# **Learning Languages with Decidable Hypotheses**

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## **Abstract**

In *language learning in the limit*, the most common type of hypothesis is to give an enumerator for a language. This so-called W-index allows for naming arbitrary computably enumerable languages, with the drawback that even the membership problem is undecidable. In this paper we use a different system which allows for naming arbitrary decidable languages, namely *programs* for characteristic functions (called C-indices). These indices have the drawback that it is now not decidable whether a given hypothesis is even a legal C-index.

In this first analysis of learning with C-indices, we give a structured account of the learning power of various restrictions employing C-indices, also when compared with W-indices. We establish a hierarchy of learning power depending on whether C-indices are required (a) on all outputs; (b) only on outputs relevant for the class to be learned and (c) only in the limit as final, correct hypotheses. Furthermore, all these settings are weaker than learning with W-indices (even when restricted to classes of computable languages). We analyze all these questions also in relation to the mode of data presentation.

Finally, we also ask about the relation of semantic versus syntactic convergence and derive the map of pairwise relations for these two kinds of convergence coupled with various forms of data presentation.

**Keywords:** language learning in the limit, inductive inference, decidable languages, characteristic index

### 1. Introduction

We are interested in the problem of algorithmically learning a description for a formal language (a computably enumerable subset of the set of natural numbers) when presented successively all and only the elements of that language; this is called *inductive inference*, a branch of (algorithmic) learning theory. For example, a learner h might be presented more and more even numbers. After each new number, h outputs a description for a language as its conjecture. The learner h might decide to output a program for the set of all multiples of h, as long as all numbers presented are

divisible by 4. Later, when h sees an even number not divisible by 4, it might change this guess to a program for the set of all multiples of 2.

Many criteria for determining whether a learner h is *successful* on a language L have been proposed in the literature. Gold (1967), in his seminal paper, gave a first, simple learning criterion,  $\mathbf{TxtGEx}$ -learning, where a learner is *successful* if and only if, on every *text* for L (listing of all and only the elements of L) it eventually stops changing its conjectures, and its final conjecture is a correct description for the input language.

Trivially, each single, describable language L has a suitable constant function as a  $\mathbf{Txt}\mathbf{GEx}$ -learner (this learner constantly outputs a description for L). Thus, we are interested in analyzing for which classes of languages  $\mathcal{L}$  is there a single learner h learning each member of  $\mathcal{L}$ . This framework is also known as language learning in the limit and has been studied extensively, using a wide range of learning criteria similar to  $\mathbf{Txt}\mathbf{GEx}$ -learning (see, for example, the textbook Jain et al. (1999)).

In this paper we put the focus on the possible descriptions for languages. Any computably enumerable language L has as possible descriptions any program enumerating all and only the elements of L, called a W-index (the language enumerated by program e is denoted by  $W_e$ ). This system has various drawbacks; most importantly, the function which decides, given e and x, whether  $x \in W_e$  is not computable. We propose to use different descriptors for languages: programs for characteristic functions (where such programs e describe the language  $C_e$  which it decides). Of course, only decidable languages have such a description, but now, given a program e for a characteristic function,  $x \in C_e$  is decidable. Additionally to many questions that remain undecidable (for example whether C-indices are for the same language or whether a C-index is for a finite language), it is not decidable whether a program e is indeed a program for a characteristic function. Thich leads to a new set of problems: learners cannot be (algorithmically) checked whether their outputs are viable (in the sense of being programs for characteristic functions).

Based on this last observation we study a range of different criteria which formalize what kind of behavior we expect form our learners. In the most relaxed setting, learners may output any number (for a program) they want, but in order to  $\mathbf{Ex}$ -learn, they need to to converge to a correct C-index; we denote this restriction with  $\mathbf{Ex}_C$ . Requiring additionally to only use C-indices in order to successfully learn we denote by  $\mathbf{CIndEx}_C$ ; requiring C-indices on all inputs (not just for successful learning, but also when seeing input from no target language whatsoever) we denote by  $\tau(\mathbf{CInd})\mathbf{Ex}_C$ . In particular, the last restriction requires the learner to be total; in order to distinguish whether the loss of learning power is due to the totality restriction or truly due to the additional requirement of outputting C-indices, we also study  $\mathcal{R}\mathbf{CIndEx}_C$ , that is, the requirement  $\mathbf{CIndEx}_C$  where additionally the learner is required to be total.

We note that  $\tau(\mathbf{CInd})\mathbf{Ex}_C$  is similar to learning *indexable families*. Indexable families are classes of languages  $\mathcal{L}$  such that there is an enumeration  $(L_i)_{i\in\mathbb{N}}$  of all and only the elements of  $\mathcal{L}$  for which the decision problem " $x\in L_i$ " is decidable. Already for such classes of languages we get a rich structure. A survey of previous work in this area can be found in Lange et al. (2008). For a learner h learning according to  $\tau(\mathbf{CInd})\mathbf{Ex}_C$  we have that  $L_x = C_{h(x)}$  gives an indexing of a family of languages, and h learns some subset thereof. We are specifically interested in the area between this setting and learning with W-indices ( $\mathbf{Ex}_W$ ).

<sup>1.</sup> **Txt** stands for learning from a *text* of positive examples; **G** for Gold, indicating full-information learning; **Ex** stands for *explanatory*.

The criteria we analyze naturally interpolate between these two settings. We show that we have the following hierarchy:  $\tau(\mathbf{CInd})\mathbf{Ex}_C$  allows for learning strictly fewer classes of languages than  $\mathcal{R}\mathbf{CIndEx}_C$ , which allow for learning the *same* classes as  $\mathbf{CIndEx}_C$ , which again are fewer than learnable by  $\mathbf{Ex}_C$ , which in turn renders fewer classes learnable than  $\mathbf{Ex}_W$ .

All these results hold for learning with full information. In order to study the dependence on the mode of information presentation, we also consider *partially set-driven* learners (**Psd**, **Blum and Blum** (1975); Schäfer-Richter (1984)), which only get the set of data presented so far and the iteration number as input; *set-driven* learners (**Sd**, Wexler and Culicover (1980)), which get only the set of data presented so far; *iterative* learners (**It**, Wiehagen (1976); Fulk (1985)), which only get the new datum and its current hypothesis and, finally, *transductive* learners (**Td**, Carlucci et al. (2007); Kötzing (2009)), which only get the current data. Note that transductive learners are mostly of interest as a proper restriction to all other modes of information presentation.

We show that full-information learners can be turned into partially set-driven learners without loss of learning power. Furthermore, iterative learning is strictly less powerful than set-driven learning, in all settings. Altogether we analyze 25 different criteria and show how each pair relates. All these results are summarized in Figure 1 as one big map stating all pairwise relations of the learning criteria mentioned, giving 300 pairwise relations in one diagram, proven with 13 theorems in Section 3. Note that the results comparing learning criteria with W-indices were previously known, and some proofs could be extended to also cover learning with C-indices.

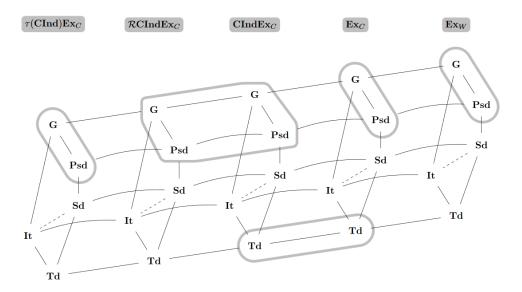


Figure 1: Relation of various requirements when to output characteristic indices paired with various memory restrictions. We omit mentioning Txt to favour readability. Black solid lines imply trivial inclusions (bottom-to-top, left-to-right). Dashed lines imply non-trivial inclusions (bottom-to-top, left-to-right). Furthermore, greyly edged areas illustrate a collapse of the enclosed learning criteria and there are no further collapses.

We derive a similar map considering a possible relaxation on  $\mathbf{Ex}_C$ -learning: while  $\mathbf{Ex}_C$  requires syntactic convergence to one single correct C-index, we consider behaviorally correct learning,  $\mathbf{Bc}_C$  for short, where the learner only has to semantically converge to correct C-indices (but may use infinitely many different such indices). We again consider the different modes of data presentation and determine all pairwise relations in Figure 2.

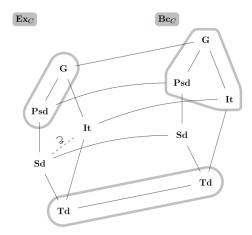


Figure 2: Relation of learning criteria under various memory restrictions. On the left-hand side we require syntactic convergence to characteristic indices and semantic on the right-hand side. We omit mentioning **T**xt in favour of readability. Black solid lines imply trivial inclusions (bottom-to-top, left-to-right). The dashed line between **Sd** and **It** indicates that [**T**xt**I**t**E**x<sub>C</sub>]<sub>**REC**</sub> ⊊ [**T**xt**SdE**x<sub>C</sub>]<sub>**REC**</sub>. Furthermore, greyly edged areas illustrate a collapse of the enclosed learning criteria and there are no further collapses.

Before getting to our results in detail, we continue with some (mathematical) preliminaries in Section 2.

### 2. Preliminaries

In this section we discuss the used notation. Unintroduced notation follows the textbook of Rogers Jr. (1987). For learning criteria we follow the system introduced by Kötzing (2009).

#### 2.1. Mathematical Notations and Learning Criteria

With  $\mathbb N$  we denote the set of all natural numbers, namely  $\{0,1,2,\ldots\}$ . We denote the subset and proper subset relation between two sets with  $\subseteq$  and  $\subsetneq$ , respectively. With  $\subseteq_{\mathbf{Fin}}$  we denote the finite subset relation. We use  $\emptyset$  and  $\varepsilon$  to denote the empty set and empty sequence, respectively. For any set A, the set of all subsets of A is denoted by  $\mathrm{Pow}(A)$ . The set of all computable functions is denoted by  $\mathcal P$ , the subset of all total computable functions by  $\mathcal R$ . If a function f is (not) defined on some argument  $x \in \mathbb N$ , we say that f converges (diverges) on x, denoting this fact with  $f(x) \downarrow (f(x) \uparrow)$ . We fix an effective numbering  $\{\varphi_e\}_{e \in \mathbb N}$  of  $\mathcal P$ . For any  $e \in \mathbb N$ , we let  $W_e$  denote the domain of  $\varphi_e$  and call e a W-index of  $W_e$ . This set we call the e-th computably enumerable set. We call  $e \in \mathbb N$  a

C-index (characteristic index) if and only if  $\varphi_e$  is a total function such that for all  $x \in \mathbb{N}$  we have  $\varphi_e(x) \in \{0,1\}$ . Furthermore, we let  $C_e = \{x \in \mathbb{N} \mid \varphi_e(x) = 1\}$ . For a computably enumerable set L, if some  $e \in \mathbb{N}$  is a C-Index with  $C_e = L$ , we write  $\varphi_e = \chi_L$ . Note that, if a set has a C-index, it is recursive. The set of all recursive sets is denoted by **REC**. For a finite set  $D \subseteq \mathbb{N}$ , we let  $\operatorname{ind}(D)$  be a C-index for D. Note that  $\operatorname{ind} \in \mathcal{R}$ . Furthermore, we fix a Blum complexity measure  $\Phi$  associated with  $\varphi$ , that is, for all  $e, x \in \mathbb{N}$ ,  $\Phi_e(x)$  is the number of steps the function  $\varphi_e$  takes on input x to converge, see (Blum, 1967). The padding function  $\operatorname{pad} \in \mathcal{R}$  is an injective function such that, for all  $e, n \in \mathbb{N}$ , we have  $\varphi_e = \varphi_{\operatorname{pad}(e,n)}$ . We use  $\langle \cdot, \cdot \rangle$  as a computable, bijective function that codes a pair of natural numbers into a single one. We use  $\pi_1$  and  $\pi_2$  as computable decoding functions for the first and section component, i.e., for all  $x, y \in \mathbb{N}$  we have  $\pi_1(\langle x, y \rangle) = x$  and  $\pi_2(\langle x, y \rangle) = y$ .

We learn computably enumerable sets L, called languages. We fix a pause symbol #, and let, for any set S,  $S_\# := S \cup \{\#\}$ . Information about languages is given from text, that is, total functions  $T : \mathbb{N} \to \mathbb{N} \cup \{\#\}$ . A text T is of a certain language L if its content is exactly L, that is,  $content(T) := range(T) \setminus \{\#\}$  is exactly L. We denote the set of all texts as  $\mathbf{Txt}$  and the set of all texts of a language L as  $\mathbf{Txt}(L)$ . For any  $n \in \mathbb{N}$ , we denote with T[n] the initial sequence of the text T of length n, that is,  $T[0] := \varepsilon$  and  $T[n] := (T(0), \ldots, T(n-1))$ . Given a language L and  $t \in \mathbb{N}$ , the set of sequences consisting of elements of  $L \cup \{\#\}$  that are at most t long, is denoted by  $L_\#^{\leq t}$ . Furthermore, we denote with  $\mathbb{S}eq$  all finite sequences over  $\mathbb{N}_\#$  and define the content of such sequences analogous to the content of texts. The concatenation of two sequences  $\sigma, \tau \in \mathbb{S}eq$  is denoted by  $\sigma \cap \tau$ . Furthermore, we write  $\subseteq$  for the extension relation on sequences and fix a order  $\le$  on  $\mathbb{S}eq$  interpreted as natural numbers.

Now, we formalize learning criteria following the system introduced by Kötzing (2009). A learner is a partial function  $h \in \mathcal{P}$ . An interaction operator  $\beta$  is an operator that takes a learner  $h \in \mathcal{P}$  and a text  $T \in \mathbf{Txt}$  as input and outputs a (possibly partial) function p. Intuitively,  $\beta$  defines which information is available to the learner for making its hypothesis. We consider Gold-style or full-information learning (Gold (1967)), denoted by  $\mathbf{G}$ , partially set-driven learning (Psd, Blum and Blum (1975); Schäfer-Richter (1984)), set-driven learning (Sd, Wexler and Culicover (1980)), iterative learning (It, Wiehagen (1976); Fulk (1985)) and transductive learning (Td, Carlucci et al. (2007); Kötzing (2009)). To define the latter formally, we introduce a symbol "?" for the learner to signalize, that the information given is insufficient. Formally, for all learners  $h \in \mathcal{P}$ , texts  $T \in \mathbf{Txt}$  and all  $i \in \mathbb{N}$ , define

$$\begin{aligned} &\mathbf{G}(h,T)(i) = h(T[i]);\\ &\mathbf{Psd}(h,T)(i) = h(\mathrm{content}(T[i]),i);\\ &\mathbf{Sd}(h,T)(i) = h(\mathrm{content}(T[i]));\\ &\mathbf{It}(h,T)(i) = \begin{cases} h(\varepsilon), & \text{if } i = 0;\\ h(\mathbf{It}(h,T)(i-1),T(i-1)), & \text{otherwise}; \end{cases}\\ &\mathbf{Td}(h,T)(i) = \begin{cases} ?, & \text{if } i = 0;\\ \mathbf{Td}(h,T)(i-1), & \text{else, if } h(T(i-1)) = ?;\\ h(T(i-1)), & \text{otherwise.} \end{cases} \end{aligned}$$

For any of the named interaction operators  $\beta$ , given a  $\beta$ -learner h, we let  $h^*$  (the *starred* learner) denote a **G**-learner simulating h, i.e., for all  $T \in \mathbf{Txt}$ , we have  $\beta(h,T) = \mathbf{G}(h^*,T)$ . For example,

let h be a Sd-learner. Then, intuitively,  $h^*$  ignores all information but the content of the input, simulating h with this information, i.e., for all finite sequences  $\sigma$ , we have  $h^*(\sigma) = h(\text{content}(\sigma))$ .

For a learner to successfully identify a language, we may oppose constraints on the hypotheses the learner makes. These are called *learning restrictions*. A famous example was given by Gold (1967). He required the learner to be *explanatory*, i.e., the learner must converge to a *single*, correct hypothesis for the target language. We hereby distinguish whether the final hypothesis is interpreted as a C-index or as a W-index, denoting this by  $\mathbf{Ex}_C$  and  $\mathbf{Ex}_W$ , respectively. Formally, for any sequence of hypotheses p and text  $T \in \mathbf{Txt}$ , we have

$$\mathbf{Ex}_{C}(h,T) \Leftrightarrow \exists n_{0} \colon \forall n \geq n_{0} \colon p(n) = p(n_{0}) \land \varphi_{p(n_{0})} = \chi_{\operatorname{content}(T)};$$
  
$$\mathbf{Ex}_{W}(h,T) \Leftrightarrow \exists n_{0} \colon \forall n \geq n_{0} \colon p(n) = p(n_{0}) \land W_{p(n_{0})} = \operatorname{content}(T).$$

We say that explanatory learning requires *syntactic* convergence. If there exists a C-index (or W-index) for a language, then there exist infinitely many. This motivates to not require syntactic but only *semantic* convergence, i.e., the learner may make mind changes, but it has to, eventually, only output correct hypotheses. This is called *behaviorally correct* learning ( $\mathbf{Bc}_C$  or  $\mathbf{Bc}_W$ , Case and Lynes (1982); Osherson and Weinstein (1982)). Formally, let p be a sequence of hypotheses and let  $T \in \mathbf{Txt}$ , then

$$\mathbf{Bc}_{C}(p,T) \Leftrightarrow \exists n_{0} \colon \forall n \geq n_{0} \colon \varphi_{p(n)} = \chi_{\operatorname{content}(T)};$$
  
$$\mathbf{Bc}_{W}(p,T) \Leftrightarrow \exists n_{0} \colon \forall n \geq n_{0} \colon W_{p(n)} = \operatorname{content}(T).$$

In this paper, we consider learning with C-indices. It is, thus, natural to require the hypotheses to consist solely of C-indices, called C-index learning, and denoted by  $\mathbf{CInd}$ . Formally, for a sequence of hypotheses p and a text T, we have

$$\mathbf{CInd}(p,T) \Leftrightarrow \forall i, x \colon \varphi_{p(i)}(x) \in \{0,1\}.$$

For two learning restrictions  $\delta$  and  $\delta'$ , their combination is their intersection, denoted by their juxtaposition  $\delta\delta'$ . We let **T** denote the learning restriction that is always true, which is interpreted as the absence of a learning restriction.

A learning criterion is a tuple  $(\alpha, \mathcal{C}, \beta, \delta)$ , where  $\mathcal{C}$  is the set of admissible learners, usually  $\mathcal{P}$  or  $\mathcal{R}$ ,  $\beta$  is an interaction operator and  $\alpha$  and  $\delta$  are learning restrictions. We denote this criterion with  $\tau(\alpha)\mathcal{C}\mathbf{T}\mathbf{x}\mathbf{t}\beta\delta$ , omitting  $\mathcal{C}$  if  $\mathcal{C}=\mathcal{P}$ , and a learning restriction if it equals  $\mathbf{T}$ . We say that an admissible learner  $h\in\mathcal{C}$   $\tau(\alpha)\mathcal{C}\mathbf{T}\mathbf{x}\mathbf{t}\beta\delta$ -learns a language L if and only if, for arbitrary texts  $T\in\mathbf{T}\mathbf{x}\mathbf{t}$ , we have  $\alpha(\beta(h,T),T)$  and for all texts  $T\in\mathbf{T}\mathbf{x}\mathbf{t}(L)$  we have  $\delta(\beta(h,T),T)$ . The set of languages  $\tau(\alpha)\mathcal{C}\mathbf{T}\mathbf{x}\mathbf{t}\beta\delta$ -learned by  $h\in\mathcal{C}$  is denoted by  $\tau(\alpha)\mathcal{C}\mathbf{T}\mathbf{x}\mathbf{t}\beta\delta(h)$ . With  $[\tau(\alpha)\mathcal{C}\mathbf{T}\mathbf{x}\mathbf{t}\beta\delta]$  we denote the set of all classes  $\tau(\alpha)\mathcal{C}\mathbf{T}\mathbf{x}\mathbf{t}\beta\delta$ -learnable by some learner in  $\mathcal{C}$ . Moreover, to compare learning with W- and  $\mathcal{C}$ -indices, these classes may only contain recursive languages, which we denote as  $[\tau(\alpha)\mathcal{C}\mathbf{T}\mathbf{x}\mathbf{t}\beta\delta]_{\mathbf{B}\mathbf{E}\mathbf{C}}$ .

### 2.2. Normal Forms

When studying language learning in the limit, there are certain properties of learner that are useful, e.g., if we can assume a learner to be total. Kötzing and Palenta (2016) and Kötzing et al. (2017) study under which circumstances learners may be assumed to be total. Importantly, this

is the case for explanatory Gold-style learners obeying delayable learning restrictions and for behaviorally correct learners obeying delayable restrictions. Intuitively, a learning restriction is *delayable* if it allows hypotheses to be arbitrarily, but not indefinitely postponed without violating the restriction. Formally, a learning restriction  $\delta$  is delayable, if and only if for all non-decreasing, unbounded functions  $r \colon \mathbb{N} \to \mathbb{N}$ , texts  $T, T' \in \mathbf{Txt}$  and learning sequences p such that for all  $n \in \mathbb{N}$ , content $(T[r(n)]) \subseteq \mathrm{content}(T'[n])$  and  $\mathrm{content}(T) = \mathrm{content}(T')$ , we have, if  $\delta(p, T)$ , then also  $\delta(p \circ r, T')$ . Note that  $\mathbf{Ex}_W$ ,  $\mathbf{Ex}_C$ ,  $\mathbf{Bc}_W$ ,  $\mathbf{Bc}_C$  and  $\mathbf{CInd}$  are delayable restrictions.

Another useful notion are *locking sequences*. Intuitively, these contain enough information such that a learner, after seeing this information, converges correctly and does not change its mind anymore whatever additional information from the target language it is given. Formally, let L be a language and let  $\sigma \in L_{\#}^*$ . Given a G-learner  $h \in \mathcal{P}$ ,  $\sigma$  is a locking sequence for h on L if and only if for all sequences  $\tau \in L_{\#}^*$  we have  $h(\sigma) = h(\sigma \tau)$  and  $h(\sigma)$  is a correct hypothesis for L, see Blum and Blum (1975). This concept can immediately be transferred to other interaction operators. Exemplary, given a Sd-learner h and a locking sequence  $\sigma$  of the starred learner  $h^*$ , we call the set content( $\sigma$ ) a locking set. Analogously, one transfers this definition to the other interaction operators. It shall not remain unmentioned that, when considering Psd-learners, we speak of locking information. In the case of  $\mathbf{Bc}_W$ -learning we do not require the learner to syntactically converge. Therefore, we call a sequence  $\sigma \in L_{\#}^*$  a  $\mathbf{Bc}_W$ -locking sequence for a  $\mathbf{G}$ -learner h on Lif, for all sequences  $\tau \in L_{\#}^*$ ,  $h(\sigma \tau)$  is a correct hypothesis for L, see Jain et al. (1999). We omit the transfer to other interaction operators as it is immediate. It is an important observation by Blum and Blum (1975), that for any learner h and any language L it learns, there exists a ( $\mathbf{Bc}_W$ -) locking sequence. These notions and results directly transfer to  $\mathbf{Ex}_{C}$ - and  $\mathbf{Bc}_{C}$ -learning. When it is clear from the context, we omit the index.

## 3. Requiring C-Indices as Output

This section is dedicated to proving Figure 1, giving all pairwise relations for the different settings of requiring C-indices for output in the various mentioned modes of data presentation. In general, we observe that the later we require C-indices, the more learning power the learner has. This holds except for transductive learners which converge to C-indices. We show that they are as powerful as CInd-transductive learners.

Although we learn classes of recursive languages, the requirement to converge to characteristic indices does heavily limit a learners capabilities. In the next theorem we show that even transductive learners which converge to W-indices can learn classes of languages which no Gold-style  $\mathbf{Ex}_{C}$ -learner can learn. We exploit the fact that C-indices, even if only conjectured eventually, must contain both positive and negative information about the guess.

**Theorem 1** We have that 
$$[\mathbf{Txt}\mathbf{TdEx}_W]_{\mathbf{REC}} \setminus [\mathbf{Txt}\mathbf{GEx}_C]_{\mathbf{REC}} \neq \emptyset$$
.

**Proof** We show this by using the Operator Recursion Theorem (**ORT**) to provide a separating class of languages. To this end, let h be the **Td**-learner with h(#) = ? and, for all  $x, y \in \mathbb{N}$ , let  $h(\langle x, y \rangle) = x$ . Let  $\mathcal{L} = \mathbf{Txt}\mathbf{TdEx}_W(h) \cap \mathbf{REC}$ . Assume  $\mathcal{L}$  can be learned by a  $\mathbf{TxtGEx}_{C}$ -learner h'. By Kötzing and Palenta (2016), we can assume  $h' \in \mathcal{R}$ . Then, by **ORT** there exist indices  $e, p, q \in \mathbb{N}$  such that

$$L := W_e = \operatorname{range}(\varphi_n);$$

$$\begin{split} \forall x \colon \tilde{T}(x) &\coloneqq \varphi_p(x) = \langle e, \varphi_q(\tilde{T}[x]) \rangle; \\ \varphi_q(\varepsilon) &= 0; \\ \forall \sigma \neq \varepsilon \colon \bar{\sigma} = \min \{ \sigma' \subseteq \sigma \mid \varphi_q(\sigma') = \varphi_q(\sigma) \}; \\ \forall \sigma \neq \varepsilon \colon \varphi_q(\bar{\sigma}) & \text{if } \forall \sigma', \bar{\sigma} \subseteq \sigma' \subseteq \sigma \colon \Phi_{h'(\sigma')}(\langle e, \varphi_q(\bar{\sigma}) + 1 \rangle) > |\sigma|; \\ \varphi_q(\bar{\sigma}) + 1, & \text{else, for min. } \sigma' \text{ contradicting the previous case, if } \\ \varphi_{h'(\sigma')}(\langle e, \varphi_q(\bar{\sigma}) + 1 \rangle) &= 0; \\ \varphi_q(\bar{\sigma}) + 2, & \text{otherwise.} \end{split}$$

Here,  $\Phi$  is a Blum complexity measure, see Blum (1967). Intuitively, to define the next  $\varphi_p(x)$ , we add the same element to content( $\tilde{T}$ ) until we know whether  $\langle e, \tilde{T}[x] + 1 \rangle \in C_{h'(\bar{\sigma})}$  holds or not. Then, we add the element contradicting this outcome.

We first show that  $L \in \mathcal{L}$  and afterwards that L cannot be learned by h'. To show the former, note that either L is finite or  $\tilde{T}$  is a non-decreasing unbounded computable enumeration of L. Therefore, we have  $L \in \mathbf{REC}$ . We now prove that h learns L. Let  $T \in \mathbf{Txt}(L)$ . For all  $n \in \mathbb{N}$  where T(n) is not the pause symbol, we have h(T(n)) = e. With  $n_0 \in \mathbb{N}$  being minimal such that  $T(n_0) \neq \#$ , we get for all  $n \geq n_0$  that  $\mathbf{Td}(h,T)(n) = e$ . As e is a correct hypothesis, h learns L from T and thus we have that  $L \in \mathbf{TxtTdEx}_W(h)$ . Altogether, we get that  $L \in \mathcal{L}$ .

By assumption, h' learns L from the text  $\tilde{T} \in \mathbf{Txt}(L)$ . Therefore, there exists  $n_0 \in \mathbb{N}$  such that, for all  $n > n_0$ ,

$$h'(\tilde{T}[n]) = h'(\tilde{T}[n_0])$$
 and  $\chi_L = \varphi_{h'(\tilde{T}[n])}$ ,

that is,  $h'(\tilde{T}[n])$  is a C-index for L. Now, as h' outputs C-indices when converging, there are  $t, t' \geq n_0$  such that

$$\Phi_{h'(\tilde{T}[t'])}(\langle e, \varphi_q(\tilde{T}[n_0]) + 1 \rangle) \le t.$$

Let  $t'_0$  and  $t_0$  be the first such found. We show that  $h'(\tilde{T}[t'_0])$  is no correct hypothesis of L by distinguishing the following cases.

- 1. Case:  $\varphi_{h'(\tilde{T}[t'_0])}(\langle e, \varphi_q(\tilde{T}[n_0]) + 1 \rangle) = 0$ . By definition of  $\varphi_q$  and by minimality of  $t'_0$ , we have that  $\langle e, \varphi_q(\tilde{T}[n_0]) + 1 \rangle \in L$ , however, the hypothesis of  $h'(\tilde{T}[t'_0])$  says differently, a contradiction.
- 2. Case:  $\varphi_{h'(\tilde{T}[t'_0])}(\langle e, \varphi_q(\tilde{T}[n_0]) + 1 \rangle) = 1$ . By definition of  $\varphi_q$  and by minimality of  $t'_0$ , we have that  $\langle e, \varphi_q(\tilde{T}[n_0]) + 2 \rangle \in L$ , but  $\langle e, \varphi_q(\tilde{T}[n_0]) + 2 \rangle \notin L$ . However, the hypothesis of  $h'(\tilde{T}[t'_0])$  conjectures the latter to be in L, a contradiction.

Furthermore, known equalities from learning W-indices directly apply in the studied setting as well. These include the following.

Theorem 2 (Kinber and Stephan (1995), Fulk (1990)) We have that

$$[\mathbf{TxtItEx}_W]_{\mathbf{REC}} \subseteq [\mathbf{TxtSdEx}_W]_{\mathbf{REC}}$$
 and  $[\mathbf{TxtPsdEx}_W]_{\mathbf{REC}} = [\mathbf{TxtGEx}_W]_{\mathbf{REC}}$ .

The remaining separations we will show in a more general way, see Theorems 11 and 12. We continue by showing that the latter result, namely that Gold-style learners may be assumed partially set-driven, transfers to all considered cases. We generalize the result by Schäfer-Richter (1984) and Fulk (1990). The idea here is to, just as in the  $\mathbf{Ex}_W$ -case, mimic the given learner and to search for minimal locking sequences. Incorporating the result of Kötzing and Palenta (2016) that unrestricted Gold-style learners may be assumed total, we even get a stronger result.

**Theorem 3** For  $\delta, \delta' \in \{CInd, T\}$ , we have that

$$[\tau(\delta)\mathbf{Txt}\mathbf{G}\delta'\mathbf{Ex}_C]_{\mathbf{REC}} = [\tau(\delta)\mathcal{R}\mathbf{Txt}\mathbf{Psd}\delta'\mathbf{Ex}_C]_{\mathbf{REC}}.$$

**Proof** We modify the proof as seen in Fulk (1990). The inclusion  $[\tau(\delta)\mathcal{R}\mathbf{TxtPsd}\delta'\mathbf{Ex}_C]_{\mathbf{REC}}\subseteq [\tau(\delta)\mathbf{TxtG}\delta'\mathbf{Ex}_C]_{\mathbf{REC}}$  follows immediately. For the other, let h be a  $\tau(\delta)\mathbf{TxtG}\delta'\mathbf{Ex}_C$ -learner, which can assumed to be total by Kötzing and Palenta (2016) and let  $\mathcal{L} = \tau(\delta)\mathbf{TxtG}\delta'\mathbf{Ex}_C(h) \cap \mathbf{REC}$ . We define, for each finite set  $D \subseteq \mathbb{N}$  and  $t \in \mathbb{N}$ ,

$$p(D,t) = \left\{ \sigma \in D^{\leq t} \ \middle| \ \forall \tau \in D^{\leq t} \colon h(\sigma) = h(\sigma\tau) \right\},$$

which, intuitively, contains potential locking sequences of h. We define a  $\tau(\delta)\mathcal{R}\mathbf{Txt}\mathbf{Psd}\delta'\mathbf{Ex}_{C}$ -learner h' for all finite sets D and  $t \in \mathbb{N}$  as

$$h'(D,t) = \begin{cases} h(\min(p(D,t))), & \text{if } p(D,t) \neq \emptyset; \\ \inf(\emptyset), & \text{otherwise.} \end{cases}$$

Note that  $h' \in \mathcal{R}$  since  $h \in \mathcal{R}$ . To show that every language learned by h is also learned by h', let  $L \in \mathcal{L}$  and  $T \in \mathbf{Txt}(L)$ . Let  $\sigma_0$  be a minimal locking sequence for h on L. Let  $n_0$  be sufficiently large such that

- content( $\sigma_0$ )  $\subseteq$  content( $T[n_0]$ ),
- $|\sigma_0| \leq n_0$ , and
- for all  $\sigma' \in L_\#^*$ , with  $\sigma' < \sigma_0$ , there exists  $\tau \in (\operatorname{content}(\sigma_0))_\#^{\leq n_0}$  with  $h(\sigma') \neq h(\sigma'\tau)$ .

Now, for all  $n \ge n_0$ , we have  $\min(p(T[n], n)) = \sigma_0$  and, thus, h' outputs a correct hypothesis on T[n] which shows that  $L \in \tau(\delta) \mathcal{R}\mathbf{TxtPsd}\delta'\mathbf{Ex}_C(h')$ .

It remains to be shown that h' preserves the restrictions imposed on h. This is clear whenever the restriction equals T. For the remaining, we consider the following cases.

- 1. Case:  $\delta = \mathbf{CInd}$ . In this case, h always outputs C-indices. Since h' mimics h or outputs  $\mathrm{ind}(\emptyset)$ , which also is an C-index, we have that h' preserves  $\delta$ .
- 2. Case:  $\delta' = \mathbf{CInd}$ . Let  $L \in \mathcal{L}$ ,  $T \in \mathbf{Txt}(L)$  and  $n \in \mathbb{N}$ . If  $p(\mathrm{content}(T[n]), n) = \emptyset$ ,  $h'(\mathrm{content}(T[n]), n)$  outputs the C-index  $\mathrm{ind}(\emptyset)$ . Otherwise, if  $p(\mathrm{content}(T[n]), n) \neq \emptyset$ , let  $\sigma = \min(p(\mathrm{content}(T[n]), n)) \in L^*$ . Then, we have that  $h'(\mathrm{content}(T[n]), n) = h(\sigma)$  which also is a C-index.

Also the former result of Theorem 2 holds in all considered cases, as the same simulating argument (where one mimics the iterative learner on ascending text with a pause symbol between two elements) suffices regardless the exact setting. We provide the general result.

**Theorem 4** *Let*  $\delta, \delta' \in \{CInd, T\}$  *and*  $C \in \{R, P\}$ . *Then, we have that* 

$$[\tau(\delta')\mathcal{C}\mathbf{Txt}\mathbf{It}\delta\mathbf{Ex}_C]_{\mathbf{REC}} \subseteq [\tau(\delta')\mathcal{C}\mathbf{Txt}\mathbf{Sd}\delta\mathbf{Ex}_C]_{\mathbf{REC}}$$

**Proof** We adapt the proof of Kinber and Stephan (1995). Let a h be a learner and let  $\mathcal{L} = \tau(\delta')\mathcal{C}\mathbf{Txt}\mathbf{It}\delta\mathbf{Ex}_C(h)$ . We show that the following learner h'  $\tau(\delta')\mathcal{C}\mathbf{Txt}\mathbf{Sd}\delta\mathbf{Ex}_C$ -learns  $\mathcal{L}$ . To that end, for any set D, let  $\mathrm{sort}_\#(D)$  be the sequence of the elements in D sorted in ascending order, with a # between each two elements, and let  $h^*$  be the starred form of h. Now, we define h' as, for all finite sets D,

$$h'(D) = \begin{cases} h^*(\operatorname{sort}_{\#}(D)), & \text{if } h^*(\operatorname{sort}_{\#}(D)) = h^*(\operatorname{sort}_{\#}(D)^{\frown}\#); \\ \operatorname{ind}(D), & \text{otherwise.} \end{cases}$$

Note that h' outputs a C-index, whenever h does so or when it outputs ind. Thus, h' preserves the **CInd**-restrictions of h. Moreover, if h is total, then so is h'. To show that h' learns  $\mathcal{L}$ , let  $L \in \mathcal{L}$ . If L is finite, then either  $h^*(\operatorname{sort}_{\#}(L)) = h^*(\operatorname{sort}_{\#}(L)^{\frown}\#)$ , in which case h converges to  $h'(L) = h^*(\operatorname{sort}_{\#}(L))$  on text  $\operatorname{sort}_{\#}(L)^{\frown}\#^{\infty}$ . Otherwise, we have  $h'(L) = \operatorname{ind}(L)$ . In both cases, h' learns L as h'(L) is a correct C-index for L.

On the other hand, if L is infinite, then h must converge to a C-index for L on the text  $\operatorname{sort}_\#(L)$ . Let  $\sigma_0$  be the start sequence of  $\operatorname{sort}_\#(L)$  after which h is converged and let  $D_0 = \operatorname{content}(\sigma_0)$ . Then, for all  $x \in L \setminus D_0$ , we have  $h^*(\sigma_0 \cap x) = h^*(\sigma_0) = h^*(\sigma_0 \cap \#)$  as h is iterative. Therefore, for all D' with  $D_0 \subseteq D' \subseteq L$ , we have  $h^*(\operatorname{sort}_\#(D')) = h^*(\operatorname{sort}_\#(D') \cap \#)$  and  $h^*(\operatorname{sort}_\#(D')) = h^*(\operatorname{sort}_\#(D_0))$ , which is a correct hypothesis for L. As  $h'(D') = h(\operatorname{sort}_\#(D'))$ , we have convergence of h' to a correct C-index for L and thus h' learns L.

Interestingly, totality is no restriction solely for Gold-style (and due to the equality also partially set-driven) learners. For the other considered learners with restricted memory, being total lessens the learning capabilities. This weakness results from the need to output some guess. A partial learner can await this guess and outperform it. This way, we obtain self-learning languages (Case and Kötzing, 2016) to show each of the three following separations.

**Theorem 5** We have that  $[\mathcal{R}\mathbf{Txt}\mathbf{SdCIndEx}_C]_{\mathbf{REC}} \subsetneq [\mathbf{Txt}\mathbf{SdCIndEx}_C]_{\mathbf{REC}}$ .

**Proof** The inclusion  $[\mathcal{R}\mathbf{Txt}\mathbf{SdCIndEx}_C]_{\mathbf{REC}} \subseteq [\mathbf{Txt}\mathbf{SdCIndEx}_C]_{\mathbf{REC}}$  is straightforward. Suppose, by way of contradiction, that  $[\mathcal{R}\mathbf{Txt}\mathbf{SdCIndEx}_C]_{\mathbf{REC}} = [\mathbf{Txt}\mathbf{SdCIndEx}_C]_{\mathbf{REC}}$ . Let h be a learner such that, for all finite sets  $D \subseteq \mathbb{N}$ 

$$h(D) = \begin{cases} \varphi_{\max(D)}(0), & \text{if } D \neq \emptyset; \\ \text{ind}(\emptyset), & \text{otherwise.} \end{cases}$$

We now show that  $\mathcal{L} = \mathbf{TxtSdCIndEx}_C(h) \cap \mathbf{REC}$  is a separating class contradicting the assumption that both classes are equally powerful. To that end, assume there exists a total learner h' with  $\mathcal{L} \subseteq \mathcal{R}\mathbf{TxtSdCIndEx}_C(h')$ . By the Operator Recursion Theorem (**ORT**) there exist an

index  $e \in \mathbb{N}$ , a strictly monotonically increasing function  $T \in \mathcal{R}$  and  $c \in \mathcal{R}$  such that, for all  $n, x \in \mathbb{N}$ ,

$$\begin{split} L &= \mathrm{range}(T); \\ \varphi_e &= \chi_L; \\ c(n) &= \mathrm{content}(T[n]); \\ \varphi_{T(n)}(x) &= \begin{cases} e, & \text{if } \forall n' \leq n \colon h'(c(n'+1)) \neq h'(c(n'+2)); \\ \mathrm{ind}(c(n+1)), & \text{otherwise.} \end{cases} \end{split}$$

Note that there is a C-index for  $\mathrm{range}(T)$  because T is strictly monotonically increasing. Intuitively, if h' always makes mind changes on the start of the text T, then  $\varphi_{T(n)}$  is a function that constantly outputs an index for a infinite set, and otherwise, if h' repeats a hypothesis, then  $\varphi_{T(n)}$  is constantly an index for a finite set.

We now show that there exists a language that is learned by h but not by h'. For this purpose, we consider the following cases.

Case 1:  $\forall n : h'(c(n+1)) \neq h'(c(n+2))$ . In this case  $L \in \mathcal{L}$  holds because h will always output e on every sequence of a text for L, which is a correct C-index for L. But h' makes infinitely many mind changes on text T and thus  $L \nsubseteq \mathcal{R}\mathbf{Txt}\mathbf{SdCIndEx}_C(h')$ .

Case 2:  $\exists n \colon h'(c(n+1)) = h'(c(n+2))$ . Let  $n_0$  be the smallest such n. Then, h learns the languages  $c(n_0+1)$  and  $c(n_0+2)$  because the maximum of these sets is  $T(n_0)$  and  $T(n_0+1)$ , respectively. Thus, h will output the correct hypothesis  $\operatorname{ind}(c(n_0+1))$  or  $\operatorname{ind}(c(n_0+2))$ , respectively. But h' cannot differentiate between those two different languages. Thus, it learn both simultaneously. Therefore, we again have  $\mathcal{L} \nsubseteq \mathcal{R}\mathbf{Txt}\mathbf{SdCIndEx}_{C}(h')$ .

# **Theorem 6** We have that $[\mathcal{R}\mathbf{TxtItCIndEx}_C]_{\mathbf{REC}} \subsetneq [\mathbf{TxtItCIndEx}_C]_{\mathbf{REC}}$ .

**Proof** The inclusion  $[\mathcal{R}\mathbf{TxtItCIndEx}_C]_{\mathbf{REC}} \subseteq [\mathbf{TxtItCIndEx}_C]_{\mathbf{REC}}$  follows immediately. We prove that we have a proper inclusion by providing a separating class using the Operator Recursion Theorem (**ORT**). Suppose now, by way of contradiction, that  $[\mathcal{R}\mathbf{TxtItCIndEx}_C]_{\mathbf{REC}} = [\mathbf{TxtItCIndEx}_C]_{\mathbf{REC}}$ . Let h be a  $\mathbf{TxtItCIndEx}_C$ -learner such that  $h(\varepsilon) = \mathrm{pad}(0,0)$  and, for all  $e, k, x \in \mathbb{N}$ ,

$$h(\operatorname{pad}(e,k),x) = \begin{cases} \uparrow, & \text{if } \varphi_x(0)\uparrow; \\ \operatorname{pad}(e,k), & \text{if } k > \pi_2(\varphi_x(0)); \\ \operatorname{pad}(\pi_1(\varphi_x(0)), \pi_2(\varphi_x(0))), & \text{otherwise.} \end{cases}$$

Recall that  $\pi_1, \pi_2$  are the inverse functions to the pairing function  $\langle \cdot, \cdot \rangle$ . Intuitively, h interprets each datum x as the index of a function and outputs the first component of  $\varphi_x(0)$  where the second component of  $\varphi_x(0)$  is maximal. Now, let  $\mathcal{L} = \mathbf{TxtItCIndEx}_C(h) \cap \mathbf{REC}$ . By our assumption there is a  $\mathcal{R}\mathbf{TxtItCIndEx}_C$ -learner h' that learns  $\mathcal{L}$ . For notational convenience, we use the starred learner  $(h')^*$ . With the **ORT** there exist an index  $e \in \mathbb{N}$  and a strictly monotonically increasing  $T \in \mathcal{R}$  such that, for all  $n, x \in \mathbb{N}$ ,

$$L := C_e = \operatorname{range}(T);$$

$$\varphi_{T(n)}(x) = \begin{cases} \langle e, 0 \rangle, & \text{if } \forall n' \leq n \colon (h')^*(T[n']) \neq (h')^*(T[n'+1]) \lor \\ (h')^*(T[n'+1]) \neq (h')^*(T[n'+2]); \\ \langle \operatorname{ind}(\operatorname{content}(T[n+1])), n \rangle, & \text{otherwise.} \end{cases}$$

Note that we can find a C-index for  $\operatorname{range}(T)$  because T is strictly monotonically increasing. We now consider the following cases.

Case 1:  $\forall n \in \mathbb{N}: (h')^*(T[n]) \neq (h')^*(T[n+1]) \vee (h')^*(T[n+1]) \neq (h')^*(T[n+2])$ . On any element  $x \in L$ , h outputs pad(e, 0), which is a correct C-Index for L. Thus, once h sees the first non-pause symbol, it converges correctly and, thus,  $L \in \mathcal{L}$ . But h' makes infinitely many mind changes on text T and thus cannot learn  $\mathcal{L}$ .

Case 2:  $\exists n \in \mathbb{N}: (h')^*(T[n]) = (h')^*(T[n+1]) \wedge (h')^*(T[n+1]) = (h')^*(T[n+2])$ . Let  $n_0$  be the smallest such n. Then, h learns the finite languages  $\operatorname{content}(T[n_0+1])$  and  $\operatorname{content}(T[n_0+2])$  because  $\varphi_{T(n_0)}(0)$  and  $\varphi_{T(n_0+1)}(0)$  have the maximum second component in the respective set and, thus, h converges to  $\operatorname{pad}(\operatorname{ind}(\operatorname{content}(T[n_0+1])), n_0)$  and  $\operatorname{pad}(\operatorname{ind}(\operatorname{content}(T[n_0+2])), n_0+1)$ , respectively. But by the assumption of this case, h' converges to same hypothesis on the texts  $T[n_0] \cap T(n_0)^{\infty}$  and  $T[n_0+1] \cap T(n_0+1)^{\infty}$ , which are texts of different languages. Thus, h cannot learn  $\mathcal{L}$ .

## **Theorem 7** We have that $[\mathcal{R}\mathbf{Txt}\mathbf{TdCIndEx}_C]_{\mathbf{REC}} \subseteq [\mathbf{Txt}\mathbf{TdCIndEx}_C]_{\mathbf{REC}}$ .

**Proof** The inclusion  $[\mathcal{R}\mathbf{Txt}\mathbf{TdCIndEx}_C]_{\mathbf{REC}} \subseteq [\mathbf{Txt}\mathbf{TdCIndEx}_C]_{\mathbf{REC}}$  follows immediately. To prove that the inclusion is proper, we provide a separating class using the Operator Recursion Theorem (**ORT**). Let h be a  $\mathbf{Td}$ -learner with h(#) = ? and, for all  $x \in \mathbb{N}$ ,  $h(x) = \varphi_x(0)$ . Let  $\mathcal{L} = \mathbf{Txt}\mathbf{TdCIndEx}_C(h)$ . Now, assume there exists a learner h' with  $\mathcal{L} \subseteq \mathcal{R}\mathbf{Txt}\mathbf{TdCIndEx}_C(h')$ . Then, with **ORT** there exists  $a \in \mathcal{R}$  such that for all  $x, n \in \mathbb{N}$ 

$$\varphi_{a(n)}(x) = \begin{cases} \inf\{a(0), a(1)\}\}, & \text{if } h'(a(0)) \neq h'(a(1)); \\ \inf\{a(n)\}\}, & \text{otherwise.} \end{cases}$$

Intuitively, if h' suggests different hypotheses for a(0) and a(1) then both are in the same language and vice versa. We now show that in both cases, there is a language learned by h which cannot be learned by h'. We distinguish the following cases.

Case 1:  $h'(a(0)) \neq h'(a(1))$ . Then, we have  $\{a(0), a(1)\} \in \mathcal{L}$ , as h outputs a C-Index for this set on both elements of the set. But h' does not converge on the text  $(a(0)a(1))^{\infty}$  and thus cannot learn this set.

Case 2: h'(a(0)) = h'(a(1)). Then, by construction, we have  $\{a(0)\}, \{a(1)\} \in \mathcal{L}$ . But h' suggests the same hypothesis on  $a(0)^{\infty}$  and  $a(1)^{\infty}$  and thus can learn at most one of these two sets.

Next, we show the gradual decrease of learning power the more we require the learners to output characteristic indices. We have already seen in Theorem 1 that converging to C-indices

lessens learning power. However, this allows for more learning power than outputting these indices during the whole learning process as shows the next theorem. The idea is that such learners have to be certain about their guesses as these are indices of characteristic functions. When constructing a separating class using self-learning languages (Case and Kötzing, 2016), one forces the CIndlearner to output C-indices on certain languages to, then, contradict its choice there. This way, the  $\mathbf{Ex}_C$ -learner learns languages the CInd-learner cannot. The following theorem holds.

**Theorem 8** We have that 
$$[\mathbf{TxtItEx}_C]_{\mathbf{REC}} \setminus [\mathbf{TxtGCIndBc}_C]_{\mathbf{REC}} \neq \emptyset$$
.

**Proof** We prove this by contradiction by providing a class of languages in  $[\mathbf{TxtItEx}_C]_{\mathbf{REC}}$  which is not in  $[\mathbf{TxtGCIndBc}_C]_{\mathbf{REC}}$ . Let h be the following  $\mathbf{It}$ -learner. Let  $p_{\mathbb{N}}$  be an index for the set of all natural numbers. For any  $e, x \in \mathbb{N}$ , we define

$$h(\varepsilon) = \operatorname{ind}(\emptyset);$$

$$h(e, x) = \begin{cases} e, & \text{if } \pi_2(e) = 1 \land \pi_2(x) = 1 \land \pi_1(x) < \pi_1(e); \\ \langle \pi_1(x), 1 \rangle, & \text{else, if } \pi_2(e) \neq 1 \land \pi_2(x) = 1; \\ \langle \pi_1(x), 2 \rangle, & \text{else, if } \pi_2(e) = 0 \lor (\pi_2(x) = 2 \land \pi_1(x) < \pi_1(e)); \\ e, & \text{otherwise.} \end{cases}$$

Without loss of generality, we may assume that  $\operatorname{ind}(\emptyset) = \langle 0, 0 \rangle$ . This way, we can distinguish whether it was the previous hypothesis or not. Intuitively, while h only sees elements with second component two, it outputs the minimal  $\langle \pi_1(x), 2 \rangle$  it has seen. Once it sees an element with second component one, it outputs the coded tuple  $\langle \pi_1(x), 1 \rangle$ , which, if no other such elements are presented, is its final hypothesis. Otherwise, h outputs the minimal  $\langle \pi_1(x), 1 \rangle$ . Now, let  $\mathcal{L} = \mathbf{TxtItEx}_C(h) \cap \mathbf{REC}$  and assume there exists a learner h' which  $\mathbf{TxtGCIndBc}_C$ -learns  $\mathcal{L}$ , that is,  $\mathcal{L} \subseteq \mathbf{TxtGCIndBc}_C(h')$ . By the Operator Recursion Theorem (**ORT**), there exist total computable increasing functions  $a, \tilde{a} \in \mathcal{R}$  and indices  $e, p \in \mathbb{N}$  such that for all  $n, x \in \mathbb{N}$ 

$$\tilde{a}(x) = \langle a(x), 2 \rangle;$$

$$L_n := \operatorname{content}(\tilde{a}[n]) \cup \{\langle a(n), 1 \rangle\};$$

$$L := C_e = \operatorname{range}(\varphi_p);$$

$$T(x) := \varphi_p(x) = \begin{cases} \langle a(2x), 2 \rangle, & \text{if } \varphi_{h'(\varphi_p[x])}(\langle a(2x), 2 \rangle) = 0; \\ \langle a(2x+1), 2 \rangle, & \text{else, if } \varphi_{h'(\varphi_p[x])}(\langle a(2x), 2 \rangle) = 1; \\ \uparrow, & \text{otherwise.} \end{cases}$$

$$\varphi_{\langle a(n), 2 \rangle}(x) = \varphi_e(x) = \begin{cases} 1, & \text{if } \langle a(2x), 2 \rangle \in \operatorname{content}(T[x+1]); \\ 0, & \text{else, if } \langle a(2x+1), 2 \rangle \in \operatorname{content}(T[x+1]); \\ \uparrow, & \text{otherwise.} \end{cases}$$

$$\varphi_{\langle a(n), 1 \rangle}(x) = \varphi_{\operatorname{ind}(L_n)}(x) = \chi_{L_n}(x);$$

First, note that, for any  $n \in \mathbb{N}$ , h learns  $L_n$  as it eventually outputs  $\langle a(n), 1 \rangle$ , a C-index for  $L_n$ , and never changes its mind again. As h' learns these as well, it outputs a C-index on every initial sequence of elements in  $\mathrm{range}(\tilde{a})$ . Thus,  $\varphi_p$  is total and there exists a C-index e for its range. We now show, that h learns the decidable language L, while h' does not. As for any  $x \in L$  there exists

 $n \in \mathbb{N}$  such that we have  $x = \langle a(n), 2 \rangle$  and  $\varphi_x = \varphi_{\langle a(n), 2 \rangle} = \varphi_e$ , we have that h identifies L correctly once it sees the minimal such element in L. On the other hand, we show that h' cannot learn L from text T. Let  $x \in \mathbb{N}$  and consider the following cases.

- 1. Case:  $\varphi_{h'(\varphi_p[x])}(\langle a(2x), 2 \rangle) = 0$ . Thus,  $\langle a(2x), 2 \rangle$  is not in the hypothesis of h', but it is in L.
- 2. Case:  $\varphi_{h'(\varphi_p[x])}(\langle a(2x), 2 \rangle) = 1$ . Here,  $\langle a(2x), 2 \rangle$  is in the hypothesis of h', but, as a is strictly monotonically increasing, it is not in L.

Thus, none of the hypothesis h'(T[x]) identifies L correctly.

Since languages which can be learned by iterative learners can also be learned by set-driven ones (see Theorem 4), this result suffices. Note that the idea above requires some knowledge on previous elements. Thus, it is no coincidence that this separation does not include transductive learners. Since these learners base their guesses on single elements, they cannot see how far in the learning process they are. Thus, they are forced to always output C-indices. The following theorem holds.

**Theorem 9** We have that  $[\mathbf{Txt}\mathbf{Td}\mathbf{CInd}\mathbf{Ex}_C]_{\mathbf{REC}} = [\mathbf{Txt}\mathbf{Td}\mathbf{Ex}_C]_{\mathbf{REC}}$ 

**Proof** The inclusion  $[\mathbf{TxtTdCIndEx}_C]_{\mathbf{REC}} \subseteq [\mathbf{TxtTdEx}_C]_{\mathbf{REC}}$  is immediate. For the other, let h be a  $\mathbf{TxtTdEx}_C$ -learner and  $\mathcal{L} = \mathbf{TxtTdEx}_C(h) \cap \mathbf{REC}$ . We show that h is, in particular, a  $\mathbf{CInd}$ -learner, i.e.,  $\mathcal{L} = \mathbf{TxtTdCIndEx}_C(h)$  holds as well. Assume the contrary, that is,  $\mathcal{L} \neq \mathbf{TxtTdCIndEx}_C(h)$ . Then there exists a  $L \in \mathcal{L}$  and a  $x \in L$  such that h(x) is no C-index. Now, given any text  $T \in \mathbf{Txt}(L)$ , consider the text, for all  $n \in \mathbb{N}$ ,

$$T'(n) = \begin{cases} T(n), & \text{if } n \text{ is even,} \\ x, & \text{otherwise.} \end{cases}$$

This text of the language L contains infinitely many occurrences of x and, therefore, the  $\mathbf{Td}$ -learner h cannot converge to a C-index on this text.

For the remainder of this section, we focus on learners which output characteristic indices on arbitrary input, that is, we focus on  $\tau(\mathbf{CInd})$ -learners. First, we show that the requirement of always outputting C-indices lessens a learners learning power, even when compared to total  $\mathbf{CInd}$ -learners. To provide the separating class of self-learning languages, one again awaits the  $\tau(\mathbf{CInd})$ -learner's decision and then, based on these, learns languages this learner cannot. The following result holds.

 $\textbf{Theorem 10} \ \textit{We have that} \ [\mathcal{R}\mathbf{Txt}\mathbf{Td}\mathbf{CInd}\mathbf{Ex}_{C}]_{\mathbf{REC}} \setminus [\tau(\mathbf{CInd})\mathbf{Txt}\mathbf{GBc}_{C}]_{\mathbf{REC}} \neq \emptyset.$ 

**Proof** We prove the result by providing a separating class of languages. Let h be the **Td**-learner with h(#) = ? and, for all  $x, y \in \mathbb{N}$ , let  $h(\langle x, y \rangle) = x$ . By construction, h is total and computable. Let  $\mathcal{L} = \mathcal{R}\mathbf{Txt}\mathbf{TdCIndEx}_{C}(h) \cap \mathbf{REC}$ . We show that there is no  $\tau(\mathbf{CInd})\mathbf{Txt}\mathbf{GBc}_{C}$ -learner learning  $\mathcal{L}$  by way of contradiction. Assume there is a  $\tau(\mathbf{CInd})\mathbf{Txt}\mathbf{GBc}_{C}$ -learner h' which learns  $\mathcal{L}$ . With the Operator Recursion Theorem (**ORT**), there are  $e, p \in \mathbb{N}$  such that for all  $x \in \mathbb{N}$ 

$$L := \operatorname{range}(\varphi_n);$$

$$\varphi_e = \chi_L;$$

$$\tilde{T}(x) := \varphi_p(x) = \begin{cases} \langle e, 2x \rangle, & \text{if } \varphi_{h'(\varphi_p[x])}(\langle e, 2x \rangle) = 0; \\ \langle e, 2x + 1 \rangle, & \text{otherwise.} \end{cases}$$

Intuitively, for all x either  $\varphi_p(x)$  is an element of L if it is not in the hypothesis of h' after seeing  $\varphi_p[x]$ , or there is an element in this hypothesis that is not in  $\operatorname{content}(\tilde{T})$ . As any hypothesis of h' is a C-index, we have that  $\varphi_p \in \mathcal{R}$  and, as  $\varphi_p$  is strictly monotonically increasing, that L is decidable.

We now prove that  $L \in \mathcal{L}$  and afterwards that L cannot be learned by h'. First, we need to prove that h learns L. Let  $T \in \mathbf{Txt}(L)$ . For all  $n \in \mathbb{N}$  where T(n) is not the pause symbol, we have h(T(n)) = e. Let  $n_0 \in \mathbb{N}$  with  $T(n_0) \neq \#$ . Then, we have, for all  $n \geq n_0$ , that  $\mathbf{Td}(h,T)(n) = e$  and, since e is a hypothesis of L, h learns L from T. Thus, we have that  $L \in \mathcal{R}\mathbf{Txt}\mathbf{TdCIndEx}_C(h) \cap \mathbf{REC}$ .

By assumption, h' learns  $\mathcal{L}$  and thus it also needs to learn L on text  $\tilde{T}$ . Hence, there is  $x_0$  such that for all  $x \geq x_0$  the hypothesis  $h'(\tilde{T}[x]) = h'(\varphi_p[x])$  is a C-index for L. We now consider the following cases.

- 1. Case:  $\varphi_{h'(\varphi_p[x])}(\langle e, 2x \rangle) = 0$ . By construction, we have that  $\tilde{T}(x) = \langle e, 2x \rangle$ . Therefore,  $\langle e, 2x \rangle \in L$ , which contradicts  $h'(\varphi_p[x])$  being a correct hypothesis.
- 2. Case:  $\varphi_{h'(\varphi_p[x])}(\langle e, 2x \rangle) = 1$ . By construction, we have that  $\tilde{T}(x) \neq \langle e, 2x \rangle$  and thus, because  $\tilde{T}$  is strictly monotonically increasing,  $\langle e, 2x \rangle \notin L = \operatorname{content}(\tilde{T})$ . This, again, contradicts  $h'(\varphi_p[x])$  being a correct hypothesis.

As in all cases  $h'(\varphi_p[x])$  is a wrong hypothesis, h' cannot learn  $\mathcal{L}$ .

It remains to be shown that memory restrictions are severe for such learners as well. First, we show that partially set-driven learners are more powerful than set-driven ones. As witnessed originally by Schäfer-Richter (1984) and Fulk (1990) (for W-indices), this is solely due to the lack of learning time. We provide the following theorem. We already separate from behaviorally correct learners, as we will need this stronger version later on.

**Theorem 11** We have that  $[\tau(\mathbf{CInd})\mathbf{TxtPsdEx}_C]_{\mathbf{REC}} \setminus [\mathbf{TxtSdBc}_W]_{\mathbf{REC}} \neq \emptyset$ .

**Proof** We prove the theorem by providing a separating class  $\mathcal{L}$ . For all  $e \in \mathbb{N}$ , we define

$$L_{e} = \{ \langle e, x \rangle \mid x \in \mathbb{N} \};$$

$$L'_{e} = \{ \langle e, x \rangle \mid \varphi_{e}(0) \downarrow \land x \leq \varphi_{e}(0) \};$$

$$\mathcal{L} = \bigcup_{e \in \mathbb{N}} (\{ L_{e} \mid \varphi_{e}(0) \uparrow \} \cup \{ L'_{e} \mid \varphi_{e}(0) \downarrow \}).$$

Note that  $\mathcal{L} \subseteq \mathbf{REC}$ . First, we provide a learner h such that  $\mathcal{L} \subseteq \tau(\mathbf{CInd})\mathbf{TxtPsdEx}_C(h) \cap \mathbf{REC}$ . To define h, we need the following auxiliary functions. Due to the S-m-n Theorem there exist  $f, p, p' \in \mathcal{R}$  such that for all finite sets D and all  $e, x \in \mathbb{N}$ 

$$f(D) = \begin{cases} \pi_1(\min(D)), & \text{if } D \neq \emptyset; \\ 0, & \text{otherwise;} \end{cases}$$

$$\varphi_{p(e)} = \chi_{L_e};$$
  
$$\varphi_{p'(e,x)} = \chi_{\{\langle e,y\rangle \mid y \leq x\}}.$$

Intuitively, we use f to recover the first component of the minimal given element. With p and p' we can generate C-Indices for  $L_e$  and  $L'_e$ , respectively. Now, we define the learner h as, for all finite sets D and all  $t \in \mathbb{N}$ ,

$$h(D,t) = \begin{cases} \operatorname{ind}(\emptyset), & \text{if } D = \emptyset; \\ p(f(D)), & \text{else, if } \Phi_{f(D)}(0) > t; \\ p'(f(D), \varphi_{f(D)}(0)), & \text{otherwise.} \end{cases}$$

Intuitively, given elements of the form  $\langle e, x \rangle$ , h suggests  $L_e$  until it witnesses  $\varphi_e(0) \downarrow$ , whereupon it suggests  $L'_e$ . Note that h is a  $\tau(\mathbf{CInd})$ -learner by construction.

To show that  $\mathcal{L} \subseteq \tau(\mathbf{CInd})\mathbf{TxtPsdEx}_C(h)$ , let  $e \in \mathbb{N}$ . If  $\varphi_e(0)\uparrow$ , h needs to learn  $L_e$ . After seeing the first non-pause symbol, h constantly outputs p(e), which is a correct index for  $L_e$ . If, otherwise,  $\varphi_e(0)\downarrow$ , h needs to learn  $L'_e$ . Let  $T \in \mathbf{Txt}(L'_e)$  and  $n_0 \in \mathbb{N}$  big enough such that  $T[n_0] \neq \emptyset$  and  $n_0 \geq \Phi_e(0)$ . Then for all  $n \geq n_0$  we have  $h(T[n], n) = p'(e, \varphi_e(0))$  and thus h learns  $L'_e$  as well.

It remains to be shown that there is no learner h' such that  $\mathcal{L} \subseteq \mathbf{TxtSdBc}_W(h')$ . Assume the opposite, i.e., let h' be a learner with  $\mathcal{L} \subseteq \mathbf{TxtSdBc}_W(h')$ . By Kleenes Recursion Theorem there exists an index  $e \in \mathbb{N}$  such that, for all  $x \in \mathbb{N}$ ,

$$\varphi_e(x) = \begin{cases} m, & \text{if } \exists m \colon \langle e, m+1 \rangle \in C_{h'(\{\langle e, x \rangle \mid x \leq m\})}; \\ \uparrow, & \text{otherwise.} \end{cases}$$

If ever, we take the first such m found. We differentiate whether  $\varphi_e(0)\downarrow$  or not.

Case 1:  $\varphi_e(0)\downarrow$ . Then h' has to learn  $L'_e$ . Let  $m=\varphi_e(0)$ . By definition of e we have  $\langle e,m+1\rangle\in C_{h'(L')}$ . As  $\langle e,m+1\rangle\notin L'_e$ , this contradicts h' learning  $L'_e$ .

Case 2:  $\varphi_e(0)\uparrow$ . Then h' has to learn  $L_e$ . Let  $T \in \mathbf{Txt}(L_e)$  be the text with, for all  $i \in \mathbb{N}$ ,  $T(i) = \langle e, i \rangle$ . By definition of e we have, for all  $m \in \mathbb{N}$ ,

$$\langle e, m+1 \rangle \notin C_{h'(\{\langle e, x \rangle \mid x \le m\})} = C_{h'(\text{content}(T[m+1]))}.$$

Therefore, h' cannot converge to a correct hypothesis for  $L_e$  on T and, thus, not learn it.

In turn, this lack of time is not as severe as lack of memory. The standard class (of recursive languages) to separate set-driven learners from iterative ones (Jain et al., 1999) can be transferred to the setting studied in this paper. We obtain the following result.

**Theorem 12** We have that  $[\tau(\mathbf{CInd})\mathbf{Txt}\mathbf{SdEx}_C]_{\mathbf{REC}} \setminus [\mathbf{TxtItEx}_W]_{\mathbf{REC}} \neq \emptyset$ .

**Proof** This is a standard proof and we include it for completeness (Jain et al., 1999). We show this theorem by stating a class of languages that can be learned by a  $\tau(\mathbf{CInd})\mathbf{Txt}\mathbf{SdEx}_C$ -learner, but

any  $\mathbf{TxtItEx}_W$ -learner fails to do so. To that end, let  $\mathcal{L} = \{D \cup \{0\} \mid D \subseteq_{\mathbf{Fin}} \mathbb{N}\} \cup \{\mathbb{N}^+\}$ . We define the  $\mathbf{Sd}$ -learner h for all finite sets D, with p being a C-Index for  $\mathbb{N}^+$ , as

$$h(D) = \begin{cases} \operatorname{ind}(D), & \text{if } 0 \in D; \\ p, & \text{otherwise.} \end{cases}$$

It is easy to verify that  $\mathcal{L} \subseteq \tau(\mathbf{CInd})\mathbf{TxtSdEx}_C(h)$ . Now, assume there is a  $\mathbf{TxtItEx}_W$ -learner h' that learns  $\mathcal{L}$  and let  $\sigma$  be a locking sequence of h' on  $\mathbb{N}^+$  with  $x = \max(\mathrm{content}(\sigma))$ . The texts  $\sigma^{\frown}(x+1)^{\frown}0^{\infty}$  and  $\sigma^{\frown}(x+2)^{\frown}0^{\infty}$  are texts for distinct languages from  $\mathcal{L}$  but h' suggests exactly the same hypotheses on both texts and can therefore not be  $\mathbf{Ex}_W$ -successful on both languages.

Lastly, we show that transductive learners, having basically no memory, do severely lack learning power. As they have to infer their conjectures from single elements they, in fact, cannot even learn basic classes such as  $\{\{0\}, \{1\}, \{0, 1\}\}\}$ . The following result holds. It concludes the map shown in Figure 1 and, therefore, also this section.

**Theorem 13** *For*  $\beta \in \{\text{It}, \text{Sd}\}$ *, we have that* 

$$[\tau(\mathbf{CInd})\mathbf{Txt}\beta\mathbf{Ex}_C]_{\mathbf{REC}}\setminus[\mathbf{Txt}\mathbf{Td}\mathbf{Ex}_W]_{\mathbf{REC}}\neq\emptyset.$$

**Proof** We include this standard proof for completeness. We follow Carlucci et al. (2007) and show that  $\mathcal{L} = \{\{0\}, \{1\}, \{0, 1\}\}$  is a separating class. Immediate, we have that  $\mathcal{L}$  can be learned by a  $\tau(\mathbf{CInd})\mathbf{Txt}\beta\mathbf{Ex}_C$ -learner and, thus,  $\mathcal{L} \in [\tau(\mathbf{CInd})\mathbf{Txt}\beta\mathbf{Ex}_C]_{\mathbf{REC}}$ . Now, assume there exists a learner h'  $\mathbf{Txt}\mathbf{TdEx}_W$ -learning  $\mathcal{L}$ . Consider the texts  $T_0 = 0^\infty \in \mathbf{Txt}(\{0\})$  and  $T_1 = 1^\infty \in \mathbf{Txt}(\{1\})$ . As h' must identify both languages on their respective text, we have that, for  $x \in \{0, 1\}$ , h'(x) must be a C-index for  $\{x\}$ . However, then h' cannot output a C-index of  $\{0, 1\}$  on the text  $T = 0^{-1}$ , a contradiction.

### 4. Syntactic versus Semantic Convergence to C-indices

In this section we investigate the effects on learners when we require them to converge to characteristic indices. We study both syntactically converging learners as well as semantically converging ones. In particular, we compare learners imposed with different well-studied memory restrictions.

Surprisingly, we observe that, although C-indices incorporate and, thus, require the learner to obtain more information during the learning process than W-indices, the relative relations of the considered restrictions remain the same. We start by gathering results which directly follow from the previous section. In particular, the following corollary holds.

### Corollary 14 We have that

$$\begin{split} [\mathbf{TxtPsdEx}_C]_{\mathbf{REC}} &= [\mathbf{TxtGEx}_C]_{\mathbf{REC}}, (\textit{Theorem 3}), \\ [\mathbf{TxtItEx}_C]_{\mathbf{REC}} &\subseteq [\mathbf{TxtSdEx}_C]_{\mathbf{REC}}, (\textit{Theorem 4}), \\ [\mathbf{TxtGEx}_C]_{\mathbf{REC}} \setminus [\mathbf{TxtSdBc}_C]_{\mathbf{REC}} &\neq \emptyset, (\textit{Theorem 11}), \\ [\mathbf{TxtSdEx}_C]_{\mathbf{REC}} \setminus [\mathbf{TxtItEx}_C]_{\mathbf{REC}} &\neq \emptyset, (\textit{Theorem 12}), \\ [\mathbf{TxtItEx}_C]_{\mathbf{REC}} \setminus [\mathbf{TxtTdEx}_C]_{\mathbf{REC}} &\neq \emptyset, (\textit{Theorem 13}). \end{split}$$

We show the remaining results. First, we show that, just as for W-indices, behaviorally correct learners are more powerful than explanatory ones. We provide a separating class exploiting that explanatory learners must converge to a single, correct hypothesis. We collect elements on which mind changes are witnessed, while maintaining decidability of the obtained language. The following result holds.

**Theorem 15** We have that 
$$[\mathbf{TxtSdBc}_C]_{\mathbf{REC}} \setminus [\mathbf{TxtGEx}_C]_{\mathbf{REC}} \neq \emptyset$$
.

**Proof** In order to provide a separating class of languages, we consider the learner, for all finite  $D \subseteq \mathbb{N}$ ,  $h(D) = \max(D)$ . Let  $\mathcal{L} = \mathbf{TxtSdBc}_C(h) \cap \mathbf{REC}$ . We show that there exists no learner h' that  $\mathbf{TxtGEx}_C$ -learns  $\mathcal{L}$ . To that end, assume there exists such a learner h', that is,  $\mathcal{L} \subseteq \mathbf{TxtGEx}_C(h')$ . Without loss of generality, we may assume h' to be total, as is shown in Kötzing and Palenta (2016).

Using the Operator Recursion Theorem (**ORT**), there exist an *interleaved increasing*  $^2$  function  $a \in \mathcal{R}$ , a sequence of sequences  $(\sigma_j)_{j \in \mathbb{N}}$  and functions  $f, i_0, s \in \mathcal{P}$  such that, for all  $i, j, k, t, x \in \mathbb{N}$  and  $b \in \{0, 1\}$ , we have

$$P_{j}(t) \Leftrightarrow h'(\sigma_{j} \cap a(0, |\sigma_{j}|)^{t}) \neq h'(\sigma_{j}) \vee h'(\sigma_{j} \cap a(1, |\sigma_{j}|)^{t}) \neq h'(\sigma_{j});$$

$$s(j) = \mu t. P_{j}(t);$$

$$\sigma_{0} = \varepsilon;$$

$$\sigma_{j+1} = \begin{cases} \uparrow, & \text{if } s(j) \uparrow; \\ \sigma_{j} \cap a(0, |\sigma_{j}|)^{s(j)}, & \text{else, if } h'(\sigma_{j} \cap a(0, |\sigma_{j}|)^{s(j)}) \neq h'(\sigma_{j}); \\ \sigma_{j} \cap a(1, |\sigma_{j}|)^{s(j)}, & \text{otherwise.} \end{cases}$$

$$\varphi_{a(b,i)}(x) = \begin{cases} 1, & \text{if } x = a(b,i); \\ 0, & \text{else, if } x = a(1-b,i); \\ f(b',k'), & \text{else, if } \exists k' \in \mathbb{N} \exists b' \in \{0,1\}: x = a(b',k'); \\ 0, & \text{otherwise.} \end{cases}$$

$$i_{0}(k) = \max\{j \mid \sigma_{j} \downarrow \wedge |\sigma_{j}| \leq k\};$$

$$f(b,k) = \begin{cases} 0, & \text{if } k > |\sigma_{i_{0}(k)}|; \\ 1, & \text{else, if } s(i_{0}(k)) \downarrow \wedge a(b,k) \in \text{content}(\sigma_{i_{0}(b,k)+1}); \\ 0, & \text{else, if } s(i_{0}(k)) \downarrow; \\ \uparrow, & \text{otherwise.} \end{cases}$$

Note that b' and k' in the third case of  $\varphi_{a(b,i)}(x)$  are, if they exist, unique as a is interleaved increasing. The intuition is the following. For  $j \in \mathbb{N}$ , we extend the sequence  $\sigma_j$  as soon as h' makes a particular mind change, if ever. This guarantees that h' cannot learn certain languages h can. Furthermore, for suitable  $b, i \in \mathbb{N}$ , every element a(b,i) of the sequence encodes the language  $\bigcup_{j \in \mathbb{N}, \sigma_j \downarrow} \operatorname{content}(\sigma_j)$  (as C-index). This encoding is done using function f which, given the right circumstances, can decide whether an element belongs to the mentioned language or not. We first provide a proof for this claim.

<sup>2.</sup> A function a is called *interleaved increasing* if, for all n, we have a(0,n) < a(1,n) < a(0,n+1).

**Claim 1** Let  $b \in \{0,1\}$  and  $j_0$  such that  $\sigma_{j_0}$  is defined. Let  $k = |\sigma_{j_0}|$ . Then, if  $\sigma_{j_0+1}$  is defined, f(b,k) correctly decides whether  $a(b,k) \in \bigcup_{j \in \mathbb{N}, \sigma_j} \text{content}(\sigma_j)$ .

**Proof** Let  $\sigma_{j_0+1}$  be defined. Then,  $j_0=i_0(k)$  and  $s(j_0)\!\downarrow$  and we have that, by definition, f correctly decides whether  $a(b,k)\in\bigcup_{j\in\mathbb{N},\sigma_j\downarrow}\operatorname{content}(\sigma_j)$ . (Claim)

We show that there exists a language h can learn, but h' cannot. To that end, we distinguish the following cases.

- 1. Case: For all  $j \in \mathbb{N}$ ,  $\sigma_j$  is defined. Let  $\tilde{T} = \bigcup_{j \in \mathbb{N}} \sigma_j$  and let  $L = \operatorname{content}(\tilde{T})$ . We first show that h learns L. Let  $T \in \mathbf{Txt}(L)$  and let  $n_0$  be minimal such that  $\operatorname{content}(T[n_0]) \neq \emptyset$ . Let  $n \geq n_0$  and  $D \coloneqq \operatorname{content}(T[n])$ . Furthermore, let  $b \in \{0,1\}$  and  $i \in \mathbb{N}$  be such that  $a(b,i) = \max(D) = h(D)$ . We show that  $C_{a(b,i)} = L$ .
  - $\supseteq$ : To show  $C_{a(b,i)}\supseteq L$ , let  $x\in L$ . We show that  $\varphi_{a(b,i)}(x)=1$ . As  $x\in L$ , there exists  $k'\in\mathbb{N}$  and  $b'\in\{0,1\}$  such that x=a(b',k'). If b=b' and i=k', then  $\varphi_{a(b,i)}(x)=1$  by definition. Otherwise, as all  $\sigma_j$  are defined, by Claim 1, we have f(b',k')=1, which is exactly the output of  $\varphi_{a(b,i)}(x)$ .
  - $\subseteq$ : To show  $C_{a(b,i)}\subseteq L$ , let  $x\notin L$ . Now, either there exist no  $k'\in\mathbb{N}$  and  $b'\in\{0,1\}$  such that x=a(b',k'). Then,  $\varphi_{a(b,i)}(x)=0$  by definition. Else, let k' and b' such that x=a(b',k'). If b'=1-b and k'=i, then  $\varphi_{a(b,i)}(x)=0$  by definition. Otherwise, again as all  $\sigma_j$  are defined, by Claim 1, f(b',k')=0, which is exactly the output of  $\varphi_{a(b,i)}(x)$ .

Thus,  $C_{a(b,i)} = L$ . So, h learns L. On the other hand, h' does not, as it makes infinitely many mind changes on text  $\tilde{T}$ .

2. Case: There exists j such that  $\sigma_j$  is defined, but  $\sigma_{j+1}$  is not. Let j' be minimal such. Let  $m := |\sigma_{j'}|$  and consider the texts

$$T_0 = \sigma_{j'} \widehat{a}(0, m)^{\infty},$$
  

$$T_1 = \sigma_{j'} \widehat{a}(1, m)^{\infty},$$

as well as the languages  $L_0=\operatorname{content}(T_0)$  and  $L_1=\operatorname{content}(T_1)$ . We show that h can learn both  $L_0$  and  $L_1$ , while h' cannot. To show that h learns  $L_0$ , let  $T\in \mathbf{Txt}(L_0)$ . As  $L_0$  is finite, there exists  $n_0$  such that  $\operatorname{content}(T[n_0])=L_0$ . Then, for all  $n\geq n_0$ , we have  $h(\operatorname{content}(T[n]))=\max(\operatorname{content}(T[n]))=a(0,m)$  as a is interleaved increasing. We show that  $C_{a(0,m)}=L_0$ .

- $\supseteq$ : To show  $C_{a(0,m)}\supseteq L_0$ , let  $x\in L_0$ . If x=a(0,m), then  $x\in C_{a(0,m)}$  by definition of  $\varphi_{a(0,m)}(x)$ . Otherwise, there exist k'< m and  $b'\in\{0,1\}$  such that x=a(b',k'). Note that  $i_0(k')< j'$ . Thus we can apply Claim 1 and get f(b',k')=1 which is exactly the output of  $\varphi_{a(0,m)}(x)$ .
- $\subseteq$ : To show  $C_{a(0,m)}\subseteq L_0$ , let  $x\notin L_0$ . Now, either there exist no  $k'\in\mathbb{N}$  and  $b'\in\{0,1\}$  such that x=a(b',k'). Then,  $\varphi_{a(0,m)}(x)=0$  by definition. Else, let k' and b' such that x=a(b',k'). We distinguish the following cases to show that  $x\notin C_{a(0,m)}$ .

- If k' = m, then  $\varphi_{a(b,i)}(x) = 0$  by definition.
- In the case of k' > m, we have  $k' > |\sigma_{i_0(k')}|$  and thus f(b', k') = 0.
- Given the case k' < m, again by Claim 1, f(b', k') = 0.

Thus,  $C_{a(0,m)} = L_0$  as desired. The reasoning for  $L_1$  is analogous.

So, h learns both  $L_0$  and  $L_1$ . However, h' converges to the same hypothesis on both  $T_0$  and  $T_1$  rendering it incapable to learn both languages simultaneously.

Next, we show that, just as for W-indices, a padding argument makes iterative behaviorally correct learners as powerful as Gold-style ones.

**Theorem 16** We have that  $[\mathbf{TxtItBc}_C]_{\mathbf{REC}} = [\mathbf{TxtGBc}_C]_{\mathbf{REC}}$ .

**Proof** The inclusion  $[\mathbf{TxtItBc}_C]_{\mathbf{REC}} \subseteq [\mathbf{TxtGBc}_C]_{\mathbf{REC}}$  follows immediately. For the other, we apply a padding argument as in the proof of  $[\mathbf{TxtItBc}_W] = [\mathbf{TxtGBc}_W]$  as given in Kötzing et al. (2017). Let  $h \in \mathcal{P}$  be a learner and let  $\mathcal{L} = \mathbf{TxtGBc}_C(h) \cap \mathbf{REC}$ . Recall that  $\mathrm{pad} \in \mathcal{R}$  is a padding function, that is, for all  $e \in \mathbb{N}$  and all finite sequences  $\sigma$  we have  $\varphi_e = \varphi_{\mathrm{pad}(e,\sigma)}$ . For any finite sequence  $\sigma$ , we define the iterative learner  $(h')^*(\sigma) = \mathrm{pad}(h(\sigma), \sigma)$ . Intuitively, the learner h' simulates h in the following way. At every iteration, given a datum x and its previous guess  $\mathrm{pad}(h(\sigma), \sigma)$ , the learner unpads  $\sigma$ , attaches x to it and makes the guess  $\mathrm{pad}(h(\sigma \cap x), \sigma \cap x)$ . While this is syntactically different hypothesis, it has the same semantics as  $h(\sigma \cap x)$ .

We show that h'  $\mathbf{TxtItBc}_C$ -learns  $\mathcal{L}$ . Let  $L \in \mathcal{L}$  and  $T \in \mathbf{Txt}(L)$ . Then, for every  $n \in \mathbb{N}$ , we have  $C_{h(T[n])} = C_{(h')^*(T[n])}$ . Thus, h' learns L as h does.

We show that the classes of languages learnable by some behaviorally correct Gold-style (or, equivalently, iterative) learner, can also be learned by partially set-driven ones. We follow the proof of Doskoč and Kötzing (2020) after a private communication with Sanjay Jain. The idea there is to search for minimal  $\mathbf{Bc}$ -locking sequences without directly mimicking the  $\mathbf{G}$ -learner. We transfer this idea to hold when converging to C-indices as well. We remark that, while doing the necessary enumerations, one needs to make sure these are characteristic. One obtains this as the original learner eventually outputs characteristic indices.

**Theorem 17** We have that  $[\mathbf{TxtPsdBc}_C]_{\mathbf{REC}} = [\mathbf{TxtGBc}_C]_{\mathbf{REC}}$ .

**Proof** The inclusion  $[\mathbf{TxtPsdBc}_C]_{\mathbf{REC}} \subseteq [\mathbf{TxtGBc}_C]_{\mathbf{REC}}$  follows immediately. For the other, we follow an idea how  $\mathbf{TxtGBc}$ -learning can be made partially set-driven, as given in  $\mathbf{Dosko\check{c}}$  and Kötzing (2020) following a private communication with Sanjay Jain. To that end, let h be a learner and let  $\mathcal{L} = \mathbf{TxtGBc}_C(h) \cap \mathbf{REC}$ . By Kötzing and Palenta (2016), we may assume h to be total. Now, define the  $\mathbf{Psd}$ -learner h' as follows. For  $x, a \in \mathbb{N}$  and for finite  $D \subseteq \mathbb{N}$  and  $t \geq 0$ , we first define the auxiliary total predicate Q(x, a, (D, t)) which holds true if and only if there exists a sequence  $\sigma \in D_\#^{\leq t}$  such that both

- (1) for all  $au \in D_\#^{\leq t}$  we have that  $\varphi_{h(\sigma au)}(x) = a$ , and
- $(2) \ \text{ for all } \sigma'<\sigma \text{, with } \sigma'\in D_\#^*, \text{ there exists } \tau'\in D_\#^{\leq t} \text{ such that } \varphi_{h(\sigma'\tau')}(x)=a.$

With the help of Q we define the learner h' such that, for finite  $D \subseteq \mathbb{N}$ ,  $t \ge 0$  and for all  $x \in \mathbb{N}$ ,

$$\varphi_{h'(D,t)}(x) = \begin{cases} 1, & \text{if } Q(x,1,(D,t)); \\ 0, & \text{otherwise.} \end{cases}$$

Intuitively, we check whether the information given is enough to witness a (minimal)  $\mathbf{Bc}_C$ -locking sequence. Then, for every element, we evaluate whether it belongs to the language or not. Note that upon correct learning, no element can be witnessed to be both part of the language and not part of it

We first show that h'(D,t) is well defined. Assume there exists some  $x \in \mathbb{N}$  and some natural number  $a \neq 1$  such that Q(x,a,(D,t)) and Q(x,1,(D,t)) simultaneously, witnessed by  $\sigma_a$  and  $\sigma_1$  respectively. Without loss of generality, suppose  $\sigma_a < \sigma_1$ . Then, by Condition (2) of  $\sigma_1$ , there exists some  $\tau' \in D_\#^{\leq t}$  such that  $\varphi_{h(\sigma_a\tau')}(x) = 1$ . However, by Condition (1) of  $\sigma_a$ , for all  $\tau \in D_\#^{\leq t}$ , we have  $\varphi_{h(\sigma_a\tau)}(x) = a$ , a contradiction.

Let  $L \in \mathcal{L}$ . We proceed by proving  $L \in \mathbf{TxtPsdBc}_C(h')$ . For that, let  $T \in \mathbf{Txt}(L)$ . By Blum and Blum (1975), there exists a  $\mathbf{Bc}_C$ -locking sequence for h on L. Let  $\alpha$  be the least such  $\mathbf{Bc}_C$ -locking sequence with respect to <. By Osherson et al. (1986), for each  $\alpha' < \alpha$  such that content $(\alpha') \subseteq L$ , there exists  $\tau_{\alpha'}$  such that  $\alpha'\tau_{\alpha'}$  is a  $\mathbf{Bc}_C$ -locking sequence for h on L. Now, let  $n_0 \in \mathbb{N}$  be large enough such that

- $n_0 \geq |\alpha|$ ,
- content( $\alpha$ )  $\subseteq$  content( $T[n_0]$ ) and
- for all  $\alpha' < \alpha$  such that  $\operatorname{content}(\alpha') \subseteq L$ , we have  $\operatorname{content}(\alpha'\tau_{\alpha'}) \subseteq \operatorname{content}(T[n_0])$  and  $|\tau_{\alpha'}| \leq n_0$ .

We claim that for  $t \geq n_0$  and  $D = \operatorname{content}(T[t])$ , we have  $C_{h'(D,t)} = L$ . Let  $x \in \mathbb{N}$  and  $a \in \mathbb{N}$  such that  $\chi_L(x) = a$ . As D and t are chosen sufficiently large,  $\alpha$  is a candidate for the enumeration of  $C_{h'(D,t)}$ . Since  $\alpha$  is a  $\operatorname{\mathbf{Bc}}_C$ -locking sequence, for every  $\tau \in D_\#^{\leq t}$ , we will witness  $\varphi_{h(\alpha\tau)}(x) = a$ . Thus, Condition (1) is witnessed. On the other hand, observe that for every  $\sigma' < \alpha$ , with  $\operatorname{content}(\sigma') \subseteq D$ , we have  $\tau_{\sigma'} \in D_\#^{\leq t}$ . So, we will witness  $\varphi_{h(\sigma'\tau_{\sigma'})}(x) = a$  for some  $\tau_{\sigma'} \in D_\#^{\leq t}$ , that is, the Condition (2).

As an element cannot be witnessed to be part of the language and not part of it simultaneously, we finally have  $\chi_L = \varphi_{h'(D,t)}$ , concluding the proof, as  $L \in \mathbf{TxtPsdBc}_C(h')$ .

Lastly, we investigate transductive learners. Such learners base their hypotheses on a single element. Thus, one would expect them to benefit from dropping the requirement to converge to a *single* hypothesis. Interestingly, this does not hold true. This surprising fact originates from *C*-indices encoding characteristic functions. Thus, one can simply search for the minimal element on which no "?" is conjectured. The next result finalizes the map shown in Figure 2 and, thus, this section.

**Theorem 18** We have that  $[\mathbf{Txt}\mathbf{TdEx}_C]_{\mathbf{REC}} = [\mathbf{Txt}\mathbf{TdBc}_C]_{\mathbf{REC}}$ .

**Proof** The inclusion  $[\mathbf{TxtTdEx}_C]_{\mathbf{REC}} \subseteq [\mathbf{TxtTdBc}_C]_{\mathbf{REC}}$  is immediate. For the other direction, let h be a learner and  $\mathcal{L} = \mathbf{TxtTdBc}_C(h) \cap \mathbf{REC}$ . We provide a learner h' such that

 $\mathcal{L} \subseteq \mathbf{TxtTdEx}_C(h')$ . Let  $L \in \mathcal{L}$ . We note that, for any  $x \in L$ , if not h(x) = ?, then h(x) is a C-index for the language L, that is,  $C_{h(x)} = L$ . Assume there exists an  $x \in L$  where  $h(x) \neq ?$  is no C-index of L. Then, on a text with infinitely many occurrences of x the language L cannot be  $\mathbf{TxtTdEx}_C$ -learned using h. Now, we define the  $\mathbf{TxtTdEx}_C$ -learner h' for all  $x \in \mathbb{N}$  as

$$h'(x) = \begin{cases} ?, & \text{if } h(x) = ?; \\ h(\min\{x \in C_{h(x)} \mid h(x) \neq ?\}), & \text{otherwise.} \end{cases}$$

It is straightforward to verify the correctness of h'.

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