to the construction of A . Also $\varphi_2(A, B) = (B', B'') \geq C$ and consequently $\varphi_1(A, B) < \varphi_2(A, B)$. Similarly, if C is not the infimum of a subset of $CoAt(\underline{S})$, $\varphi_1(A, B) = (A'', A') \leq C$, $\varphi_2(A, B) = (B', B'') > C$ and $\varphi_1(A, B) < \varphi_2(A, B)$. \square

Lemma 16. *Let \mathbb{K} be a formal context, $\underline{S} \in \mathcal{SOB}(\mathfrak{B}(\mathbb{K}))$. Then $\varphi_1(\psi(\underline{S}))$ is a sub- \vee -semilattice and $\varphi_2(\psi(\underline{S}))$ is a sub- \wedge -semilattice of $\mathfrak{B}(\mathbb{K})$.*

Proof. Let $\mathbb{S} = [H, N] := \psi(\underline{S})$. H is the set of all minimal generators of the atoms of \underline{S} . Due to the Boolean structure, all concepts in \mathbb{K} that are generated by a subset of H are exactly the supremum of a subset of $At(\mathbb{S})$. Since this generation corresponds to mapping the concepts $C \in \mathfrak{B}(\mathbb{S})$ with φ_1 , $\varphi_1(\mathbb{S})$ is a sub- \vee -semilattice. The second part of the statement is proved similarly. \square

Definition 5. *Let \mathbb{K} be a formal context, $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$. We call $\varphi_1(\psi(\underline{S}))$ the sub- \vee -sublattice of $\mathfrak{B}(\mathbb{K})$ associated to \underline{S} and $\varphi_2(\psi(\underline{S}))$ the sub- \wedge -sublattice of $\mathfrak{B}(\mathbb{K})$ associated to \underline{S} .*

The statement in Lemma 16 holds especially for a \underline{S} being a Boolean sub-semilattice or a Boolean sublattice of $\mathfrak{B}(\mathbb{K})$ and provides $\varphi_1(\psi(\underline{S})) = \underline{S}$ and $\varphi_2(\psi(\underline{S})) = \underline{S}$, respectively, as follows.

Lemma 17. *Let \mathbb{K} be a formal context and $\underline{S} \in \mathcal{SOB}_k(\mathfrak{B}(\mathbb{K}))$. If \underline{S} is a sub- \vee -semilattice, $\varphi_1(\psi(\underline{S})) = \underline{S}$. If \underline{S} is a sub- \wedge -semilattice, $\varphi_2(\psi(\underline{S})) = \underline{S}$.*

Proof. Let \underline{S} be a sub- \vee -semilattice and $\mathbb{S} = [H, N] := \psi(\underline{S})$. H is the set of minimal generators of the atoms of \underline{S} . Due to the Boolean structure all concepts in $\mathfrak{B}(\mathbb{K})$ that are generated by a subset of H are exactly the supremums of a subset of the atoms of \underline{S} . Since this generation corresponds to mapping the concepts $C \in \mathfrak{B}(\mathbb{S})$ with φ_1 , every image of $\varphi_1(C)$ is contained in \underline{S} . The second statement is proved similarly. \square

Proposition 3. *Let \mathbb{K} be a formal context and $\underline{S} \in \mathcal{SLB}_k(\mathfrak{B}(\mathbb{K}))$ a sublattice. Then $\varphi_1(\psi(\underline{S})) = \varphi_2(\psi(\underline{S})) = \underline{S}$.*

Our research can be concluded in the following theorems. They give an insight into the interplay of φ_1, φ_2 and ψ and the structural properties they transfer.

Theorem 2. *Let \mathbb{K} be a formal context and $\mathbb{S} \in \mathcal{SB}(\mathbb{K})$. Then:*

- i) $\psi(\varphi_1(\mathbb{S})) = \mathbb{S}$ iff a sub- \vee -semilattice $\underline{S} \in \mathcal{SOB}(\mathfrak{B}(\mathbb{K}))$ exists with $\psi(\underline{S}) = \mathbb{S}$.
- ii) $\psi(\varphi_2(\mathbb{S})) = \mathbb{S}$ iff a sub- \wedge -semilattice $\underline{S} \in \mathcal{SOB}(\mathfrak{B}(\mathbb{K}))$ exists with $\psi(\underline{S}) = \mathbb{S}$.
- iii) $\psi(\varphi_1(\mathbb{S})) = \psi(\varphi_2(\mathbb{S})) = \mathbb{S}$ iff a $\underline{S} \in \mathcal{SLB}(\mathfrak{B}(\mathbb{K}))$ exists with $\psi(\underline{S}) = \mathbb{S}$.

Furthermore, if \mathbb{S} is reduced, $\varphi_1(\mathbb{S}) = \varphi_1(\psi(\varphi_1(\mathbb{S})))$ and $\varphi_2(\mathbb{S}) = \varphi_2(\psi(\varphi_2(\mathbb{S})))$.

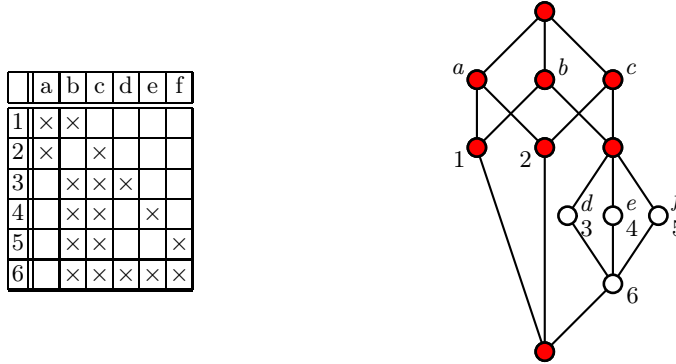


Fig. 5. Example of a formal context that shows that neither φ_1 and ψ nor φ_2 and ψ are (dually) adjoint mappings.

Proof. Consider i): (\Rightarrow) follows directly from Lemma 16 since \mathbb{S} is the subcontext corresponding to the suborder $\varphi_1(\mathbb{S})$. (\Leftarrow) is presented in Lemma 17. ii) is proved similarly and iii) follows from the combination of i) and ii). The last statement follows from the combination of Lemma 9 and Lemma 15. \square

Theorem 3. Let \mathbb{K} be a formal context and $\underline{S} \in \text{SOB}(\mathfrak{B}(\mathbb{K}))$.

- i) Then $\varphi_1(\psi(\underline{S})) = \underline{S}$ iff \underline{S} is a sub- \vee -semilattice.
- ii) Then $\varphi_2(\psi(\underline{S})) = \underline{S}$ iff \underline{S} is a sub- \wedge -semilattice.
- iii) Then $\varphi_1(\psi(\underline{S})) = \varphi_2(\psi(\underline{S})) = \underline{S}$ iff \underline{S} is a sublattice.

Proof. Consider i): (\Rightarrow) follows directly from Lemma 16. (\Leftarrow) is presented in Lemma 17, ii) is proved similarly, iii) follows from combining i) and ii). \square

Although φ_1 and ψ (or φ_2 and ψ) seem to be (dually) adjoint mappings, they are not. E.g., in Figure 5 consider the subcontexts $\mathbb{S}_1 = [\{1, 2, 3, 4\}, \{a, b, c\}]$, $\mathbb{S} = [\{1, 2, 3, 4, 5\}, \{a, b, c\}]$, and $\mathbb{S}_2 = [\{1, 2, 3, 4, 5, 6\}, \{a, b, c\}]$. It holds $\varphi_1(\mathbb{S}_1) = \varphi_1(\mathbb{S}_2) = \varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S}_2) = \varphi_2(\mathbb{S}_1)$ – the image is highlighted in the line diagram, and its associated context is \mathbb{S} . This shows that $\psi \circ \varphi_1$ is neither monotonic nor anti-monotonic, and the same holds for $\psi \circ \varphi_2$.

7 Interplay of both approaches

In the previous sections, two approaches to relate Boolean substructures of a formal context \mathbb{K} with those of the corresponding concept lattice $\mathfrak{B}(\mathbb{K})$ were introduced. In this section, we set both of them in relation.

In Section 5 a one-to-one correspondence between the closed-subcontexts of a formal context \mathbb{K} and the sublattices of $\mathfrak{B}(\mathbb{K})$ is presented. However, subsemilattices and suborders are not addressed. In addition, the closed-subcontexts restrict

not only the object set and the attribute set of a formal context but also its incidence relation, whereby they could be understood as a more substantial altering of \mathbb{K} compared to the approach presented in Section 6. It provides different maps to associate specific Boolean suborders on the one side with Boolean subcontexts on the other side while transferring some structural information.

The intersection of both approaches is localised in the Boolean subcontexts that are closed-subcontexts as well and in general the subcontexts $\mathbb{S} \leq \mathbb{K}$ with $C \in \mathfrak{B}(\mathbb{K})$ for all $C \in \mathfrak{B}(\mathbb{S})$.

Lemma 18. *Let \mathbb{K} be a formal context. $\mathbb{S} \leq \mathbb{K}$ is a closed-subcontext of \mathbb{K} iff $\varphi_1(C) = \varphi_2(C) = C$ for all $C \in \mathfrak{B}(\mathbb{S})$.*

This statement can be restricted to Boolean subcontexts. E.g., the Boolean subcontext $\mathbb{S} = [G, \{a, b, c\}]$ in Figure 2 fulfils the requirement. In general, the set of the Boolean subcontexts of \mathbb{K} that are closed-subcontexts is smaller than the set of all Boolean sublattices of $\mathfrak{B}(\mathbb{K})$. So not every Boolean sublattice of $\mathfrak{B}(\mathbb{K})$ can be reached by an embedding of a subcontext of such a structure. Referring to those structures we expand the statement of Lemma 11 as follows:

Lemma 19. *Let \mathbb{K} be a formal context and $\mathbb{S} \in \mathcal{SB}_k(\mathbb{K})$ with \mathbb{S} a closed-subcontext of \mathbb{K} . Then $\underline{\mathbb{S}} := \varphi_1(\mathbb{S}) = \varphi_2(\mathbb{S}) \in \mathcal{SLB}_k(\mathfrak{B}(\mathbb{K}))$.*

However, in general the subcontext $\tilde{\mathbb{S}}$ associated to $\underline{\mathbb{S}}$ is not equal to \mathbb{S} . E.g. in Figure 2 the subcontext $\mathbb{S} = [G, \{a, b, c\}]$ is embedded to a Boolean sublattice $\underline{\mathbb{S}}$ but the sublattice, that is associated to $\underline{\mathbb{S}}$ is $\tilde{\mathbb{S}} = [\{1, 2, 3, 4\}, \{a, b, c\}]$.

8 Conclusion

This work relates Boolean substructures in a formal context \mathbb{K} with those in its concept lattice $\mathfrak{B}(\mathbb{K})$. The notion of closed-subcontexts of \mathbb{K} is presented to generalize closed relations and provide a one-to-one correspondence to the set of all sublattices of $\mathfrak{B}(\mathbb{K})$ using a direct construction. In particular, this relationship can be restricted to the set of all Boolean closed-subcontexts of \mathbb{K} , that can be generated based on the set of all reduced Boolean subcontexts of \mathbb{K} , and all Boolean sublattices of $\mathfrak{B}(\mathbb{K})$. Moreover, we investigated two embeddings of Boolean subcontexts of \mathbb{K} into $\mathfrak{B}(\mathbb{K})$. The images of those embeddings are, in general, not sub(semi)lattices but only Boolean suborders and do not cover $\mathcal{SOB}(\mathbb{K})$ completely. Through the introduction of the subcontext \mathbb{S} associated to a Boolean suborder $\underline{\mathbb{S}}$ of $\mathfrak{B}(\mathbb{K})$, the investigated connection is investigated the other way around. The combination of both approaches give an insight of their interplay and the structural information they transfer. Through this every subsemilattice $\underline{\mathbb{S}}$ can be associated with a concrete subcontext, that can be mapped to $\underline{\mathbb{S}}$ by one of the two embeddings.

We conclude this work with two open questions. First, we are curious to which amount the presented findings can be transferred to general substructures of (not necessarily finite) formal contexts and their corresponding concept lattices. Secondly, we are interested in consideration of other special substructures, e.g., the

subcontexts of a concept lattice isomorphic to a nominal scale, as those scales also contain nearly identical objects that differ only in one attribute.

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