# Approximation algorithms for connectivity augmentation problems 

Zeev Nutov

The Open University of Israel, nutov@openu.ac.il


#### Abstract

In Connectivity Augmentation problems we are given a graph $H=\left(V, E_{H}\right)$ and an edge set $E$ on $V$, and seek a min-size edge set $J \subseteq E$ such that $H \cup J$ has larger edge/node connectivity than $H$. In the Edge-Connectivity Augmentation problem we need to increase the edge-connectivity by 1. In the Block-Tree Augmentation problem $H$ is connected and $H \cup S$ should be 2-connected. In Leaf-to-Leaf Connectivity Augmentation problems every edge in $E$ connects minimal deficient sets. For this version we give a simple combinatorial approximation algorithm with ratio $5 / 3$, improving the 1.91 approximation of [6] (see also [23]), that applies for the general case. We also show by a simple proof that if the Steiner Tree problem admits approximation ratio $\alpha$ then the general version admits approximation ratio $1+\ln (4-x)+\epsilon$, where $x$ is the solution to the equation $1+\ln (4-x)=\alpha+(\alpha-1) x$. For the currently best value of $\alpha=\ln 4+\epsilon[7$ this gives ratio 1.942 . This is slightly worse than the ratio 1.91 of [6, but has the advantage of using STEINER TREE approximation as a "black box", giving ratio $<1.9$ if ratio $\alpha \leq 1.35$ can be achieved.

In the Element Connectivity Augmentation problem we are given a graph $G=(V, E)$, $S \subseteq V$, and connectivity requirements $r=\{r(u, v): u, v \in S\}$. The goal is to find a min-size set $J$ of new edges on $S$ (any edge is allowed and parallel edges are allowed) such that for all $u, v \in S$ the graph $G \cup J$ contains $r(u, v) u v$-paths such that no two of them have an edge or a node in $V \backslash S$ in common. The problem is NP-hard even when $r_{\max }=\max _{u, v \in S} r(u, v)=2$. We obtain approximation ratio $3 / 2$, improving the previous ratio $7 / 4$ of [21]. For the case of degree bounds on $S$ we obtain the same ratio with just +1 degree violation, which is tight, since deciding whether there exists a feasible solution is NP-hard even when $r_{\max }=2$. A similar result is shown for the more general problem of covering a skew-supermodular set function by a min-size set of edges.


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## 1 Introduction

A graph is $k$-connected if it contains $k$ internally disjoint paths between every pair of nodes; if the paths are only required to be edge disjoint then the graph is $k$-edge-connected. In Connectivity Augmentation problems we are given an "initial" graph $G_{0}=\left(V, E_{0}\right)$ and an edge set $E$ on $V$, and seek a min-size edge set $J \subseteq E$ such that $G_{0} \cup J=\left(V, E_{0} \cup J\right)$ has larger edge/node connectivity than $G_{0}$.

- In the Edge-Connectivity Augmentation problem we seek to increase the edge connectivity by one, so $G_{0}$ is $k$-edge-connected and $G_{0} \cup J$ should be $(k+1)$-edge connected.
- In the 2-Connectivity Augmentation problem we seek to make a connected graph 2-connected, so $G_{0}$ is connected and $G_{0} \cup J$ should be 2-connected.

A cactus is a "tree-of-cycles", namely, a 2-edge-connected graph in which every block is a cycle (equivalently - every edge belongs to exactly one simple cycle). By [8], the Edge-Connectivity Augmentation problem is equivalent to the following problem:

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Cactus Augmentation
Input: A cactus $T=\left(V, E_{T}\right)$ and an edge set $E$ on $V$.
Output: A min-size edge set $J \subseteq E$ such that $T \cup J$ is 3-edge-connected.
It is also known (c.f. [16]) that the 2-Connectivity Augmentation problem is equivalent to the following problem:

## Block-Tree Augmentation

Input: A tree $T=\left(V, E_{T}\right)$ and an edge set $E$ on $V$.
Output: A min-size edge set $F \subseteq E$ such that $T \cup F$ is 2-connected.
A more general problem than Cactus Augmentation is as follows. Two sets $A, B$ cross if $A \cap B \neq \emptyset$ and $A \cup B \neq V$. A set family $\mathcal{F}$ on a groundset $V$ is a crossing family if $A \cap B, A \cup B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ cross; $\mathcal{F}$ is a symmetric family if $V \backslash A \in \mathcal{F}$ whenever $A \in \mathcal{F}$. The 2-edge-cuts of a cactus form a symmetric crossing family, with the additional property that whenever $A, B \in \mathcal{F}$ cross and $A \backslash B, B \backslash A$ are both non-empty, the set $(A \backslash B) \cup(B \backslash A)$ is not in $\mathcal{F}$; such a symmetric crossing family is called proper [9]. Dinitz, Karzanov, and Lomonosov [8] showed that the family of minimum edge cuts of a graph $G$ can be represented by 2-edge cuts of a cactus. Furthermore, when the edge-connectivity of $G$ is odd, the min-cuts form a laminar family and thus can be represented by a tree. Dinitz and Nutov [9, Theorem 4.2] (see also [20, Theorem 2.7]) extended this by showing that an arbitrary symmetric crossing family $\mathcal{F}$ can be represented by 2 -edge cuts and specified 1 -node cuts of a cactus; when $\mathcal{F}$ is a proper crossing family this reduces to the cactus representation of [8]. We say that an edge $f$ covers a set $A$ if $f$ has exactly one end in $A$. The following problem combines the difficulties of the Cactus Augmentation and the Block-Tree Augmentation problems, see [23].

## Crossing Family Augmentation

Input: A graph $G=(V, E)$ and a symmetric crossing family $\mathcal{F}$ on $V$.
Output: A min-size edge set $J \subseteq E$ that covers $\mathcal{F}$.
In this problem, the family $\mathcal{F}$ may not be given explicitly, but we require that certain queries related to $\mathcal{F}$ can be answered in polynomial time, see [23]. Block-Tree Augmentation and Crossing Family Augmentation admit ratio 2 [25, 11], that applies also for the min-cost versions of the problems.

The inclusion minimal members of a set family $\mathcal{F}$ are called leaves. In the Leaf-to-Leaf Crossing Family Augmentation problem, every edge in $E$ connects two leaves of $\mathcal{F}$. In the Leaf-to-Leaf Block-Tree Augmentation problem, every edge in $E$ connects two leaves of the input tree $T$.

- Theorem 1. The leaf-to leaf versions of Crossing Family Augmentation and BlockTree Augmentation admit ratio $5 / 3$.

Better ratios are known for two special cases. In the Tree Augmentation problem the family $\mathcal{F}$ is laminar, namely, any two sets in $\mathcal{F}$ are disjoint or one contains the other; this problem can be also defined in connectivity terms - make a spanning tree 2-edge-connected by adding a min-size edge set $J \subseteq E$. This problem was vastly studied; see [1, 15, 18, 10,22 and the references therein for additional literature on the Tree Augmentation problem. In the Leaf-to-Leaf Tree Augmentation problem, every edge in $E$ connects two leaves of the tree; this problem admits ratio 17/12 [19]. The Cycle Augmentation problem is a particular case of the Cactus Augmentation problem when the cactus is a cycle; in this
case the leaves are the singleton nodes. The Cycle Augmentation problem admits ratio $\frac{3}{2}+\epsilon$ [14]; our algorithm from Theorem 1] uses some ideas from [14].

Byrka, Grandoni, and Ameli [6] showed that Cactus Augmentation admits ratio $2 \ln 4-\frac{967}{1120}+\epsilon<191$, breaching the natural 2 approximation barrier. This was extended to Crossing Family Augmentation and Block Tree Augmentation in 23.

In the Steiner Tree problem we are given a graph $G=(V, E)$ with edge costs and a set $R \subseteq V$ of terminals, and seek a min-cost subtree of $G$ that spans $R$. We prove the following.

- Theorem 2. If Steiner Tree admits ratio $\alpha$ then Crossing Family Augmentation and Block-Tree Augmentation admit ratio $1+\ln (4-x)+\epsilon$, where $x$ is the solution to the equation $1+\ln (4-x)=\alpha+(\alpha-1) x$.

Currently, $\alpha=\ln 4+\epsilon[7]$; in this case we have ratio 1.942 for the problems in the theorem. This is slightly worse than the ratio 1.91 of [6] (see also [23]), but our algorithm is very simple and has the advantage of using Steiner Tree approximation as a "black box". E.g., if ratio $\alpha=1.35$ can be achieved, then we immediately get ratio $1.895<1.9$.

We also consider the following problem:

## Element Connectivity Augmentation

Input: An undirected graph $G=(V, E)$, a set $S \subseteq V$ of terminals, and connectivity requirements $\{r(u, v): u, v \in S\}$ on pairs of terminals.
Output: A minimum size set $J$ of new edges on $S$ (any edge is allowed and parallel edges are allowed) such that the graph $G \cup J$ contains $r(u, v) u v$-paths such that no two of them have an edge or a node in $V \backslash S$ in common.

A particular case when the graph $G$ is bipartite with sides $S$ and $V \backslash S$ is known as the Hypergraph Edge-Connectivity Augmentation problem; here $S$ is the set of nodes of the hypergraph and $V \backslash S$ is the set of the hyperedges. This problem is solvable in polynomial time for uniform requirements when $r(u, v)=k$ for all $u, v \in S$ [2] (see also [4] and [5] for a simpler algorithm and proof), and when $r_{\max }=1$, where $r_{\max }$ is the maximum requirement. See also [12, 13, 5] for additional polynomially solvable cases. The non-uniform version of the problem is NP-hard even when the initial graph $G$ is connected and $r_{\max }=2$ [17]. The previous best approximation ratio for the general version was $7 / 4$, and $3 / 2$ when $r_{\max }=2$ [21].

In the degree bounded version of the problem we also have degree bounds $\{b(v): v \in S\}$ and require that $d_{J}(v) \leq b(v)$ for all $v \in S$, where $d_{J}(v)$ is the degree of $v$ w.r.t. $J$. We show that Element Connectivity Augmentation admits ratio $3 / 2$, and that this ratio can be achieved also for the degree bounded version with only additive +1 degree violation; a better degree approximation is unlikely, since deciding whether there exists a feasible solution is NP-hard even when $r_{\max }=2$ and $b_{\max }=1$ [17].

- Theorem 3. Element Connectivity Augmentation admits approximation ratio 3/2. Moreover, the degree bounded version admits a bicriteria approximation algorithm that computes a solution $J$ of size at most $3 / 2$ times the optimal such that $d_{J}(v) \leq b(v)+1$ for all $v \in S$.

The proof of this theorem is based on a generic algorithm for covering a skew-supermodular set function, as is explained in Section 5

## 2 The leaf-to-leaf case (Theorem 1)

We prove Theorem 1 for the Crossing Family Augmentation problem, and later indicate the changes needed to adopt the proof for the Block-Tree Augmentation problem. We need some definition to describe the algorithm. Let $\mathcal{F}$ be a set family on $V$. We say that $A \in \mathcal{F}$ separates $u, v \in V$ if $|A \cap\{u, v\}|=1 ; u, v$ are $\mathcal{F}$-separable if such $A$ exists and $u, v$ are $\mathcal{F}$-inseparable otherwise. Similarly, $A$ separates edges $f, g$ if one of $f, g$ has both ends in $A$ and the other has no end in $A ; f, g$ are $\mathcal{F}$-separable if such $A \in \mathcal{F}$ exists, and $\mathcal{F}$-inseparable otherwise. The relation $\{(u, v) \in V \times V: u, v$ are $\mathcal{F}$-inseparable $\}$ is an equivalence, and we call its equivalence classes $\mathcal{F}$-classes. W.l.o.g. we will assume that all $\mathcal{F}$-classes are singletons and that no edge in $E$ has both ends in the same class; in particular, the leaves of $\mathcal{F}$ are singletons, and we denote the leaf set of $\mathcal{F}$ by $L$. We will also often abbreviate the notation for singleton sets and write $v, e$ instead of $\{v\},\{e\}$. Given $J \subseteq E$, the residual instance $\left(\left(V^{J}, E^{J}\right), \mathcal{F}^{J}\right)$ is defined as follows.

- The residual family $\mathcal{F}^{J}$ of $\mathcal{F}$ w.r.t. $J$ consists of all members of $\mathcal{F}$ that are uncovered by the edges in $J$. It is known that $\mathcal{F}^{J}$ is crossing (and symmetric) if $\mathcal{F}$ is.
- $V^{J}$ is the set of $\mathcal{F}^{J}$-classes (w.l.o.g, each of them can be shrunk into a single element).
- $E^{J}$ is obtained from $E \backslash J$ by removing all edges that have both ends in the same $\mathcal{F}^{J}$-class.

In addition, given a set $R \subseteq V$ of terminals, the residual set of terminals $R^{J}$ is the set of $\mathcal{F}^{J}$-classes that contain some member of $R$. For illustration see Fig. 1 (a,b,c).

For any edge $e=u v$, there is an $\mathcal{F}^{e}$-class that contains both $u$ and $v$; denote this class by $C(\mathcal{F}, e)$. Given a set $R$ of terminals (a subset of $\mathcal{F}$-classes), the ( $R, E, \mathcal{F}$ )-incidence graph $H=\left(U, E_{H}\right)$ has node set $U=E \cup R$ and edge set

$$
E_{H}=\left\{e e^{\prime}: e, e^{\prime} \in E \text { are } \mathcal{F} \text {-inseparable }\right\} \cup\{e r: r \in R, e \in E, r \in C(\mathcal{F}, e)\}
$$

Let $R \subseteq V$ and let $H$ be the $(R, E, \mathcal{F})$-incidence graph. Note that $R$ is an independent set in $H$. It was shown in [23] that for $R=L$ being the set of leaves of $\mathcal{F}$, an edge set $J \subseteq E$ is a feasible solution to Crossing Family Augmentation if and only if the subgraph $H[J \cup R]$ of $H$ induced by $J \cup R$ is connected. The proof in [23] extends to any $R \subseteq V$ that contains $L$. This implies that Crossing Family Augmentation admits an approximation ratio preserving reduction to the following problem (see [23, 3] for more details).

Subset Steiner Connected Dominating Set (SS-CDS)
Input: A graph $H=\left(U, E_{H}\right)$ and a set $R \subseteq U$ of independent terminals.
Output: A min-size node set $S \subseteq U \backslash R$ such that $H[S]$ is connected and $S$ dominates $R$.

Given a SS-CDS instance and $s \in S=U \backslash R$ let $R(s)=R_{H}(s)$ denote the set of neighbors of $s$ in $H$ that belong to $R$. Let opt be the optimal solution value of a problem instance at hand. Before describing the algorithm, we will prove the following lemma.

- Lemma 4. Let $\mathcal{I}=(H, R)$ be a SS-CDS instance such that $|R(s)|=2$ for all $s \in S=U \backslash R$. Then one of the following holds:
(i) There are adjacent $a, b \in S$ with $R(a) \cap R(b)=\emptyset$.
(ii) opt $\geq|R|-1$.

Proof. Assume that (i) does not hold for $\mathcal{I}$; we will prove that then (ii) holds. The proof is by induction on $|R|$. In the base case $|R|=2$ (ii) holds. Assume that the statement is true for $|R|-1 \geq 2$. Let $T$ be an optimal solution tree and $S$ the set of non-terminals in $T$. Root $T$ at some node and let $s \in S$ be a non-terminal farthest from the root. The children of $s$ are


Figure 1 Illustration of definitions for a Crossing Family Augmentation instance where $\mathcal{F}$ is represented by a cactus. Here $A \in \mathcal{F}$ if and only if $A$ is a connected component obtained by removing a pair of edges that belong to the same cycle of the cactus. The edges in $E$ are shown by dashed arcs and the terminals in $R$ are shown by gray circles. The cactus of the residual family w.r.t. to a single edge is obtained by "squeezing" the cycles along the path of cycles between the ends of the edge. (a) The original instance. (b) The residual instance w.r.t. e. (c) The residual instance w.r.t. $f$. (d) The $(R, E, \mathcal{F})$-incidence graph of the instance in (a).


Figure 2 Illustration to the proof of Lemma 4
terminal leaves, and assume w.l.o.g. that $R(s)=\{u, v\}$ is the set of children of $s$; if $s$ has just one child in $T$, then it has another terminal neighbor in $H$, that can be attached to $s$.

Consider the residual instance $\mathcal{I}^{\prime}=\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), R^{\prime}\right)$ and the tree $T^{\prime}$ obtained by contracting $R(s)$ into the new terminal $s^{\prime}$, and deleting any $z \in U \backslash(R+s)$ with $R(z)=R(s)$. Then $\left|R^{\prime}\right|=|R|-1,\left|R^{\prime}(z)\right|=2$ for all $z \in R^{\prime}, T^{\prime}$ is an optimal solution for $\mathcal{I}^{\prime}$, and $S^{\prime}=S-s$ is the set of non-terminals of $T^{\prime}$.

If (i) does not hold for the new instance $\mathcal{I}^{\prime}$ then (ii) holds for $\mathcal{I}^{\prime}$, by the induction hypothesis. Then $|S|=\left|S^{\prime}\right|+1 \geq\left(\left|R^{\prime}\right|-1\right)+1=|R|-1$, and we get that (ii) holds for $\mathcal{I}$. Assume henceforth that (i) holds for $\mathcal{I}^{\prime}$. We obtain a contradiction by showing that then (i) holds for $\mathcal{I}$. Let $a, b \in V^{\prime} \backslash R^{\prime}$ be such that $R^{\prime}(a) \cap R^{\prime}(b)=\emptyset$, see Fig. 2. If $s^{\prime} \notin R^{\prime}(a) \cup R^{\prime}(b)$ then clearly (ii) holds for $\mathcal{I}$. Otherwise, if say $s^{\prime} \in R^{\prime}(a)$, then we have two cases. If one of $u$, $v$, say $v$, is a neighbor of $a$ in $G$ (see Fig. 2(a)) then $R(a) \cap R(b)=\emptyset$. Otherwise (see Fig. $2($ a) $), R(a) \cap R(s)=\emptyset$. In both cases, we obtain a contradiction to the assumption that (i) does not hold for $\mathcal{I}$.

We also need the following known lemma.

- Lemma 5. Any inclusion minimal cover $J$ of a set family $\mathcal{F}$ is a forest.

Proof. Suppose to the contrary that $J$ contains a cycle $C$. Since $P=C \backslash\{e\}$ is a $u v$-path, then for any $A$ covered by $e$, there is $e^{\prime} \in P$ that covers $A$. This implies that $J \backslash\{e\}$ also covers $\mathcal{F}$, contradicting the minimality of $J$.

The algorithm starts with a partial solution $J=\emptyset$ and has two phases. Phase 1 consists of iterations. At the beginning of each iteration, construct the $\left(E, R^{J}, \mathcal{F}^{J}\right)$-incidence graph $H^{J}$, where initially $R$ is the set of leaves of $\mathcal{F}$. Then, do one of the following:

1. If $H^{J}$ has a node $e \in E$ with $\left|R^{J}(e)\right| \geq 3$, then add $e$ to $J$.
2. Else, if there are $e, f \in E$ with $R^{J}(e) \cap R^{J}(f)=\emptyset$, then add both $e, f$ to $J$.

If none of the above two cases occurs, then we apply Phase 2 , in which we add to $J$ an inclusion minimal cover of $\mathcal{F}^{J}$; note that all edges in $E^{J}$ have both endnodes in $R^{J}$. A more formal description is given in Algorithm 1

We show that the algorithm achieves ratio 5/3. Note that:

- Adding an edge $e$ as in step 4 reduces the number of terminals by at least 2 .
- Adding an edge pair $e, f$ as in step 5 reduces the number of terminals by at least 3 .

```
Algorithm 1: \((G=(V, E), \mathcal{F}, R)\)
    \(J \leftarrow \emptyset\)
    repeat
        let \(H^{J}\) be the \(\left(E^{J}, R^{J}, \mathcal{F}^{J}\right)\)-incidence graph
        if \(H^{J}\) has a node \(e \in E\) with \(\left|R^{J}(e)\right| \geq 3\) then do \(J \leftarrow J \cup\{e\}\)
        else if \(H^{J}\) has node pair \(e, f \in E\) with \(R^{J}(e) \cap R^{J}(f)=\emptyset\) then do \(J \leftarrow J \cup\{e, f\}\)
    until no edge \(e\) or an edge pair \(e, f\) as above exists;
    find an inclusion minimal \(\mathcal{F}^{J}\)-cover and add it to \(J\)
    return \(J\)
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Hence the reduction in the number of terminals per added edge is at least $3 / 2$. Let $\ell=|L|$ be the initial number of terminals. Let $\ell^{\prime}=\left|R^{J}\right|$ be the number of terminals at the end of Phase 1 (steps 2-6 in Algorithm 1). Let $k$ be the number of edges added during Phase 1. Then $\ell^{\prime} \leq \ell-\frac{3}{2} k$, hence $k \leq \frac{2}{3}\left(\ell-\ell^{\prime}\right)$. The number of edges added at the second phase is at most $\ell^{\prime}-1$, by Lemma 5 note that every edge in $E^{J}$ has both ends in $R^{J}$ and that $\left|R^{J}\right|=\ell^{\prime}$.

On the other hand, opt $\geq \frac{\ell}{2}$, and opt $\geq \ell^{\prime}-1$, by part (ii) of Lemma 4 Summarizing, we have the following:

- The solution size is at most $k+\ell^{\prime}-1 \leq \frac{2}{3}\left(\ell-\ell^{\prime}\right)+\ell^{\prime}-1=\left(2 \ell+\ell^{\prime}-3\right) / 3$.
- opt $\geq \ell / 2$ and opt $\geq \ell^{\prime}-1$.

Thus the approximation ratio is bounded by $\frac{\left(2 \ell+\ell^{\prime}-3\right) / 3}{\max \left\{\ell / 2, \ell^{\prime}-1\right\}}$. If $\ell / 2 \geq \ell^{\prime}-1$ then

$$
\frac{\left(2 \ell+\ell^{\prime}-3\right) / 3}{\max \left\{\ell / 2, \ell^{\prime}-1\right\}} \leq \frac{(2 \ell+(\ell / 2+1)-3) / 3}{\ell / 2}=\frac{(5 \ell / 2-2) / 3}{\ell / 2}<\frac{5}{3}
$$

Else, $\ell / 2<\ell^{\prime}-1$, and then

$$
\frac{\left(2 \ell+\ell^{\prime}-3\right) / 3}{\max \left\{\ell / 2, \ell^{\prime}-1\right\}}<\frac{\left(4\left(\ell^{\prime}-1\right)+\ell^{\prime}-3\right) / 3}{\ell^{\prime}-1}=\frac{\left(5 \ell^{\prime}-7\right) / 3}{\ell^{\prime}-1}<\frac{5}{3}
$$

In both cases the ratio is bounded by $5 / 3$.
We now adjust the proof to the Block-Tree Augmentation problem. Let $G=(V, E)$ be a connected graph. A node $v$ is a cutnode of $G$ if $G \backslash\{v\}$ is disconnected; an inclusion maximal node subset whose induced subgraph is connected and has no cutnodes is a block of $G$; equivalently, $B$ is a block if it is the node set of an inclusion maximal 2-connected subgraph or of a bridge. The block-tree $T$ of $G$ has node set $C_{G} \cup \mathcal{B}_{G}$, where $C_{G}$ is the set of cutnodes of $G$ and $\mathcal{B}_{G}$ is the set of blocks of $G ; T$ has an edge for each pair of a block and a cutnode that belongs to that block. It is known that every $v \in V \backslash C_{G}$ belongs to a unique block, and that $T$ is a tree. The block-tree mapping $\psi: V \rightarrow C_{G} \cup \mathcal{B}_{G}$ of $G$ is defined by $\psi(v)=v$ is $v \in C_{G}$ and $\psi(v)$ is the block that contains $v$ if $v \in V \backslash C_{G}$.

Given a Block-Tree Augmentation instance $\left(T=\left(V, E_{T}\right), E\right)$ and $J \subseteq E$, the residual instance $\left(T^{J}=\left(V^{J}, E_{T}^{J}\right), E^{J}\right)$ is defined as follows.

- $T^{J}$ is the block tree of $T \cup J$.
- $E^{J}=\{\psi(u) \psi(v): u v \in E \backslash J, \psi(u) \neq \psi(v)\}$, where $\psi$ is the the block-tree mapping of $T \cup J$.
For a set $R \subseteq V$ of terminals, the residual set of terminals is $R^{J}=\psi(R)=\cup_{r \in R} \psi(r)$. For an edge $e=u v$ let $T_{e}$ denote the unique $u v$-path in $T$. We say that $e, f \in E$ are $T$ inseparable if the paths $T_{e}, T_{f}$ have an edge in common. The ( $R, E, T$ )-incidence graph $H=\left(U, E_{H}\right)$ has node set $U=E \cup R$ and edge set

$$
E_{H}=\{e f: e, f \in E \text { are } T \text {-inseparable }\} \cup\left\{e r: r \in R, e \in E, r \in T_{e}\right\}
$$

It was shown in [23] that for $R=L$ being the set of leaves of $\mathcal{F}$, an edge set $J \subseteq E$ is a feasible solution to Block-Tree Augmentation if and only if the subgraph $H[J \cup R]$ of $H$ induced by $J \cup R$ is connected. The proof in [23] extends to any $R \subseteq V$ that contains $L$. This implies that Crossing Family Augmentation admits an approximation ratio preserving reduction to SS-CDS, see [23] for details. Lemma 5 also extends to this case, as it is known that an if $J$ is an inclusion minimal edge set whose addition makes a connected graph 2-connected, then $J$ is a forest.

With these definitions and facts, the rest of the proof for the Block-Tree Augmentation coincides with the proof given for Crossing Family Augmentation, concluding the proof of Theorem 1 .

## 3 The general case (Theorem 2)

Recall that each of the problems Crossing Family Augmentation and Block-Tree Augmentation admits an approximation ratio preserving reduction to the SS-CDS problem
with $R=L$ being the set of terminals. The SS-CDS instances that arise from this reduction have the following property, see [6, 23]:
(*) The neighbors of every $r \in R$ induce a clique.
In fact, SS-CDS with property $(*)$ is equivalent to the Node Weighted Steiner Tree problem with property ( $*$ ) with unit node weights for non-terminals (the terminals have weight zero). Clearly, any SS-CDS solution is a feasible Node Weighted Steiner Tree solution; for the other direction, note that if property $(*)$ holds, then the set of non-terminals in any feasible Node Weighted Steiner Tree solution is a feasible SS-CDS solution. The relation to the ordinary Steiner Tree problem is given in following lemma.

- Lemma 6 ([6]). Let $S$ be a SS-CDS solution and $T=(U, J)$ a Steiner Tree solution on instance $(G, R)$ with unit edge costs. Then:
(i) If $(*)$ holds then $T$ can be converted into a SS-CDS solution $S_{J}$ with $\left|S_{J}\right|=|J|-|R|+1$.
(ii) $S$ can be converted into a Steiner Tree solution $T_{S}=\left(U_{S}, J_{S}\right)$ with $\left|J_{S}\right|=|S|+|R|-1$.

Proof. We prove (i). Any Steiner Tree solution $T^{\prime}=\left(U^{\prime}, J^{\prime}\right)$ can be converted into a solution $T=(U, J)$ such that $|J|=\left|J^{\prime}\right|$ and $R$ is the leaf set of $T^{\prime}$. For this, for each $r \in R$ that is not a leaf of $T^{\prime}$, among the edges incident to $r$ in $T^{\prime}$, choose one and replace the other edges by a tree on the neighbors of $r$; this is possible by $(*)$. The non-leaf nodes of such $T$ form a a SS-CDS as required. For (ii), taking a tree on $S$ and for each $r \in R$ adding an edge from $r$ to $S$ gives a Steiner Tree solution as required.

Let $J^{*}$ be an optimal and $J$ an $\alpha$-approximate Steiner Tree solutions. Let $S_{J}, S^{*}$ be SS-CDS solutions, where $S_{J}$ is derived from $J$ and $S^{*}$ is an optimal one. Then

$$
\left|S_{J}\right|+R-1=|J| \leq \alpha\left|J^{*}\right| \leq \alpha\left|J_{S^{*}}\right|=\alpha\left(\left|S^{*}\right|-1+|R|\right)=\alpha\left|S^{*}\right|+\alpha(|R|-1) .
$$

This implies that if Steiner Tree admits ratio $\alpha$ then SS-CDS with property ( $*$ ) admits a polynomial time algorithm that computes a solution $S$ of size $|S| \leq \alpha$ opt $+(\alpha-1)|L|$ and achieves ratio $\alpha+(\alpha-1) \frac{|L|}{\mathrm{opt}}=\alpha+(\alpha-1) x$, where $x=\frac{|L|}{\mathrm{opt}}, 0<x \leq 2$. We will prove the following.

- Theorem 7. Crossing Family Augmentation and Block-Tree Augmentation admit ratio $1+\ln \left(4-\frac{|L|}{\mathrm{opt}}\right)+\epsilon$.

From Lemma 6 and Theorem 7 it follows that we can achieve ratio

$$
\max \{\alpha+(\alpha-1) x, 1+\ln (4-x)\}+\epsilon \text { where } x=\frac{|L|}{\mathrm{opt}} .
$$

The worse case is when these two ratios are equal, which gives the Theorem 2 ratio. In the case $\alpha=\ln 4+\epsilon[7$, we have $x \approx 1.4367$, so $L \approx 1.4367$ opt and opt $\approx 0.69 L$. The ratio in this case is $1+\ln (4-x)+\epsilon<1.942$.

## 4 Proof of Theorem 7

A set function $f$ is increasing if $f(A) \leq f(B)$ whenever $A \subseteq B ; f$ is decreasing if $-f$ is increasing, and $f$ is sub-additive if $f(A \cup B) \leq f(A)+f(B)$ for any subsets $A, B$ of the ground-set. Let us consider the following algorithmic problem:

## Min-Covering

Input: Non-negative set functions $\nu, \tau$ on subsets of a ground-set $U$ such that $\nu$ is decreasing, $\tau$ is sub-additive, and $\tau(\emptyset)=0$.
Output: $A \subseteq U$ such that $\nu(A)+\tau(A)$ is minimal.
We call $\nu$ the potential and $\tau$ the payment. The idea behind this interpretation and the subsequent greedy algorithm is as follows. Given an optimization problem, the potential $\nu(A)$ is the (bound on the) value of some "simple" augmenting feasible solution for $A$. We start with an empty set solution, and iteratively try to decrease the potential by adding a set $B \subseteq U \backslash A$ of minimum "density" - the price paid for a unit of the potential. The algorithm terminates when the price $\geq 1$, since then we gain nothing from adding $B$ to $A$. The ratio of such an algorithm is bounded by $1+\ln \frac{\nu(\nmid)}{\text { opt }}$ (assuming that during each iteration a minimum density set can be found in polynomial time). So essentially, the greedy algorithm converts ratio $\alpha=\frac{\nu(\emptyset)}{\text { opt }}$ into ratio $1+\ln \alpha$.

Fix an optimal solution $A^{*}$. Let $\nu^{*}=\nu\left(A^{*}\right), \tau^{*}=\tau\left(A^{*}\right)$, so opt $=\tau^{*}+\nu^{*}$. The quantity $\frac{\tau(B)}{\nu(A)-\nu(A \cup B)}$ is called the density of $B$ (w.r.t. $A$ ); this is the price paid by $B$ for a unit of potential. The Greedy Algorithm (a.k.a. Relative Greedy Heuristic) for the problem starts with $A=\emptyset$ and while $\nu(A)>\nu^{*}$ repeatedly adds to $A$ a non-empty augmenting set $B \subseteq U$ that satisfies the following condition, while such $B$ exists:
Density Condition: $\frac{\tau(B)}{\nu(A)-\nu(A \cup B)} \leq \min \left\{1, \frac{\tau^{*}}{\nu(A)-\nu^{*}}\right\}$.
Note that since $\nu$ is decreasing, $\nu(A)-\nu\left(A \cup A^{*}\right) \geq \nu(A)-\nu\left(A^{*}\right)=\nu(A)-\nu^{*}$; hence if $\nu(A)>\nu^{*}$, then $\frac{\tau\left(A^{*}\right)}{\nu(A)-\nu\left(A \cup A^{*}\right)} \leq \frac{\tau^{*}}{\nu(A)-\nu^{*}}$ and there exists an augmenting set $B$ that satisfies the condition $\frac{\tau(B)}{\nu(A)-\nu(A \cup B)} \leq \frac{\tau^{*}}{\nu(A)-\nu^{*}}$, e.g., $B=A^{*}$. Thus if $B^{*}$ is a minimum density set and $\frac{\tau\left(B^{*}\right)}{\nu(A)-\nu\left(A \cup B^{*}\right)} \leq 1$, then $B^{*}$ satisfies the Density Condition; otherwise, the density of $B^{*}$ is larger than 1 so no set can satisfy the Density Condition. The following statement is known, c.f. an explicit proof in [24].

- Theorem 8. The Greedy Algorithm achieves approximation ratio $1+\frac{\tau^{*}}{\mathrm{opt}} \ln \frac{\nu(\emptyset)-\nu^{*}}{\tau^{*}}$.

This applies also in the case when we can only compute a $\rho$-approximate minimum density augmenting set, while invoking an additional factor $\rho$ in the ratio.

To use the framework of Theorem 8 we need to define $\tau$ and $\nu$. Let $J \subseteq E$ be an edge set. The payment $\tau(J)=|J|$ is just the size of $J$. The potential of $J$ is defined by $\nu(J)=\left|R^{J}\right|-1$, where $R$ is a set of terminals such that $L \subseteq R \subseteq V$, defined in the following lemma. For an edge set $F$ let $F_{L L}$ be the set of edges in $F$ with both ends in $L$, and $F_{L}$ the set of edges in $F$ that have exactly one end in $L$.

- Lemma 9. Let $F$ be an optimal solution to Crossing Family Augmentation instance and $c$ be a cost function on $E$ defined by $c(e)=0$ if $e \in E_{L L}, c(e)=1$ if $e \in E_{L}$, and $c(e)=2$ otherwise. Let $J$ be a 2-approximate $c$-costs solution and let $R$ be the set of ends of the edges in $J$. Then $|R| \leq c(J)+L \leq 4|F|-|L|=4$ opt $-|L|$.

Proof. Clearly, $|R| \leq c(J)+|L|$. We show that $c(J) \leq 4|F|-2|L|$. Let $F^{\prime}$ be the set of edges in $F$ that have no end in $L$. Since $\left|F^{\prime}\right|=|F|-\left|F_{L}\right|-\left|F_{L L}\right|$ and $2\left|F_{L L}\right|+\left|F_{L}\right| \geq L$

$$
c(F)=\left|F_{L}\right|+2\left|F^{\prime}\right|=\left|F_{L}\right|+2\left(|F|-\left|F_{L}\right|-\left|F_{L L}\right|\right)=2|F|-\left(\left|F_{L}\right|+2\left|F_{L L}\right|\right) \leq 2|F|-|L|
$$

Since $c(J) \leq 2 c(F)$, the lemma follows.

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It is easy to see that $\nu$ is decreasing and $\tau$ is subadditive. The next lemma shows that the obtained Min-covering instance is equivalent to the Crossing Family Augmentation instance, and that we may assume that $\tau^{*}=$ opt and $\nu^{*}=0$.

- Lemma 10. If $J$ is a feasible solution to Crossing Family Augmentation then $\nu(J)=0$. If $J$ is a feasible Min-Covering solution then one can construct in polynomial time a feasible Crossing Family Augmentation solution of size $\leq \tau(J)+\nu(J)$. In particular, both problems have the same optimal value, and Min-Covering has an optimal solution $J^{*}$ such that $\nu\left(J^{*}\right)=0$ and $\tau\left(J^{*}\right)=\mathrm{opt}$.

Proof. If $J$ is a feasible Crossing Family Augmentation solution then $\left|R^{J}\right|=1$ and thus $\nu(J)=0$. Let $I$ be a Min-Covering solution such that every edge in $I$ has both ends in $R$; e.g., $I$ can be as in Lemma 9. Then $I^{J}$ is a feasible solution to the residual problem w.r.t. $J$ and every edge in $I^{J}$ has both ends in $R^{J}$. Let $I^{\prime} \subseteq I^{J}$ be an inclusion minimal edge set such that $J \cup I^{\prime}$ is a feasible solution. By Lemma 5, $I^{\prime}$ is a forest, hence $|I| \leq\left|R^{J}\right|-1$. Consequently, $J \cup I^{\prime}$ is a feasible solution of size at most $|J|+\left|I^{\prime}\right| \leq|J|+\left|R^{J}\right|-1=\tau(J)+\nu(J)$.

Recall also that $\nu(\emptyset) \leq 4$ opt $-|L|$, by Lemma 9 . We will show how to find for any $\epsilon>0$, a $(1+\epsilon)$-approximate best density set in polynomial time. It follows therefore that we can apply the greedy algorithm to produce a solution of value $1+\epsilon$ times of

$$
1+\frac{\tau^{*}}{\mathrm{opt}} \ln \frac{\nu(\emptyset)-\nu^{*}}{\tau^{*}}=1+\ln \frac{4 \mathrm{opt}-|L|}{\mathrm{opt}}=1+\ln \left(4-\frac{|L|}{\mathrm{opt}}\right) .
$$

In what follows note that if $a_{1}, \ldots, a_{q}$ and $b_{1}, \ldots b_{q}$ are positive reals, then by an averaging argument there exists an index $1 \leq i \leq q$ such that $a_{i} / b_{i} \leq \sum_{j=1}^{q} a_{j} / \sum_{i=1}^{q} b_{j}$.

Given a Crossing Family Augmentation instance, a set $R \supseteq L$ of terminals, and $F \subseteq E$, consider the corresponding SS-CDS instance $\left(H=\left(U, E_{H}\right), R\right)$ and the set of non-terminals $Q$ that corresponds to $F$. The density of $F$ is $\frac{|F|}{|R|-\left|R^{F}\right|}$, and in the SS-CDS instance this is computed by taking a maximal forest in the graph induced by $Q$ and the terminals that have a neighbor in $Q$; then the density is $|Q|$ over the number of trees in this forest. So in what follows we may speak of a density of a subforest of $H$. Let $T_{i}=\left(S_{i} \cup R_{i}, E_{i}\right), i=1, \ldots, q$, be the connected components of such a forest, $\left(R_{i}\right.$ is the set of terminals in $T_{i}$ ) and let $s_{i}=\left|S_{i}\right|$ and $r_{i}=\left|R_{i}\right|$, where $r_{i} \geq 2$. The density of the forest is $\sum_{i=1}^{q} s_{i} / \sum_{i=1}^{q}\left(r_{i}-1\right)$ while the density of each $T_{i}$ is $s_{i} /\left(r_{i}-1\right)$. By an averaging argument, some $T_{i}$ has density not larger than that of the forest. Consequently, we may assume that the minimum density is attained for a tree, say $T$.

Let $T=(S \cup R, E)$ be a tree with leaf set $R$. The density of $T$ is $\frac{s}{r-1}$, where $r=|R|$ is the number of terminals ( $R$-nodes) and $s$ is the number of non-terminals ( $S$-nodes) in $T$. The usual approach is to show that for any $k$ there exists a subtree $T^{\prime}$ of $T$ with $k$ terminals (or $k$ non-terminals) such that the density of $T^{\prime}$ is at most $1+f(k)$ times the density of $T$, where $\lim _{k \rightarrow \infty} f(k)=0$. The decomposition lemma that we prove is not a standard one. The difficulty can be demonstrated by the following examples. Consider the case when $T$ is a star with $n$ leaves. Then the density of $T$ is $1 /(n-1)$, while a subtree with $k$ leaves has density $1 /(k-1)$. If $T$ is a path with $n$ non-terminals, then the density of $T$ is $n$, while a subtree with $k<n$ non-terminals has density $k / 0=\infty$. In both cases, the density of the subtree may be arbitrarily larger than that of $T$. To overcome this difficulty, we will decompose $T$ w.r.t. a certain subset $P$ of the non-terminals.

Let $P \subseteq S$. Let $s=|S|, r=|R|$, and $p=|P|$. For a subtree $T^{\prime}$ of $T$ let $S\left(T^{\prime}\right), R\left(T^{\prime}\right)$, and $P\left(T^{\prime}\right)$ denote the set of $S$-nodes, $R$-nodes, and $P$-nodes in $T^{\prime}$, respectively. We prove the following.

Lemma 11. Let $k \geq 2$. If $p \geq 3 k+1$ then there exists subtrees $T_{1}, \ldots, T_{q}$ of $T$ such that the following holds.

- $\sum_{i=1}^{q}\left|S\left(T_{i}\right)\right| \leq s+q$.
- Every $R$-node belongs to exactly one subtree, hence $\sum_{i=1}^{q}\left|R\left(T_{i}\right)\right|=r$.
- $\left|P\left(T_{i}\right)\right| \in[k, 3 k]$ for all $i$ and $q \leq \frac{p}{k-1}$.

Proof. Root $T$ at some node in $S$. For any $v \in S$ chosen as a "local root", the subtree $T^{v}$ rooted at $v$ is a subtree of $T$ that consist of $v$ and its descendants. Let $T^{v}$ be an inclusion minimal rooted subtree of $T$ such that $\left|P\left(T^{v}\right)\right| \geq k+1$. Note that $v \in P$. Let $B_{1}, \ldots, B_{m}$ be the branches hanging on $v$ and let $p_{j}=\left|P\left(B_{j}\right)\right|$. By the definition of $T_{v}$, each $p_{j}$ is in the range $[0, k]$ and $\sum_{j=1}^{m} p_{j} \geq k$. We claim that $\left\{p_{1}, \ldots, p_{m}\right\}$ can be partitioned such that the sum of each part plus 1 is in the range $[k, 3 k]$. To see this, apply a greedy algorithm for Multi-Bin Packing with bins of capacity $2 k$; at the end there is at most one bin with sum $\leq k-1$ (as two such bins can be joined), and joining this bin to any other bin gives a partition as required. Now we remove $T^{v}$ and the $S$-nodes on the path from $v$ to its closest terminal ancestor, and apply the same procedure on the remaining tree. If the last rooted subtree $T^{v}$ considered has $\left|P\left(T^{v}\right)\right| \leq k-1$, then this tree can be joined to a subtree $T_{i}$ with $\left|P\left(T_{i}\right)\right| \leq 2 k$ derived in previous iteration. Finally, $q \leq \frac{p+q}{k}$ by the construction and since $\left|P\left(T_{i}\right)\right| \geq k$ for all $i$; this implies $q \leq \frac{p}{k-1}$.

Now we let $P=P_{1} \cup P_{2}$, where $P_{1}$ is the set of nodes that have degree at least 3 in $T$ and $P_{2}$ is the of nodes that have a terminal neighbor in $T$. Note that $\left|P_{1}\right| \leq r$ and $\left|P_{2}\right| \leq r$. Hence $p \leq 2 r$, and clearly $p \leq s$. By an averaging argument and Lemma 11 the density of some $T_{i}$ is bounded by $s_{i} /\left(r_{i}-1\right) \leq \sum_{j=1}^{q} s_{j} / \sum_{j=1}^{q}\left(r_{j}-1\right) \leq(s+q) /(r-q)$. Thus for $k \geq 3$ we get

$$
\frac{s_{i}}{r_{i}-1} \cdot \frac{r-1}{s} \leq \frac{s+q}{r-q} \cdot \frac{r}{s} \leq \frac{s+p /(k-1)}{r-p /(k-1)} \cdot \frac{r}{s}=\frac{1+1 /(k-1)}{1-2 /(k-1)}=\frac{k}{k-3}=1+\frac{3}{k-3} .
$$

This implies that we can find a $(1+\epsilon)$-approximate min-density tree by searching over all trees $T^{\prime}$ with $\left|P\left(T^{\prime}\right)\right| \in[k, 3 k]$, where given $\epsilon>0$ we let $k=\lceil 3 / \epsilon\rceil+3$. Specifically, for every $P^{\prime} \subseteq S$ with $\left|P^{\prime}\right| \in[k, 3 k]$, we find an MST $T^{\prime}$ in the metric completion of the current incidence graph, and then add to $T^{\prime}$ all the terminals that have a neighbor in $P^{\prime}$. Among all subtrees we choose one of minimum density. The time complexity is $n^{3 k}$ which is polynomial for any fixed $\epsilon>0$.

The process of adjusting the proof to the Block-Tree Augmentation is identical to the one in the proof of Theorem 1 This concludes the proof of Theorem 7 and thus also the proof of Theorem 2 is complete.

## 5 Covering skew-supermodular functions (Theorem 3)

Let $p: 2^{S} \rightarrow \mathbb{Z}$ be a set function and $J$ an edge set on a finite groundset $S$. We say that $J$ covers $p$ if $d_{J}(A) \geq p(A)$ for all $A \subseteq S$, where $d_{J}(A)$ denote the set of edges with exactly one end in $A$. $p$ is symmetric if $p(A)=p(S \backslash A)$ for all $A \subseteq S$, and $p$ is skew-supermodular (a.k.a. weakly supermodular) if for all $A, B \subseteq S$ at least one of the following two inequalities holds:

$$
p(A)+p(B) \leq p(A \cap B)+p(A \cup B) \quad p(A)+p(B) \leq p(A \backslash B)+p(B \backslash A)
$$

Element Connectivity Augmentation can be reduced to the following problem, with skew-supermodular set function $p$, c.f. [13, 21, 5].

## Set Function Edge Cover

Input: A set function $p$ on a ground-set $S$.
Output: A minimum size set $J$ of edges that covers $p$.

In this problem, the function $p$ may not be given explicitly, and a polynomial time implementation of algorithms requires that some queries related to $p$ can be answered in polynomial time. But the problem is also NP-hard for skew-supermodular $p$ even if $p$ given explicitly, specifically when $p_{\max }=1$ and $|A|=3$ for every set $A$ with $p(A)=1$ [21].

In the degree bounded version of this problem we are also given degree bounds $\{b(v)$ : $v \in S\}$ and require that $d_{J}(v) \leq b(v)$ for all $v \in S$.

- Definition 12. A function $g: S \rightarrow \mathbb{Z}_{+}$is a $p$-transversal if $g(A) \geq p(A)$ for all $A \subseteq S$. Let $T_{g}=\{v \in S: g(v) \geq 1\}$ denote the support of $g$. We say that $g$ is a minimal $p$-transversal if for any $v \in T_{g}$ reducing $g(v)$ by 1 results in a function that is not a p-transversal.

The following was proved by Benczú and Frank in [4], see also [21] Lemmas 1.1 and 3.2].

- Lemma 13. Let $g$ be a transversal of a skew-supermodular set function $p$. Then:
- $g(S)=\max \left\{\sum_{A \in \mathcal{F}} p(A): \mathcal{F}\right.$ is a subpartition of $\left.S\right\}$ if $g$ is minimal.
- There exists an optimal p-cover $J$ such that every $e \in J$ has both ends in $T_{g}$.

Let $\tau(p)$ denote the size of a minimal $p$-cover. As $g=\left\{d_{J}(v): v \in S\right\}$ is a $p$-transversal for any $p$-cover $J, \tau(p) \geq g(S) / 2$ for any minimal $p$-transversal $g$. Thus a natural approach to compute a small $p$-cover is: repeatedly choose an edge $u v$ with $u, v \in T_{g}$, such that updating $p$ and reducing $g(u)$ and $g(v)$ by 1 , keeps $g$ being a $p$-transversal. This approach works for many interesting special cases, c.f. [5], but in general such an edge $u v$ may not exist. Formally, given $u, v \in T_{g}$ define $p^{u v}$ and $g^{u v}$ by:

$$
\begin{aligned}
& p^{u v}(A)=\max \{p(A)-1,0\} \text { if }|A \cap\{u, v\}|=1 \text { and } p^{u v}(A)=p(A) \text { otherwise } \\
& g^{u v}(w)=g(w)-1 \text { if } w=u \text { or if } w=v \text { and } g^{u v}(w)=g(w) \text { otherwise }
\end{aligned}
$$

It is easy to see that if $p$ is (symmetric) skew-supermodular, so is $p^{u v}$. However, $g^{u v}$ may not be a $p^{u v}$-transversal if $g$ is. We say that a pair $u, v \in T_{g}$ is $(p, g)$-legal if $g^{u v}$ is a $p^{u v_{-}}$ transversal; then replacing $p, g$ by $p^{u v}, g^{u v}$ is the splitting-off operation at $u, v$. Intuitively, splitting-off is an attempt to add the edge $u v$ to a partial solution, and to consider the residual problem of covering $p^{u v}$ with the residual lower bound $\left\lceil g^{u v}(S) / 2\right\rceil=\lceil g(S) / 2\rceil-1$. We need the following result due to [21], see also [5] for a short and elegant proof.

- Lemma 14 ([21]). Let $p$ be symmetric skew-supermodular and $g$ a $p$-transversal. If $p_{\max } \geq 2$ then there exists a $(p, g)$-legal pair.

Lemma 14 implies that if no $(p, g)$-legal pair exists, then any inclusion minimal solution on $T_{g}$ is a forest, and that any tree on $T_{g}$ is a feasible solution. In [21, 5] was considered a simple greedy algorithm which repeatedly splits-off legal pairs as long as such exist, and then adds to the partial solution a tree (or any inclusion minimal solution) on $T_{g}$.

```
Algorithm 2: \(\operatorname{GreEDY}(p, g)\)
( \(p\) is symmetric skew-supermodular, \(g\) is a \(p\)-transversal)
    \(M \leftarrow \emptyset\)
    while there exists a \((p, g)\)-legal pair \(u, v\) do
        \(g \leftarrow g^{u v}, p \leftarrow p^{u v}, M \leftarrow M+u v\)
    let \(F\) be a tree on \(T^{\prime}=\{v \in S: g(v)=1\}\)
    return \(M \cup F\)
```

In the degree bounded version we let $g=\{b(v): v \in S\}$. If this $g$ is not a $p$-transversal, then the problem has no feasible solution. To get a degree violation +1 , at step 4 of the algorithm we choose $F$ to be a path on $T^{\prime}$.

In [21] is was shown that for skew-supermodular $p$ this algorithm achieves ratio $7 / 4$, by characterizing those pairs $p, g$ for which no $(p, g)$-legal pair exists and deriving a lower bound on $\tau(p)$. We establish a better lower bound than that of [21] and prove the following.

- Theorem 15. Algorithm Greedy achieves approximation ratio 3/2. Moreover, if $F$ is chosen to be a path at step 4, then $d_{J}(v) \leq g(v)+1$ for all $v \in S$.

Theorem 15 second statement is obvious, so in the rest of this section we prove the first statement. The following can be deduced from Lemma 14, see [21.

- Corollary 16 ([21]). Let $p^{\prime}$ be symmetric skew-supermodular and $g^{\prime}$ a minimal $p^{\prime}$-transversal with non-empty support $T^{\prime}=\left\{v \in V: g^{\prime}(v) \geq 1\right\}$, and suppose that no $\left(p^{\prime}, g^{\prime}\right)$-legal pair exists. Then $p_{\max }^{\prime}=g_{\max }^{\prime}=1,\left|T^{\prime}\right| \geq 3$, and for every $A^{\prime} \subseteq T^{\prime}$ with $\left|A^{\prime}\right| \in\{1,2\}$ there is $A \subseteq V$ with $p^{\prime}(A)=1$ such that $A \cap T^{\prime}=A^{\prime}$. Furthermore, $\tau\left(p^{\prime}\right) \geq \frac{2}{3}\left|T^{\prime}\right|$.

We now describe the analysis of the $7 / 4$-approximation [21]. Let $k$ be the number of edges accumulated in $M$ during the while-loop. Let $t=g(S)$ be the initial value of the $p$-transversal $g$ and let $t^{\prime}=\left|T^{\prime}\right|$ be the the transversal value at the beginning of step 4. Note that $t-t^{\prime}=2 k$ and that $|F|=\left|T^{\prime}\right|-1=t-2 k-1$. Consequently, $|M \cup F| \leq k+(t-2 k-1) \leq t-k$. On the other hand we have the lower bounds $\tau(p) \geq t / 2$ and $\tau(p) \geq \frac{2}{3}\left|T^{\prime}\right|=\frac{2}{3}(t-2 k)$. Thus the approximation ratio is bounded by $\frac{t-k}{\max \{t / 2,2(t-2 k) / 3\}} \leq 7 / 4$, with $k=t / 8$ being the worse case.

One can observe that if $k \geq t / 4$ then $t / 2 \geq \frac{2}{3}(t-2 k)$, and thus in this case the ratio is bounded by $\frac{t-k}{t / 2} \leq \frac{3 / 4}{1 / 2}=3 / 2$. To get ratio $3 / 2$ for the range $k \leq t / 4$ we give a better lower bound on $\tau(p)$. For this, we need the following lemma.

- Lemma 17. Let $g$ be a minimal transversal of a skew-supermodular symmetric set function $p$. Then there exists an optimal p-cover $J$ such that $d_{J}(v) \geq g(v)$ if $v \in T_{g}$ and $d_{J}(v)=0$ otherwise.

Proof. By induction on $\tau(p)$. The base case $\tau(p)=1$ is trivial. For $\tau(p) \geq 2$, let $J$ be an optimal $p$-cover such that every $e \in J$ has both ends in $T_{g}$; such exists by Lemma 13 Choose some $e=u v \in J$ and let $p^{\prime}=p^{u v}$. Let $g^{u}$ be obtained from $g$ by decreasing $g(u)$ by 1 , and similarly $g^{v}$ is defined. Then one of $\left\{g, g^{u}, g^{v}, g^{u v}\right\}$ is a minimal $p^{\prime}$-transversal; denote it by $g^{\prime}$. By the induction hypothesis there exists a $p^{\prime}$-cover $J^{\prime}$ such that: $d_{J^{\prime}}(w) \geq g^{\prime}(w)$ if $w \in T_{g^{\prime}}$ and $d_{J^{\prime}}(w)=0$ otherwise. It is easy to see that $J=J^{\prime} \cup\{e\}$ has the required property.

- Lemma 18. $\tau(p) \geq \frac{2}{3}(t-k)$.


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Proof. Let $J$ be a $p$-cover as in Lemma 17. Let $X$ be the set of edges in $J$ with both ends in $T_{g} \backslash T^{\prime}$ and $Y$ the set of edges in $J$ with exactly one end in $T_{g} \backslash T^{\prime}$; let $x=|X|$ and $y=|Y|$. Since $d_{J}(v) \geq g(v)$ for all $v \in T, 2 x+y \geq t-t^{\prime}=2 k$, hence $y \geq 2 k-2 x$. Let $Q$ be the set of nodes in $T^{\prime}$ that are uncovered by edges in $X \cup Y$ and let $z$ be the number of edges in $J$ with both ends in $Q$. Note that $|Q| \geq t^{\prime}-y=t-2 k-y$. By Corollary 16 , for every $A^{\prime} \subseteq Q$ with $\left|A^{\prime}\right| \in\{1,2\}$ there is $A \subseteq V$ with $p(A)>0$ such that $A \cap T^{\prime}=A^{\prime}$. This implies that $z \geq \frac{2}{3}|Q| \geq \frac{2}{3}(t-2 k-y)$. Consequently, since $|J| \geq x+y+z$ and $y \geq 2(k-x)$
$|J| \geq x+y+\frac{2}{3}(t-2 k-y)=x+\frac{1}{3} y+\frac{2}{3}(t-2 k) \geq x+\frac{2}{3}(k-x)+\frac{2}{3}(t-2 k)=\frac{1}{3} x+\frac{2}{3}(t-k)$.
Since $x \geq 0$ we get $|J| \geq \frac{2}{3}(t-k)$.
From Lemma 18 it follows that the approximation ratio of the algorithm is bounded by $\frac{t-k}{\max \{t / 2,2(t-k) / 3\}} \leq 3 / 2$, with $k=t / 4$ being the worse case.

This concludes the proof of Theorem 15 and thus also the proof of Theorem 3 is complete.

## References

1 D. Adjiashvili. Beating approximation factor two for weighted tree augmentation with bounded costs. In SODA, pages 2384-2399, 2017.
2 J. Bang-Jensen and B. Jackson. Augmenting hypergraphs by edges of size two. Math. Programming, 84:457-481, 1999.
3 M. Basavaraju, F. V. Fomin, P. A. Golovach, P. Misra, M. S. Ramanujan, and S. Saurabh. Parameterized algorithms to preserve connectivity. In ICALP, Part I, pages 800-811, 2014.
4 A. Benczúr and A. Frank. Covering symmetric supermodular functions by graphs. Math. Programming, 84:483-503, 1999.
5 A. Bernáth and T. Király. A unifying approach to splitting-off. Combinatorica, 32(4):373-401, 2012.

6 J. Byrka, F. Grandoni, and A. J. Ameli. Breaching the 2-approximation barrier for connectivity augmentation: a reduction to steiner tree. In STOC, pages 815-825, 2020. For the full version see. URL: https://arxiv.org/abs/1911.02259
7 J. Byrka, F. Grandoni, T. Rothvoß, and L. Sanità. Steiner tree approximation via iterative randomized rounding. J. ACM, 60(1):6:1-6:33, 2013.
8 E. A. Dinic, A. V. Karzanov, and M. V. Lomonosov. On the structure of a family of minimal weighted cuts in a graph. Studies in Discrete Optimization, page 290-306, 1976.
9 Y. Dinitz and Z. Nutov. A 2-level cactus model for the system of minimum and minimum +1 edge-cuts in a graph and its incremental maintenance. In STOC, pages 509-518, 1995.
10 S. Fiorini, M. Groß, J. Könemann, and L. Sanitá. A $\frac{3}{2}$-approximation algorithm for tree augmentation via chvátal-gomory cuts. In $S O D A$, pages 817-831, 82018.
11 L. Fleischer, K. Jain, and D. P. Williamson. Iterative rounding 2-approximation algorithms for minimum-cost vertex connectivity problems. J. Comput. Syst. Sci., 72(5):838-867, 2006.
12 A. Frank. Connections in Combinatorial Optimization. Oxford University Press, 2011.
13 A. Frank and T. Jordán. Graph connectivity augmentation. In K. Thulasiraman, S. Arumugam, A. Brandstadt, and T. Nishizeki, editors, Handbook of Graph Theory, Combinatorial Optimization, and Algorithms, chapter 14, pages 313-346. CRC Press, 2015.
14 W. Gálvez, F. Grandoni, A. J. Ameli, and K. Sornat. On the cycle augmentation problem: Hardness and approximation algorithms. In WAOA, pages 138-153, 2019.
15 F. Grandoni, C. Kalaitzis, and R. Zenklusen. Improved approximation for tree augmentation: saving by rewiring. In STOC, pages 632-645, 2018.
16 S. Khuller. Approximation algorithms for finding highly connected subgraphs. In D. Hochbaum, editor, Approximation Algorithms for NP-hard problems, chapter 6, pages 236-265. PWS, 1995.

17 Z. Király, B. Cosh, and B. Jackson. Local edge-connectivity augmentation in hypergraphs is NP-complete. Discrete Applied Mathematics, 158(6):723-727, 2010.
18 G. Kortsarz and Z. Nutov. A simplified 1.5-approximation algorithm for augmenting edgeconnectivity of a graph from 1 to 2. ACM Transactions on Algorithms, 12(2):23, 2016.
19 Y. Maduel and Z. Nutov. Covering a laminar family by leaf to leaf links. Discrete Applied Mathematics, 158(13):1424-1432, 2010.
20 Z. Nutov. Structures of Cuts and Cycles in Graphs; Algorithms and Applications. PhD thesis, Technion, Israel Institute of Technology, 1997.
21 Z. Nutov. Approximating connectivity augmentation problems. ACM Trans. Algorithms, 6(1):5:1-5:19, 2009.
22 Z. Nutov. On the tree augmentation problem. In ESA, pages 61:1-61:14, 2017. To appear in Algorithmica.
23 Z. Nutov. 2-node-connectivity network design. CoRR, abs/2002.04048, 2020. To appear in WAOA20. URL: https://arxiv.org/abs/2002.04048
24 Z. Nutov, G. Kortsarz, and E. Shalom. Approximating activation edge-cover and facility location problems. In MFCS, pages 20:1-20:14, 2019.
25 R. Ravi and D. P. Williamson. An approximation algorithm for minimum-cost vertexconnectivity problems. Algorithmica, 18(1):21-43, 1997.

