Königsberg Sightseeing: Eulerian Walks in Temporal Graphs

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Abstract. An Eulerian walk (or Eulerian trail) is a walk (resp. trail) that visits every edge of a graph G at least (resp. exactly) once. This notion was first discussed by Leonhard Euler while solving the famous Seven Bridges of Königsberg problem in 1736. What if Euler had to take a bus? In a temporal graph (G, λ) , with $\lambda : E(G) \to 2^{[\tau]}$, an edge $e \in E(G)$ is available only at the times specified by $\lambda(e) \subset [\tau]$, in the same way the connections of the public transportation network of a city or of sightseeing tours are available only at scheduled times. In this scenario, even though several translations of Eulerian trails and walks are possible in temporal terms, only a very particular variation has been exploited in the literature, specifically for infinite dynamic networks (Orlin, 1984). In this paper, we deal with temporal walks, local trails, and trails, respectively referring to edge traversal with no constraints, constrained to not repeating the same edge in a single timestamp, and constrained to never repeating the same edge throughout the entire traversal. We show that, if the edges are always available, then deciding whether (G, λ) has a temporal walk or trail is polynomial, while deciding whether it has a local trail is NP-complete even if it has lifetime 2. In contrast, in the general case, solving any of these problems is NP-complete, even under very strict hypothesis.

1 Introduction

An Eulerian walk (or Eulerian trail) is a walk (resp. trail) that visits every edge of a graph G at least (resp. exactly) once. The Eulerian trail notion was first discussed by Leonhard Euler while solving the famous Seven Bridges of Königsberg problem in 1736, where one wanted to pass by all the bridges over the river Preger without going twice over the same bridge. Imagine now a similar problem, where you have a set of sights linked by possible routes. If the routes themselves are also of interest, a sightseeing tourism company might want to plan visits on different days that cover all the routes. One could do that with no constraints at all (thus performing a walk), or with the very strict constraint of never repeating a route (thus getting a trail), or constraining oneself to at least not repeting the same route on the same day (thus getting what we called a local

trail). If we further assume that some routes might not be always accessible, we then get distinct problems defined on temporal graphs.

In a temporal graph (G, λ) , with $\lambda : E(G) \to 2^{[\tau]}$, an edge $e \in E(G)$ is available only at the times specified by $\lambda(e) \subseteq [\tau]$, in the same way the connections of the public transportation network of a city or of sightseeing tours are available only at scheduled times. In this scenario, paths and walks are valid only if they traverse a sequence of adjacent edges e_1, \ldots, e_k such that for each $i \in [k-1]$, $\lambda(e_i) \leq \lambda(e_{i+1})$, i.e. whose time sequence is non-decreasing (similarly, strictly increasing sequences, i.e. with $\lambda(e_i) < \lambda(e_{i+1})$, can be considered).

Several translations of Eulerian trails and walks are possible in temporal terms, depending on the constraints we consider. In particular, we study the following variations. Below, all the walks and trails are implicitly considered to be temporal, as defined in the previous paragraph.

Problem 1. Given a temporal graph (G, λ) , we consider the following problems:

- EULERIAN WALK: deciding whether (G, λ) has an Eulerian walk, i.e. a walk traversing each edge at least once.
- EULERIAN LOCAL TRAIL: deciding whether (G, λ) has an Eulerian local trail, i.e. a walk traversing each edge at least once and at most once in each time stamp.
- EULERIAN TRAIL: deciding whether (G, λ) has an Eulerian trail, i.e. a walk traversing each edge exactly once.

We also consider the related problems where the walks/trails are closed (first vertex equal to the last one), respectively referring to them as EULERIAN CLOSED WALK, EULERIAN LOCAL TOUR, and EULERIAN TOUR.

The research on temporal graphs have attracted a lot of attention in the past decade (we refer the reader to the surveys [18,16] and the seminal paper [15]), and they appear also under different names, e.g. as time-varying graphs [3], as evolving networks [2], and as link streams [16]. Even after all the received attention, surprisingly enough none of the above problems have been previously considered.

Concerning EULERIAN WALK, one of the closest concepts is the one defined by Orlin in [20], where he gives a polynomial algorithm to check the existence of an Eulerian closed walk (i.e. a *tour*) in dynamic graphs. However, the dynamic graph model is quite different from the temporal graph model used in this paper, as pointed out in [18]. Indeed, looking at the corresponding time-expanded graph, temporal edges can go back in time and the graph is infinite. Nevertheless, the results presented there seemed to point towards the polinomiality of the problems investigated here, as observed in [18]: "the results proved for it [the dynamic graph model] are resounding and possibly give some first indications of what to expect when adding to combinatorial optimization problems a time dimension". We found however that this is not the case, as we will show that even EULERIAN WALK turns out to be much harder on temporal graphs (for more details about dynamic graphs and the result proved by Orlin in [20] see Appendix A). Taking inspiration in [20], we also define a dynamic-based temporal graph as a temporal graph whose edges are always available, and we analyze the complexity of the above problems on these particular instances.

Still concerning EULERIAN WALK, a closely related problem is the TEMPO-RAL EXPLORATION PROBLEM (TEXP) [19], which consists of, given a temporal graph (G, λ) , finding a temporal walk that visits all vertices in G (possibly, more than once) whose arrival time is minimum. In [19], they prove that this problem is NP-complete and even not approximable unless P = NP; this is in stark contrast with the static version of the problem, which can be trivially solved in linear time. However this is quite different wrt our EULERIAN WALK, as we are considering walks passing through all the edges instead of all the vertices, and this difference is crucial as a transformation from TEXP to EULERIAN WALK does not seem to be easy. Indeed, by simply transforming each vertex into an edge, we would get two types of edges, connection-edges and vertex-edges, and a negative answer to EULERIAN WALK could be just motivated by the impossibility of visiting all the connection-edges, which are actually not required to be visited to solve TEXP. This is not surprising as we are trying to transform a Hamiltonian walk (i.e. a walk passing by each vertex) into an Eulerian walk. We mention that, unlike the problems investigated here, a lot of research has been devoted to temporal node exploration, e.g. bounding the arrival time of such walks in special instances [9,8] and extending previous results in the case of non-strictly increasing paths [10].

Concerning EULERIAN TRAIL and EULERIAN LOCAL TRAIL, observe that, when $\tau = 1$, then both of them degenerate into the original formulation of the Seven Bridges of Königsberg problem. This is why we think they appear to be more natural adaptations of the static version of the problem. Nevertheless they have never been investigated before, up to our knowledge.

Our results Our results are summarized in Table 1 and detailed in Theorem 1.

Theorem 1. Given a temporal graph (G, λ)

- 1. EULERIAN WALK is NP-complete, even if each snapshot of (G, λ) is a forest of constant size, while it is polynomial for bounded τ . It is also polynomial if (G, λ) is dynamic-based.
- 2. EULERIAN LOCAL TRAIL is NP-complete for each $\tau \geq 2$ and in the case (G, λ) is dynamic-based.
- 3. EULERIAN TRAIL is NP-complete for each $\tau \geq 2$. It is polynomial if (G, λ) is dynamic-based.

Same applies to tours, i.e. EULERIAN CLOSED WALK, EULERIAN LOCAL TOUR, and EULERIAN TOUR.

Theorem 1 gives a complete taxonomy of our problems, also focusing on the possibility of getting polynomial algorithms when we have a small lifetime τ . In particular, for EULERIAN TRAIL and EULERIAN LOCAL TRAIL, since they become polynomial when $\tau = 1$, the bound for τ is optimal, giving us a complete

	Eulerian Walk	Eulerian Local Trail	Eulerian Trail
General	$NP\text{-}c$ for τ unbounded	NP-c [from below]	NP-complete
	Poly for τ fixed		for $\tau = 2$
DYNAMIC-BASED	$Poly^{\dagger}$	NP-c for $\tau = 2$	Poly [*]

Table 1: **Our results concerning Problem 1.** For general temporal graphs (first row) and for dynamic-based temporal graphs (second row). \dagger corresponds to deciding whether *G* is connected. * corresponds to deciding whether *G* has an Eulerian trail.

dichotomy with respect to the lifetime of (G, λ) . In contrast EULERIAN WALK is easily solvable for every fixed τ , showing that walks are easier than trails even on the temporal context.

It is important to remark that none of the above variations immediately implies any of the others. We will show indeed that the property of being Eulerian for the static base graph G is in general a necessary but not sufficient condition for the existence of an Eulerian trail, becoming sufficient only if we restrict to dynamic-based temporal graphs. In the case of Eulerian local trail, we will see that this property is not even necessary.

Finally, as a by product of our reductions we get the following result about static graphs, which can be of independent interest.

Corollary 1. Given a graph G, deciding whether all the edges of G can be covered with two trails is NP-complete.

Further Related Work. Other than the papers on TEXP previously mentioned, there is a vast literature about finding special paths or walks in temporal graphs, and some interesting papers include [4,24,21,19]. Also, it is largely known that a static graph G has an Eulerian tour (trail) if and only if G has at most one non-trivial component and all the vertices have even degree (at most two vertices have odd degree). A graph is called Eulerian if it has an Eulerian tour.

When considering dynamic-based temporal graphs, as edges are assumed to be always available during the lifetime τ , we could relate our problems to several other problems on static graphs. A closely related one would be the Chinese Postman problem, where the edges of the graph have positive weights and one wants to find an Eulerian closed walk on G with minimum weight; in other words, one wants to add copies of existing edges in order to obtain an Eulerian graph of minimum sum weight. Even if we regard the Chinese Postman problem where the weights are all equal to 1, this is very different from our approach since for us, repetition of a long common trail in different snapshots does not make the solution worse, while it would considering the Chinese Postman problem. It is easy to see though that the solution for the Chinese Postman would give us an upper bound for the amount of time spent on an Eulerian local tour of a dynamic-based graph, as we could start a new trail on a new snapshot whenever an edge repetition was detected. The Chinese Postman problem is largely known to be polynomial [13], and some variations that take time into consideration have been investigated, mostly from the practical point of view (see e.g. [5,22,23]), but none of which is equivalent to our problem.

The problem of trying to obtain an Eulerian subgraph (as opposed to a supergraph) has also been studied. In [6], the authors study a family of problems where the goal is to make a static graph Eulerian by a minimum number of deletions. They completely classify the parameterized complexity of various versions of the problem: vertex or edge deletions, undirected or directed graphs, with or without the requirement of connectivity. Also in [11], the parameterized complexity of the following Euler subgraph problems is studied: (i) Largest Euler Subgraph: for a given graph G and integer parameter k, does G contain an induced Eulerian subgraph with at least k vertices?; and (ii) Longest Circuit: for a given graph G and integer parameter k, does G contain an Eulerian subgraph with at least k edges?

EULERIAN LOCAL TRAIL on dynamic-based graphs is actually more closely related to the problem of covering the edges of a graph with the minimum number of (not necessarily disjoint) trails, whereas the aforementioned problems are more concerned with either minimizing edge repetitions or maximizing the subgraph covered by a single trail. Even if the trail cover problem can be so naturally defined and involve such a basic structure as trail, up to our knowledge it has not yet been previously investigated. Note that EULERIAN LOCAL TRAIL is slightly different from trail cover, since we also require that together the trails form a walk. In any case, a small modification of our proof of Theorem 1.2 implies that deciding whether the edges of a graph can be covered with at most two trails is NP-complete (Corollary 1). Interestingly enough, the vertex version of this problem, namely the path cover problem, has been largely investigated (see e.g. [1,14,17]).

Preliminaries. We use and extend the notation in [18]. A temporal graph is a graph together with a function on the edges saying when each edge is active; more formally, a temporal graph is a pair (G, λ) , where $\lambda : E(G) \to 2^{\mathbb{N}-\{0\}}$. Here, we consider only finite temporal graphs, i.e., graphs such that $\max \bigcup_{e \in E(G)} \lambda(e)$ is defined. This value is called the *lifetime of* (G, λ) and denoted by τ . Given $i \in [\tau]$, we define the snapshot G_i as being the subgraph of G containing exactly the edges active in time i; more formally, $V(G_i) = V(G)$ and $E(G_i) = \{e \in E(G) \mid i \in \lambda(e)\}$.

Given vertices v_0, v_k in a graph G, a v_0, v_k -walk in G is an alternating sequence $(v_0, e_1, v_1, \ldots, e_k, v_k)$ of vertices and edges such that e_i goes from v_{i-1} to v_i for $i \in \{1, \ldots, k\}$. We define a walk in a temporal graph similarly, except that a walk cannot go back in time. More formally, given a temporal graph (G, λ) and a v_0, v_k -walk $W = (v_0, e_1, v_1, \ldots, e_k, v_k)$, we say that W is a temporal v_0, v_k -walk if $\lambda(e_1) \leq \lambda(e_2) \leq \ldots \leq \lambda(e_k)$. It is closed if it starts and finishes on the same vertex of G, i.e., if $v_0 = v_k$.

We say that a temporal walk W is a *local trail* if there are no two occurrences of the same edge of G in the same snapshot, i.e., if W restricted to G_i is a trail in G for every $i \in [\tau]$. We say that W is a *trail* if there are no two occurrences of the same edge of G in W. A closed (local) trail is also called a *(local) tour*. Finally, a temporal walk W is called *Eulerian* if at least one copy of each edge of G appears at least once in W. Observe that, by definition, an Eulerian trail visits every edge exactly once.

A dynamic-based graph is a temporal graph (G, λ) where the edges are always available.³ We denote a dynamic-based graph simply by $(G, [\tau])$ where τ is the lifetime of the temporal graph.

2 Eulerian Walk

In this section we focus on EULERIAN WALK, i.e. deciding if there is a temporal walk passing by each edge at least once, proving the results in Item 1 in Theorem 1, summarized in the first column of Table 1.

In particular, a preliminary result, whose formal proof is reported in Appendix B.1, concerns the case where the lifetime τ is bounded. It consists basically of checking whether there is a choice of connected components H_1, \ldots, H_{τ} , one for each timestamp *i*, that together cover all the edges of *G* and is such that H_i intersects H_{i+1} , for each $i \in [\tau - 1]$.

Lemma 1. Given a temporal graph (G, λ) with fixed lifetime τ , solving EULE-RIAN WALK on (G, λ) can be done in time $O((n+m) \cdot n^{\tau-1})$, where n = |V(G)|and m = |E(G)|.

In the following, we show that when τ is unbounded, deciding whether (G, λ) admits an Eulerian walk is NP-complete by reducing from 3-SAT. This is best possible because of the above lemma.

Theorem 2. Given a temporal graph (G, λ) , deciding whether (G, λ) admits an Eulerian walk is NP-complete, even if each snapshot of (G, λ) is a forest of constant size.

Proof. We make a reduction from 3-SAT. Let ϕ be a 3-CNF formula on variables $\{x_1, \dots, x_n\}$ and clauses $\{c_1, \dots, c_m\}$, and construct G as follows. For each clause c_i , add vertices $\{a_i, b_i\}$ to G and edge $a_i b_i$. Now consider a variable x_i , and let c_{i_1}, \dots, c_{i_p} be the clauses containing x_i positively, and c_{j_1}, \dots, c_{j_q} be the clauses containing x_i positively, and c_{j_1}, \dots, c_{j_q} be the clauses containing x_i positively, and c_{j_1}, \dots, c_{j_q} be the clauses containing x_i negatively. Add two new vertices $x_i, \overline{x_i}$ to G, and edges $\{x_i a_{i_k} \mid k \in [p]\} \cup \{\overline{x_i} a_{j_k} \mid k \in [q]\}$; denote the spanning subgraph of G formed by these edges by H_i , and let H'_i be equal to H_i together with edges $\{a_i b_i \mid i \in \{i_1, \dots, i_p, j_1, \dots, j_q\}$. We can suppose that $\{i_1, \dots, i_p\} \cap \{j_1, \dots, j_q\} = \emptyset$ as otherwise the clauses in the intersection would always be trivially valid; thus we get that H_i, H'_i are forests. Finally, add a new vertex T and make it adjacent to every vertex in $\{x_i, \overline{x_i} \mid i \in [n]\}$.

³ This is the reason why we use the term dynamic-based, as they are similar to the dynamic networks used in [20] when studying Eulerian trails, except that edges cannot go back in time and the lifetime is finite.

We now describe the snapshots of (G, λ) . See Figure 1 to follow the construction. We first build 2 consecutive snapshots in (G, λ) related to x_i , for each $i \in [n]$. The first one is equal to H'_i , and the second one contains exactly the edges $\{Tx_i, T\overline{x}_i, Tx_{i+1}, T\overline{x}_{i+1}\}$ if i < n, and if i = n, then the second snapshot is equal to $G - \{a_j b_j \mid j \in [m]\}$; this can be done because this subgraph is connected. Denote by S_i^1, S_i^2 the first and second snapshot of x_i , for each $i \in [n]$. Put these snapshots consecutively in timestamps 1 through 2n, in the order of the indexing of the variables. For now, observe that only the last snapshot might not be a forest; this will be fixed later. We now prove that ϕ is a satisfiable formula if and only if (G, λ) admits an Eulerian walk.

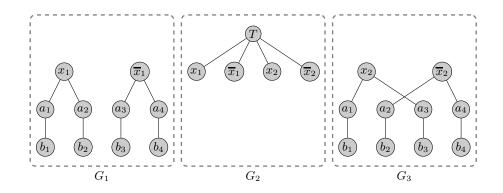


Fig. 1: First three snapshots of the construction. In this example, we have c_1 containing $(x_1 \vee x_2)$, c_2 containing $(x_1 \vee \overline{x}_2)$, c_3 containing $(\overline{x}_1 \vee x_2)$, and c_4 containing $(\overline{x}_1 \vee \overline{x}_2)$.

Observe that, since we are dealing with a walk, we are allowed to repeat edges as many times as we want; hence, if we visit any vertex inside a component of a snapshot, we can also visit the entire component. At the same time, it is not possible to visit more than one component within a snapshot. This is implicitly used below.

First consider a satisfying assignment of ϕ . Now, construct an Eulerian walk as follows. Start by visiting all the edges in the component of H'_1 containing x_1 , if x_1 is true, or the one containing \overline{x}_1 , otherwise. Then, in S_1^2 , jump to x_2 if x_2 is true, or to \overline{x}_2 , otherwise. Repeat the process until reaching S_n^1 , and at the last snapshot, visit all the edges in $G - \{a_j b_j \mid j \in [m]\}$. Because each clause c_i contains at least one true variable, we know that edge $a_i b_i$ is visited.

Now, consider an Eulerian walk W of (G, λ) , and denote by W_i^j the walk W restricted to S_i^j , for each $i \in [n]$ and $j \in [2]$. We set x_i to true if W_i^1 contains x_i , and to false otherwise. Now, consider a clause c_i containing variables $x_{k_1}, x_{k_2}, x_{k_3}$. Because $a_i b_i$ appears only in snapshots $S_{k_1}^1, S_{k_2}^1, S_{k_3}^1$, and only in the component containing the literal that appears in c_i , we get that at least one of the three literals must be set to true.

Finally, observe that we could repeat the same pattern as the one in the first 2n-1 snapshots in order to visit the remaining edges in $G - \{a_j b_j \mid j \in [m]\}$. As long as the edges in $\{a_i b_i \mid i \in [m]\}$ appear only in the first 2n-1 snapshots, the same argument as before still applies, and we get the further constraint that each snapshot is a forest. Additionally, they can also be considered to have constant size since 3-SAT NP-complete even if each variable appears at most three times [7].

Now, if we consider a dynamic-based graph (G, λ) , since all the edges are active throughout its lifetime, we clearly have that there exists an Eulerian walk if and only if G is connected, as highlighted by the following Lemma.

Lemma 2. EULERIAN WALK is polynomial for dynamic-based temporal graphs.

By Lemma 1, Theorem 2, and Lemma 2, we obtain Item 1 of Theorem 1. Finally, note that if one is interested in closed walks instead, not only our NPcompleteness reduction can be adapted in order to ensure that we can always go back to the initial vertex, but also the complexity results still hold.

3 Eulerian Local Tours and Trails

In this section we focus on Item 2 of Theorem 1. In the whole section, we will focus on dynamic-based temporal graphs as the hardness results for general temporal graphs are implied by the ones we prove for this restricted class. After the preliminary result in Lemma 3, we focus on proving the hardness result for the problem of deciding whether (G, [2]) has an Eulerian local tour, explaining the construction behind our reduction from NAE 3-SAT, whose correctness is proved in Theorem 3. We also argue that, if G is a cubic graph, then being Hamiltonian is a necessary but not sufficient condition for (G, [2]) to admit an Eulerian local tour, arguing the need of an *ad hoc* reduction for our problem. As the reduction in Theorem 3 focuses on solving EULERIAN LOCAL TOUR for $\tau = 2$, in Corollary 2 we extend this result to each fixed τ and to trails, thus completing the proof of Item 2 of Theorem 1.

The following Lemma, whose proof is given in Appendix B.2, will help our proof.

Lemma 3. Let G be a graph. If (G, [2]) has an Eulerian local tour T, then T restricted to timestamp i must pass by all vertices of odd degree in G, for each $i \in [2]$.

A simple consequence of the above lemma is that, as previously said, if G is cubic, then G must Hamiltonian in order for (G, [2]) to have an Eulerian local tour. Since deciding whether a cubic graph is Hamiltonian is NP-complete [12], this hints towards the NP-completeness of the problem. However, since the other way around is not necessarily true (see e.g. the graph in Figure 2), we need an explicit reduction. Indeed, the construction in Figure 2 shows us that we might

need an arbitrarily large lifetime in order to be able to visit all the edges of $(G, [\tau])$ even if G is a 2-connected outerplanar cubic graph (which is trivially hamiltonian).

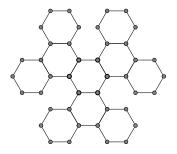


Fig. 2: Example of outerplanar graph G such that (G, [2]) does not have an Eulerian local tour.

In the following we explain the construction behind our reduction from NAE 3-SAT. Let ϕ be a CNF formula on variables $\{x_1, \dots, x_n\}$ and clauses $\{c_1, \dots, c_m\}$. We start by presenting a meta-construction, in the sense that part of the constructed graph will be presented for now as black boxes and the actual construction is done later, as depicted in Figure 3. The meta part concerns the clauses; so for now, denote by C_i the black box related to clause c_i . Without going into details, C_i will contain exactly one entry vertex for each of its literal. So, given a literal ℓ contained in c_i , denote by $I_i(\ell)$ the entry vertex for ℓ in C_i . All defined three vertices are distinct.

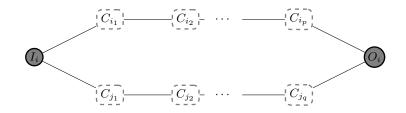


Fig. 3: Edge gadget with clause black boxes.

Now, for each variable x_i , let c_{i_1}, \dots, c_{i_p} be the clauses containing x_i positively and c_{j_1}, \dots, c_{j_q} containing x_i negatively. Add two new vertices, I_i and O_i (these will be the entry and exit vertices for the variable gadget), and add the

following edges (these compose the paths shown in Figure 3):

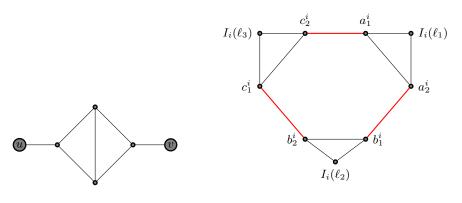
$$E_{i} = \{I_{i}I_{i_{1}}(x_{i}), I_{i}I_{j_{1}}(\overline{x}_{i}), I_{i_{p}}(x_{i})O_{i}, I_{j_{q}}(\overline{x}_{i})O_{i}\} \cup \{I_{i_{h}}(x_{i})I_{i_{h+1}}(x_{i}) \mid h \in [p-1]\} \cup \{I_{j_{h}}(\overline{x}_{i})I_{j_{h+1}}(\overline{x}_{i}) \mid h \in [q-1]\}$$

The paths will function as a switch, telling us whether the variable is true or false within the considered snapshot; we then denote by P_i the set of edges in the path $(I_i, I_{i_1}(x_i), \dots, I_{i_p}(x_i), O_i)$, and by \overline{P}_i the set of edges in the path $(I_i, I_{j_1}(\overline{x}_i), \dots, I_{j_q}(\overline{x}_i), O_i)$. Now, to link the variable gadgets and to construct the clause gadgets, we will need a gadget that will function as an edge that must appear in the trail performed in G_1 and the one performed in G_2 . For this, we use Lemma 3 applied to the gadget in Figure 4a; when adding such a gadget between a pair u, v, we simply say that we are adding the *forced edge uv*.

Now, to link the variable gadgets, we add three new vertices s_1, s_2, t and the following forced edges.

$$E' = \{s_i t \mid i \in [2]\} \cup \{tI_1, O_n t\} \cup \{O_i I_{i+1} \mid i \in [n]\}.$$

The new vertices simply help us assume where the trail starts and finishes. Now, let T be an Eulerian local tour of (G, [2]) and denote by T_i the trail in G defined by T restricted to G_i , for $i \in [2]$. It is fairly easy to see (and we will prove it shortly) that if we can ensure that T_1 uses P_i if and only if T_2 uses $\overline{P_i}$, then we can prove equivalence with NAE 3-SAT. In other words, the clause gadget must be so that, for every clause c_j containing x_i (or equivalently $\overline{x_i}$), we get that either both edges incident to $I_j(x_i)$ in P_i (or equivalently $I_j(\overline{x_i})$ in $\overline{P_i}$) are used, or none of them is used. Such a gadget is presented in Figure 4b, where the red edges are forced.



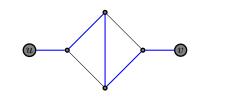
(a) Gadget related to a forced edge uv.

(b) Gadget related to clause c_i . Red edges represent forced edges.

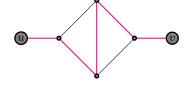
Fig. 4: Gadgets for the reduction in Theorem 3.

Theorem 3. Let G be a graph, with degree is at most 4. Then EULERIAN LOCAL TOUR is NP-complete on (G, [2]).

Proof. Let ϕ and G be as previously stated. First, consider a truth NAE assignment f to ϕ . We construct $T_1, T_2 \subseteq E(G)$ and prove that they form an Eulerian local tour of G. Start by putting P_i in T_1 and \overline{P}_i in T_2 if x_i is true, and the other way around if x_i is false. From now on, whenever we add a forced edge to T_1 and T_2 , we are actually adding the trails depicted in Figure 5.



(a) Trail added to T_1 when forced edge uv is added to T_1 .



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(b) Trail added to T_2 when forced edge uv is added to T_2 .

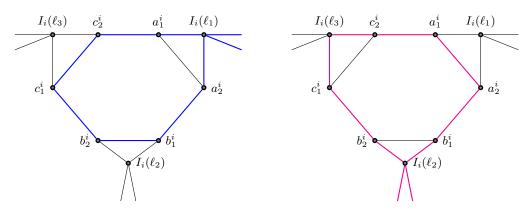
Fig. 5: Trails related to forced edges.

Now, add E' to both T_1 and T_2 , and consider c_i with literals ℓ_1, ℓ_2, ℓ_3 . Suppose, without loss of generality, that ℓ_1 is true and ℓ_2 is false. We then add to T_1 the trail depicted in Figure 6a, and to T_2 the one depicted in Figure 6b. Observe that all internal edges of C_i are covered; also note that the edges incident to $I_i(\ell_3)$ that come from the variable must be either in T_1 or in T_2 by construction. We know that the remaining edges are also covered by $T_1 \cup T_2$ by construction. Finally, notice that both T_1 and T_2 touch all odd-degree vertices in a way that every vertex (including the even-degree ones) has even degree in T_1 and in T_2 , except s_1, s_2 which have degree exactly 1. Also note that they form a connected graph; indeed they are formed by the cycle passing through the variable gadgets and t, together with some pending trails passing by the clause gadgets. Therefore, we can find an s_1, s_2 -trail passing by all edges of T_1 , and an s_2, s_1 -trail passing by all edges of T_2 , thus getting our Eulerian local tour.

Now, let T be an Eulerian local tour of (G, [2]), and for each $i \in [2]$, denote by T_i the trail in G defined by T restricted to G_i . First observe that Lemma 3 indeed ensures that T_1 and T_2 restricted to the gadget related to a forced edge uv must be exactly as the trails depicted in Figure 5; so in what follows we treat them exactly like edges that must appear in T_1 and T_2 . Since there are 2 vertices of degree 1, namely s_1 and s_2 , by Lemma 3 we can suppose that T_1 starts in s_1 and finishes in s_2 , while T_2 starts in s_2 and finishes in s_1 . Therefore, each of T_1 and T_2 contains a tour in $G - \{s_1, s_2\}$, and hence:

(I) Every vertex $u \in V(G) \setminus \{s_1, s_2\}$ has even degree in both T_1 and T_2 .

Now, we prove that for each x_i , if T_1 intersects P_i , then T_1 does not intersect \overline{P}_i , the same holding for T_2 . For this, consider c_i with literals ℓ_1 , ℓ_2 , ℓ_3 , and for



(a) Added to T_1 when ℓ_1 is true. (b) Added to T_2 when ℓ_2 is false.

Fig. 6: Trails in C_i related to a given NAE assignment.

each $j \in [3]$, denote by $e_i^1(\ell_j), e_i^2(\ell_j)$ the edges incident to $I_i(\ell_j)$ not contained in C_i . We first prove that, for each $j \in [3]$ and $k \in [2]$:

(II) Edge $e_i^1(\ell_j)$ is used in T_k if and only if edge $e_i^2(\ell_j)$ is used in T_k .

Without loss of generality, assume j = 1 and let $k \in [2]$; suppose by contradiction that $e_i^1(\ell_1)$ is in T_k , while $e_i^2(\ell_1)$ is not in T_k . By (I) we get that either $a_1^i I_i(\ell_1)$ or $a_2^i I_i(\ell_1)$ is in T_k , say $a_1^i I_i(\ell_1)$. But then we get that $a_1^i a_2^i$ and $a_2^i I_i(\ell_1)$ are not in T_k , as otherwise either a_1^i or $I_i(\ell_1)$ would have odd degree in T_k . This is a contradiction since we then get a_2^i with degree 1 in T_k . The same argument can be analogously applied when $a_2^i I_i(\ell_1)$ is in T_k or when $j \in \{2, 3\}$.

Consider now a variable x_i , and let c_{i_1}, \dots, c_{i_p} be the clauses containing x_i positively and c_{j_1}, \dots, c_{j_q} containing x_i negatively. Because I_i has degree 3 in Gand $E' \setminus \{s_i t \mid i \in [2]\}$ is contained in T_1 , we get that exactly one between $I_1 I_{i_1}(x_i)$ and $I_1 I_{j_1}(\overline{x}_i)$ is contained in T_1 . From (II), we then get that either P_i is contained in T_1 or \overline{P}_i is contained in T_1 . Observe that this implies that P_i is contained in T_k , while \overline{P}_i is contained in T_{3-k} , for some $k \in [2]$. We then set x_i to be true if and only if T_1 contains P_i . Because the edges in $E_i = \{e_i^1(\ell_j), e_i^2(\ell_j) \mid j \in [3]\}$ separate C_i from the rest of the graph and by (I), we get that both T_1 and T_2 must intersect E_i . Finally by (II) we get that this assignment is a NAE truth assignment for ϕ .

Observe that if we add two new vertices of degree one adjacent to vertex t, then we get a reduction to the problem of deciding whether the edges of G can be covered by two trails, proving Corollary 1.

Corollary 2. EULERIAN LOCAL TOUR and EULERIAN LOCAL TRAIL are NPcomplete on temporal graphs with lifetime τ for every fixed $\tau \geq 2$. This also holds on dynamic-based graphs. **Proof.** We first make a reduction from EULERIAN LOCAL TOUR on (G, [2]) to EULERIAN LOCAL TRAIL on $(G', [\tau])$. Given (G, [2]), let G' be obtained from G by adding a star on $\tau + 1$ vertices and identifying one of its leaves with a vertex $s \in V(G)$. We argue that (G, [2]) has an Eulerian local tour starting and finishing in s if and only if $(G', [\tau])$ has an Eulerian local trail. The lemma follows because we can then obtain a Turing reduction by building a distinct instance for each $s \in V(G)$. Denote the vertices of the initial star by $u, v_1, \cdots, v_{\tau+1}$, where u is the central vertex, and v_2 is the vertex where G is pending. Let T be an Eulerian local tour of (G, [2]) starting and finishing in s, and T_1, T_2 be the trails in G defined by T. Build an Eulerian local trail of $(G', [\tau])$ by visiting v_1u, v_2u and T_1 in G'_1 , then performing T_2 and visiting v_2u and uv_3 in G'_2 , and finish to visit the remaining edges of the star in the obvious way.

Now, let T be an Eulerian local trail of $(G', [\tau])$, and denote by T_i the trail in G' defined by T restricted to G'_i , for each $i \in [\tau]$. Observe that because we have $\tau + 1$ cut edges, we get that each T_i contains at most 2 of them, and in case it contains exactly 2, say v_1u, uv_2 , then T_{i+1} either does not contain any cut edge, or must intersect T_i in v_1u, uv_2 . This means that the best we can do in order to finish by time τ is to visit exactly two of them in the first snapshot, and exactly one more in each of the subsequent snapshots. We can therefore suppose, without loss of generality that T_i contains $v_iu, v_{i+1}u$ for each $i \in [\tau]$. Note that this implies that every edge of (G, [2]) must be visited in T_1 and T_2 , with T_1 starting in v_2 and T_2 finishing in v_2 , as we wanted to prove.

Finally, note that $(G', [\tau])$ constructed above has an Eulerian local trail if and only if $(G', \tau + 1)$ has an Eulerian local tour. This completes our proof. \Box

4 Eulerian Tours and Trails

We finally focus on EULERIAN TRAIL and EULERIAN TOUR, proving that in the general case they are both NP-complete, hence, proving Item 3 in Theorem 1. To this aim, we make an adaptation of the construction in Theorem 3. Observe that here the base graph needs to be Eulerian as otherwise the answer to EU-LERIAN TRAIL is trivially NO. This also implies that the problem restricted to dynamic-based graphs is trivial: if the base graph is Eulerian, then the answer to EULERIAN TOUR is YES; otherwise, then the answer is NO. The trick now is to take advantage of the function λ in order to enforce the edges.

Theorem 4. EULERIAN TOUR and EULERIAN TRAIL are NP-complete, even on temporal graphs with fixed lifetime $\tau \geq 2$.

Proof. We first prove the case $\tau = 2$. For this, we simply replace the gadget to enforce an edge uv in the construction of Section 3 by two paths of length 2, P_{uv}^1 and P_{uv}^2 , where the edges in P_{uv}^i are active only in snapshot G_i , for each $i \in [2]$. Because the arguments used in Section 3 depended only on the fact of uv be indeed an enforced edge, we can apply the same arguments here. The only difference is that the trails in G_1 and G_2 now cannot intersect, which indeed is the case since the intersection between T_1 and T_2 in Section 3 is exactly the set

of forced edges, and since here each appearance of a forced edge uv is actually related either to P_{uv}^1 or to P_{uv}^2 .

Now, in order to prove the NP-completeness for higher values of τ , we can simply add new vertices v_3, \dots, v_{τ} and edges $\{s_1v_3\} \cup \{v_iv_{i+1} \mid i \in \{3, \dots, \tau\}\}$. This gives us that EULERIAN TRAIL is NP-complete on (G, λ) with lifetime τ for every fixed $\tau \geq 2$. And if we want a closed trail, it suffices to identify v_{τ} with s_1 , if $\tau \geq 4$, and if $\tau = 3$, we add a new vertices v_4 and edges v_3v_4, v_4s_1 active in snapshot G_3 . This concludes our proof.

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A A brief summary about Eulerian tours in dynamic graphs

A dynamic graph is a pair (G, T) where G is a finite digraph and T is a function $T : E(G) \to \mathbb{Z}$, called transit time function. A dynamic graph can also be seen as a special type of infinite digraph \mathcal{G} , where $V(\mathcal{G}) = V(G) \times \mathbb{Z}$, and $(u, i)(v, j) \in E(\mathcal{G})$ if and only if $uv \in E(G)$ and T(uv) = j - i. Observe that the transit time of an arc can also be negative, and therefore there might exist arcs going from a vertex (u, i) to a vertex (v, j) with j < i, which in the temporal graph context would be considered as going back in time. An Eulerian trail in (G, T) is a trail that passes through all the edges of \mathcal{G} . More formally, it is a function $f : \mathbb{Z} \to V(G) \times \mathbb{Z}$ such that $f(i)f(i+1) \in E(\mathcal{G})$ for every $i \in \mathbb{Z}$, and

for every $(u, i)(v, j) \in E(\mathcal{G})$, there exists a unique ℓ such that (u, i)(v, j) is equal to $f(\ell)f(\ell+1)$.

Recall that a digraph G is Eulerian if and only: (i) G has at most one nontrivial component; and (ii) the indegree of u, denoted with $d^{-}(u)$, is equal to its outdegree, denoted as $d^{+}(u)$, for every $u \in V(G)$. Observe that these conditions are also trivially necessary for the infinite case. Also, note that, given $u \in V(G)$, each in-arc uv of G incident to u gives rise to exactly one in-arc (u, i)(v, i + T(u, v)) incident to (u, i), for every $i \in \mathbb{Z}$. The same clearly holds for every outarc. Therefore, one can see that \mathcal{G} satisfies (i) and (ii) if and only if G satisfies (i) and (ii). However, as proved in [20], these are not the only necessary conditions. Nevertheless, a characterization is still possible, with an additional, also easy to test, condition.

Theorem 5 ([20]). Let (G,T) be a dynamic graph. Then (G,T) has an Eulerian trail if and only the following conditions hold:

- 1. $d^{-}(u) = d^{+}(u)$ for every $u \in V(G)$;
- 2. G is connected; and
- 3. $\sum_{e \in E(G)} T(e) \in \{-1, 1\}$

Even if the necessary part of the proof is more technical, the sufficiency part is quite natural, because condition (3) tells us that, given an eulerian tour $T = (v_1, \dots, v_m, v_1)$ of G and fixing a time i, we can use T to traverse all the edges incident to (v_1, i) in a way that we arrive in $(v_1, i+1)$ (or $(v_1, i-1)$ if the sum is -1) just in time to apply the same process to $(v_1, i+1)$. Because (v_1, i) is chosen arbitrarily, we are ensured to visit all edges of \mathcal{G} .

B Proofs

B.1 Proof of Lemma 1

Proof. Let G_1, \dots, G_{τ} be the snapshots of G; note first that if $E(G_i)$ is empty, then this snapshot can be suppressed. Our problem reduces to deciding whether there is a choice of connected components H_1, \dots, H_{τ} , one for each timestamp i, that together cover all the edges of G and is such that H_i intersects H_{i+1} , for each $i \in [\tau - 1]$. As for each $i \in [\tau]$, there are at most n nodes in the intersections, there are at most $O(n^{\tau-1})$ choices. For each choice the test can be done in $O(\tau(n+m))$, obtaining $O(\tau(n+m)n^{\tau-1})$ running time, which is $O((n+m)n^{\tau-1})$.

B.2 Proof of Lemma 3

Proof. For each $i \in [2]$, denote by T_i the trail in G equal to T restricted to timestamp i, and suppose, by contradiction, that $u \in V(G)$ is a vertex with odd degree not contained in T_1 . Because T is a temporal tour, observe that T_1 is a trail in G starting at some s and finishing at some t, and T_2 is a trail in

G starting at *t* and finishing at *s*, with possibly s = t. This means that the subgraph of *G* formed by the edges of T_2 is such that every $x \in V(G) \setminus \{s, t\}$ has even degree. This is a contradiction because, since no edge incident to *u* is visited in T_1 , we get that all the edges incident to *u* must be visited in T_2 , i.e., *u* would have odd degree in T_2 . The same argument holds in case *u* is not in T_2 , and the lemma follows.