NONPARAMETRIC ESTIMATIONS AND THE DIFFEOLOGICAL FISHER METRIC

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ABSTRACT. In this paper, first, we survey the concept of diffeological Fisher metric and its naturality, using functorial language of probabilistic morphisms, and slightly extending Lê's theory in [Le2020] to include weakly C^k -diffeological statistical models. Then we introduce the resulting notions of the diffeological Fisher distance, the diffeological Hausdorff–Jeffrey measure and explain their role in classical and Bayesian nonparametric estimation problems in statistics.

1. Introduction

In the present paper we survey the concept of the diffeological Fisher metric, introduced in [Le2020], and explain its role in frequentist and Bayesian nonparametric density estimations. Diffeological Fisher metric is a natural extension of the Fisher metric to singular statistical models, which are ubiquitous in machine learning [Watanabe 2009], [Amari 2016]. Among different approaches to singular spaces, we find the Souriau theory of diffeological spaces [Souriau1980] best suitable for our study of statistical models, parameterized statistical models and dynamics on them. The role of the diffeological Fisher metric in frequentist nonparametric estimation is expressed via the Cramér–Rao inequality (Theorem 3.4, Remark 3.5). The role of the diffeological Fisher metric in Bayesian estimations is expressed via the choice of the objective a prior Hausdorff–Jeffrey measure on 2-integrable diffeological statistical models (Definition 4.2, Theorem 4.3). The Hausdorff-Jeffrey measure is a natural generalization of the Jeffrey measure, using the concept of the diffeological Fisher distance that is introduced in the present paper, and combining with the concept of the Hausdorff measure in geometric measure theory. Geometric measure theory could be described as differential geometry, generalized through measure theory to deal with singular mappings and singular spaces and applied to the calculus of variations [Federer1969], [AT2004], [Morgan2009]. Hausdorff measures play an important role in several areas of mathematics, e.g., in the theory of fractals, in the theory of stochastic processes. We also refer the reader to [JLT2021] for

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a categorical treatment of the Dirichlet (a prior) measure on the set $\mathcal{P}(\mathcal{X})$ of all probability measures on a measurable space \mathcal{X} whose σ -algebra will be denoted by $\Sigma_{\mathcal{X}}$.

The remaining part of our paper is organized as follows. In Section 2 we recall the concept of a C^k -diffeological space, adapted from [IZ2013], and the resulting concepts of a C^k -diffeological statistical model and a weakly C^k -diffeological statistical model (Definitions 2.1, 2.6, Example 2.4, Lemma 2.7, Remarks 2.8, 2.10), the notion of the diffeological Fisher metric, slightly extending the concepts introduced by Lê in [Le2020]. Then we introduce the notion of the diffeological Fisher distance (Definition 2.14, Theorem 2.15, Remark 2.17). In the last part of Section 2 we recall the concept of probabilistic morphisms and the monotonicity (resp. the invariance) of the diffeological Fisher metric under probability morphisms (resp. sufficient probabilistic morphisms). Then we deduce similar functorial properties for the diffeological Fisher distance. In Section 3 we recall the concept of a nonparametric φ -estimator introduced in [Le2020] and the related diffeological Cramér-Rao inequality, proved in [Le2020], see also Remark 3.5, where we discuss the validity of the diffeological Crámer-Rao for weakly C^k -diffeological statistical models. In Section 4 we introduce the resulting notion of the Hausdorff-Jeffrey measure (Definition 4.2). Then we derive their monotonicity and invariance property from the corresponding properties of the Fisher distance (Theorem 4.3). In the last section we discuss some open questions and future directions.

2. Diffeological Fisher metric, diffeological Fisher distance and probabilistic morphisms

First let us recall the notion of a C^k -diffeological space.

- **Definition 2.1.** [Le2020, Definition 3], cf. [IZ2013, §1.5] For $k \in \mathbb{N} \cup \infty$ and a nonempty set X, a C^k -diffeology of X is a set \mathcal{D} of mappings $\mathbf{p}: U \to X$, where U is an open domain in \mathbb{R}^n , and n runs over nonnegative integers, such that the three following axioms are satisfied.
- D1. Covering. The set \mathcal{D} contains the constant mappings $\mathbf{x}: r \mapsto x$, defined on \mathbb{R}^n , for all $x \in X$ and for all $n \in \mathbb{N}$.
- D2. Locality. Let $\mathbf{p}: U \to X$ be a mapping. If for every point $r \in U$ there exists an open neighborhood V of r such that $\mathbf{p}_{|V}$ belongs to \mathcal{D} then the map \mathbf{p} belongs to \mathcal{D} .
- D3. Smooth compatibility. For every element $\mathbf{p}: U \to X$ of \mathcal{D} , for every real domain V, for every $\psi \in C^k(V, U)$, $\mathbf{p} \circ \psi$ belongs to \mathcal{D} .
- A C^k -diffeological space X is a nonempty set X equipped with a C^k -diffeology \mathcal{D} . Elements $\mathbf{p}: U \to X$ of \mathcal{D} will be called C^k -maps from U to X
- A map $f:(X,\mathcal{D})\to (X',\mathcal{D}')$ between two C^k -diffeological spaces is called a C^k -map, if for any $\mathbf{p}\in\mathcal{D}$ we have $f\circ\mathbf{p}\in\mathcal{D}'$.

<u>Digression.</u> Recall that a map $\varphi: U \to V$ is called weakly $(Fr\acute{e}chet)^1$ differentiable in $u_0 \in U$ if there exists a bounded linear operator $d\varphi_{u_0}: \mathbb{R}^n \to V$ such that [AJLS2017, p. 384]

$$w - \lim_{v \to 0} \frac{\varphi(u_0 + v) - \varphi(u_0) - d\varphi_{u_0}(v)}{||v||} = 0,$$

where w-lim denotes the weak limit. In this case $d\varphi_{u_0}$ is called the weak differential of φ at u_0 . Denote by $\operatorname{Lin}(E,V)$ the Banach space of bounded linear maps from a Banach space E to a Banach space V with the induced norm. A map $\varphi: U \to V$ is called a weak C^k -map, if it is weakly differentiable, and if the inductively defined maps $d^1 := d\varphi: U \to Lin(\mathbb{R}^n, V)$, and

$$d^{r+1}\varphi: \mathbb{R}^n \to \operatorname{Lin}((\mathbb{R}^n)^r, V), u \mapsto d(d^r\varphi)_u \in \operatorname{Lin}((\mathbb{R}^n)^{r-1}, V)$$

are weakly differentiable for $r=1,\dots,k-1$ and weakly continuous for r=k [AJLS2017, p. 384]. Clearly the composition of weak C^k -maps is a weak C^k -map and a weak C^k -map between finite dimensional smooth manifolds is a C^k -map. We also write shorthand "w- C^k -map" for "weak C^k -map".

The concept of a weak C^k -map is a natural extension of the concept of weak convergence. The weak convergence of measures is one of most important tools in applied and theoretical statistics [Bogachev2018]. It is known that the class of weakly differentiable maps is strictly larger than the class of differentiable maps [Kaliaj2016].

Example 2.2. (1) Let V be a Banach space. Then V has the canonical C^k -diffeology \mathcal{D}^k_{can} that consists of all C^k -mappings $\mathbf{p}:U\to V$, where U is an open domain in \mathbb{R}^n . The space V has also another C^k -diffeology \mathcal{D}^k_w that consists of all weak C^k -mappings $U\to V$, where U is an open domain in \mathbb{R}^n .

(2) Assume that (X', \mathcal{D}') is a C^k -diffeological space and $f: X \to X'$ is a map. Then the *pullback diffeology* $f^*(\mathcal{D}')$ is the C^k -diffeology on X defined as follows [IZ2013, p. 14],

$$f^*(\mathcal{D}') := \{ \mathbf{p} : U \to X | f \circ \mathbf{p} \in \mathcal{D}' \},$$

where U is an open subset of \mathbb{R}^n .

(3) Let (X, \mathcal{D}) be a C^k -diffeological space and $f: X \to X'$ a map. Then the pushforward diffeology $f_*(\mathcal{D})$ is the diffeology on X' that consists of all mappings $\mathbf{p}: U \to X'$ where $U \subset \mathbb{R}^n$ is an open subset and \mathbf{p} satisfies the following property [IZ2013, p. 24]. For every $u \in U$, there exists an open neighborhood $O(u) \subset U$ of u such that, either $\mathbf{p}|_{O(u)}$ is a constant map, or there exists a map $\mathbf{q}: O(u) \to X$ such that $\mathbf{p}|_{O(u)} = f \circ \mathbf{q}$.

Let us recall that the space $\mathcal{S}(\mathcal{X})$ of all finite signed measures on a measurable space \mathcal{X} is a Banach space, denoted by $\mathcal{S}(\mathcal{X})_{TV}$, with the total

¹in this paper we shall consider only (possibly weakly) Fréchet differentiable mappings and we shall omit "Fréchet" in the remaining part of this paper

variation norm $\|\cdot\|_{TV}$. For any statistical model $\mathcal{P}_{\mathcal{X}}$, which is, by definition, any subset in $\mathcal{P}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$ [McCullagh2002], [Le2020], we denote by $i: \mathcal{P}_{\mathcal{X}} \to \mathcal{S}(\mathcal{X})$ the natural inclusion.

Definition 2.3. cf. [Le2020, Definition 3] (1) A statistical model $\mathcal{P}_{\mathcal{X}}$ endowed with a C^k -diffeology $\mathcal{D}_{\mathcal{X}}$ is called a C^k -diffeological statistical model or a weakly C^k -diffeological statistical model, respectively, if $i_*(\mathcal{D}_{\mathcal{X}}) \subset \mathcal{D}_{can}^k$ or $i_*(\mathcal{D}_{\mathcal{X}}) \subset \mathcal{D}_w^k$, respectively. A C^k -diffeology on a weakly C^k -diffeological statistical model will be called a weak C^k -diffeology.

(2) Let $\mathcal{D}_{\mathcal{X}}$ be a C^k -diffeology on a statistical model $\mathcal{P}_{\mathcal{X}}$. For $l \in \mathbb{N} \cup \infty$ we shall call an element $\mathbf{p}: U \to \mathcal{P}_{\mathcal{X}}$ in $\mathcal{D}_{\mathcal{X}}$ of class C^{k+l} or of class w- C^{k+1} , respectively, if $i \circ \mathbf{p} \in \mathcal{D}_{can}^{k+l}$ or $i \circ \mathbf{p} \in \mathcal{D}_{w}^{k+l}$, respectively. In other words, there is a filtration of diffeologies $(i^*(\mathcal{D}_{can}^{\infty}) \cap \mathcal{D}_{\mathcal{X}}) \subset \cdots \subset (i^*(\mathcal{D}_{can}^k) \cap \mathcal{D}_{\mathcal{X}}) = \mathcal{D}_{\mathcal{X}}$ or $(i^*(\mathcal{D}_{w}^{\infty}) \cap \mathcal{D}_{\mathcal{X}}) \subset \cdots \subset (i^*(\mathcal{D}_{w}^k) \cap \mathcal{D}_{\mathcal{X}}) = \mathcal{D}_{\mathcal{X}})$, respectively.

Examples of C^k -diffeological statistical models are the image $(\mathbf{p}(M), \mathbf{p}_*(\mathcal{D}^k_{can}))$ of parameterized statistical models $(M, \mathcal{X}, \mathbf{p})$, where M is a smooth Banach manifold and $i \circ \mathbf{p} : M \to \mathcal{S}(\mathcal{X})$ is a C^k -map, see [Le2020, Example 8.2]. There are many parameterized statistical models $(M, \mathcal{X}, \mathbf{p})$ whose image $\mathbf{p}(M)$ are singular statistical models [Amari2016], [Watanabe2009], see also Example 2.11 below. We shall provide an example of an weakly C^1 -diffeological statistical model, which is not a C^1 -diffeological statistical model.

Example 2.4. Let $\mathcal{X} = [-\pi, \pi]$ with the Lebesgue measure dx. For $t \in [-1, 1] \setminus \{0\}$ we set

$$f_t(x) = \sin(\frac{x}{t})$$

and we let $f_0(x) = 0$. Then for all t we have $f_t \in L^1(\mathcal{X}, dx)$ and f_t is weakly continuous in $L^1(\mathcal{X}, dx)$ but not strongly continuous. Next we define a function $F_t(x)$ for $t \in (-1, 1)$ and $x \in [-\pi, \pi]$ as follows.

$$F_t(x) := \int_0^t f_s(x) ds.$$

Since $f_s(x) = -f_s(-x)$, for all $t \in [-1, 1]$

(2.1)
$$\int_{-\pi}^{\pi} F_t(x) \, dx = 0.$$

Since $F_t(x)$ is continuous in t and in x there exists a number A > 0 such that

$$2\pi |F_t(x)| \le A \text{ for all } (t,x) \in [-1,1] \times [-\pi,\pi].$$

Finally we define a map $c:(-1,1)\to \mathcal{P}(\mathcal{X})\subset \mathcal{S}(\mathcal{X})$

$$c(t) := \left(\frac{1}{2\pi} + \frac{F_t(x)}{2A}\right) dx.$$

Clearly c(t) is differentiable, but its derivative $c'(t) = \frac{1}{2A} f_t(x) dx$ is only weakly continuous, therefore the map c is a weak C^1 -map but not a C^1 -map. Hence the image of c is a weakly C^1 -diffeological statistical model, which is not a C^1 -diffeological statistical model.

Concerning weakly C^k -diffeological statistical models we have the following local structure result.

Let us recall that a finite signed measure $\nu \in \mathcal{S}(\mathcal{X})$ is said to be dominated by a non-negative measure μ on \mathcal{X} , if $\mu(A) = 0$ implies $\nu(A) = 0$ for any $A \in \Sigma_{\mathcal{X}}$. Alternatively, ν is called absolutely continuous w.r.t. μ , see e.g. [Neveu1970, Chapter IV].

Lemma 2.5. cf. [AJLS2017, Proposition 3.3, p. 150] Assume that $U \subset \mathbb{R}^n$ is an open connected domain and $\varphi : U \to \mathcal{P}(\mathcal{X})$ is a map such that $i \circ \varphi : U \to \mathcal{S}(\mathcal{X})$ is a weak C^1 -map. Then there exists $\mu_0 \in \mathcal{P}(\mathcal{X})$ that dominates $\varphi(u)$ for all $u \in U$.

Proof. Let $U_{\mathbb{Q}} \subset U$ be the subset of all points in U with rational coordinates in \mathbb{R}^n . Then $U_{\mathbb{Q}}$ is a countable set. By [AJLS2017, Lemma 3.1, p. 146], cf. [Neveu1970, Ex.IV.1.3] there is a measure $\mu_0 \in \mathcal{P}(\mathcal{X})$ that dominates $\varphi(u)$ for all $u \in U_{\mathbb{Q}}$. Now let $u \in U$. We shall prove that $\varphi(u) \ll \mu_0$. Assume that $A \in \Sigma_{\mathcal{X}}$ is a null-set of μ_0 . Then for all k we have $\varphi(u_k)(A) = 0$. Since $\varphi: U \to \mathcal{S}(\mathcal{X})$ is weakly continuous, and $u_{\mathbb{Q}}$ is dense in U, it follows that $\varphi(u)(A) = 0$. Hence $\varphi(u) \ll \mu_0$. This completes the proof of Lemma 2.7.

The concept of the tangent space of a C^k -diffeological statistical model $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ at a point $\xi \in \mathcal{P}_{\mathcal{X}}$ [Le2020, Remark 2]) extends naturally to the case of weakly C^k -diffeological statistical models, see Definition 2.6 below. We also refer the reader to [Souriau1980, (5.1)], [IZ2013, p. 166] for a bit more abstract approach. Note that any C^k -diffeological statistical model is a weakly C^k -diffeological statistical model.

Definition 2.6. cf. [Le2020, Remark 2] Let $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a weakly C^k -diffeological statistical model. Let $c: (-\varepsilon, \varepsilon) \to (\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a C^k -map. The tangent vector $\partial_t c(0)$ at c(0) is the image of the map $dc_0(\partial t) \in \mathcal{S}(\mathcal{X})$, where dc_0 is the weak differential of c at 0. For $\xi \in \mathcal{P}_{\mathcal{X}}$, the tangent cone $C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ consists of all tangent vectors $\partial_t c(0)$ at $c(0) = \xi$, where $c: (0,1) \to (\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a C^k -map, and the tangent space $T_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is the linear hull of $C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$.

Lemma 2.7. Let v be a tangent vector at ξ in a weakly C^k -diffeological statistical model $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$. Then v is dominated by ξ .

Proof. The proof of Lemma 2.7 uses the same argument in the proof for the case of tangent vectors of C^k -diffeological statistical models [Le2020, Remark 2], [Bogachev2010, Corollary 3.3.2, p.77], [AJLS2017, Theorem 3.1, p. 142]. Let $v = \partial_t c(0)$, where $c: (-\varepsilon, \varepsilon) \to (\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is a weak C^k -map. Let $A \in \Sigma_{\mathcal{X}}$

such that c(0)(A) = 0. Since the map $I_A : \mathcal{S}(\mathcal{X}) \to \mathbb{R}$, $: \mu \mapsto \mu(A)$, is a linear bounded map, the map $I_A \circ c :\to \mathbb{R}$ is a C^1 -map, see e.g. [AJLS2017, Proposition C.2, p. 385]. It follows that

$$\frac{d}{dt}_{|t=0}I_A \circ c(t) = I_A(v) = 0$$

since $I_A \circ c(t) \geq 0$. Hence $v \ll c(0) = \xi$. This completes the proof of Lemma 2.7.

Lemma 2.7 implies that for any tangent vector v at a point ξ of a weakly C^k -diffeological statistical model $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$, the logarithmic representation of v

$$(2.2) \log v := dv/d\xi$$

is an element of $L^1(\mathcal{X}, \xi)$. The set $\{\log v | : v \in C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})\}$ is a subset in $L^1(\mathcal{X}, \xi)$. We denote it by $\log(C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}}))$ and will call it the logarithmic representation of $C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$. In [AJLS2017, Definition 3.6, p. 152], for a C^1 -map $c:(0,1) \to \mathcal{P}_{\mathcal{X}} \subset \mathcal{S}(\mathcal{X})$ we call $dc(\partial t)/dc(t)$ the logarithmic derivative of c in the direction $\partial t \in T_t(0,1)$, since in the classical case where $c(t) = f(t) \cdot \mu_0$ is a dominated measure family with differentiable density function f(t), then $dc(\partial t)/dc(t) = (d/dt)\log f(t)$.

Remark 2.8. Any bounded function H on \mathcal{X} defines a continuous linear function I_H on the Banach space $\mathcal{S}(\mathcal{X})_{TV}$ as follows

$$I_H: \mathcal{S}(\mathcal{X})_{TV} \to \mathbb{R}, : \mu \mapsto \int_{\mathcal{X}} H d\mu.$$

Assume that a map $\varphi:(0,1)\to \mathcal{P}(\mathcal{X}),:t\mapsto \mu_t$, is weakly differentiable. Let $\mu'_t:=\partial_t(\varphi(t))\in \mathcal{S}(\mathcal{X})$. Then we have

(2.3)
$$\frac{d}{dt} \int_{t=0}^{t} H d\mu_t = \int_{\mathcal{X}} H d\mu_t = \int_{\mathcal{X}} H d(\mu_0').$$

The identity (2.3) is central for many applications, see e.g. [Pflug1996] and Remark 3.5, and therefore the concept of weakly C^k -diffeological statistical models is useful. Note that measure valued weak differentiable maps from an open subset of \mathbb{R}^n have been first introduced by Pflug [Pflug1988], see also [Pflug1996, Definition 3.25, p. 158] in the case \mathcal{X} is a metric space with Borel σ -algebra, using (2.3) as the definition (with H bounded and continuous).

Definition 2.9. [Le2020, Definition 4] A C^k -diffeological statistical model $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ will be called *almost 2-integrable*, if $\log(C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})) \subset L^2(\mathcal{X}, \xi)$ for all $\xi \in \mathcal{P}_{\mathcal{X}}$. In this case we define the diffeological Fisher metric \mathfrak{g} on $\mathcal{P}_{\mathcal{X}}$ as follows. For each $v, w \in C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ we set

(2.4)
$$\mathfrak{g}_{\xi}(v,w) := \langle \log v, \log w \rangle_{L^{2}(\mathcal{X},\xi)} = \int_{\mathcal{X}} \log v \cdot \log w \, d\xi.$$

The Fisher metric on $C_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ extends naturally to a positive quadratic form on $T_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$, which is also called the Fisher metric.

An almost 2-integrable C^k -diffeological statistical model $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ will be called 2-integrable, if for any C^k -map $\mathbf{p}: U \to P_{\mathcal{X}}$ in $\mathcal{D}_{\mathcal{X}}$, the function $v \mapsto |d\mathbf{p}(v)|_{\mathfrak{g}}$ is continuous on TU.

- **Remark 2.10.** (1) As in Definition 2.9, we shall say that a weakly C^k -diffeological statistical model is almost 2-integrable, if we can define the Fisher metric on its tangent cone as in (2.4). We shall say that an almost 2-integrable weakly C^k -diffeological statistical model $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is 2-integrable, if for any weak C^k -map $\mathbf{p}: U \to \mathcal{P}_{\mathcal{X}}$ in $\mathcal{D}_{\mathcal{X}}$, the function $v \mapsto |d\mathbf{p}(v)|_{\mathfrak{g}}$ is continuous on TU.
- (2) On C^k -diffeological spaces, in particular on (weakly) C^k -diffeological statistical models $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$, we can define the notion of C^k -functions. If the dimension of its tangent spaces $T_{\xi}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is finite for all $\xi \in \mathcal{P}_{\mathcal{X}}$, then we can define the notion of a gradient of a C^k -differentiable function on $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$.

Example 2.11. Let us consider an example of a 2-integrable C^{∞} -diffeological statistical model which is the image of a parameterized statistical model $(W, \mathbb{R}, \mathbf{p} = p \cdot \mu_0)$ where

$$W = \{(a, b) \in \mathbb{R}^2 | a \in [0, 1], b \in \mathbb{R}\},\$$

 μ_0 is the Lebesgue measure on \mathbb{R} , and

$$p(x|a,b) := \frac{(1-a)e^{-x^2/2} + ae^{-(x-b)^2/2}}{\sqrt{2\pi}}.$$

This family is a typical example of Gaussian mixture models [Watanabe2009, Example 1.2, p. 14], which comprise also the changing time model (the Nile River model) and the ARMA model in time series [Amari2016, $\S12.2.6$, p. 311]. We decompose W as a disjoint union of its subsets as follows

$$W = W_- \cup W_0 \cup W_+$$

where

$$W_{-} = \{(a,b) \in W \mid a \in (0,1), b < 0\},\$$

$$W_{0} = \{(a,b) \in W \mid a \in (0,1) \& b = 0 \text{ or } a = 0 \& b \in \mathbb{R}\},\$$

$$W_{+} = \{(a,b) \in W \mid a \in (0,1), b > 0\}.$$

The restriction of \mathbf{p} to $W_- \cup W_+$ is injective, and $\mathbf{p}(W_0) = \mathbf{p}(0,0)$. We compute

$$\partial_a p(x|a,b) = \frac{-e^{-x^2/2} + e^{-(x-b)^2/2}}{\sqrt{2\pi}},$$
$$\partial_b p(x|a,b) = \frac{a(x-b)e^{-(x-b)^2/2}}{\sqrt{2\pi}}.$$

In [Watanabe2009, p. 14], using the expression of the Fisher metric via the Kullback-Leibler divergence, Watanabe showed that the Fisher metric on the parameterized statistical model $(W, \mathbb{R}, \mathbf{p} = p \cdot \mu_0)$ exists and is continuous. It follows that $(\mathbf{p}_*(W), \mathbf{p}_*(\mathcal{D}^k_{can}))$ is a 2-integrable C^k -diffeological statistical model for any $k \in \mathbb{N}$.

Lemma 2.12. The statistical model $\mathbf{p}_*(W)$ has two different C^1 -diffeologies $\mathbf{p}_*(\mathcal{D}^1_{can})$ and $i^*(\mathcal{D}^1_{can})$.

Proof. Since $\partial_a p(x|0,0) = \partial_b p(x|0,0)$ we have $T_{\mathbf{p}(0,0)}(\mathbf{p}(W),\mathbf{p}_*(\mathcal{D}^1_{can})) = \{0\}$. Now we shall show that $T_{\mathbf{p}(0,0)}(\mathbf{p}(W),i^*(\mathcal{D}^1_{can}))$ contains a nonzero vector. Let us consider a C^1 -curve $c: (-1,1) \to \mathbf{p}(W) \stackrel{i}{\to} \mathcal{S}(\mathbb{R})$ defined as follows

$$c(t) := \frac{(1 - \alpha(t))e^{-x^2/2} + \alpha(t)e^{-(x - \beta(t))^2/2}}{\sqrt{2\pi}},$$

where $\alpha(t), \beta(t)$ are the following functions on (-1, 1):

$$\alpha(t) = \int_0^t \frac{d\tau}{\log(\tau^2)} \text{ for } t \neq 0 \text{ and } \alpha(0) = 0,$$

$$\beta(t) = t \log(t^2) \text{ for } t \neq 0 \text{ and } \beta(0) = 0.$$

Clearly α, β are continuous functions. Moreover α is a C^1 -function and β is a C^1 -function outside the point $0 \in (-1,1)$ and

$$\dot{c}(t) = \frac{-\dot{\alpha}(t)e^{-x^2/2} + \dot{\alpha}(t)e^{-(x-\beta(t))^2/2} + \alpha(t)(x-\beta(t))\dot{\beta}(t)e^{-(x-\beta(t))^2/2}}{\sqrt{2\pi}}$$

Since

$$\dot{\alpha}(t) = \frac{1}{\log(t^2)}, \quad \dot{\beta}(t) = \log(t^2) + \frac{2t^2}{\log(t^2)},$$

we have

$$\lim_{t \to 0} \dot{c}(t) = \frac{xe^{-x^2/2}}{\sqrt{2\pi}}.$$

This implies that c(t) is a C^1 -curve in $(\mathbf{p}(W), i^*(\mathcal{D}_{can}^1))$ and $c(0) = 0, \dot{c}(0) \neq 0$. This completes the proof of Lemma 2.12.

Example 2.13. Let \mathcal{X} be a measurable space and λ be a σ -finite measure. In [Friedrich1991, p. 274] Friedrich considered a family $P(\lambda) := \{\mu \in \mathcal{P}(\mathcal{X}) | \mu \ll \lambda\}$ that is endowed with the following diffeology $\mathcal{D}(\lambda)$. A curve $c : \mathbb{R} \to P(\lambda)$ is a C^1 -curve, iff

$$\log \dot{c}(t) \in L^2(\mathcal{X}, c(t)).$$

Hence $(P(\lambda), \mathcal{D}(\lambda))$ is an almost 2-integrable C^1 -diffeological statistical model, see [Le2020, Example 10]. Next we shall prove that $P(\lambda)$ is not a 2-integrable C^1 -diffeological statistical model for $\mathcal{X} = (-1,1)$ and λ being the Lebesgue measure dx. It suffices to show a C^1 -curve $c: (-1,1) \to P(dx)$ such that $\dot{c}(t) \in L_2(\mathcal{X}, c(t))$ for all $t \in (-1,1)$ but $|\dot{c}(t)|_{\mathfrak{g}}$ is not continuous at t=0.

We shall construct such a curve using [AJLS2017, Example 3.4, p. 155]. First we consider a smooth function $f:[0,\infty)\to\mathbb{R}$ such that

$$f(u) > 0, f'(u) < 0$$
 for $u \in [0, 1)$, and $f(u) = 0$ for $u \ge 1$.

For $t \in (-1,1)$ we define $\mathbf{p}: (-1,1) \to \mathcal{S}(\mathcal{X}), : \mathbf{p}(t) = p(t,x)dx$, where

$$p(t,x) := \begin{cases} 1 & \text{if } x \le 0 \text{ and } t \in \mathbb{R} \\ |t|f(\frac{x}{t})^2 dx & \text{if } x > 0 \text{ and } t \ne 0 \\ 0 & \text{otherwise} \end{cases}$$

Then for all $t \in (-1,1)$ we have $\mathbf{p}(t)(\mathcal{X}) = \|\mathbf{p}(t)\|_{TV} \ge 1$. By op. cit.,

(2.5)
$$|\mathbf{p}(t) - \mathbf{p}(0)||_{TV} = t^2 \int_0^1 f(u)^2 du \le A$$

for some finite constant A, hence $\|\mathbf{p}(t)\|_{TV} \leq 2A$ for all $t \in (-1,1)$. It has been shown ibid. that $\mathbf{p}: (-1,1) \to \mathcal{S}(\mathcal{X})$ is a C^1 -map. Now we set

$$c(t) := \frac{\mathbf{p}(t)}{\|\mathbf{p}(t)\|_{TV}}.$$

Then $c:(-1,1)\to \mathcal{S}(\mathcal{X})$ is a C^1 -curve lying on $\mathcal{P}(\mathcal{X})$. By local. cit. we have for $t\neq 0$

$$\dot{\mathbf{p}}(t) = \chi_{(0,1)}(x)\operatorname{sign}(t) \Big(f(u)^2 - 2uff'(u) \Big)|_{u=x|t|^{-1}} dt$$

and $\dot{\mathbf{p}}(0) = 0$. It follows that $\dot{\mathbf{p}}(t) \in L^2(\mathcal{X}, \mathbf{p}(t))$. Furthermore we have

$$\dot{c}(t) = \frac{\dot{\mathbf{p}}(t)}{\|\mathbf{p}(t)\|_{TV}} + \frac{\mathbf{p}(t)(d/dt)\|\mathbf{p}(t)\|_{TV}}{\|\mathbf{p}(t)\|_{TV}^2},$$

$$\|\dot{c}_t\|_{\mathfrak{g}}^2 = \int_{\mathcal{X}} \left| \frac{\dot{c}(t)}{c(t)} \right|^2 c(t) dt = \frac{1}{\|\mathbf{p}(t)\|_{TV}} \int_{\mathcal{X}} \left| \frac{\dot{\mathbf{p}}(t)}{\mathbf{p}(t)} + \frac{(d/dt)\|\mathbf{p}(t)\|_{TV}}{\|\mathbf{p}(t)\|_{TV}} \right|^2 \mathbf{p}(t) dt < \infty.$$

Thus $\dot{c}(t) \in L^2(\mathcal{X}, c(t))$ for all $t \in (-1, 1)$ and $\dot{c}(0) = 1$. Since $\lim_{t\to 0} \|\dot{\mathbf{p}}(t)\|_{TV} = 0$, it follows

$$\lim_{t \to 0} \frac{d}{dt} \|\mathbf{p}(t)\|_{TV} = 0.$$

Since $\|\mathbf{p}(0)\|_{TV} = 1$, it follows that

$$\lim_{t\to 0} \|\dot{c}_t\|_g = \lim_{t\to 0} \|\dot{\mathbf{p}}(t)\|_{\mathfrak{g}}$$

which is positive by Ay-Jost-Lê-Schwachhöfer's result loc. cit. This proves our claim that P(dx) is not a 2-integrable C^1 -diffeological statistical model.

We shall use the diffeological Fisher metric to define the Fisher distance on 2-integrable C^k -diffeological statistical models $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$. Recall that $\mathcal{P}_{\mathcal{X}}$ is a topological space with the strong topology induced from the strong topology on the Banach space $\mathcal{S}(\mathcal{X})$.

Definition 2.14. Let $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a 2-integrable C^k -diffeological statistical model.

- (1) A map $c:[a,b] \to (\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ will be called a C^k -curve, if there exists $\varepsilon > 0$ and a C^k -map: $c_{\varepsilon}:(a-\varepsilon,b+\varepsilon) \to (\mathcal{P}_{\mathcal{X}},\mathcal{D}_{\mathcal{X}})$ such that the restriction of c_{ε} to [a,b] is c.
- (2) A continuous map $c: [0,1] \to (\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ will be called a piece-wise C^k -curve, if there exists a finite number of points $0 = a_0 < a_1 < a_2 \cdots < a_m = 1$ such that the restriction of c to $[a_{i-1}, a_i]$ is a C^k -curve for $i \in [1, m]$.
- (3) Let $c:[0,1] \to (\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a C^k -curve connecting $q_1, q_2 \in \mathcal{P}_{\mathcal{X}}$ such that $c(0) = q_1$ and $c(1) = q_2$. We define the length of c by

$$L(c) = \int_0^1 |\dot{c}(t)|_{\mathfrak{g}} dt$$

where $|\cdot|_{\mathfrak{g}}$ denotes the length defined by the diffeological Fisher metric \mathfrak{g} . The length of a piece-wise C^k -curve will be defined as the sum of the lengths of its C^k -smooth sub-intervals.

(4) The diffeological Fisher distance $\rho_{\mathfrak{g}}(x,y)$ between two points $x,y\in\mathcal{P}_{\mathcal{X}}$ will be defined as the infimum of the length over the space of piece-wise C^k -curves connecting x,y. In particular, if there is no C^k -path connecting x,y then $\rho_{\mathfrak{g}}(x,y)=\infty$.

Theorem 2.15. The distance function $\rho_{\mathfrak{g}}(x,y)$ is an extended metric, i.e., it can be infinite somewhere.

Proof. Clearly $\rho_{\mathfrak{g}}(x,y)$ is a symmetric nonnegative function and $\rho_{\mathfrak{g}}(x,y)$ satisfies the triangle inequality. It remains to show that $\rho_{\mathfrak{g}}(x,y)=0$ iff x=y. Since constant maps belong to $\mathcal{D}_{\mathcal{X}}$, it follows that $\rho_{\mathfrak{g}}(x,x)=0$ for all $x\in\mathcal{P}_{\mathcal{X}}$. To prove that $\rho_{\mathfrak{g}}(x,y)=0$ implies x=y, it suffices to prove the following

Lemma 2.16. For any $x, y \in \mathcal{P}_{\mathcal{X}}$ we have

$$\rho_{\mathfrak{g}}(x,y) \ge ||x-y||_{TV}.$$

Proof. Let $\gamma:[a,b]\to \mathcal{P}_{\mathcal{X}}\subset \mathcal{S}(\mathcal{X})$ be a C^k -curve joining x and y. Since

$$\frac{d\dot{\gamma}(t)}{d\gamma(t)} \in L^2(\gamma(t))$$

for all t, we have

$$||y - x||_{TV} = ||\gamma(b) - \gamma(b)||_{TV} = ||\int_{a}^{b} \dot{\gamma}(t) dt||_{TV}$$

$$\leq \int_{a}^{b} ||\dot{\gamma}(t)||_{TV} dt \leq \int_{a}^{b} |\dot{\gamma}(t)|_{\mathfrak{g}} dt.$$

This proves Lemma 2.16 for C^k -curves γ .

Next we assume that $\gamma:[0,1]\to \mathcal{P}_{\mathcal{X}}\to \mathcal{S}(\mathcal{X})$ is a piece-wise C^k -curve. Combining the previous argument and the triangle inequality for the total variation norm, we complete the proof of Lemma 2.16 immediately.

This completes the proof of Theorem 2.15,

Remark 2.17. Note that Definition 2.14 also works for weakly C^k -diffeological statistical models, but the proof of Lemma 2.16 does not work for weak C^1 -maps $\gamma:[a,b]\to \mathcal{P}_{\mathcal{X}}\subset \mathcal{S}(\mathcal{X})$. Since any weakly differentiable map $\gamma:[a,b]\to \mathcal{S}(\mathcal{X})$ is a.e. differentiable [Kaliaj2016, Theorem 3.2], we conjecture that Lemma 2.16 and Theorem 2.15 also hold for weakly C^k -diffeological statistical models.

Note that our definition of the diffeological Fisher metric and the diffeological Fisher distance is coordinate-free. In the remainder of this section we shall show the naturality of the diffeological Fisher metric and the diffeological Fisher distance, using the language of probabilistic morphisms.

In 1962 Lawvere proposed a categorical approach to probability theory, where morphisms are Markov kernels, and most importantly, he supplied the space $\mathcal{P}(\mathcal{X})$ with a natural σ -algebra Σ_w , making the notion of Markov kernels and hence many constructions in probability theory and mathematical statistics functorial [Lawvere1962].

Let us recall the definition of Σ_w . Given a measurable space \mathcal{X} , let $\mathcal{F}_s(\mathcal{X})$ denote the linear space of simple functions on \mathcal{X} . There is a natural homomorphism $I: \mathcal{F}_s(\mathcal{X}) \to \mathcal{S}^*(\mathcal{X}) := \operatorname{Hom}(S(\mathcal{X}), \mathbb{R}), \ f \mapsto I_f$, defined by integration: $I_f(\mu) := \int_{\mathcal{X}} f d\mu$ for $f \in \mathcal{F}_s(\mathcal{X})$ and $\mu \in \mathcal{S}(\mathcal{X})$. Following Lawvere [Lawvere1962], we define Σ_w to be the smallest σ -algebra on $\mathcal{S}(\mathcal{X})$ such that I_f is measurable for all $f \in \mathcal{F}_s(\mathcal{X})$. Let $\mathcal{M}(\mathcal{X})$ denote the space of all finite nonnegative measures on \mathcal{X} . We also denote by Σ_w the restriction of Σ_w to $\mathcal{M}(\mathcal{X})$, $\mathcal{M}^*(\mathcal{X}) := \mathcal{M}(\mathcal{X}) \setminus \{0\}$, and $\mathcal{P}(\mathcal{X})$.

Definition 2.18. [JLT2021, Definition 1] A probabilistic morphism (or an arrow) from a measurable space \mathcal{X} to a measurable space \mathcal{Y} is an measurable mapping from \mathcal{X} to $(\mathcal{P}(\mathcal{Y}), \Sigma_w)$.

We shall denote by $\overline{T} \colon \mathcal{X} \to (\mathcal{P}(\mathcal{Y}), \Sigma_w)$ the measurable mapping defining/generating a probabilistic morphism $T \colon \mathcal{X} \leadsto \mathcal{Y}$. Similarly, for a measurable mapping $\mathbf{p} \colon \mathcal{X} \to \mathcal{P}(\mathcal{Y})$ we shall denote by $\underline{\mathbf{p}} \colon \mathcal{X} \leadsto \mathcal{Y}$ the generated probabilistic morphism. Note that a probabilistic morphism is denoted by a curved arrow and a measurable mapping by a straight arrow.

From now on we shall always assume that $\mathcal{P}(\mathcal{X})$ is a measurable space with the σ -algebra Σ_w . Let $\delta_x \in \mathcal{P}(\mathcal{X})$ denote the Dirac measure concentrated at x on \mathcal{X} . Giry proved that the inclusion $i: \mathcal{X} \to \mathcal{P}(\mathcal{X}), x \mapsto \delta_x$, is a measurable mapping [Giry1982]. It follows that any measurable mapping $\kappa: \mathcal{X} \to \mathcal{Y}$ assigns a probabilistic morphism $\underline{i \circ \kappa}: \mathcal{X} \leadsto \mathcal{Y}$, which we shall write shorthand as $\underline{\kappa}: \mathcal{X} \leadsto \mathcal{Y}$. Hence the set of probabilistic mappings between \mathcal{X} and \mathcal{Y} contains a subset of measurable mappings between \mathcal{X} and \mathcal{Y} .

Given a probabilistic mapping $T: \mathcal{X} \leadsto \mathcal{Y}$, we define a linear map $S_*(T): \mathcal{S}(\mathcal{X}) \to \mathcal{S}(\mathcal{Y})$, called *Markov morphism*, as follows [Chentsov1972, Lemma

5.9, p. 72]

(2.6)
$$S_*(T)(\mu)(B) := \int_{\mathcal{X}} \overline{T}(x)(B) d\mu(x)$$

for any $\mu \in \mathcal{S}(\mathcal{X})$ and $B \in \Sigma_{\mathcal{Y}}$. We also denote by T_* the map $S_*(T)$ if no confusion can arise. It is known that $T_*(\mathcal{P}(\mathcal{X})) \subset \mathcal{P}(\mathcal{Y})$ [Le2020, Proposition 1]. Abusing notation, given a probabilistic mapping $T : \mathcal{X} \leadsto \mathcal{Y}$ and a C^k -diffeological statistical model $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ we define a C^k -diffeological space $(T_*(\mathcal{P}_{\mathcal{X}}), T_*(\mathcal{D}_{\mathcal{X}}))$ as the image of $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ under the smooth map $T_* : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{Y})$.

Diffeological (almost/2-integrable) statistical models are preserved under probabilistic morphisms.

Proposition 2.19. [Le2020, Theorem 1] Let $T: \mathcal{X} \leadsto \mathcal{Y}$ be a probabilistic morphism and $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ a C^k -diffeological statistical model.

- (1) Then $(T_*(\mathcal{P}_{\mathcal{X}}), T_*(\mathcal{D}_{\mathcal{X}}))$ is a C^k -diffeological statistical model.
- (2) If $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is an almost 2-integrable C^k -diffeological statistical model, then $(T_*(\mathcal{P}_{\mathcal{X}}), T_*(\mathcal{D}_{\mathcal{X}}))$ is also an almost 2-integrable C^k -diffeological statistical model.
- (3) If $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is a 2-integrable C^k -diffeological statistical model, then $(T_*(\mathcal{P}_{\mathcal{X}}), T_*(\mathcal{D}_{\mathcal{X}}))$ is also a 2-integrable C^k -diffeological statistical model.

Remark 2.20. Proposition 2.19 also holds for weakly C^k -diffeological statistical models, because the transformation $T_*: \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$, where $T: \mathcal{X} \leadsto \mathcal{Y}$ is a probabilistic morphism, is the restriction of a linear bounded map $T_* = S_*(T)$ from $\mathcal{S}(\mathcal{X})$ to itself.

Furthermore, the diffeological Fisher metric (and hence the diffeological Fisher distance) is decreasing under probabilistic morphisms and invariant under sufficient probabilistic morphisms. Denote by $L(\mathcal{X})$ the space of bounded measurable functions on \mathcal{X} . Recall that a probabilistic morphism $T: \mathcal{X} \to \mathcal{Y}$ is called sufficient for $\mathcal{P}_{\mathcal{X}}$ if there exists a probabilistic morphism $\mathbf{p}: \mathcal{Y} \to \mathcal{X}$ such that for all $\mu \in \mathcal{P}_{\mathcal{X}}$ and $h \in L(\mathcal{X})$ we have ([JLT2021, Definition 2.22], cf. [MS1966])

(2.7)
$$T_*(h\mu) = \underline{\mathbf{p}}^*(h)T_*(\mu), \text{ i.e., } \underline{\mathbf{p}}^*(h) = \frac{dT_*(h\mu)}{dT_*(\mu)} \in L^1(\mathcal{Y}, T_*(\mu)).$$

Examples of probabilistic morphisms $T: \mathcal{X} \sim \mathcal{Y}$ that are sufficient for a statistical model $\mathcal{P}_{\mathcal{X}} \subset \mathcal{X}$ are 1-1 measurable mappings [Le2020, Example 20], and measurable mappings $\kappa: \mathcal{X} \to \mathcal{Y}$ that are "regular" and satisfying the Fisher-Neymann condition, see [JLT2021, Example 4], [Le2020, Example 19] for more details.

Proposition 2.21. [Le2020, Theorem 2] Let $T: \mathcal{X} \leadsto \mathcal{Y}$ be a probabilistic morphism and $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ an almost 2-integrable C^k -diffeological statistical

model. Then for any $\mu \in \mathcal{P}_{\mathcal{X}}$ and any $v \in T_{\mu}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ we have the following monotonicity

$$\mathfrak{g}_{\mu}(v,v) \ge \mathfrak{g}_{T_*\mu}(T_*v,T_*v)$$

with the equality if T is sufficient for $\mathcal{P}_{\mathcal{X}}$.

Remark 2.22. Proposition 2.21 also holds for almost 2-integrable weakly C^k -diffeological statistical models, since the monotonicity assertion follows from the fact that, given a probabilistic morphism $T: \mathcal{X} \leadsto \mathcal{Y}$, the norm of the associated linear bounded map $T_*: \mathcal{S}(\mathcal{X}) \to \mathcal{S}(\mathcal{Y})$ in Remark 2.20 is less than or equal to 1. From the monotonicity assertion we obtain the second assertion concerning sufficient probabilistic morphisms, since if $T: \mathcal{X} \leadsto \mathcal{Y}$ is sufficient w.r.t. $\mathcal{P}_{\mathcal{X}}$ then by [JLT2021, Theorem 2.8.2] there exists a probabilistic morphism $\mathbf{p}: \mathcal{Y} \to \mathcal{X}$ such that $\mathbf{p}_*(T_*(\mathcal{P}_{\mathcal{X}})) = \mathcal{P}_{\mathcal{X}}$ and therefore $\mathbf{p}_*(T_*(\mathcal{D}_{\mathcal{X}})) = \mathcal{D}_{\mathcal{X}}$.

The monotonicity (and the invariance under sufficient probabilistic morphisms) of the diffeological Fisher metric suggests that the diffeological Fisher metric can be regarded as information metric on almost 2-integrable (weakly) C^k -diffeological statistical models cf. [AJLS2015, AJLS2017, AJLS2018], [Le2017].

3. Diffeological Cramér-Rao inequality

For a locally convex topological vector space V we denote by $Map(\mathcal{P}_{\mathcal{X}}, V)$ the space of all mappings $\varphi : \mathcal{P}_{\mathcal{X}} \to V$ and by V' the topological dual of V. Sometime we need to estimate only a "coordinate" $\varphi(\xi)$ of a probability measure $\xi \in \mathcal{P}_{\mathcal{X}}$, which determines ξ uniquely if φ is an embedding.

Definition 3.1. [Le2020, Definition 8] Let $\mathcal{P}_{\mathcal{X}}$ be a statistical model and $\varphi \in Map(\mathcal{P}_{\mathcal{X}}, V)$. A nonparametric φ -estimator $\hat{\sigma}_{\varphi}$ is a composition $\varphi \circ \hat{\sigma} : \mathcal{X} \xrightarrow{\hat{\sigma}} \mathcal{P}_{\mathcal{X}} \xrightarrow{\varphi} V$.

Example 3.2. (1) In supervised learning with an input space \mathcal{X} and a label space \mathcal{Y} we are interested in the stochastic relation between $x \in \mathcal{X}$ and its label $y \in \mathcal{Y}$, which is expressed via a measure $\mu \in (\mathcal{P}(\mathcal{X} \times \mathcal{Y}))$ that governs the distribution of labelled pair $(x,y) \in \mathcal{X} \times \mathcal{Y}$. Finding μ is a density estimation problem, assuming that we are given a sequence of i.i.d. labelled pairs $\{(x_1,y_1),\cdots,(x_n,y_n)\}$. In practice, we are interested only in knowing the conditional probability $\mu_{\mathcal{Y}|\mathcal{X}}(\cdot|x)$, which is regular under very general assumptions [Faden1985]. Then finding the conditional probability $\mu_{\mathcal{Y}|\mathcal{X}}(\cdot|x)$ is equivalent to finding a measurable mapping $\overline{T}: \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, or equivalently, a probabilistic morphism $T: \mathcal{X} \leadsto \mathcal{Y}$. Usually \mathcal{Y} is represented as a subset in \mathbb{R}^n and the knowledge of $\mu_{\mathcal{Y}|\mathcal{X}}(\cdot|x)$ is often not required, it is sufficient to determine one of its characteristics, for example the regression function

$$r_{\mu}(x) = \int_{\mathcal{V}} y d\mu_{\mathcal{Y}|\mathcal{X}}(y|x).$$

In this case, the map $\varphi : \mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to Map(\mathcal{X}, \mathbb{R}), \mu \mapsto r_{\mu}$, is defined as the composition of the mappings defined above

$$\mathcal{P}(\mathcal{X} \times \mathcal{Y}) \to \operatorname{Probm}(\mathcal{X}, \mathcal{Y}) \to Map(\mathcal{X}, \mathbb{R}),$$

where $\operatorname{Probm}(\mathcal{X}, \mathcal{Y})$ denotes the space of probabilistic morphisms from \mathcal{X} to \mathcal{Y} .

(2) A classical example of a φ -map is the moment of a probability measure in a 1-dimensional statistical model $\mathbf{p}(\Theta)$, where Θ is an interval or the real line. Given a real function g(x), we define

$$\varphi(\mathbf{p}(\theta)) := \int g(x)d\mathbf{p}(\theta).$$

Under a certain condition this map is 1-1 [Borovkov1998, p. 55].

Now we shall define an admissible class of φ -estimators, introduced in [Le2020]. Let $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a C^k -diffeological statistical model and V a locally convex vector space. For $\varphi \in Map(\mathcal{P}_{\mathcal{X}}, V)$ and $l \in V'$ we denote by φ^l the composition $l \circ \varphi$. Then we set

$$L^2_{\varphi}(\mathcal{X}, \mathcal{P}_{\mathcal{X}}) := \{ \hat{\sigma} \colon \mathcal{X} \to P_{\mathcal{X}} \mid \varphi^l \circ \hat{\sigma} \in L^2_{\xi}(\mathcal{X}) \text{ for all } \xi \in P_{\mathcal{X}} \text{ and } l \in V' \}.$$

For $\hat{\sigma} \in L^2_{\varphi}(\mathcal{X}, \mathcal{P}_{\mathcal{X}})$ we define the φ -mean value of $\hat{\sigma}$, denoted by $\varphi_{\hat{\sigma}} : \mathcal{P}_{\mathcal{X}} \to V''$, as follows (cf. [AJLS2017, (5.54), p. 279])

$$\varphi_{\hat{\sigma}}(\xi)(l) := \mathbb{E}_{\xi}(\varphi^l \circ \hat{\sigma}) \text{ for } \xi \in \mathcal{P}_{\mathcal{X}} \text{ and } l \in V',$$

where \mathbb{E}_{ξ} denoted the mathematical expectation w.r.t. the probability measure $\xi \in \mathcal{P}(\mathcal{X})$. Let us identify V with a subspace in V'' via the canonical pairing.

The difference $b_{\hat{\sigma}}^{\varphi} := \varphi_{\hat{\sigma}} - \varphi \in Map(\mathcal{P}_{\mathcal{X}}, V'')$ will be called the bias of the φ -estimator $\varphi \circ \hat{\sigma}$.

For all $\xi \in \mathcal{P}_{\mathcal{X}}$ we define a quadratic function $MSE_{\xi}^{\varphi}[\hat{\sigma}]$ on V', which is called the *mean square error quadratic function* at ξ , by setting for $l, h \in V'$ (cf. [AJLS2017, (5.56), p. 279])

(3.1)
$$\operatorname{MSE}_{\xi}^{\varphi}[\hat{\sigma}](l,h) := \mathbb{E}_{\xi} \Big[\big(\varphi^{l} \circ \hat{\sigma} - \varphi^{l}(\xi) \big) \cdot \big(\varphi^{h} \circ \hat{\sigma} - \varphi^{h}(\xi) \big) \Big].$$

Similarly we define the variance quadratic function of the φ -estimator $\varphi \circ \hat{\sigma}$ at $\xi \in \mathcal{P}_{\mathcal{X}}$ is the quadratic form $V_{\xi}^{\varphi}[\hat{\sigma}]$ on V' such that for all $l, h \in V'$ we have (cf. [AJLS2017, (5.57), p.279])

$$V_{\xi}^{\varphi}[\hat{\sigma}](l,h) = \mathbb{E}_{\xi}[(\varphi^{l} \circ \hat{\sigma} - \mathbb{E}_{\xi}(\varphi^{l} \circ \hat{\sigma})) \cdot (\varphi^{h} \circ \hat{\sigma} - \mathbb{E}_{\xi}(\varphi^{h} \circ \hat{\sigma}))].$$

Then it is known that (cf.[AJLS2017, (5.58), p. 279])

(3.2)
$$\mathrm{MSE}_{\xi}^{\varphi}[\hat{\sigma}](l,h) = V_{\xi}^{\varphi}[\hat{\sigma}](l,h) + b_{\hat{\sigma}}^{\varphi}(\xi)(l) \cdot b_{\hat{\sigma}}^{\varphi}(\xi)(h).$$

Now we assume that $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is an almost 2-integrable C^k -diffeological statistical model. For any $\xi \in \mathcal{P}_{\mathcal{X}}$ let $T_{\xi}^{\mathfrak{g}}(P_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be the completion of

 $T_{\xi}(P_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ w.r.t. the Fisher metric \mathfrak{g} . Since $T_{\xi}^{\mathfrak{g}}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ is a Hilbert space, the map

$$L_{\mathfrak{g}}: T_{\varepsilon}^{\mathfrak{g}}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}}) \to (T_{\varepsilon}^{\mathfrak{g}}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}}))', \, L_{\mathfrak{g}}(v)(w) := \langle v, w \rangle_{\mathfrak{g}},$$

is an isomorphism. Then we define the inverse \mathfrak{g}_{ξ}^{-1} of the Fisher metric \mathfrak{g}_{ξ} on $(T_{\xi}^{\mathfrak{g}}(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}}))'$ as follows

(3.3)
$$\mathfrak{g}_{\xi}^{-1}(L_g v, L_g w) := \mathfrak{g}_{\xi}(v, w)$$

Definition 3.3. [Le2020, Definition 9], cf. [AJLS2017, Definition 5.18, p. 281] Assume that $\hat{\sigma} \in L^2_{\varphi}(\mathcal{X}, \mathcal{P}_{\mathcal{X}})$. We shall call $\hat{\sigma}$ a φ -regular estimator, if for all $l \in V'$ the function $\xi \mapsto \|\varphi^l \circ \hat{\sigma}\|_{L^2(\mathcal{X},\xi)}$ is locally bounded, i.e., for all $\xi_0 \in \mathcal{P}_{\mathcal{X}}$

$$\lim_{\xi \to \xi_0} \sup \|\varphi^l \circ \hat{\sigma}\|_{L^2(\mathcal{X},\xi)} < \infty.$$

For any $\xi \in \mathcal{P}_{\mathcal{X}}$ we denote by $(\mathfrak{g}_{\hat{\sigma}}^{\varphi})_{\xi}^{-1}$ to be the following quadratic form on V':

$$(3.4) (\mathfrak{g}_{\hat{\sigma}}^{\varphi})_{\xi}^{-1}(l,k) := \mathfrak{g}_{\xi}^{-1}(d\varphi_{\hat{\sigma}}^{l},d\varphi_{\hat{\sigma}}^{k}) = \mathfrak{g}_{\xi}(\operatorname{grad}_{\mathfrak{g}}(\varphi_{\hat{\sigma}}^{l}),\operatorname{grad}_{\mathfrak{g}}(\varphi_{\hat{\sigma}}^{k})).$$

If $\varphi : \mathcal{P}_{\mathcal{X}} \to V$ is a local coordinate chart around a point $\xi \in \mathcal{P}_{\mathcal{X}}$ and $\hat{\sigma}$ is φ -unbiased then $(\mathfrak{g}_{\hat{\sigma}}^{\varphi})_{\xi}^{-1}$ is the inverse of the Fisher metric at ξ , see [AJLS2017, §5.2.3 (A), p. 286].

In [Le2020] Lê proved the following diffeological Cramér–Rao inequality

Theorem 3.4. [Le2020, Theorem 3] Let $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a 2-integrable C^k -diffeological statistical model, φ a V-valued function on $\mathcal{P}_{\mathcal{X}}$ and $\hat{\sigma} \in L^2_{\varphi}(\mathcal{X}, \mathcal{P}_{\mathcal{X}})$ a φ -regular estimator. Then the difference $\mathcal{V}^{\varphi}_{\xi}[\hat{\sigma}] - (\hat{\mathfrak{g}}^{\varphi}_{\hat{\sigma}})^{-1}_{\xi}$ is a positive semi-definite quadratic form on V' for any $\xi \in \mathcal{P}_{\mathcal{X}}$.

Remark 3.5. The proof of Theorem 3.4 does not extend to the case of 2-integrable weakly C^k -diffeological statistical models $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$. The main problem is the validity of the differentiation under integral sign for a C^k -curve $c:(0,1)\to(\mathcal{P}_{\mathcal{X}},\mathcal{D}_{\mathcal{X}})$, $t\mapsto\mu_t$,

(3.5)
$$\frac{d}{dt} \int_{\mathcal{X}} l \circ \varphi \circ \hat{\sigma} \, d\mu_t = \int_{\mathcal{X}} l \circ \varphi \circ \hat{\sigma} \, d\mu_t'$$

where $\mu'_t = \partial_t^w c(t)$. This identity is valid if $i \circ c : (0,1) \to \mathcal{S}(\mathcal{X})$ is a C^1 -map and if the function $\xi \to \|\varphi^l \circ \hat{\sigma}\|_{L^2(\mathcal{X},\xi)}$ is locally bounded, see [AJLS2017, Lemma 5.2, p. 282], whose proof involves estimations using the total variation norm. This local boundedness condition has been stated in Definition 3.3 and Theorem 3.4. The identity (3.5) has been used in the proof of [Le2020, Proposition 2], which is an important ingredient of the proof of the diffeological Cramér–Rao inequality [Le2020, Theorem 3]. If instead of the condition that the function $\xi \to \|\varphi^l \circ \hat{\sigma}\|_{L^2(\mathcal{X},\xi)}$ is locally bounded, we assume a stronger condition that $l \circ \varphi \circ \hat{\sigma} : \mathcal{X} \to \mathbb{R}$ is a bounded function for all $l \in V'$, by Remark 2.8 the identity (3.5) holds. Under this

stronger assumption, the Crámer-Rao equality holds for 2-integrable weakly C^k -diffeological statistical models $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$, since other arguments used in the proof of the diffeological Crámer-Rao inequality [Le2020, Theorem 3] also hold for this general case. We conjecture that Theorem 3.4 is also valid for weakly C^k -diffeological statistical models, since any weakly C^1 -map $[0,1] \to \mathcal{S}(\mathcal{X})_{TV}$ is a.e. differentiable by [Kaliaj2016, Theorem 3.2].

4. Diffeological Hausdorff–Jeffrey measure

In the previous sections we demonstrated that the diffeological Fisher metric is a natural extension of the Fisher metric, and the diffeological Fisher metric plays the same role in frequentist nonparametric estimation as the Fisher metric in frequentist parametric estimation. In this section we shall introduce the concept of the Hausdorff–Jeffrey measure, using the diffeological Fisher metric and the concept of the Hausdorff measure, which plays a fundamental role in geometric measure theory [Federer1969].

Let us first recall the concept of the Hausdorff measure on a metric space (E,d), following [Federer1969], [AT2004], [Morgan2009]. Recall that for any subset $S \subset E$ the diameter of S is

$$\operatorname{diam}(S) = \sup\{d(x, y) | x, y \in S\}.$$

For $k \in \mathbb{N}$ let α_k denote the Lebesgue measure of the closed unit ball $B^k(0,1)$ of radius 1 and centered at 0 in \mathbb{R}^k . Let $A \subset E$. For small δ cover A efficiently by countably many sets A_j with diam $(A_j) \leq \delta$, and the k-dimensional Hausdorff measure of A is defined as follows (4.1)

$$\mathcal{H}^{k}(A) := \lim_{\delta \to 0} \alpha_{k} \inf \left\{ \sum_{j \in I} \left(\frac{\operatorname{diam}(A_{j})}{2} \right)^{k} \mid \operatorname{diam}(A_{j}) \leq \delta \& A \subset \bigcup_{j \in I} A_{i} \right\}.$$

It is known that \mathcal{H}^k is a regular Borel measure [Federer1969, p. 171], see also [AT2004, Theorem 2.1.4, p.21]. Furthermore, \mathcal{H}^0 is the counting measure.

The definition of the k-dimensional Hausdorff measure extends to any nonnegative real dimension k, by extending the definition of α_k with the following definition

$$\alpha_k := \frac{\pi^{k/2}}{\Gamma(1+k/2)}$$
 where $\Gamma(t) := \int_0^\infty x^{t-1} e^{-x} dx$.

The Hausdorff dimension $\mathcal{H}\text{-}\dim(A)$ of a nonempty set $A\subset (E,d)$ is defined as

$$\mathcal{H}\text{-}\dim(A) := \inf\{m \ge 0 | \mathcal{H}^m(A) = 0\}.$$

It is known that if $k > \mathcal{H}\text{-}\dim(A)$ then $\mathcal{H}^k(B) = 0$ and if $k < \mathcal{H}\text{-}\dim(A)$ then $\mathcal{H}^k(A) = \infty$.

The Hausdorff measure enjoys the following natural properties.

Proposition 4.1. (1) Let (M^m, g) be a Riemannian manifold, regarded as a metric space with the Riemannian distance d_g . Then the Hausdorff measure \mathcal{H}^m on M^m coincides with the standard volume.

(2) Let $\varphi: A \subset (M^k, g) \to (N^n, g')$ be a Lipschitz map from an open domain A in a Riemannian manifold (M^k, g) of dimension k to a Riemannian manifold (N^n, g') of dimension n and $n \geq k$. By Rademacher's theorem $d\varphi$ and its area factor

$$\mathbf{J}\,d\varphi := \sqrt{\det\bigl((d\varphi)^* \circ (d\varphi)\bigr)}$$

are defined \mathcal{H}^k -almost everywhere on A. If k = n then we have the following area formula

$$\mathcal{H}^{n}(\varphi(A)) = \int_{A} \mathbf{J} d\varphi d\mathcal{H}^{n}(x).$$

For a proof of Proposition 4.1(1) see [AT2004, p. 29,30]. For a proof of Proposition 4.1(2) see [AT2004, p. 44, 45].

Definition 4.2. Let $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ be a 2-integrable C^k -diffeological statistical model with the diffeological Fisher distance $d_{\mathfrak{g}}$ and $m \in \mathbb{R}$ the Hausdorff dimension of $(\mathcal{P}_{\mathcal{X}}, d_{\mathfrak{g}})$. Then the Hausdorff measure $\mathcal{H}^m_{\mathfrak{g}}$ on $(\mathcal{P}_{\mathcal{X}}, d_{\mathfrak{g}})$ will be called the diffeological Hausdorff–Jeffrey measure of $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$.

Now we shall relate the diffeological Hausdorff–Jeffrey measure $\mathcal{H}^m_{\mathfrak{g}}$ with the unnormalized Jeffrey prior measure $\mathcal{J}^m_{\mathfrak{g}}$ defined on a 2-integrable parameterized statistical model $(M^m, \mathcal{X}, \mathbf{p})$, where $\mathbf{p}: M^m \to \mathcal{S}(\mathcal{X})$ is an injective C^1 -map [Jeffrey1946]. Recall that $J_{\mathfrak{g}}$ is equal to the Riemannian volume of the (possibly degenerate) Riemannian manifold (M^k, \mathfrak{g}) , whose density is zero at the points of M where the Fisher metric \mathfrak{g} is degenerate.

Theorem 4.3. (1) Let $(M^m, \mathcal{X}, \mathbf{p})$ be a 2-integrable parameterized statistical model, where M^m is a smooth manifold of dimension m and $\mathbf{p} \colon M^m \to \mathcal{S}(\mathcal{X})$ is an injective C^1 -map. We regard \mathbf{p} as a C^1 -map from M^m to the 2-integrable C^1 -diffeological statistical model $(\mathbf{p}(M), \mathbf{p}_*(\mathcal{D}_{can}))$. Then

$$\mathbf{p}_*(\mathcal{J}^m_{\mathfrak{g}}) = \mathcal{H}^m_{\mathfrak{g}}.$$

(2) Let $T: \mathcal{X} \sim \mathcal{Y}$ be a probability morphism and $(\mathcal{P}_{\mathcal{X}}, \mathcal{D}_{\mathcal{X}})$ a 2-integrable C^k -diffeological statistical model. Then for any $k \in \mathbb{R}$ and any Borel set in $(\mathcal{P}_{\mathcal{X}}, d_{\mathfrak{g}})$ we have

(4.2)
$$\mathcal{H}_{\mathfrak{g}}^{k}(A) \geq \mathcal{H}_{\mathfrak{g}}^{k}(\mathbf{p}_{*}(A)).$$

The inequality in (4.2) turns to inequality if T is sufficient w.r.t. $\mathcal{P}_{\mathcal{X}}$.

Proof. The first assertion of Theorem 4.3 follows from Proposition 4.1. The second assertion is a consequence of Proposition 2.21. \Box

According to Jordan [Jordan2011], to justify the choice of an a prior probability measure is one of main theoretical problems in Bayesian statistics. Theorem 4.3 justifies the choice of the Hausdorff–Jeffrey measure as natural objective a prior measure on 2-integrable C^k -diffeological statistical models.

5. Conclusion and outlook

In this paper we showed that the concept of the diffeological Fisher metric is a natural extension of the notion of the Fisher metric, originally defined on parameterized statistical models. There are two advantages of the nonparametric diffeological Fisher metric: (1) it can be defined on singular statistical models, (2) it turns a 2-integrable C^k -diffeological statistical model to a length space, on which the Hausdorff measure is a natural generalization of the Jeffrey measure. We also discussed some open questions concerning extending results from C^k -diffeological statistical models to weakly C^k diffeological statistical models. To make more use of the diffeological Fisher metric we expect that we need to put certain assumptions on the singularities of underlying 2-integrable C^k -diffeological statistical models. In the case a C^k -diffeological statistical model does not admit a diffeological Fisher metric, we might consider instead diffeological Finsler metric as in [Amari1984]. In view of recent developments of Barbaresco's and Gay-Balmaz' geometric theory of Gibbs probability densities with promising applications in machine learning [BG2020], we plan to develop a theory of diffeological exponential models for a diffeological treatment of infinite dimensional families of Gibbs probability densities.

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