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Abstract

In this paper we discuss the construction of probability distributions on the group $\mathrm{SL}(2, \mathbb{C})$ and the Möbius group using the exponential map. In particular, we describe the injectivity and surjectivity domains of the exponential map and provide its Jacobian determinant. We also show that on $\mathrm{SL}(2, \mathbb{C})$ and the Möbius group, there are no isotropic densities in the group sense.

Keywords: statistics on Lie groups, exponential map, wrapped distributions, Killing form

1 Introduction

Modelling and estimating probability densities on manifolds raises several difficulties and is still an active research topic, see for instance [1, 2]. In most cases the manifolds studied are endowed with a Riemannian metric. It is for instance the case for the manifold of positive definite matrices, or the rotations group of an Euclidean space. In these contexts all the structures used in the statistical analysis are related to the distance. The other important structure addressed in the literature is the Lie group structure. Despite being a large class of manifolds, a reason why it is less often studied than the Riemannian setting, is because compact Lie groups admit a bi-invariant metric. In that case, the statistical analysis based on the Riemannian distance satisfies all the group requirements. Most examples of statistics problems on non-compact groups studied in the literature arise from rigid and affine deformations of the physical space $\mathbb{R}^3$. Due to their role in polarization optics, see [7, 8], we study here the group $\mathrm{SL}(2, \mathbb{C})$ and its quotient, the Möbius group.

In section 2, we review the main important facts about $\mathrm{SL}(2, \mathbb{C})$ and the Möbius group. In section 3, we describe the construction of exponential-wrapped distributions. In particular, we show that the non-surjectivity of the exponential map on $\mathrm{SL}(2, \mathbb{C})$ is not a major obstacle and give an expression of its Jacobian. In section 4, we make a parallel between the notion of isotropy on a Riemannian manifold and a notion of isotropy on a group. We show that unfortunately, except the Dirac on the identity, there are no isotropic probability distributions on $\mathrm{SL}(2,\mathbb{C})$ and the Möbius group in that sense.

2 The group $\mathrm{SL}(2,\mathbb{C})$ and the Möbius group

$\mathrm{SL}(2,\mathbb{C})$ is the group of 2 by 2 complex matrices of determinant 1. Since it is defined be the polynomial equation

$$\det(M) = 1, \quad M \in M_2(\mathbb{C}), \tag{1}$$

it is a complex Lie group. Recall that a complex Lie group is a complex manifold such that the group operations are holomorphic. Recall also that a complex manifold is a manifold whose charts are open sets of $\mathbb{C}^d$ and whose transition are holomorphic.

In this paper, we define the Möbius group, noted $M\ddot{o}b$, as the quotient of $\mathrm{SL}(2,\mathbb{C})$ by the group $+I, -I$, where I is the identity matrix. Since

$$M\ddot{o}b = \mathrm{SL}(2,\mathbb{C}) / \{+I, -I\} \sim \mathrm{PGL}(2,\mathbb{C}),$$

the Möbius group can also be seen as a projective linear group. Recall that $\mathrm{PGL}(2,\mathbb{C})$ is defined as a quotient of $GL(2,\mathbb{C})$ by the multiples of the identity matrix.

Since $M\ddot{o}b$ is a quotient of $\mathrm{SL}(2,\mathbb{C})$ by a discrete subgroup, they have the same Lie algebra, noted $\mathfrak{sl}(2,\mathbb{C})$. The complex structure of $\mathrm{SL}(2,\mathbb{C})$ makes $\mathfrak{sl}(2,\mathbb{C})$ a complex vector space. By differentiating Eq.1 around the identity matrix, we can check that $\mathfrak{sl}(2,\mathbb{C})$ is the vector space of complex matrices with zero trace. It is easy to see that $\mathfrak{sl}(2,\mathbb{C})$ is the complexification of real traceless matrices, hence

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

generates $\mathfrak{sl}(2,\mathbb{C})$ and it is of complex dimension 3. The Lie brackets are given by

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h,$$

hence their adjoints in the basis (e,h,f) are

$$ad_e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad ad_h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad ad_f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}.$$

The Killing form on a Lie algebra is given by

$$\kappa(X,Y) = \mathrm{Tr}(ad_X ad_Y).$$

It can be checked that on $\mathfrak{sl}(2, \mathbb{C})$,

$$\kappa(X, Y) = 4\,\mathrm{Tr}(XY),$$

and that for a matrix $X = a.e + b.h + c.f$,

$$\kappa(X, X) = 8(b^2 + ac) = -8\det(X) = 8\lambda^2,$$

where λ is an eigenvalue of X. Recall that the eigenvalues of $X \in \mathrm{SL}(2, \mathbb{C})$ are either distinct and opposite or null.

3 Exponential-wrapped distributions

In this section, we determine domains on the Lie groups and Lie algebra involved in the construction of exponential-wrapped distributions on $\mathrm{SL}(2, \mathbb{C})$ and the Möbius group, and provide the expression of their densities.

Let us recall how the definition of the exponential map is defined on the Lie algebra, and how it is extended to every tangent spaces. The exponential map exp on a Lie group G maps a vector in the Lie algebra $X \in T_eG$ to $\gamma(1)$, where γ is the one parameter subgroup with $\gamma'(0) = X$. This map is defined on T_eG but can be extended to every tangent spaces T_gG using pushforwards of the group multiplications. The following identity

$$g \exp_e(X) g^{-1} = \exp_e(dL_g dR_g^{-1}(X)) = \exp_e(dR_g^{-1} dL_g(X)),$$

ensures that the definition of the exponential at g is independent of the choice of left or right multiplication:

$$\begin{aligned} \exp_g : \quad & T_gG \to G \\ & u \mapsto \exp_g(X) = g.\exp\left(dL_{g^{-1}}X\right) = \exp\left(dR_{g^{-1}}X\right) g. \end{aligned}$$

In the rest of the paper, exponentials without subscript refer to exponentials at identity. The definition of exponential maps on arbitrary tangent space has a deep geometric interpretation, as exponentials maps of an affine connection. See [5] for a detailed description of this point of view.

Exponential-wrapped distributions refers to distributions pushed forward from a tangent space to the group, by an exponential map. Given a probability distribution $\tilde{\mu}$ on T_gG,

$$\mu = \exp_{g*}(\tilde{\mu})$$

is a probability distribution on G. Consider a left or right invariant field of basis on G and the associated Haar measure. Recall that since $\mathrm{SL}(2, \mathbb{C})$ is unimodular, left invariant and right invariant measure are bi-invariant. When $\tilde{\mu}$ has a density $\tilde{f}$, the density f of μ with respect to the Haar measure is the density $\tilde{f}$ divided by the absolute value of the Jacobian determinant of the differential of the exponential map. If $\tilde{f}$ is vanishing outside an injectivity domain,

$$f(\exp_g(X)) = \frac{\tilde{f}(X)}{J(X)}, \quad \text{with } J(X) = |\det(d\exp_{g,X})|. \tag{2}$$

The interest of this construction is dependent on the injectivity and surjectivity properties of the exponential map. The situation is ideal when the exponential map is bijective: exponential-wrapped distributions can model every distribution on the group. Non-surjectivity is a potentially bigger issue that non-injectivity. As we will see, the injectivity can be forced by restricting the tangent space to an injectivity domain. However when the ranges of the exponential maps are too small, arbitrary distributions on the group might only be modeled by a mixture of exponential-wrapped distributions involving a large number of tangent spaces. On $\mathrm{SL}(2,\mathbb{C})$, the exponential map is unfortunately neither injective nor surjective. However, we have the following.

Fact 1. *The range of the exponential map of the group* $\mathrm{SL}(2,\mathbb{C})$ *is*

$$U = \{M \in \mathrm{SL}(2,\mathbb{C}) |\, \mathrm{Tr}(M) \neq -2\} \cup \{-I\},$$

where I *is the identity matrix. As a result,* $\mathrm{SL}(2,\mathbb{C}) \setminus \exp(\mathfrak{sl}(2,\mathbb{C}))$ *has zero measure.*

Fact.1. can be checked using the Jordan decomposition of $\mathrm{SL}(2,\mathbb{C})$ matrices. A detailed study of the surjectivity of the exponential for $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{C})$ can be found in [6]. This fact shows that the non surjectivity does not significantly affect the modeling capacities of exponential-wrapped distributions. As a direct consequence of Fact.1, we have

Corollary 1.

- $\mathrm{SL}(2,\mathbb{C})$ *is covered by* $\exp_I$ *and* $\exp_{-I}$
- *The exponential map of the Möbius group is surjective.*

Proof. For $M \in \mathrm{SL}(2,\mathbb{C})$, M or $-IM$ has a positive trace and is in the range of the $\mathrm{SL}(2,\mathbb{C})$ exponential. Since the Möbius group is $\mathrm{SL}(2,\mathbb{C})$ quotiented by the multiplication by $-I$, at least one element of the equivalent classes is reached by the exponential. □

We now provide injectivity domains of the exponentials, which enable the definition of the inverses.

Fact 2. *The exponential of* $\mathrm{SL}(2,\mathbb{C})$ *is a bijection between* $\mathfrak{U}$ *and* U *with,*

$$\mathfrak{U} = \{X \in \mathfrak{sl}(2,\mathbb{C}) | \quad \mathrm{Im}(\lambda) \in]-\pi,\pi] \text{ for any eigenvalue } \lambda \text{ of } X\}.$$

Recall that eigenvalues of X *are always opposite. The exponential of the Möbius group is bijective on*

$$\mathfrak{U}_{M\ddot{o}b} = \left\{X \in \mathfrak{sl}(2,\mathbb{C}) | \quad \mathrm{Im}(\lambda) \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right] \text{ for any eigenvalue } \lambda \text{ of } X\right\}.$$

This is a direct consequence of theorem 1.31 of [4], which defines the principal matrix logarithm for matrices with no eigenvalues in $\mathbb{R}_-$, by setting the eigenvalues of the logarithm to the interval $]-\pi, \pi[$. In order to have a bijective maps, the intervals should contains exactly one of their extremities. $\mathfrak{U}_{M\ddot{o}b}$ is obtained by noting that the quotient by $\{I, -I\}$ in $\mathrm{SL}(2, \mathbb{C})$ translates to an quotient by $i\pi\mathbb{Z}$ on eigenvalues in the Lie algebra. Note that since $\sqrt{\frac{\kappa(X,X)}{2}} = \pm\lambda$, the injectivity domains can be expressed as the inverse image of subset of $\mathbb{C}$ by the quadratic form $X \mapsto \kappa(X, X)$.

Hence, when the support of $\tilde{f}$ is included in $\mathfrak{U}$ or $\mathfrak{U}_{M\ddot{o}b}$, Eq.2 holds. We address now the computation of the volume change term. Note that in Eq.2, the determinant is seen as a volume change between real vector spaces. Recall also that for a complex linear map A on $\mathbb{C}^n$, the real determinant over $\mathbb{R}^{2n}$ is given by $\det_{\mathbb{R}}(A) = |\det_{\mathbb{C}}(A)|^2$.

The differential in a left invariant field of basis of the exponential map at identity e evaluated on the vector $X \in T_eG$ is given by the following formula, see [3],

$$d\exp_X = dL_{\exp(X)} \circ \left(\sum_{k \geq 0} \frac{(-1)^k}{(k+1)!} ad_X^k \right),$$

where subscript e is dropped. Using the Jordan decomposition of ad_X, authors of [2] pointed out the fact that this Jacobian can be computed for every Lie group, even when the adjoint endmorphisms are not diagonalizable. We have,

$$\det_{\mathbb{C}}(d\exp_X) = \prod_{\lambda \in \Lambda_X} \left(\frac{1 - e^{\lambda}}{\lambda} \right)^{d_\lambda},$$

where Λ_X is the set of nonzero eigenvalues of ad_X and d_λ the algebraic multiplicity of the eigenvalue λ. For the group $\mathrm{SL}(2, \mathbb{C})$, a calculation shows at the matrix X of coordinates (a, b, c) in the basis (e, h, f), the eigenvalues of

$$ad_X = a. \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + b. \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} + c. \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix},$$

are

$$\lambda_1 = 0, \quad \lambda_2 = 2\sqrt{ac + b^2} = \sqrt{\frac{\kappa(X, X)}{2}}, \quad \lambda_3 = -\lambda_2.$$

It is interesting to note that eigenvalues of X are also eigenvalues of ad_X. The absolute value of the Jacobian becomes

$$J(a, b, c) = |\det_{\mathbb{C}}(d\exp_X)|^2 = \left| \frac{(1 - e^{-\lambda_2})(1 - e^{\lambda_2})}{\lambda_2^2} \right|^2 = \left| 2\frac{1 - \cosh(\lambda_2)}{\lambda_2^2} \right|^2.$$

J is extended by continuity by $J(0,0,0)=1$: it is not surprising since the differential of the exponential at zero is the identity. Eq.2 can be rewritten as,

$$f(\exp(X)) = \left| \frac{\kappa(X,X)}{4\left(1-\cosh\left(\sqrt{\frac{1}{2}\kappa(X,X)}\right)\right)} \right| \tilde{f}(X). \tag{3}$$

Recall that $\lambda_{2,3}$ are complex numbers, and that $\cosh(ix)=\cos(x)$. It is interesting to note that on the one parameter subgroup generated by $(0,1,0)$ the Jacobian is increasing, which is a sign of geodesic spreading. On the other hand on the one parameter subgroup generated by $(0,i,0)$ the Jacobian is decreasing over $[0,\pi]$, which is a sign of geodesic focusing.

4 There are no group-isotropic probability distributions

When the underlying manifold is equipped with a Riemannian metric, it is possible to define the notion of isotropy of a measure. A measure μ is isotropic if there is a point on the manifold such that μ is invariant by all the isometries which preserve the point. They form an important class of probability, due to their physical interpretation, and to the fact that their high degree of symmetries enable to parametrize them with a small number of parameters.

On compact Lie groups, there exists Riemannian metrics such that left and right translations are isometries, and the notion of isotropy can hence be defined in term of the distance. Unfortunately there are no Riemannian metric on $\mathrm{SL}(2,\mathbb{C})$ compatible with the group multiplications. This comes from the fact that the scalar product at identity of such a metric should be invariant by the adjoint action of the group. Since the adjoint representation of $\mathrm{SL}(2,\mathbb{C})$ is faithful, this scalar product should be invariant by a non compact group, which is not possible. Hence, isotropy cannot be defined by Riemannian distance.

However the role of the distance in the definition of isotropy is not crucial: isotropy is defined by the invariance with respect to a set of transformations. When the manifold is Riemannian, this set is the set of isometries that fix a given point, when the manifold is a Lie group, the relevant set becomes a set of group operations.

[group-isotropy] Let G be a Lie group and $\mathcal{T}$ be the set of all maps $T: G \to G$ obtained by arbitrary compositions of left multiplications and right multiplication. A measure μ on a Lie group G is group-isotropic with respect to an element g if

$$T_*\mu = \mu, \quad \forall T \in \mathcal{T}, T(g) = g,$$

where $T_*\mu$ is the pushforward of μ by T.

It is easy to checked that the elements of $\mathcal{T}$ which preserve the identity are the conjugations. Recall that the Killing form is invariant under the differential of conjugations, and that the Jacobian of the exponantial is a function of the Killing form, see Eq.3. It can be checked that if a density $\tilde{f}$ on the Lie algebra is a function of the Killing form, the push forward on the group is group-isotropic. Hence wrapping measures to the group with the exponential map is a natural way to construct group-isotropic measures on $\mathrm{SL}(2,\mathbb{C})$. However, the following results shows that unfortunately, the group-isotropy notion is not relevant on $\mathrm{SL}(2,\mathbb{C})$ for probability distributions: it contains only Dirac distributions.

If μ is a finite positive measure on $\mathrm{SL}(2,\mathbb{C})$ isotropic with respect to the identity, then μ is a Dirac at the identity.

Proof. Assume that μ is a measure on $\mathrm{SL}(2,\mathbb{C})$ whose support contains a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ different from the identity matrix. Since

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a+tc & b+t(d-a)-ct^2 \\ c & -ct+d \end{pmatrix},$$

M is always conjugated to a matrix whose upper right coefficient is not zero. By the isotropic assumption, this matrix is still in the support of μ. Hence we can suppose that $b \neq 0$. Let $B(M)$ be the open ball centered on M:

$$B(M) = \left\{ \begin{pmatrix} a+\epsilon_1 & b+\epsilon_2 \\ c+\epsilon_3 & d+\epsilon_4 \end{pmatrix}, |\epsilon_i| < \frac{|b|}{2} \right\}.$$

Since M is in the support of μ, $\mu(B(M)) > 0$. Let $g = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C})$. If we can show that $g^n B(M) g^{-n}$ and $g^{n+k} B(M) g^{-(n+k)}$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_*$ are disjoint, μ has to be infinite since there are countable disjoint sets of identical nonzero mass. We have

$$\begin{aligned} g^n M g^{-n} &= \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}^{-n} \\ &= \begin{pmatrix} a & 2^{2n} b \\ \frac{c}{2^{2n}} & d \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} & g^n B(M) g^{-n} \cap g^{n+k} B(M) g^{-(n+k)} \neq \emptyset \\ \Rightarrow \quad & \exists \epsilon, \epsilon', \text{ with } |\epsilon| \text{ and } |\epsilon'| < \tfrac{|b|}{2}, \text{ such that, } 2^{2n}(b+\epsilon) = 2^{2n} 2^{2k} (b+\epsilon') \end{aligned}$$

Using the triangular inequalities $|b+\epsilon| < |b|+|\epsilon|$ and $|b|-|\epsilon'| < |b+\epsilon'|$, we see that such ϵ and ϵ' do not exist when k is a positive integer. Hence the images of $B(M)$ by the conjugations by $g^{n\in\mathbb{N}}$ are disjoints and the measure is infinite.

$\square$

5 Conclusion

In this paper, we laid the foundations for density modeling using exponential-wrapped distributions on $\mathrm{SL}(2,\mathbb{C})$ and the Möbius group. The Möbius group plays an important role in polarization optics due to its action on wave polarization states. Future works will focus on applications of density modeling on the Möbius group to the propagation of light through random media.

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