On the degrees of constructively immune sets

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Abstract. Xiang Li (1983) introduced what are now called constructively immune sets as an effective version of immunity. Such have been studied in relation to randomness and minimal indices, and we add another application area: numberings of the rationals. We also investigate the Turing degrees of constructively immune sets and the closely related Σ_1^0 -dense sets of Ferbus-Zanda and Grigorieff (2008).

Keywords: constructively immune, Turing degrees, theory of numberings

1 Introduction

Effectively immune sets, introduced by Smullyan in 1964 [12], are well-known in computability as one of the incarnations of diagonal non-computability, first made famous by Arslanov's completeness criterion. A set $A\subseteq \omega$ is effectively immune if there is a computable function h such that $|W_e|\leq h(e)$ whenever $W_e\subseteq A$, where $\{W_e\}_{e\in\omega}$ is a standard enumeration of the computably enumerable (c.e.) sets.

There is a more obvious effectivization of immunity (the lack of infinite computable subsets), however: *constructive immunity*, introduced by Xiang Li [8] who actually (and inconveniently) called it "effective immunity".

Definition 1. A set A is constructively immune if there exists a partial recursive ψ such that for all x, if W_x is infinite then $\psi(x) \downarrow$ and $\psi(x) \in W_x \setminus A$.

The Turing degrees of constructively immune sets and the related Σ_1^0 -dense sets have not been considered before in the literature, except that Xiang Li implicitly showed that they include all c.e. degrees. We prove in Section 3 that the Turing degrees of Σ_1^0 -dense sets include all non- Δ_2^0 degrees, all high degrees, and all c.e. degrees. We do not know whether they include *all* Turing degrees.

The history of the study of constructive immunity seems to be easily summarized. After Xiang Li's 1983 paper, Odifreddi's 1989 textbook [9] included Li's results as exercises, and Calude's 1994 monograph [2] showed that the set

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 $RAND_t^C = \{x: C(x) \geq |x| - t\}$ is constructively immune, where C is Kolmogorov complexity. Schafer 1997 [11] further developed an example involving minimal indices, and Brattka 2002 [1] gave one example in a more general setting than Cantor space. Finally in 2008 Ferbus-Zanda and Grigorieff proved an equivalence with constructive Σ_1^0 -density.

Definition 2 (Ferbus-Zanda and Grigorieff [6]). A set $A \subseteq \omega$ is Σ_1^0 -dense if for every infinite c.e. set C, there exists an infinite c.e. set D such that $D \subseteq C$ and $D \subseteq A$.

If there is a computable function $f: \omega \to \omega$ such that for each W_e , $W_{f(e)} \subseteq A \cap W_e$, and $W_{f(e)}$ is infinite if W_e is infinite, then A is constructively Σ_1^0 -dense.

We should note that while the various flavors of immune sets are always infinite by definition, Ferbus-Zanda and Grigorieff do not require Σ^0_1 -dense sets to be co-infinite.

The Σ_1^0 -dense sets form a natural Π_4^0 class in 2^{ω} that coincides with the simple sets on Δ_2^0 but is prevalent (in fact exists in every Turing degree) outside of Δ_2^0 by Theorem 8 below.

2 Σ_1^0 -density

To show that there exists a set that is Σ_1^0 -dense, but not constructively so, we use Mathias forcing. A detailed treatment of the computability theory of Mathias forcing can be found in [3].

Definition 3. A Mathias condition is a pair (d, E) where $d, E \subseteq \omega$, d is a finite set, E is an infinite computable set, and $\max(d) < \min(E)$. A condition (d_2, E_2) extends a condition (d_1, E_1) if

- $-d_1=d_2\cap (\max d_1+1), i.e., d_1 \text{ is an initial segment of } d_2,$
- $-E_2$ is a subset of E_1 , and
- d_2 is contained in $d_1 \cup E_1$.

A set A is Mathias generic if it is generic for Mathias forcing.

Theorem 1. If A is Mathias generic, then

- 1. $\omega \setminus A$ is Σ_1^0 -dense.
- 2. $\omega \setminus A$ is not constructively Σ_1^0 -dense.

Proof. 1. Let W_e be an infinite c.e. set. Let (d, E) be a Mathias condition.

Case (i): $E \cap W_e$ is finite. Then for any Mathias generic A extending the condition (d, E), $\omega \setminus A$ contains an infinite subset of W_e , in fact a set of the form $W_e \setminus F$ where F is finite.

Case (ii): $E \cap W_e$ is infinite. Then $E \cap W_e$ is c.e., hence has an infinite computable subset D. Write $D = D_1 \cup D_2$ where D_1, D_2 are disjoint infinite c.e. sets. The condition (d, D_1) extends (d, E) and forces a Mathias generic A

extending it to be such that $\omega \setminus A$ has an infinite subset in common with W_e , namely D_2 .

We have shown that for each infinite c.e. set W_e , each Mathias condition has an extension forcing the statement that a Matias generic A satisfies

$$\omega \setminus A$$
 has an infinite c.e. subset in common with W_e . (*)

Thus by standard forcing theory it follows that each Mathias generic satisfies (*).

2. Let f be a computable function. It suffices to show that for each Mathias generic A, there exists an i such that W_i is infinite and $W_{f(i)}$ is either finite, or not a subset of W_i , or not a subset of \overline{A} . For this, as in (1) above it suffices to show that for each condition (d, D) there exists a condition (d', E') extending (d, E) and an i such that W_i is infinite and $W_{f(i)}$ is either finite, or not a subset of W_i , or not a subset of \overline{A} for any A extending (d', D').

Let (d, E) be a Mathias condition and write $D = W_i$. If $W_{f(i)}$ is finite or not a subset of W_i then we are done. Otherwise there exists a condition (d', E') extending (d, E) such that $E' \cap W_{f(i)}$ is nonempty. This can be done by a finite extension (making only finitely many changes to the condition).

Theorem 2 ([6, Proposition 3.3]). A set $Z \subseteq \omega$ is constructively immune if and only if it is infinite and $\omega \setminus Z$ is constructively Σ_1^0 -dense.

Since Ferbus-Zanda and Grigorieff's paper has not gone through peer review, we provide the proof.

Proof. \Leftarrow : Let the function g witness that $\omega \setminus Z$ is constructively Σ_1^0 -dense. Define a partial recursive function φ by stipulating that $\varphi(i)$ is the first number in the enumeration of $W_{g(i)}$, if any.

 \Rightarrow : Define a partial recursive function $\mu(i,n)$ by

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 \begin{aligned} & - \ \mu(i,0) = \varphi(i); \\ & - \ \mu(i,n+1) = \varphi(i_n), \text{ where } i_n \text{ is such that } W_{i_n} = W_i \setminus \{\mu(i,m) : m \leq n\}. \end{aligned}
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Let g be total recursive so that $W_{g(i)} = \{\mu(i, m) : m \in \omega\}$. If W_i is infinite then all $\mu(i, m)$'s are defined and distinct and belong to $W_i \cap Z$. Thus, $W_{g(i)}$ is an infinite subset of $W_i \cap Z$.

Recall that a c.e. set is *simple* if it is co-immune.

Theorem 3 (Xiang Li [8]). Let A be a set and let $\{\phi_x\}_{x\in\omega}$ be a standard enumeration of the partial computable functions.

- 1. If A is constructively immune then A is immune and \overline{A} is not immune.
- 2. If A is simple then \overline{A} is constructively immune.
- 3. $\{x: (\forall y)(\phi_x = \phi_y \to x \leq y)\}\ is\ constructively\ immune.$

2.1 Numberings

A numbering of a countable set \mathcal{A} is an onto function $\nu : \omega \to \mathcal{A}$. The theory of numberings has a long history [5]. Numberings of the set of rational numbers \mathbb{Q} provide an application area for Σ_1^0 -density. Rosenstein [10, Section 16.2: Looking at \mathbb{Q} effectively] discusses computable dense subsets of \mathbb{Q} . Here we are mainly concerned with noncomputable sets.

Proposition 1. Let $A \subseteq \omega$. The following are equivalent:

- 1. $\nu(A)$ is dense for every injective computable numbering ν of \mathbb{Q} ;
- 2. A is co-immune.
- *Proof.* (1) \Longrightarrow (2): We prove the contrapositive. Suppose \overline{A} contains an infinite c.e. set W_e . Consider a computable numbering ν that maps W_e onto $[0,1] \cap \mathbb{Q}$. Then $\nu(A)$ is disjoint from [0,1] and hence not dense.
- $(2) \Longrightarrow (1)$: We again prove the contrapositive. Assume that $\nu(A)$ is not dense for a certain computable ν . Let $\{x_n : n \in \omega\}$ be a converging infinite sequence of rationals disjoint from $\nu(A)$. Then $\{\nu^{-1}(x_n) : n \in \omega\}$ is an infinite c.e. subset of \overline{A} .

Definition 4. A subset A of \mathbb{Q} is co-nowhere dense if for each interval $[a,b] \subseteq \mathbb{Q}$, $[a',b'] \subseteq A$ for some $[a',b'] \subseteq [a,b]$.

Proposition 2. A set is co-nowhere dense under every numbering iff it is co-finite.

Proof. Only the forward direction needs to be proven; the other direction is immediate. Let A be a co-infinite set, and define ν by letting ν map $\omega \setminus A$ onto [0,1]. Then A is not co-nowhere dense.

Proposition 3. A is infinite and non-immune iff there exists a computable numbering with respect to which A is co-nowhere dense.

Proof. Let A be infinite and not immune. Thus, there is an infinite $W_e \subseteq A$ for some e. Let ν be a computable numbering that maps W_e onto $\mathbb{Q} \setminus \omega$. Then A is co-nowhere dense under ν .

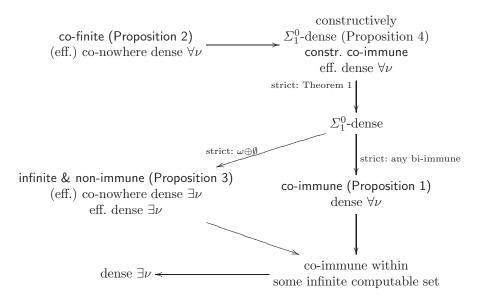
Conversely, let A be co-nowhere dense under some computable numbering ν . Then $\nu^{-1}([a,b])$ is an infinite c.e. subset of A for some suitable a,b.

A set $D \subseteq \mathbb{Q}$ is *effectively dense* if there is a computable function f(a,b) giving an element of $D \cap (a,b)$ for $a < b \in \mathbb{Q}$.

Proposition 4. A set A is constructively Σ_1^0 -dense iff it is effectively dense for all computable numberings.

Proof. By Theorem 2, A is constructively Σ_1^0 -dense iff it is infinite and $\omega \setminus A$ is constructively immune. Constructive immunity of $\omega \setminus A$ implies effective density of A since the witnessing function for constructive immunity can be be used to witness effective density. For the converse we exploit the assumption that we get to choose a suitable ν .

Let A and B be sets, with B computable. We say that A is co-immune within B if there is no infinite computable subset of $A^c \cap B$. The following diagram includes some claims not proved in the paper, whose proof (or disproof) may be considered enjoyable exercises. The quantifiers $\exists \nu, \forall \nu$ range over computable numberings of \mathbb{Q} .



3 Prevalence of Σ_1^0 -density

In this section we investigate the existence of Σ_1^0 -density in the Turing degrees at large.

3.1 Closure properties and Σ_1^0 -density

Proposition 5. 1. The intersection of two Σ_1^0 -dense sets is Σ_1^0 -dense. 2. The intersection of two constructively Σ_1^0 -dense sets is constructively Σ_1^0 -dense.

Proof. Let A and B be Σ_1^0 -dense sets. Let W_e be an infinite c.e. set. Since A is Σ_1^0 -dense, there exists an infinite c.e. set $W_d \subseteq A \cap W_e$. Since B is Σ_1^0 -dense, there exists an infinite c.e. set $W_a \subseteq B \cap W_d$. Then $W_a \subseteq (A \cap B) \cap W_e$, as desired. This proves (1). To prove (2), let f and g witness the effective Σ_1^0 -density of A and B, respectively. Given W_e , we have $W_{f(e)} \subseteq A \cap W_e$ and then

$$W_{g(f(e))} \subseteq B \cap W_{f(e)} \subseteq A \cap B \cap W_e.$$

In other words, $g \circ f$ witnesses the effective Σ_1^0 -density of $A \cap B$.

Corollary 1. $Bi-\Sigma_1^0$ -dense sets do not exist.

Proof. If A and A^c are both Σ_1^0 -dense then by Proposition 5, $A \cap A^c$ is Σ_1^0 -dense, which is a contradiction.

For sets A and B, $A \subseteq^* B$ means that $A \setminus B$ is a finite set.

Proposition 6. 1. If A is Σ₁⁰-dense and A ⊆* B, then B is Σ₁⁰-dense.
2. If A is constructively Σ₁⁰-dense and A ⊆* B, then B is constructively Σ₁⁰-dense.

Proof. Let W_e be an infinite c.e. set. Since A is Σ_1^0 -dense, there exists an infinite c.e. set W_d such that $W_d \subseteq A \cap W_e$. Let $W_c = W_d \setminus (A \setminus B)$. Since $A \setminus B$ is finite, W_c is an infinite c.e. set. Since $W_d \subseteq A$, we have $W_c = W_d \cap (B \cup A^c) = W_d \cap B$. Then, since $W_d \subseteq W_e$, we have $W_c \subseteq B \cap W_e$, and we conclude that B is Σ_1^0 -dense. This proves (1). To prove (2), if f witnesses that A is constructively Σ_1^0 -dense then a function g with $W_{g(e)} = W_{f(e)} \setminus (A \setminus B)$ witnesses that B is constructively Σ_1^0 -dense.

Proposition 7. Let B be a co-finite set. Then B is constructively Σ_1^0 -dense.

Proof. The set ω is constructively Σ_1^0 -dense as witnessed by the identity function f(e) = e. Thus by Item 2 of Proposition 6, B is as well.

As usual we write $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$.

Proposition 8. 1. If X_0 and X_1 are Σ_1^0 -dense sets then so is $X_0 \oplus X_1$. 2. If X_0 and X_1 are constructively Σ_1^0 -dense sets then so is $X_0 \oplus X_1$.

Proof. Let $W_e = W_{c_0} \oplus W_{c_1}$ be an infinite c.e. set. For i = 0, 1, since X_i is Σ_1^0 -dense there exists $W_{d_i} \subseteq X_i \cap W_{c_i}$ such that W_{d_i} is infinite if W_{c_i} is infinite. Then $W_{d_0} \oplus W_{d_1}$ is an infinite c.e. subset of $(X_0 \oplus X_1) \cap W_e$.

This proves (1). To prove (2), if d_i are now functions witnessing the effective Σ_1^0 -density of X_i then $W_{d_i(c_i)} \subseteq X_i \cap W_{c_i}$, and $W_{d_0(c_0)} \oplus W_{d_1(c_1)}$ is an infinite c.e. subset of $(X_0 \oplus X_1) \cap W_e$. Thus a function g satisfying

$$W_{g(e)} = W_{d_0(c_0)} \oplus W_{d_1(c_1)},$$

where $W_e = W_{c_0} \oplus W_{c_1}$, witnesses the effective Σ_1^0 -density of $X_0 \oplus X_1$.

Theorem 4. There is no Σ_1^0 -dense set A such that all Σ_1^0 -dense sets B satisfy $A \subseteq^* B$.

Proof. Suppose there is such a set A. Let W_d be an infinite computable subset of A. Let G be a Mathias generic with $G \cap W_d^c = \emptyset$, i.e., $G \subseteq W_d$. Then $B := G^c$ is Σ_1^0 -dense by Theorem 1. Thus $A \cap G^c$ is also Σ_1^0 -dense by Proposition 5. And $G \subseteq W_d \subseteq A$ and by assumption $A \subseteq^* G^c$ so we get $G \subseteq^* G^c$, a contradiction.

These results show that the Σ_1^0 -dense sets under \subseteq^* form a non-principal filter whose Turing degrees form a join semi-lattice.

Theorem 5. Let A be a c.e. set. The following are equivalent:

- 1. A is co-infinite and constructively Σ_1^0 -dense.
- 2. A is co-infinite and Σ_1^0 -dense.
- 3. A is co-immune.

Proof. $1 \implies 2 \implies 3$ is immediate from the definitions, and $3 \implies 1$ is immediate from Theorem 2 and Theorem 3.

Theorem 6. Every c.e. Turing degree contains a constructively Σ_1^0 -dense set.

Proof. Let **a** be a c.e. degree. If $\mathbf{a} > \mathbf{0}$ then **a** contains a simple set A, see, e.g., [13], so Theorem 5 finishes this case. The degree **0** contains all the co-finite sets, which are constructively Σ_1^0 -dense by Proposition 7.

3.2 Cofinality in the Turing degrees of constructive Σ_1^0 -density

Definition 5. For $k \ge 0$, let I_k be intervals of length k+2 such that $\min(I_0) = 0$ and $\max(I_k) + 1 = \min(I_{k+1})$.

Let $V_e = \bigcup_{s \in \omega} V_{e,s}$ be a subset of W_e defined by the condition that $x \in I_k$ enters V_e at a stage s where x enters W_e if this makes $|V_{e,s} \cap I_k| \leq 1$, and for all j > k, $V_{j,s} \cap I_k = \emptyset$.

Lemma 1. There exists a c.e., co-infinite, constructively Σ_1^0 -dense, and effectively co-immune set.

Proof. Let $A = \bigcup_{e \in \omega} V_e$. V_e is c.e. by construction, and if W_e is infinite, V_e is also infinite. So $V_e = W_{f(e)}$ is the set witnessing that A is constructively Σ_1^0 -dense.

Moreover A is coinfinite since $|A \cap I_k| \le k+1 < k+2 = |I_k|$ gives $I_k \not\subseteq A$ for each k and

$$|\omega \setminus A| = \left| \left(\bigcup_{k \in \omega} I_k \right) \setminus A \right| = \left| \bigcup_{k \in \omega} (I_k \setminus A) \right| = \sum_{k \in \omega} |I_k \setminus A| \ge \sum_{k \in \omega} 1 = \infty.$$

The set A is effectively co-immune because if W_e is disjoint from A then since as soon as a number in I_k for $k \ge e$ enters W_e then that number is put into A, $W_e \subseteq \bigcup_{k < e} I_k$ so $|W_e| \le \sum_{k < e} (k+2) = \sum_{k \le e+1} k = \frac{(e+1)(e+2)}{2}$.

Theorem 7. For each set R there exists a constructively Σ_1^0 -dense, effectively co-immune set S with $R \leq_T S$.

Proof. Let R be any set, which we may assume is co-infinite. Let A be as in the proof of Lemma 1. Let $S \supseteq A$ be defined by

$$S = A \cup \bigcup_{k \in R} I_k.$$

Since $A \subseteq S$ and S is co-infinite, S is constructively Σ_1^0 -dense and effectively co-immune. Since $k \in R \iff I_k \subseteq S$, we have $R \leq_T S$.

3.3 Non- Δ_2^0 degrees

Lemma 2. Suppose that $T \subseteq 2^{<\omega}$ is a tree with only one infinite path. Then for each length n there exists a length m > n such that exactly one string of length n has an extension of length m in T.

Proof. Suppose not, i.e., there is a length n such that for all m > n there are at least two strings σ_m, τ_m of length n with extensions of length m in T. By the pigeonhole principle there is a pair (σ, τ) that is a choice of (σ_m, τ_m) for infinitely many m. Then by compactness both σ and τ must be extendible to infinite paths of T.

Lemma 3. Suppose that $T \subseteq 2^{<\omega}$ is a tree with only one infinite path A, and that T is a c.e. set of strings. Then A is Δ_2^0 .

Proof. By Lemma 2, for each length n there exists a length m > n such that exactly one string of length n has an extension of length m in T. Using 0' as an oracle we can find that m and define $A \upharpoonright n$ by looking for such a string. In fact, $T \leq_T 0'$ and so its unique path $A \leq_T 0'$ as well.

Theorem 8. Given $A \in 2^{\omega}$, let $\hat{A} := \{ \sigma \in 2^{<\omega} \mid \sigma \prec A \}$ be the set of finite prefixes of A. If A is not Δ_2^0 then \hat{A} is $co-\Sigma_1^0$ -dense.

Proof. Let A^* be the complement of \hat{A} . Let $W_e \subseteq 2^{<\omega}$ be an infinite c.e. set of strings. Let T be the set of all prefixes of elements of W_e . Then T is an infinite tree, hence by compactness it has at least one infinite path. That is, there is at least one real B such that all its prefixes are in T.

Case 1: The only such real is B = A. Then by Lemma 3, A is Δ_2^0 .

Case 2: There is a $B \neq A$ such that all its prefixes are in T. Let σ be a prefix of B that is not a prefix of A. Let $W_d = [\sigma] \cap W_e$. Since all prefixes of B are prefixes of elements of W_e , there are infinitely many extensions of σ that are prefixes of elements of W_e . Consequently W_d is infinite. Thus, W_d is our desired infinite subset of $A^* \cap W_e$.

3.4 High degrees

Definition 6. A set A is co-r-cohesive if its complement is r-cohesive. This means that for each computable (recursive) set W_d , either $W_d \subseteq^* A$ or $W_d^c \subseteq^* A$.

Definition 7 (Odifreddi [9, Exercise III.4.8], Jockusch and Stephan [7]). A set A is strongly hyperhyperimmune (s.h.h.i.) if for each computable $f: \omega \to \omega$ for which the sets $W_{f(e)}$ are disjoint, there is an e with $W_{f(e)} \subseteq \omega \setminus A$. A set A is strongly hyperimmune (s.h.i.) if for each computable $f: \omega \to \omega$ for which the sets $W_{f(e)}$ are disjoint and computable, with $\bigcup_{e \in \omega} W_{f(e)}$ also

computable, there is an e with $W_{f(e)} \subseteq \omega \setminus A$.

Proposition 9. Every s.h.i. set is co- Σ_1^0 -dense.

Proof. Let A be s.h.i. Let W_e be an infinite c.e. set. Let W_d be an infinite computable subset of W_e . Effectively decompose W_d into infinitely many disjoint infinite computable sets,

$$W_d = \bigcup_{i \in \omega} W_{g(d,i)}.$$

For instance, if $W_d = \{a_0 < a_1 < \dots\}$ then we may let $W_{g(e,i)} = \{a_n : n = 2^i(2k+1), i \geq 0, k \geq 0\}$. Since A is s.h.i., there exists some i_e such that $W_{g(d,i_e)} \subseteq A^c$. The sets $W_{g(d,i_e)}$ witness that A^c is Σ_1^0 -dense.

Clearly r-cohesive implies s.h.i., and s.h.h.i. implies s.h.i. It was shown by Jockusch and Stephan [7, Corollary 2.4] that the cohesive degrees coincide with the r-cohesive degrees and (Corollary 3.10) that the s.h.i. and s.h.h.i. degrees coincide.

Proposition 10. Every high degree contains a Σ_1^0 -dense set.

Proof. Let **h** be a Turing degree. If $\mathbf{h} \leq \mathbf{0}'$, then **h** contains a Σ_1^0 -dense set by Theorem 8.

If $\mathbf{h} \leq \mathbf{0}'$ and \mathbf{h} is high then since the strongly hyperhyperimmune and cohesive degrees coincide, and are exactly the high degrees [4], \mathbf{h} contains a strongly hyperimmune set. Hence by Theorem 9, \mathbf{h} contains a Σ_1^0 -dense set.

3.5 Progressive approximations

Definition 8. Let A be a Δ_2^0 set. A computable approximation $\{\sigma_t\}_{t\in\omega}$ of A, where each σ_t is a finite string and $\lim_{t\to\infty} \sigma_t = A$, is progressive if for each t,

- if $|\sigma_t| \leq |\sigma_{t-1}|$ then $\sigma_t \upharpoonright (|\sigma_t| 1) = \sigma_{t-1} \upharpoonright (|\sigma_t| 1)$ (the last bit of σ_t is the only difference with σ_{t-1});
- $-if |\sigma_t| > |\sigma_{t-1}| then \sigma_{t-1} \prec \sigma_t; and$
- if $\sigma_t \not\prec \sigma_s$ for some s > t then $\sigma_t \not\prec \sigma_{s'}$ for all $s' \geq s$ (once an approximation looks wrong, it never looks right again).

If A has a progressive approximation then we say that A is progressively approximable.

Note that a progressively approximable set must be h-c.e. where $h(n) = 2^n$.

Theorem 9. Let A be a progressively approximable and noncomputable set. Let $\{\sigma_t\}_{t\in\omega}$ be a progressive approximation of A. Then $\{t:\sigma_t\prec A\}$ is constructively immune.

Proof. Let W_e be an infinite c.e. set and let T be an infinite computable subset of W_e . Since A is noncomputable, we do not have $T \subseteq \{t : \sigma_t \prec A\}$. Since the approximation $\{\sigma_t\}_{t \in \omega}$ is progressive, once we observe a t for which $\sigma_t \not\prec \sigma_s$, for some s > t, then we know that $\sigma_t \not\prec A$. Then we define $\varphi(e) = t$, and φ witnesses that $\{t : \sigma_t \prec A\}$ is constructively immune.

A direction for future work may be to find new Turing degrees of progressively approximable sets.

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