

KL-randomness and effective dimension under strong reducibility

Bjørn Kjos-Hanssen¹[0000–0002–1825–0097]* and David J. Webb¹[0000–0002–5031–7669]

University of Hawai'i at Mānoa, Honolulu HI 96822, U.S.A.,
 bjoern.kjos-hanssen@hawaii.edu, dwebb42@hawaii.edu,
 WWW home page: <http://math.hawaii.edu/wordpress/bjoern/>

Abstract. We show that the (truth-table) Medvedev degree KLR of Kolmogorov–Loveland randomness coincides with that of Martin–Löf randomness, MLR, answering a question of Miyabe. Next, an analogue of complex packing dimension is studied which gives rise to a set of weak truth-table Medvedev degrees isomorphic to the Turing degrees.

Keywords: algorithmic randomness, effective dimension, Medvedev reducibility

1 Introduction

Computability theory is concerned with the relative computability of reals, and of collections of reals. The latter can be compared by various means, including Medvedev and Muchnik reducibility. Among the central collections considered are those of completions of Peano Arithmetic, Turing complete reals, Cohen generic reals, random reals, and various weakenings of randomness such as reals of positive effective Hausdorff dimension.

Perhaps the most famous open problem in algorithmic randomness [2,9] is whether Kolmogorov–Loveland randomness is equal to Martin–Löf randomness. Here we show that at least they are Medvedev equivalent.

Randomness extraction in computability theory concerns whether reals that are close (in some metric) to randoms can compute random reals. A recent example is [4]. That paper does for Hausdorff dimension what was done for a notion intermediate between packing dimension and Hausdorff dimension in [3]. That intermediate notion, *complex packing dimension*, has a natural dual which we introduce in this article. Whereas our result on KL-randomness is positive, we establish some negative (non-reduction) results for our new *inescapable dimension* and for relativized complex packing dimension (in particular Theorem 14). These results are summarized in Figure 1.

Let CR, SR, KLR, and MLR be the classes of computably random, Schnorr random, Kolmogorov–Loveland random, and Martin–Löf random reals, respectively. For basic definitions from algorithmic randomness, the reader may consult

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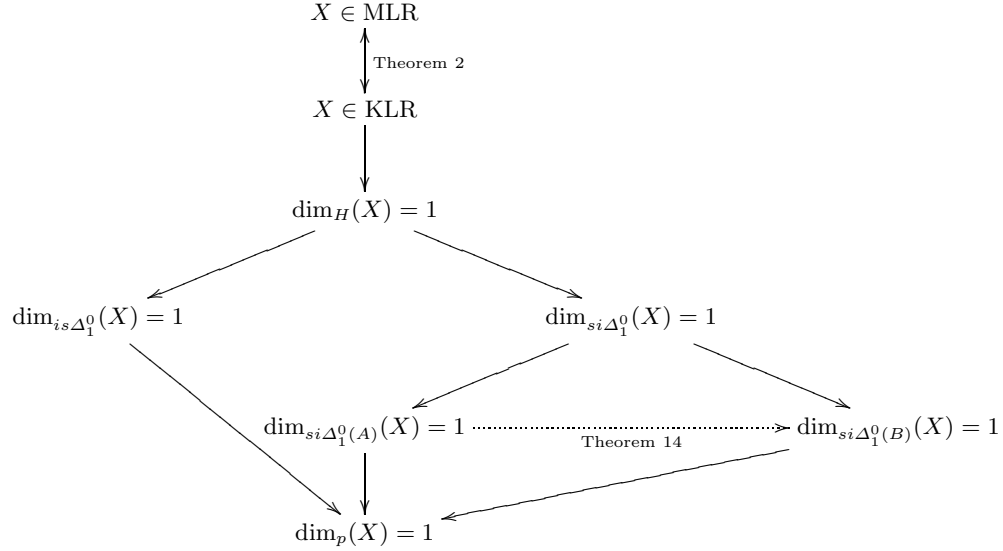


Fig. 1. Truth-table Medvedev degrees of mass problems associated with randomness and dimension. Here $\mathcal{C} \rightarrow \mathcal{D}$ means $\mathcal{C} \geq_s \mathcal{D}$, dotted arrow means $\mathcal{C} \not\geq_s \mathcal{D}$ and we assume $A \not\leq_T B$.

two recent monographs [2,9]. Let \leq_s denote the uniform (strong) reducibility of mass problems known as Medvedev reducibility, and let \leq_w denote the non-uniform (weak) version known as Muchnik reducibility. It was shown by Nies, Stephan and Terwijn [10] that $\text{CR} \leq_w \text{SR}$. Miyabe [8] obtains an interesting counterpoint by showing as his main theorem that $\text{CR} \not\leq_s \text{SR}$.

Theorem 1 ([7]). *Given a KL-random set $A = A_0 \oplus A_1$, at least one of A_0 , A_1 is ML-random.*

As a corollary, $\text{MLR} \leq_w \text{KLR}$. Miyabe [8] posed one open problem — is $\text{MLR} \leq_s \text{KLR}$? — which we answer in Theorem 2.

Let $K(\sigma)$ denote the prefix-free Kolmogorov complexity of a string $\sigma \in 2^{<\omega}$, and let $K_s(\sigma)$ be a computable nonincreasing approximation of $K(\sigma)$ in stages $s \in \omega$. The prefix of A of length n is denoted $A \upharpoonright n$.

Theorem 2. $\text{MLR} \leq_s \text{KLR}$.

Proof. Given a KL-random set $A = A_0 \oplus A_1$, we output bits of either A_0 or A_1 , switching whenever we notice that the smallest possible randomness deficiency (c such that $\forall n (K(A_i \upharpoonright n) \geq n - c)$) increases.

This constant c depends on s and changes at stage $s + 1$ if

$$(\exists n \leq s + 1) \quad K_{s+1}(A_i \upharpoonright n) < n - c_s.$$

By Theorem 1, one of A_0, A_1 is ML-random, hence switching will occur only finitely often. Thus our output will have an infinite tail that is ML-random, and hence be itself ML-random. \square

Inspection of the proof of Theorem 2 shows that we do not need the full power of Turing reductions, but have a truth-table reduction with use $\varphi(n) \leq 2n$.

2 Complex packing dimension and its analogue

Let $K(\sigma)$ denote the prefix-free Kolmogorov complexity of a string $\sigma \in 2^{<\omega}$. The prefix of A of length n is denoted $A \upharpoonright n$.

Viewed in terms of complexity [1,6], the Hausdorff and packing dimensions are dual to one another:

Definition 1. Let $A \in 2^\omega$. The effective Hausdorff dimension of A is defined by

$$\dim_H(A) = \sup_{m \in \mathbb{N}} \inf_{n \geq m} \frac{K(A \upharpoonright n)}{n}.$$

The effective packing dimension of A is

$$\dim_p(A) = \inf_{m \in \mathbb{N}} \sup_{n \geq m} \frac{K(A \upharpoonright n)}{n}.$$

Another notion of dimension was defined in previous work by Kjos-Hanssen and Freer [3], which we review here. Let \mathfrak{D} denote the collection of all infinite Δ_1^0 elements of 2^ω . The complex packing dimension is defined as

Definition 2. $\dim_{cp}(A) = \sup_{N \in \mathfrak{D}} \inf_{n \in N} \frac{K(A \upharpoonright n)}{n}$.

This leads naturally to a new notion, the dual of complex packing dimension:

Definition 3. $\dim_i(A) = \inf_{N \in \mathfrak{D}} \sup_{n \in N} \frac{K(A \upharpoonright n)}{n}$.

This is the *inescapable* dimension of A , so named because if $\dim_i(A) = \alpha$, every infinite computable collection of prefixes of A must contain prefixes with relative complexity arbitrarily close to α . For such a real, there is no (computable) escape from high complexity prefixes.

As Freer and Kjos-Hanssen show in [3], for any $A \in 2^\omega$,

$$0 \leq \dim_H(A) \leq \dim_{cp}(A) \leq \dim_p(A) \leq 1.$$

The expected analogous result also holds:

Theorem 3. For any $A \in 2^\omega$, $0 \leq \dim_H(A) \leq \dim_i(A) \leq \dim_p(A) \leq 1$.

Proof. As the sets $[n, \infty)$ are computable subsets of \mathbb{N} , $\dim_i(A) \leq \dim_p(A)$. For the second inequality, notice that for all $m \in \mathbb{N}$ and all $N \in \Delta_1^0$,

$$\begin{aligned} \inf_{n \in [m, \infty)} \frac{K(A \upharpoonright n)}{n} &\leq \inf_{n \in N \cap [m, \infty)} \frac{K(A \upharpoonright n)}{n} \\ &\leq \sup_{n \in N \cap [m, \infty)} \frac{K(A \upharpoonright n)}{n} \leq \sup_{n \in N} \frac{K(A \upharpoonright n)}{n}. \quad \square \end{aligned}$$

Unexpectedly, this is the best one can do. As we will see in the next section, while the Hausdorff dimension of a real is always lower than its packing dimension, any permutation is possible for the complex packing and inescapable dimensions of a real.

3 Incomparability for inescapable dimension

We begin with a proof that the inescapable and complex packing dimensions are incomparable in the following sense: $\dim_{cp}(A) \leq \dim_{cp}(B)$ does not imply $\dim_i(A) \leq \dim_i(B)$, nor vice versa. In fact we show a stronger statement:

Theorem 4. *There exist A and B in 2^ω such that $\dim_{cp}(A) < \dim_{cp}(B)$, but $\dim_i(B) < \dim_i(A)$.*

Recall that a real A *meets* a set of strings S if there is some $\sigma \in S$ such that σ is a prefix of A . Moreover, A is weakly 2-generic if for each dense Σ_2^0 set of strings S , A meets S [5].

For a real A , let us write $A[m, n]$ to denote the string $A(m)A(m+1) \dots A(n-1)$. For two functions $f(n), g(n)$ we write $f(n) \leq^+ g(n)$ to denote $\exists c \forall n f(n) \leq g(n) + c$. We write $f(n) = \mathcal{O}(g(n))$ to denote $\exists M \exists n_0 \forall n > n_0 f(n) \leq M g(n)$. It will also be useful to have the following theorem of Schnorr at our disposal:

Theorem 5. *A is Martin-Löf random iff $n \leq^+ K(A \upharpoonright n)$.*

Finally, for a real A and $n \in \omega$ we use the indicator function 1_A defined by

$$1_A(n) = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Proof (of Theorem 4). Let A be a weakly 2-generic real, and let R be a Martin-Löf random real. Let $s_k = 2^{k^2}$, $k_n = \max\{k \mid s_k \leq n\}$, $C = (01)^\omega$. Define

$$B(n) = R(n - s_{k_n}) \cdot 1_C(k_n).$$

Unpacking this slightly, this is

$$B(n) = \begin{cases} R(n - s_k) & \text{if } s_k \leq n < s_{k+1} \text{ for some even } k, \\ 0 & \text{otherwise.} \end{cases}$$

In this proof, let us say that an R -segment is a string of the form $B \upharpoonright [s_{2m}, s_{2m+1})$ for some m , and say that a 0-segment is a string of the form $B \upharpoonright [s_{2m+1}, s_{2m+2})$ for some m . These are named so that a 0-segment consists of zeros, and an R -segment consists of random bits. Notice that by construction, each such segment is much longer than the combined length of all previous segments. This guarantees certain complexity bounds at the segments' right endpoints. For instance, B has high complexity at the end of R -segments: for any even $k \in \mathbb{N}$,

$$\begin{aligned} s_{k+1} - s_k &\leq^+ K(B[s_k, s_{k+1}]) \\ &\leq^+ K(B \upharpoonright s_k) + K(B \upharpoonright s_{k+1}) \leq^+ 2s_k + K(B \upharpoonright s_{k+1}). \end{aligned}$$

The first inequality holds by Theorem 5 because $B[s_k, s_{k+1}] = R \upharpoonright (s_{k+1} - s_k)$. The second (rather weak) inequality holds because from descriptions of $B \upharpoonright s_k$ and $B \upharpoonright s_{k+1}$ we can recover $B[s_k, s_{k+1}]$. Finally, $K(\sigma) \leq^+ 2|\sigma|$ is a property of prefix-free Kolmogorov complexity K . Combining and dividing by s_{k+1} gives

$$\begin{aligned} s_{k+1} - 3s_k &\leq^+ K(B \upharpoonright s_{k+1}) \\ 1 - 3 \cdot 2^{-(2k+1)} &\leq \frac{K(B \upharpoonright s_{k+1})}{s_{k+1}} + \mathcal{O}\left(2^{-(k+1)^2}\right) \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (1)$$

Dually, the right endpoints of 0-segments have low complexity: for odd $k \in \mathbb{N}$,

$$K(B \upharpoonright s_{k+1}) \leq^+ K(B \upharpoonright s_k) + K(B[s_k, s_{k+1}]) \leq^+ 2s_k + 2\log(s_{k+1} - s_k).$$

The first inequality is again the weak bound that $B \upharpoonright s_{k+1}$ can be recovered from descriptions of $B \upharpoonright s_k$ and $B[s_k, s_{k+1}]$. For the second, we apply the $2|\sigma|$ prefix-free complexity bound to $B \upharpoonright s_k$, but also notice that since $B[s_k, s_{k+1}] = 0^{s_{k+1}-s_k}$, it can be recovered effectively from a code for its length. Combining and dividing by s_{k+1} , we have

$$\begin{aligned} K(B \upharpoonright s_{k+1}) &\leq^+ 2s_k + 2(k+1)^2 \\ \frac{K(B \upharpoonright s_{k+1})}{s_{k+1}} &\leq 2^{-(2k+1)} + \mathcal{O}\left(2^{-(k+1)^2}\right) \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (2)$$

Now we can examine the dimensions of A and B .

Claim 1: $\dim_{cp}(B) = 1$.

Let R_n be the set of right endpoints of R -segments of B , except for the first n of them — that is, $R_n = \{s_{2k+1}\}_{k=n}^\infty$. Then the collection of these R_n is a subfamily of \mathfrak{D} , so that a supremum over \mathfrak{D} will be at least the supremum over this family.

$$\begin{aligned} \sup_{N \in \mathfrak{D}} \inf_{n \in N} \frac{K(B \upharpoonright n)}{n} &\geq \sup_{n \in \mathbb{N}} \inf_{s \in R_n} \frac{K(B \upharpoonright s)}{s} \\ &\geq \sup_{n \in \mathbb{N}} \inf_{s \in R_n} 1 - 3 \cdot 2^{-(2s+1)} = \sup_{m \in \mathbb{N}} 1 - 3 \cdot 2^{-(2m+1)} = 1 \end{aligned}$$

by (1).

Claim 2: $\dim_i(B) = 0$.

Let Z_n be the set of right endpoints of 0-segments of B , except for the first n of them: $Z_n = \{s_{2k}\}_{k=n}^\infty$. Similarly to Claim 1, we obtain

$$\begin{aligned} \inf_{N \in \mathfrak{D}} \sup_{n \in N} \frac{K(B \upharpoonright n)}{n} &\leq \inf_{n \in \mathbb{N}} \sup_{s \in Z_n} \frac{K(B \upharpoonright s)}{s} \\ &\leq \inf_{n \in \mathbb{N}} \sup_{s \in Z_n} 2^{-(2s+1)} = \inf_{m \in \mathbb{N}} 2^{-(2m+1)} = 0 \end{aligned}$$

by (2).

Claim 3: $\dim_{cp}(A) = 0$.

For each natural k and N in \mathfrak{D} , the following sets are dense Σ_1^0 :

$$\{\sigma \in 2^{<\omega} : |\sigma| \in N \text{ and } (\exists s) K_s(\sigma) < |\sigma|/k\}.$$

As A is weakly 2-generic, it meets all of them. Hence

$$\sup_{N \in \mathfrak{D}} \inf_{m \in N} \frac{K(\sigma \upharpoonright m)}{m} = 0.$$

Claim 4: $\dim_i(A) = 1$.

For each natural k and N in \mathfrak{D} ,

$$\{\sigma \in 2^{<\omega} : |\sigma| \in N \text{ and } (\forall s) K_s(\sigma) > |\sigma|(1 - 1/k)\}$$

is a dense Σ_2^0 set. As A is weakly 2-generic, it meets all of these sets. Hence

$$\inf_{N \in \mathfrak{D}} \sup_{m \in N} \frac{K(A \upharpoonright m)}{m} = 1. \quad \square$$

We say that A is finite-to-one reducible to B if there is a total computable function $f : \omega \rightarrow \omega$ such that the preimage of each $n \in \omega$ is finite and for all n , $n \in A \iff f(n) \in B$.

Definition 4. Let \mathfrak{B} be a class of infinite sets downward closed under finite-to-one reducibility. For $A \in 2^\omega$, we define

$$\dim_{is\mathfrak{B}}(A) = \inf_{N \in \mathfrak{B}} \sup_{n \in N} \frac{K(A \upharpoonright n)}{n} \quad \text{and} \quad \dim_{si\mathfrak{B}}(A) = \sup_{N \in \mathfrak{B}} \inf_{n \in N} \frac{K(A \upharpoonright n)}{n}.$$

Notice that for any oracle X , the classes of infinite sets that are $\Delta_n^0(X)$, $\Sigma_n^0(X)$ or $\Pi_n^0(X)$ are downward closed under finite-to-one reducibility, and so give rise to notions of dimension of this form. We will label these $\mathfrak{D}_n(X)$, $\mathfrak{S}_n(X)$, and $\mathfrak{P}_n(X)$ respectively, leaving off X when X is computable. Interestingly, for fixed n , the first two give the same notion of dimension.

Theorem 6. For all $A \in 2^\omega$ and $n \in \mathbb{N}$, $\dim_{is\Sigma_n^0}(A) = \dim_{is\Delta_n^0}(A)$.

Proof. We prove the unrelativized version of the statement, $n = 1$.

[\leq] As $\Delta_1^0 \subseteq \Sigma_1^0$, this direction is trivial.

[\geq] As every infinite Σ_1^0 set N contains an infinite Δ_1^0 set N' , we have

$$\begin{aligned} \dim_{is\Sigma_1^0}(A) &= \inf_{N \in \mathfrak{S}_1} \sup_{n \in N} \frac{K(A \upharpoonright n)}{n} \geq \inf_{N \in \mathfrak{S}_1} \sup_{n \in N'} \frac{K(A \upharpoonright n)}{n} \\ &\geq \inf_{N \in \mathfrak{D}_1} \sup_{n \in N} \frac{K(A \upharpoonright n)}{n} = \dim_{is\Delta_1^0}(A). \square \end{aligned}$$

By a similar analysis, the analogous result for *si* dimensions is also true.

Theorem 7. For all $A \in 2^\omega$ and $n \in \mathbb{N}$, $\dim_{si\Sigma_n^0}(A) = \dim_{si\Delta_n^0}(A)$.

What about the Π_n^0 dimensions? Unlike the Σ_1^0 case, these do not collapse to any other dimension. Two lemmas will be useful in proving this. The first (which was implicit in Claims 1 and 2 of Theorem 3) will allow us to show that an *si*-dimension of a real is high by demonstrating a sequence that witnesses this. The second is a generalization of the segment technique, forcing a dimension to be 0 by alternating 0- and *R*-segments in a more intricate way, according to the prescriptions of a certain real. The constructions below proceed by selecting a real that will guarantee that one dimension is 0 while leaving room to find a witnessing sequence for another.

Lemma 1 (Sequence Lemma). Let \mathfrak{B} be a class of infinite sets downward closed under finite-to-one reducibility, and let $N = \{n_k \mid k \in \omega\} \in \mathfrak{B}$.

1. If $\lim_{k \rightarrow \infty} \frac{K(X \upharpoonright n_k)}{n_k} = 1$, then $\dim_{si\mathfrak{B}}(X) = 1$.
2. If $\lim_{k \rightarrow \infty} \frac{K(X \upharpoonright n_k)}{n_k} = 0$, then $\dim_{is\mathfrak{B}}(X) = 0$.

Proof. We prove (1); (2) is similar.

Form the infinite family of sets $\{N_m\}$ defined by $N_m = \{n_k \mid k \geq m\}$. From the definition of the limit, for any $\varepsilon > 0$ there is an l such that

$$\inf_{N_l} \frac{K(X \upharpoonright n_k)}{n_k} > 1 - \varepsilon.$$

As ε was arbitrary,

$$\sup_m \inf_{N_m} \frac{K(X \upharpoonright n_m)}{n_m} = 1.$$

Thus as \mathfrak{B} is closed under finite-to-one reduction, the N_m form a subfamily of \mathfrak{B} , so that $\sup_{N \in \mathfrak{B}} \inf_{n \in N} K(X \upharpoonright n)/n = 1$. \square

Recall that an infinite real A is said to be *immune* to a class \mathfrak{B} if there is no infinite member $B \in \mathfrak{B}$ such that $B \subseteq A$ as sets, or *co-immune* to a class \mathfrak{B} if its complement is immune to \mathfrak{B} . We will sometimes refer to these properties as \mathfrak{B} -immunity or \mathfrak{B} -co-immunity, respectively.

Lemma 2 (Double Segment Lemma). *Let $X_0 \in 2^\omega$ be such that X_0 is \mathfrak{B} -immune for a class \mathfrak{B} of infinite sets downward closed under finite-to-one reducibility. Set $X = X_0 \oplus X_0$. Let $s_k = 2^{k^2}$, and $k_n = \max\{\text{odd } k \mid s_k \leq n\}$. Let A be an arbitrary real and let R be Martin-Löf random.*

1. *If $B = A(n - s_{k_n}) \cdot 1_X(k_n)$, then $\dim_{si\mathfrak{B}}(B) = 0$.*
2. *If $B = R(n - s_{k_n}) \cdot 1_{\overline{X}}(k_n)$, then $\dim_{is\mathfrak{B}}(B) = 1$.*

Again, we will give a detailed proof of only the $\dim_{si\mathfrak{B}}$ result (though the necessary changes for $\dim_{is\mathfrak{B}}$ are detailed below). Unpacking the definition of B ,

$$B(n) = \begin{cases} A(n - s_k) & \text{if } k_n \in X \\ 0 & \text{otherwise.} \end{cases}$$

B is here built out of segments of the form $B[s_{k_n}, s_{k_n+2}]$ for odd k . Here a segment is a 0-segment if $k_n \notin X$, or an A -segment if $k_n \in X$, which by definition is a prefix of A . These segments are now placed in a more intricate order according to X , with a value n being contained in a 0-segment if $X(k_n) = 0$, and in an A -segment if $X(k_n) = 1$. With some care, this will allow us to leverage the \mathfrak{B} -immunity of X_0 to perform the desired complexity calculations.

Specifically, we want to show that for any $N \in \mathfrak{B}$, $\inf_N K(B \upharpoonright n)/n = 0$. It is tempting to place the segments according to X_0 and invoke its \mathfrak{B} -immunity to show that for any $N \in \mathfrak{B}$, there are infinitely many $n \in N$ such that n is in a 0-segment, and argue that complexity will be low there. The problem is that we have no control over *where* in the 0-segment n falls. Consider in this case the start of any segment following an A -segment: $n = s_{k_n}$ for $k_n - 1 \in X_0$ and $k_n \in X_0$. We can break A and B into sections to compute

$$\begin{aligned} K(A \upharpoonright n) &\leq^+ K(A \upharpoonright (n - s_{k_n-1})) + K(A[n - s_{k_n-1}, n]) \\ &= K(B[s_{k_n-1}, n]) + K(A[n - s_{k_n-1}, n]) \quad (k_n - 1 \in X_0) \\ &\leq^+ K(B \upharpoonright n) + K(B \upharpoonright s_{k_n-1}) + K(A[n - s_{k_n-1}, n]) \\ K(A \upharpoonright n) &\leq^+ K(B \upharpoonright n) + 4s_{k_n-1} \quad (K(\sigma) \leq^+ 2|\sigma|) \end{aligned}$$

Even if n is the start of a 0-segment, if $K(A \upharpoonright n)$ is high, $K(B \upharpoonright n)$ may not be as low as needed for the proof. Our definition of X avoids this problem:

Proof (of Theorem 2). Suppose for the sake of contradiction that for some $N \in \mathfrak{B}$, there are only finitely many $n \in N$ with $k_n, k_n - 1 \in \overline{X}$, i.e., that are in a 0-segment immediately following another 0-segment. Removing these finitely many counterexamples we are left with a set $N' \in \mathfrak{B}$ such that for all $n \in N'$, $\neg[(k_n \notin X) \wedge (k_n - 1 \notin X)]$. As k_n is odd, the definition of X gives that $\lfloor k_n/2 \rfloor \in X_0$. By a finite-to-one reduction from N' , the infinite set $\{\lfloor k_n/2 \rfloor\}_{n \in N'}$ is a member of \mathfrak{B} and is contained in X_0 - but $\overline{X_0}$ is immune to such sets.

Instead it must be the case that there are infinitely many $n \in N$ in a 0-segment following a 0-segment, where the complexity is

$$\begin{aligned} K(B \upharpoonright n) &\leq^+ K(B \upharpoonright s_{n_{k-1}}) + K(B[s_{n_{k-1}}, n]) \\ &\leq^+ 2 \cdot s_{n_{k-1}} + 2 \log(n - s_{n_{k-1}}). \end{aligned}$$

Here the second inequality follows from the usual $2|\sigma|$ bound and the fact that $B[s_{n_{k-1}}, n]$ contains only 0s. As $2^{k_n^2} \leq n$, we can divide by n to get

$$\frac{K(B \upharpoonright n)}{n} \leq + \frac{2^{k_n^2 - 2k_n}}{2^{k_n^2}} + \frac{2 \log(n)}{n} = 2^{-2k_n} + \frac{2 \log(n)}{n}.$$

As there are infinitely many of these n , it must be that $\inf_{n \in N} K(B \upharpoonright n)/n = 0$. This holds for every real N with property \mathfrak{B} , so taking a supremum gives the result.

The $\dim_{is\mathfrak{B}}$ version concerns reals B constructed in a slightly different way. Here, the same argument now shows there are infinitely many $n \in N$ in an R -segment following an R -segment. At these locations, the complexity $K(B \upharpoonright n)$ can be shown to be high enough that $\sup_N K(B \upharpoonright n)/n = 1$, as desired. \square

With these lemmas in hand, we are ready to prove the following theorem:

Theorem 8. *For all natural n there is a set A with $\dim_{si\Pi_n^0}(A) = 1$ and $\dim_{si\Delta_n^0}(A) = 0$.*

Proof. We prove the $n = 1$ case, as the proofs for higher n are analogous.

Let S_0 be a co-c.e. immune set, and let R be Martin-Löf random. Set $S = S_0 \oplus S_0$, and define $k_n = \max\{\text{odd } k \mid 2^{k^2} \leq n\}$. To build A out of 0-segments and R -segments, define $A(n) = R(n - 2^{k_n^2}) \cdot 1_S(k_n)$.

As S_0 is Π_1^0 , so is S . Thus the set of right endpoints of R -segments, $M = \{2^{k^2} \mid k \text{ is odd and } k - 1 \in S\}$ is also Π_1^0 . By construction $\lim_{m \in M} K(A \upharpoonright m)/m = 1$ and thus the Sequence Lemma 1 gives that $\dim_{si\Pi_1^0}(A) = 1$.

As the complement of a simple set is immune, the Double Segment Lemma 2 shows that $\dim_{si\Delta_1^0}(A) = 0$. \square

The proof of analogous result for the is -dimensions is similar, using the same S_0 and S , and the real defined by $B(n) = R(n - 2^{k_n^2}) \cdot 1_{\overline{S}}(k_n)$.

Theorem 9. *For all $n \geq 1$ there exists a set B with $\dim_{is\Pi_n^0}(B) = 1$ and $\dim_{is\Delta_n^0}(B) = 0$.*

It remains to show that the Δ_{n+1}^0 and Π_n^0 dimensions are all distinct. We can use the above lemmas for this, so the only difficulty is finding sets of the appropriate arithmetic complexity with the relevant immunity properties.

Lemma 3. *For all $n \geq 1$, there is an infinite Δ_{n+1}^0 set S that is Π_n^0 -immune.*

Proof. We prove the unrelativized version, $n = 1$. Let C be a Δ_2^0 cohesive set that is not co-c.e., i.e., for all e either $W_e \cap C$ or $\overline{W_e} \cap C$ is finite. As \overline{C} is not c.e. it cannot finitely differ from any W_e , so for all e , $W_e \setminus \overline{C} = W_e \cap C$ is infinite. Hence if $\overline{W_e} \subseteq C$, then by cohesiveness, $\overline{W_e} \cap C = \overline{W_e}$ is finite. \square

Theorem 10. *For all $n \geq 1$ there exists a set A with $\dim_{si\Delta_{n+1}^0}(A) = 1$ and $\dim_{si\Pi_n^0}(A) = 0$.*

Proof. This is exactly like the proof of Theorem 8, but S_0 is now the Π_1^0 -immune set guaranteed by Lemma 3. \square

Again, the analogous result for *is*-dimensions is similar:

Theorem 11. *For all $n \geq 1$ there exists a set B with $\dim_{is\Pi_n^0}(B) = 1$ and $\dim_{is\Delta_{n+1}^0}(B) = 0$.*

After asking questions about the arithmetic hierarchy, it is natural to turn our attention to the Turing degrees. As the familiar notion of B -immunity for an oracle is exactly $\Delta_1^0(B)$ -immunity for a class, we have access to the usual lemmas. We shall embed the Turing degrees into the $si\text{-}\Delta_1^0(A)$ dimensions (and dually, $is\text{-}\Delta_1^0(A)$). First, a helpful lemma:

Lemma 4 (Immunity Lemma). *If $A \not\leq_T B$, there is an $S \leq_T A$ such that S is B -immune.*

Proof. Let S be the set of finite prefixes of A . If S contains a B -computable infinite subset C , then we can recover A from C , but then $A \leq_T C \leq_T B$. \square

Theorem 12 ($si\text{-}\Delta_1^0$ Embedding Theorem). *Let $A, B \in 2^\omega$. Then $A \leq_T B$ iff for all $X \in 2^\omega$, $\dim_{si\Delta_1^0(A)}(X) \leq \dim_{si\Delta_1^0(B)}(X)$.*

Proof. $[\Rightarrow]$ Immediate, as $\Delta_1^0(A) \subseteq \Delta_1^0(B)$.

$[\Leftarrow]$ This is again exactly like the proof of Theorem 8, now using the set guaranteed by the Immunity Lemma 4 as S_0 . \square

The result for *is*-dimensions is again similar:

Theorem 13 ($is\text{-}\Delta_1^0$ Embedding Theorem). *Let $A, B \in 2^\omega$. Then $A \leq_T B$ iff for all $X \in 2^\omega$, $\dim_{is\Delta_1^0(A)}(X) \geq \dim_{is\Delta_1^0(B)}(X)$.*

We can push this a little further by considering weak truth table reductions:

Definition 5. A is weak truth table reducible to B ($A \leq_{wtt} B$) if there exists a computable function f and an oracle machine Φ such that $\Phi^B = A$, and the use of $\Phi^X(n)$ is bounded by $f(n)$ for all n ($\Phi^X(n)$ is not guaranteed to halt).

Theorem 14. *If $A \not\leq_T B$, then for all wtt-reductions Φ there exists an X such that $\dim_{si\Delta_1^0(A)}(X) = 1$ and, either Φ^X is not total or $\dim_{si\Delta_1^0(B)}(\Phi^X) = 0$.*

Proof. Let $A \not\leq_T B$, and let Φ be a wtt-reduction. Let f be a computable bound on the use of Φ , and define $g(n) = \max\{f(i) \mid i \leq n\}$, so that $K(\Phi^X \upharpoonright n) \leq^+ K(X \upharpoonright g(n)) + 2\log(n)$. For notational clarity, for the rest of this proof we will denote inequalities that hold up to logarithmic (in n) terms as \leq^{\log} .

Next, we define two sequences ℓ_k and λ_k which play the role 2^{k^2} played in previous constructions:

$$\ell_0 = \lambda_0 = 1, \quad \lambda_k = \lambda_{k-1} + \ell_{k-1}, \quad \ell_k = \min \left\{ 2^{n^2} \mid g(\lambda_k) < 2^{n^2} \right\}.$$

These definitions have the useful consequence that $\lim_k \ell_{k-1}/\ell_k = 0$. To see this, suppose $\ell_{k-1} = 2^{(n-1)^2}$. As g is an increasing function, the definitions give

$$\ell_k > g(\lambda_k) \geq \lambda_k = \lambda_{k-1} + \ell_{k-1} \geq \ell_{k-1} = 2^{(n-1)^2}.$$

Hence $\ell_k \geq 2^{n^2}$, so that $\ell_{k-1}/\ell_k \leq 2^{-2n+1}$. As $\ell_k > \ell_{k-1}$ for all k , this ratio can be made arbitrarily small, giving the limit.

A triple recursive join operation is defined by

$$\bigoplus_{i=0}^2 A_i = \{3k + j \mid k \in A_j, \quad 0 \leq j \leq 2\}, \quad A_0, A_1, A_2 \subseteq \omega.$$

Let $S_0 \leq_T A$ be as guaranteed by Lemma 4, and define $S = \bigoplus_{i=0}^2 S_0$. Let R be Martin-Löf random, and define $X(n) = R(n - \ell_{k_n}) \cdot 1_S(k_n)$, where $k_n = \max\{k = 2 \pmod{3} \mid \ell_k \leq n\}$. This definition takes an unusual form compared to the previous ones we have seen in order to handle the interplay between λ_k and ℓ_k - specifically the growth rate of $g(n)$. We are effectively “tripling up” bits of S_0 (rather than doubling them as before) to account for the possibility that $g(n)$ grows superexponentially, with the condition that $k = 2 \pmod{3}$ replacing the condition that k is odd.

Claim 1: $\dim_{si\Delta_1^0(A)}(X) = 1$.

Proof: As $N = \{\ell_k\}_{k \in S}$ is an A -computable set, by the Sequence Lemma 1 it suffices to show that $\lim_{k \in S} K(X \upharpoonright \ell_k)/\ell_k = 1$. For $\ell_k \in N$,

$$\begin{aligned} K(X \upharpoonright \ell_k) &\geq^{\log} K(X[\ell_{k-1}, \ell_k]) - K(X \upharpoonright \ell_{k-1}) \\ &\geq K(R \upharpoonright (\ell_k - \ell_{k-1})) - 2\ell_{k-1} && (\text{as } k \in S) \\ &\geq^{\log} \ell_k - \ell_{k-1} - 2\ell_{k-1} && (\text{as } R \text{ is Martin-Löf random}) \\ \frac{K(X \upharpoonright \ell_k)}{\ell_k} &\geq^{\log} \frac{\ell_k - 3\ell_{k-1}}{\ell_k} = 1 - 3\frac{\ell_{k-1}}{\ell_k}. \end{aligned}$$

which gives the desired limit by the above.

Claim 2: $\dim_{si\Delta_1^0(B)}(\Phi^X) = 0$.

Proof: Suppose $N \leq_T B$. For notation, define $a = k_{g(n)}$. By mimicking the proof of Lemma 2, we can use the B -immunity of S to show that there are infinitely many $n \in N$ such that $g(n)$ is in a 0-segment following two 0-segments, i.e., $a-2, a-1, a \notin S$. By the definition of X ,

$$X[\ell_{a-2}, \ell_{a+1}] = 0^{\ell_{a+1} - \ell_{a-2}}.$$

Suppose the value $X(m)$ is queried in the course of computing $\Phi^X \upharpoonright n$. By the definitions of g , a , and ℓ_k , $m \leq g(n) < \ell_{a+1}$. Hence either $m < \ell_{a-2}$ or $m \in [\ell_{a-2}, \ell_{a+1}]$, so that $X(m) = 0$. Thus to compute $\Phi^X \upharpoonright n$, up to a constant it suffices to know $X \upharpoonright \ell_{a-2}$. Thus

$$K(\Phi^X \upharpoonright n) \leq^+ K(X \upharpoonright \ell_{a-2}) \leq^+ 2\ell_{a-2}$$

As $g(n) > \ell_a$ it must be that $n > \lambda_a$. Dividing by n , we find that

$$\frac{K(\Phi^X \upharpoonright n)}{n} \leq + \frac{2\ell_{a-2}}{\lambda_a} < \frac{2\ell_{a-2}}{\lambda_{a-1} + \ell_{a-1}} < \frac{2\ell_{a-2}}{\ell_{a-1}}.$$

As there are infinitely many of these n , it must be that $\inf_{n \in N} K(\Phi^X \upharpoonright n)/n = 0$. This holds for every $N \leq_T B$, so taking a supremum gives the result. \square

Remark. We only consider *si*-dimensions for this theorem, as it is not clear what an appropriate analogue for *is*-dimensions would be. The natural dual statement for *is*-dimensions would be that for all reductions Φ there is an X such that $\dim_{is\Delta_1^0(A)}(X) = 0$, and either Φ^X is not total or $\dim_{is\Delta_1^0(B)}(\Phi^X) = 1$. But many reductions use only computably much of their oracle, so that Φ^X is a computable set. This degenerate case is not a problem for the *si* theorem, as its conclusion requires $\dim_{\Delta_1^0(B)}(\Phi^X) = 0$. But for an *is* version, it is not even enough to require that Φ^X is not computable - consider the reduction that repeats the n th bit of X $2n - 1$ times, so that n bits of X suffice to compute n^2 bits of Φ^X . Certainly $\Phi^X \equiv_{wtt} X$, so that Φ^X is non-computable iff X is. But

$$\frac{K(\Phi^X \upharpoonright n)}{n} \leq + \frac{K(X \upharpoonright \sqrt{n})}{n} \leq + \frac{2\sqrt{n}}{n}$$

for all n , so that $\dim_p(\Phi^X) = 0$, and hence all other dimensions are 0 as well.

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