SCHRÖDINGER ENCOUNTERS FISHER AND RAO: A SURVEY

LÉONARD MONSAINGEON AND DMITRY VOROTNIKOV

ABSTRACT. In this short note we review the dynamical Schrödinger problem on the non-commutative Fisher-Rao space of positive semi-definite matrixvalued measures. The presentation is meant to be self-contained, and we discuss in particular connections with Gaussian optimal transport, entropy, and quantum Fisher information.

The abstract Schrödinger problem on a generic Riemannian manifold (M, g) can be formulated as the dynamical minimization problem

$$\min_{q} \left\{ \int_{0}^{1} |\dot{q}_{t}|^{2} \mathrm{d}t + \varepsilon^{2} \int_{0}^{1} |\nabla V(q_{t})|^{2} \mathrm{d}t \qquad \text{s.t.} \qquad q|_{t=0,1} = q_{0}, q_{1} \right\}.$$
(1)

The unknown curves $q = (q_t)_{t \in [0,1]}$ take values in M and interpolate between the prescribed endpoints $q_0, q_1 \in M$, the potential $V : M \to \mathbb{R}$ is given, and $\varepsilon > 0$ is a regularization parameter. Clearly when $\varepsilon \to 0$ this ε -problem is expected to converge in some sense to the geodesic problem on M.

The original Schrödinger problem [17], or rather, its dynamical equivalent (sometimes referred to as the Yasue problem [9, 5]), can be rewritten as a particular instanciation of (1) when (M, g) is the Wasserstein space of probability measures, the potential V is given by the Boltzmann entropy $H(\rho) = \int \rho \log \rho$, and its Wasserstein gradient $I(\rho) = ||\nabla_W H(\rho)||^2 = \int |\nabla \log \rho|^2 \rho$ is the Fisher information functional. This was only recently recognized as an *entropic regularization* of the Monge-Kantorovich problem [10] and led to spectacular developments in computational optimal transport [16]. Recently, attempts have been made to develop an optimal transport theory for *quantum* objects, namely (symmetric positive-definite) matrixvalued measures, see [6] and references therein. In [13] we studied (1) precisely in the corresponding noncommutative Fisher-Rao space of matrix-measures. In this note we aim at providing a comprehensive introduction to this specific setting, in particular we wish to emphasize the construction based on Gaussian optimal transport that will lead to some specific entropy and quantum Fisher information functionals later on.

Section 1 briefly reviews classical optimal transport and the original Schrödinger problem. In Section 2 we discuss Gaussian optimal transport and the corresponding Bures-Wasserstein geometry. This then suggests the natural construction of the Fisher-Rao space detailed in Section 3. We introduce in Section 4 the resulting entropy and Fisher information functionals dictated by the previous construction. We also compute explicitly the induced heat flow, and finally discuss the Fisher-Rao-Schrödinger problem.

Key words and phrases. Schrödinger problem; Fisher-Rao space; optimal transport; Fisher information; entropy.

1. Optimal transport and the Schrödinger problem

In its fluid-mechanical *Benamou-Brenier* formulation [3], the quadratic *Wasser*stein distance between probability measures $\rho_0, \rho_1 \in \mathcal{P}(\Omega)$ over a smooth domain $\Omega \subset \mathbb{R}^d$ reads

$$\mathcal{W}^{2}(\rho_{0},\rho_{1}) = \min_{\rho,v} \left\{ \int_{0}^{1} \int_{\Omega} |v_{t}(x)|^{2} \rho_{t}(x) \mathrm{d}x \, \mathrm{d}t \right.$$

s.t. $\partial_{t} \rho_{t} + \operatorname{div}(\rho_{t} v_{t}) = 0$ in $(0,1) \times \Omega$ and $\rho|_{t=0,1} = \rho_{0,1} \right\}$ (2)

(supplemented with homogeneous no-flux boundary conditions on $\partial\Omega$ if needed in order to ensure mass conservation). This can be seen as an optimal control problem, where the control $v = v_t(x)$ is used to drive the system from ρ_0 to ρ_1 while minimizing the overall kinetic energy. We refer to [19] for a gentle yet comprehensive introduction to optimal transport.

As originally formulated by E. Schrödinger himself in [17], the Schrödinger problem roughly consists in determining the most likely evolution of a system for $t \in [0, 1]$ in an ambient noisy environment at temperature $\varepsilon > 0$, given the observation of its statistical distribution at times t = 0 and t = 1. Here we shall rather focus on the equivalent dynamical problem

$$\min_{\rho,\upsilon} \left\{ \int_0^1 \int_\Omega |v_t(x)|^2 \rho_t(x) \mathrm{d}x \, \mathrm{d}t + \varepsilon^2 \int_0^1 \int_\Omega |\nabla \log \rho_t(x)|^2 \rho_t(x) \mathrm{d}x \, \mathrm{d}t \\
\text{s.t.} \quad \partial_t \rho_t + \operatorname{div}(\rho_t v_t) = 0 \text{ in } (0,1) \times \Omega \quad \text{and} \quad \rho|_{t=0,1} = \rho_{0,1} \right\} \quad (3)$$

(again supplemented with Neumann boundary conditions on $\partial\Omega$ if needed). This is sometimes called the Yasue problem, cf. [9, 5]. It is known [11, 10, 2] that the noisy problem Gamma-converges towards deterministic optimal transport in the smalltemperature limit $\varepsilon \to 0$, and this actually holds in a much more general metric setting than just optimal transport [14]. We also refer to [20, 1] for connections with Euclidean quantum mechanics, and to the survey [5] for an optimal-control perspective.

Following F.Otto [15], an important feature of optimal transport is that one can view $\mathcal{P}(\Omega)$ as a (formal) Riemannian manifold, whose Riemannian distance coincides with the Wasserstein distance (2). This relies upon the identification of infinitesimal variations ξ with (gradients of) Kantorovich potentials $\phi : \Omega \to \mathbb{R}$ by uniquely selecting the velocity field $v = \nabla \phi$ with minimal kinetic energy $\int_{\Omega} \rho |v|^2$, given $-\operatorname{div}(\rho v) = \xi$. Practically speaking, this means that tangent vectors $\xi \in T_{\rho}\mathcal{P}$ at a point ρ are identified with scalar potentials ϕ_{ξ} via the Onsager operator $\Delta_{\rho} = \operatorname{div}(\rho \nabla \cdot)$ and the elliptic equation

$$-\operatorname{div}(\rho\nabla\phi_{\xi}) = \xi,\tag{4}$$

see [19] for details. Accordingly, the linear heat flow $\partial_t \rho = \Delta \rho$ can be identified [8] as the Wasserstein gradient flow $\frac{d\rho}{dt} = -\operatorname{grad}_{\mathcal{W}} H(\rho)$ of the *Boltzmann entropy*

$$H(\rho) = \int_{\Omega} \rho(x) \log(\rho(x)) dx.$$

Here $\operatorname{grad}_{\mathcal{W}}$ denotes the gradient computed with respect to Otto's Riemannian structure. Moreover, the *Fisher information* functional

$$F(\rho) = \int_{\Omega} |\nabla \log \rho(x)|^2 \rho(x) \, \mathrm{d}x = \|\operatorname{grad}_{\mathcal{W}} H(\rho)\|^2$$
(5)

appearing in (3) coincides with the squared gradient of the entropy, or equivalently with the dissipation rate $-\frac{dH}{dt} = F$ of H along its own gradient flow. This identification (5) shows that (3) can indeed be written as a particular case of (1) with respect to the Wasserstein geometry.

2. The Bures-Wasserstein distance and Gaussian optimal transport

When $\rho_0 = \mathcal{N}(m_0, A_0), \rho_1 = \mathcal{N}(m_1, A_1)$ are Gaussian measures with means $m_i \in \mathbb{R}^d$ and covariance matrices $A_i \in \mathcal{S}_d^{++}(\mathbb{R})$ (the space of symmetric positive definite matrices), the optimal transport problem (2) is explicitly solvable [18] and the Wasserstein distance can be computed as

$$\mathcal{W}^2(\rho_0, \rho_1) = |m_1 - m_0|^2 + \mathfrak{B}^2(A_0, A_1).$$
(6)

Here \mathfrak{B} is the *Bures distance*

$$\mathfrak{B}^{2}(A_{0}, A_{1}) = \min_{RR^{t} = \mathrm{Id}} |A_{1}^{\frac{1}{2}} - RA_{0}^{\frac{1}{2}}|^{2} = \mathrm{tr} A_{0} + \mathrm{tr} A_{1} - 2 \,\mathrm{tr} \left(\left(A_{0}^{\frac{1}{2}} A_{1} A_{0}^{\frac{1}{2}} \right)^{\frac{1}{2}} \right), \quad (7)$$

and $|M|^2 = \operatorname{tr}(MM^t)$ is the Euclidean norm of a matrix $M \in \mathbb{R}^{d \times d}$ corresponding to the Frobenius scalar product $\langle M, N \rangle = \operatorname{tr}(MN^t)$. The Bures distance, sometimes also called the *Helstrom metric*, can be considered as a quantum generalization of the usual Fisher information metric [4]. Since the Euclidean geometry of the translational part $|m_1 - m_0|^2$ in (6) is trivially flat, we restrict for simplicity to centered Gaussians $m_i = 0$. We denote accordingly

$$\mathfrak{S}_0 = \left\{ \rho \in \mathcal{P}(\mathbb{R}^d) : \qquad \rho = \mathcal{N}(0, A) \text{ for some } A \in \mathcal{S}_d^{++}(\mathbb{R}) \right\}$$

the statistical manifold of centered Gaussians and simply write $\rho = \mathcal{N}(0, A) = \mathcal{N}(A)$. Note that, for positive-definite matrices $A \in \mathcal{S}_d^{++}(\mathbb{R})$, the space of infinitesimal perturbations (the tangent plane $T_A \mathcal{S}_d^{++}(\mathbb{R})$) is the whole space of symmetric matrices $U \in \mathcal{S}_d(\mathbb{R})$. Semi-definite matrices are somehow degenerate (extremal) points of $\mathcal{S}_d^{++}(\mathbb{R})$, but it is worth stressing that that (7) makes sense even for semi-definite $A_0, A_1 \in \mathcal{S}_d^+(\mathbb{R})$.

Going back to optimal transport, it is well-known [18, 12] that the statistical manifold \mathfrak{S}_0 is a totally geodesic submanifold in the Wasserstein space. In other words, if $\rho_0 = \mathcal{N}(A_0)$ and $\rho_1 = \mathcal{N}(A_1)$ are Gaussians, then minimizers of the dynamical problem (2) remain Gaussian, i-e $\rho_t = \mathcal{N}(0, A_t)$ for some suitable $A_t \in \mathcal{S}_d^{++}$. Thus one expects that \mathfrak{S}_0 , or rather $\mathcal{S}_d^{++}(\mathbb{R})$, can be equipped with a well-chosen, finite-dimensional Riemannian metric such that the induced Riemannian distance coincides with the Wasserstein distance between corresponding Gaussians. This is indeed the case [18], and the Riemannian metric is explicitly given at a point $\rho = \mathcal{N}(A)$ by

$$g_A(U,V) = \operatorname{tr}(UAV) = \langle AU, V \rangle = \langle AV, U \rangle, \qquad U, V \in T_A \mathcal{S}_d^{++}(\mathbb{R}) \cong \mathcal{S}_d(\mathbb{R}).$$

Note that, if $g_{Eucl}(\xi,\zeta) = \langle \xi,\zeta \rangle = \operatorname{tr}(\xi\zeta^t)$ is the Euclidean scalar product, the corresponding Riesz isomorphism $U \mapsto \xi_U$ (at a point $A \in \mathcal{S}_d^{++}$) identifying

 $g_{Eucl}(\xi_U,\xi_V) = g_A(U,V)$ is given by $\xi_U = (AU)^{sym} = \frac{AU+UA}{2}$. (This is the exact equivalent of the elliptic correspondence (4) for optimal transport.) As a consequence the Bures-Wasserstein distance can be computed [18, 13] as

$$4\mathfrak{B}^2(A_0, A_1) = \min_{A, U} \left\{ \int_0^1 \langle A_t U_t, U_t \rangle \,\mathrm{d}t \quad \text{s.t.} \quad \frac{dA_t}{dt} = (A_t U_t)^{sym} \right\}, \quad A_0, A_1 \in \mathcal{S}_d^{++}$$
(8)

Up to the scaling factor 4 this an exact counterpart of (2), where $\frac{dA_t}{dt} = (A_t U_t)^{sym}$ plays the role of the continuity equation $\partial_t \rho_t + \operatorname{div}(\rho_t v_t) = 0$ and $\langle A_t U_t, U_t \rangle$ substitutes for the kinetic energy density $\int_{\Omega} \rho_t |v_t|^2$.

3. The non-commutative Fisher-Rao space

Given two probability densities $\rho_0, \rho_1 \in \mathcal{P}(\mathcal{D})$ on a domain $\mathcal{D} \subset \mathbb{R}^N$ (not to be confused with the previous $\Omega = \mathbb{R}^d$ for Gaussian optimal transport, codomain over which our matrices $A \in \mathcal{S}_d^{++}$ were built), the *scalar* Fisher-Rao distance is classically defined as the Riemannian distance induced by the Fisher information metric

$$\mathsf{FR}^{2}(\rho_{0},\rho_{1}) = \min_{\rho} \left\{ \int_{0}^{1} \int_{\mathcal{D}} \left| \frac{\partial \log \rho_{t}(x)}{\partial t} \right|^{2} \rho_{t}(x) \, \mathrm{d}x \, \mathrm{d}t, \quad \text{s.t.} \ \rho_{t} \in \mathcal{P}(\mathcal{D}) \right\}$$
$$= \min_{\rho,u} \left\{ \int_{0}^{1} \int_{\mathcal{D}} \left| u_{t}(x) \right|^{2} \rho_{t}(x) \, \mathrm{d}x \, \mathrm{d}t \quad \text{s.t.} \ \partial_{t} \rho_{t} = \rho_{t} u_{t} \text{ and } \rho_{t} \in \mathcal{P}(\mathcal{D}) \right\}.$$
(9)

Note that the unit-mass condition $\rho_t \in \mathcal{P}(\mathcal{D})$ is enforced here as a hard constraint. (Without this constraint, (9) actually defines the *Hellinger distance* H between arbitrary nonnegative measures [13].)

The strong structural similarity between (8) (9) suggest a natural extension of the latter scalar setting to *matrix-valued* measures. More precisely, the *Fisher-Rao* space is the space of (*d*-dimensional) positive *semi*-definite valued measures over \mathcal{D} with unit mass

$$\mathbb{P}(\mathcal{D}) = \left\{ A \in \mathcal{M}(\mathcal{D}; \mathcal{S}_d^+(\mathbb{R})) \quad \text{s.t.} \quad \int_{\mathcal{D}} \operatorname{tr} A(x) \, \mathrm{d}x = 1 \right\}.$$

This can be thought of as the space of noncommutative probability measures (see [13] for a short discussion on the connections with *free probability* theory and C^* -algebras). The *matricial* Fisher-Rao distance is then defined as

$$\mathsf{FR}^{2}(A_{0}, A_{1}) = \min_{A, U} \left\{ \int_{0}^{1} \int_{\mathcal{D}} \langle A_{t}(x) U_{t}(x), U_{t}(x) \rangle \, \mathrm{d}x \, \mathrm{d}t \right.$$

s.t. $\partial_{t} A_{t} = (A_{t} U_{t})^{sym} \text{ in } (0, 1) \times \mathcal{D} \text{ and } A_{t} \in \mathbb{P}(\mathcal{D}) \left. \right\},$ (10)

which is indeed a higher-dimensional extension of (9) and should be compared again with (2). This clearly suggests viewing $\mathbb{P}(\mathcal{D})$ as a (formal, infinite-dimensional) Riemannian manifold with tangent space and norm

$$T_A \mathbb{P} = \left\{ \xi_U = (AU)^{sym} : \quad U \in L^2 \left(A(x) dx; \, \mathcal{S}_d^+ \right) \text{ and } \int_{\mathcal{D}} \langle A(x), U(x) \rangle \, \mathrm{d}x = 0 \right\}$$

$$\|\xi_U\|_A^2 = \|U\|_{L^2_A}^2 = \int_{\mathcal{D}} \langle A(x)U(x), U(x)\rangle \,\mathrm{d}x.$$
(11)

Note once again that we imposed the unit-mass condition $A_t \in \mathbb{P}$ as a hard constraint in (10), which results in the zero-average condition $\int \langle A, U \rangle = 0$ for tangent vectors U. Removing the mass constraint leads to (an extension of) the Hellinger distance H between arbitrary PSD matrix-valued measures. (The Hellinger space $(\mathbb{H}^+, \mathsf{H})$ is a metric cone over the "unit-sphere" (\mathbb{P}, FR) but we shall not discuss this rich geometry any further, see again [13].)

The corresponding Riemannian gradients of internal-energy functionals $\mathcal{F}(A) = \int_{\mathcal{D}} F(A(x)) dx$ can be explicited [13] as

$$\operatorname{grad}_{\mathsf{FR}}\mathcal{F}(A) = \left(A(x)F'(A(x))\right)^{sym} - \operatorname{tr}\left(\int_{\mathcal{D}} A(y)F'(A(y))\mathrm{d}y\right)A(x).$$
(12)

Here F'(A) stands for the usual first variation of the matricial function F(A), computed with respect to the standard Euclidean (Frobenius) scalar product.

4. EXTENDED ENTROPY, THE HEAT FLOW, AND THE SCHRÖDINGER PROBLEM

From now on we always endow \mathbb{P} with the Riemannian metric given by (11). Once this metric is fixed, the geometric Schrödinger problem (1) will be fully determined as soon as we choose a driving potential V on \mathbb{P} . One should ask now:

What is a good choice of "the" canonical entropy on the Fisher-Rao space?

A first natural guess would be based on the classical (negative) von Neumann entropy $S(A) = tr(A \log A)$ from quantum statistical mechanics. However, the functional $\mathcal{S}(A) = \int_{\mathcal{D}} S(A(x)) \, dx$ lacks geodesic convexity with respect to our ambient Fisher-Rao metric [13], and this makes it unfit for studying a suitable Schrödinger problem (see [14] for the connection between the validity of the Gamma-convergence in the limit $\varepsilon \to 0$ in (1) and the necessity of geodesic convexity of V in a fairly general metric context).

It turns out that there is a second natural choice, dictated by our previous construction based on Gaussian optimal transport and the canonical Boltzmann entropy $H(\rho) = \int \rho \log \rho$ on $\mathcal{P}(\mathbb{R}^d)$. Indeed, consider $A \in \mathcal{S}_d^{++}$ and write $\rho_A = \mathcal{N}(A)$ for the corresponding Gaussian. Assuming for simplicity that \mathcal{D} has measure $|\mathcal{D}| = \frac{1}{d}$ so that tr $\int_{\mathcal{D}} \operatorname{Id} dx = 1$, we choose as a reference measure the generalized uniform Lebesgue measure $\operatorname{Id} \in \mathbb{P}$ (more general reference measures can also be covered, see [13]). An explicit computation then gives the Boltzmann entropy of ρ_A relatively to $\rho_I = \mathcal{N}(\operatorname{Id})$ as

$$E(A) = H(\rho_A|\rho_I) = \int_{\mathbb{R}^d} \frac{\rho_A(y)}{\rho_I(y)} \log\left(\frac{\rho_A(y)}{\rho_I(y)}\right) \rho_I(y) \,\mathrm{d}y = \frac{1}{2} \operatorname{tr}[A - \log A - \operatorname{Id}].$$
(13)

Note carefully that $E(A) = +\infty$ as soon as A is only positive *semi*-definite (due to $-\operatorname{tr} \log A = -\sum \log \lambda_i = +\infty$ if any of the eigenvalues $\lambda_i = 0$), and that by convexity $E(A) \ge E(\operatorname{Id}) = 0$ is minimal only when $A = \operatorname{Id}$. Our canonical definition of *the* entropy on the Fisher-Rao space is then simply

$$\mathcal{E}(A) = \int_{\mathcal{D}} E(A(x)) \, \mathrm{d}x = -\frac{1}{2} \operatorname{tr} \int_{\mathcal{D}} \log A(x) \, \mathrm{d}x, \quad \text{for } A = A(x) \in \mathbb{P}.$$
(14)

(The terms $\int \text{tr} A = 1 = \int \text{tr} \text{Id}$ cancel out in (13) due to our mass normalization.) Since the (Euclidean) first variation of (13) is $E'(A) = \frac{1}{2}(\text{Id} - A^{-1})$, (12) shows that the Fisher-Rao gradient flow $\frac{dA_t}{dt} = -\operatorname{grad}_{\mathsf{FR}}\mathcal{E}(A_t)$ reads

$$\partial_t A_t = -\left(A_t \frac{1}{2} [\mathrm{Id} - A_t^{-1}]\right)^{sym} + \mathrm{tr}\left(\int_{\mathcal{D}} A_t \frac{1}{2} [\mathrm{Id} - A_t^{-1}] \mathrm{d}y\right) A_t$$
$$= \frac{1}{2} (\mathrm{Id} - A_t)^{sym} + \frac{1}{2} \mathrm{tr}\left(\int_{\mathcal{D}} (A_t - \mathrm{Id}) \mathrm{d}y\right) A_t = \frac{1}{2} (\mathrm{Id} - A_t). \quad (15)$$

This generalized "heat flow" is consistent with the observation that, whenever $\rho_0 = \mathcal{N}(A_0)$ is Gaussian, the solution $\rho_t = \mathcal{N}(A_t)$ of the Fokker-Planck equation (the Wasserstein gradient flow of the relative entropy $\rho \mapsto H(\rho|\rho_I)$)

$$\partial_t \rho = \Delta \rho - \operatorname{div}(\rho \nabla \log \rho_I) \tag{16}$$

remains Gaussian with precisely $\frac{dA_t}{dt} = \frac{1}{2}(\mathrm{Id} - A_t)$. Our extended Fisher information functional can then be defined as the dissipation rate of the entropy \mathcal{E} along the "heat flow" $\frac{dA_t}{dt} = -\operatorname{grad}_{\mathsf{FR}} \mathcal{E}(A_t)$ given by (15), namely

$$\mathcal{F}(A_t) = -\frac{d}{dt} \mathcal{E}(A_t) = \frac{1}{2} \frac{d}{dt} \operatorname{tr} \int_{\mathcal{D}} \log A_t(x) \, \mathrm{d}x = \frac{1}{2} \operatorname{tr} \int_{\mathcal{D}} A_t^{-1}(x) \partial_t A_t(x) \, \mathrm{d}x = \frac{1}{2} \operatorname{tr} \int_{\mathcal{D}} A_t^{-1}(x) \frac{1}{2} [\operatorname{Id} - A_t(x)] \, \mathrm{d}x = \frac{1}{4} \left(\operatorname{tr} \int_{\mathcal{D}} A_t^{-1}(x) \, \mathrm{d}x - 1 \right).$$
(17)

Equivalently and consistently, $\mathcal{F}(A) = \| \operatorname{grad}_{\mathsf{FR}} \mathcal{E}(A) \|^2$. Note that $\mathcal{F}(A_t) > F(\operatorname{Id}) = 0$ and $\mathcal{E}(A_t) > \mathcal{E}(\operatorname{Id}) = 0$ unless $A_t(x) \equiv \operatorname{Id}$, which is of course consistent with the expected long-time behavior $A_t \to \operatorname{Id}$ for (15) as $t \to \infty$ (or equivalently $\rho_t \to \rho_I = \mathcal{N}(\operatorname{Id})$ for Gaussian solutions of the Fokker-Planck equation (16)).

With these explicit representations (11)(17) of the quantum (i-e matricial) Fisher-Rao metric and Fisher information, we can now make sense of the geometric Schrödinger problem (1) in the noncommutative Fisher-Rao space as

$$\min_{A} \left\{ \int_{0}^{1} \int_{\mathcal{D}} \langle A_{t}(x) U_{t}(x), U_{t}(x) \rangle \, \mathrm{d}x \mathrm{d}t + \frac{\varepsilon^{2}}{4} \int_{0}^{1} \left(\mathrm{tr} \int_{\mathcal{D}} A_{t}^{-1}(x) \, \mathrm{d}x - 1 \right) \\
\text{s.t.} \qquad \partial_{t} A_{t} = (A_{t} U_{t})^{sym} \quad \text{and} \quad A_{t} \in \mathbb{P} \text{ for all } t \in [0, 1] \right\}, \quad (18)$$

with fixed endpoints $A_0, A_t \in \mathbb{P}(\mathcal{D})$. Let us mention at this stage that a different entropic regularization of Gaussian optimal transport was investigated in [7] in a static framework and for a much simpler setting, namely when $\mathcal{D} = \{x\}$ is a single point

As could be expected from the above geometric machinery, we have now

Theorem ([13]). In the limit $\varepsilon \to 0$ and for fixed endpoints $A_0, A_1 \in \mathbb{P}$, the ε -functional in (18) Gamma-converges towards the kinetic functional in (10) for the uniform convergence on $C([0,1];\mathbb{P})$.

We omit the details for the sake of brevity, but let us point out that our proof leverages fully explicit properties of the geometric heat flow (15). A particular byproduct of our analysis is the $\frac{1}{2}$ -geodesic convexity of the entropy (14) in the Fisher-Rao geometry, which was already established in [12] by formal Riemannian computations. The key argument builds up on a Lagrangian construction originally due to A. Baradat in [2]. Finally, we recently extended the result to arbitrary metric spaces and general entropy functionals [14] (provided a suitable heat flow is available), and we showed moreover that geodesic convexity is actually necessary (and almost sufficient) for the Γ -convergence.

References

- S. Albeverio, K. Yasue, J.C. Zambrini. Euclidean quantum mechanics: analytical approach. Annales de l'IHP Physique théorique 50(3):259–308, 1989.
- [2] A. Baradat and L. Monsaingeon. Small Noise Limit and Convexity for Generalized Incompressible Flows, Schrödinger Problems, and Optimal Transport. Archive for Rational Mechanics and Analysis 235:1357–1403, 2020.
- [3] J.D. Benamou, Y. Brenier. A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem. *Numerische Mathematik*, 84(3):375–393, 2000.
- [4] D. Bures. An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite ω^{*}-algebras. Transactions of the AMS, 135:199–212, 1969.
- [5] Y. Chen, T.T. Georgiou, M. Pavon. Stochastic control liasons: Richard Sinkhorn meets Gaspard Monge on a Schroedinger bridge. arXiv preprint arXiv:2005.10963, 2020.
- [6] Y. Chen, W. Gangbo, T. Georgiou, A. Tannenbaum. On the matrix Monge-Kantorovich problem. European Journal of Applied Mathematics, 31(4):574–600, 2020.
- [7] H. Janati, B. Muzellec, G. Peyré, M. Cuturi. Entropic optimal transport between (unbalanced) Gaussian measures has a closed form. arXiv preprint arXiv:2006.02572, 2020.
- [8] R. Jordan, D. Kinderlehrer, F. Otto. The variational formulation of the Fokker-Planck equation. SIAM journal on mathematical analysis, 29(1):1–17, 1998.
- [9] F. Léger. A Geometric Perspective on Regularized Optimal Transport. Journal of Dynamics and Differential Equations 31:1777–1791, 2019.
- [10] C. Léonard. A survey of the Schrödinger problem and some of its connections with optimal transport. Discrete Contin. Dyn. Syst. A, 34:1533–1574, 2014.
- [11] T. Mikami. Monge's problem with a quadratic cost by the zero-noise limit of h-path processes. Probability Theory and Related Fields, 129(2):245-260, 2004.
- [12] K. Modin. Geometry of matrix decompositions seen through optimal transport and information geometry. Journal of Geometric Mechanics, 9(3):335-390, 2017.
- [13] L. Monsaingeon and D. Vorotnikov. The Schrödinger problem on the non-commutative Fisher-Rao space. Calculus of Variations and Partial Differential Equations, 60(1), art. no. 14, 2021.
- [14] L. Monsaingeon, D. Vorotnikov, and L. Tamanini. The dynamical Schrödinger problem in abstract metric spaces. *Preprint* arXiv:2012.12005, 2020.
- [15] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. Communications in Partial Differential Equations, 23(1-2):101-174, 2001.
- [16] G. Peyré, M. Cuturi. Computational optimal transport: With applications to data science. Foundations and Trends[®] in Machine Learning, 11(5-6):355–607, 2019.
- [17] E. Schrödinger. Über die umkehrung der naturgesetze. Angewandte Chemie, 44(30):636–636 1931.
- [18] A. Takatsu. Wasserstein geometry of Gaussian measures. Osaka Journal of Mathematics, 48(4):1005–1026, 2011.
- [19] C. Villani. Topics in optimal transportation. AMS Graduate Studies in Mathematics (No. 58), 2003.
- [20] J.C. Zambrini. Variational processes and stochastic versions of mechanics. Journal of Mathematical Physics, 27(9):2307–2330, 1986.

(L. Monsaingeon) GFM UNIVERSIDADE DE LISBOA, CAMPO GRANDE, EDIFÍCIO C6, 1749-016 LISBOA, PORTUGAL & IECL UNIVERSITÉ DE LORRAINE, F-54506 VANDOEUVRE-LÈS-NANCY CEDEX, FRANCE

Email address: leonard.monsaingeon@univ-lorraine.fr

(D. Vorotnikov) University of Coimbra, CMUC, Department of Mathematics, 3001-501 Coimbra, Portugal

Email address: mitvorot@mat.uc.pt