# Information cohomology of classical vector-valued observables 

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#### Abstract

We provide here a novel algebraic characterization of two information measures associated with a vector-valued random variable, its differential entropy and the dimension of the underlying space, purely based on their recursive properties (the chain rule and the nullity-rank theorem, respectively). More precisely, we compute the information cohomology of Baudot and Bennequin with coefficients in a module of continuous probabilistic functionals over a category that mixes discrete observables and continuous vector-valued observables, characterizing completely the 1-cocycles; evaluated on continuous laws, these cocycles are linear combinations of the differential entropy and the dimension.


Keywords: Information cohomology • Entropy • Dimension • Information measures • Topos theory

## 1 Introduction

Baudot and Bennequin [2] introduced information cohomology, and identified Shannon entropy as a nontrivial cohomology class in degree 1. This cohomology has an explicit description in terms of cocycles and coboundaries; the cocycle equations are a rule that relate different values of the cocycle. When its coefficients are a module of measurable probabilistic functionals on a category of discrete observables, Shannon's entropy defines a 1-cocycle and the aforementioned rule is simply the chain rule; moreover, the cocycle equations in every degree are systems of functional equations and one can use the techniques developed by Tverberg, Lee, Knappan, Aczél, Daróczy, etc. [1] to show that, in degree one, the entropy is the unique measurable, nontrivial solution. The theory thus gave a new algebraic characterization of this information measure based on topos theory $\grave{a}$ la Grothendieck, and showed that the chain rule is its defining algebraic property.

It is natural to wonder if a similar result holds for the differential entropy. We consider here information cohomology with coefficients in a presheaf of continuous probabilistic functionals on a category that mixes discrete and continuous (vector-valued) observables, and establish that every 1-cocycle, when evaluated
on probability measures absolutely continuous with respect to the Lebesgue measure, is a linear combination of the differential entropy and the dimension of the underlying space (the term continous has in this sentence three different meanings). We already showed that this was true for gaussian measures [9]; in that case, there is a finite dimensional parametrization of the laws, and we were able to use Fourier analysis to solve the 1-cocycle equations. Here we exploit that result, expressing any density as a limit of gaussian mixtures (i.e. convex combinations of gaussian densities), and then using the 1-cocycle condition to compute the value of a 1-cocycle on gaussian mixtures in terms of its value on discrete laws and gaussian laws. The result depends on the conjectural existence of a "well-behaved" class of probabilities and probabilistic functionals, see Section 3.2 .

The dimension appears here as an information quantity in its own right: its "chain rule" is the nullity-rank theorem. In retrospective, its role as an information measure is already suggested by old results in information theory. For instance, the expansion of Kolmogorov's $\varepsilon$-entropy $H_{\varepsilon}(\xi)$ of a continuous, $\mathbb{R}^{n}$ valued random variable $\xi$ "is determined first of all by the dimension of the space, and the differential entropy $h(\xi)$ appears only in the form of the second term of the expression for $H_{\varepsilon}(\xi)$." [7, Paper 3, p. 22]

## 2 Some known results about information cohomology

Given the severe length constraints, it is impossible to report here the motivations behind information cohomology, its relationship with traditional algebraic characterizations of entropy, and its topos-theoretic foundations. For that, the reader is referred to the introductions of [9] and [10]. We simply remind here a minimum of definitions in order to make sense of the characterization of 1cocycles that is used later in the article.

Let $\mathbf{S}$ be a partially ordered set (poset); we see it as a category, denoting the order relation by an arrow. It is supposed to have a terminal object $T$ and to satisfy the following property: whenever $X, Y, Z \in \mathrm{Ob} \mathbf{S}$ are such that $X \rightarrow Y$ and $X \rightarrow Z$, the categorical product $Y \wedge Z$ exists in $\mathbf{S}$. An object of $X$ of $\mathbf{S}$ (i.e. $X \in \mathrm{Ob} \mathbf{S})$ is interpreted as an observable, an arrow $X \rightarrow Y$ as $Y$ being coarser than $X$, and $Y \wedge Z$ as the joint measurement of $Y$ and $Z$.

The category $\mathbf{S}$ is just an algebraic way of encoding the relationships between observables. The measure-theoretic "implementation" of them comes in the form of a functor $\mathcal{E}: \mathbf{S} \rightarrow$ Meas that associates to each $X \in \mathrm{Ob} \mathbf{S}$ a measurable set $\mathcal{E}(X)=\left(E_{X}, \mathfrak{B}_{X}\right)$, and to each arrow $\pi: X \rightarrow Y$ in $\mathbf{S}$ a measurable surjection $\mathcal{E}(\pi): \mathcal{E}(X) \rightarrow \mathcal{E}(Y)$. To be consistent with the interpretations given above, one must suppose that $E_{\top} \cong\{*\}$ and that $\mathcal{E}(Y \wedge Z)$ is mapped injectively into $\mathcal{E}(Y) \times \mathcal{E}(Z)$ by $\mathcal{E}(Y \wedge Z \rightarrow Y) \times \mathcal{E}(Y \wedge Z \rightarrow Z)$. We consider mainly two examples: the discrete case, in which $E_{X}$ finite and $\mathfrak{B}_{X}$ the collection of its subsets, and the Euclidean case, in which $E_{X}$ is a Euclidean space and $\mathfrak{B}_{X}$ is its Borel $\sigma$-algebra. The pair $(\mathbf{S}, \mathcal{E})$ is an information structure.

Throughout this article, conditional probabilities are understood as disintegrations. Let $\nu$ a $\sigma$-finite measure on a measurable space $(E, \mathfrak{B})$, and $\xi$ a $\sigma$-finite measure on $\left(E_{T}, \mathfrak{B}_{T}\right)$. The measure $\nu$ has a disintegration $\left\{\nu_{t}\right\}_{t \in E_{T}}$ with respect to a measurable map $T: E \rightarrow E_{T}$ and $\xi$, or a $(T, \xi)$-disintegration, if each $\nu_{t}$ is a $\sigma$-finite measure on $\mathfrak{B}$ concentrated on $\{T=t\}$-i.e. $\nu_{t}(T \neq t)=0$ for $\xi$-almost every $t$ - and for each measurable nonnegative function $f: E \rightarrow \mathbb{R}$, the mapping $t \mapsto \int_{E} f \mathrm{~d} \nu_{t}$ is measurable and $\int_{E} f \mathrm{~d} \nu=\int_{E_{T}}\left(\int_{E} f(x) \mathrm{d} \nu_{t}(x)\right) \mathrm{d} \xi(t)$ [3].

We associate to each $X \in \mathrm{Ob} \mathbf{S}$ the set $\Pi(X)$ of probability measures on $\mathcal{E}(X)$ i.e. of possible laws of $X$, and to each arrow $\pi: X \rightarrow Y$ the marginalization map $\pi_{*}:=\Pi(\pi): \Pi(X) \rightarrow \Pi(Y)$ that maps $\rho$ to the image measure $\mathcal{E}(\pi)_{*}(\rho)$. More generally, we consider any subfunctor $\mathcal{Q}$ of $\Pi$ that is stable under conditioning: for all $X \in \operatorname{ObS}, \rho \in \mathcal{Q}(X)$, and $\pi: X \rightarrow Y,\left.\rho\right|_{Y=y}$ belongs to $\mathcal{Q}(X)$ for $\pi_{*} \rho$-almost every $y \in E_{Y}$, where $\left\{\left.\rho\right|_{Y=y}\right\}_{y \in E_{Y}}$ is the $\left(\mathcal{E} \pi, \pi_{*} \rho\right)$-disintegration of $\rho$.

We associate to each $X \in \mathrm{Ob} \mathbf{S}$ the set $\mathcal{S}_{X}=\{Y \mid X \rightarrow Y\}$, which is a monoid under the product $\wedge$ introduced above. The assignment $X \mapsto \mathcal{S}_{X}$ defines a contravariant functor (presheaf). The induced algebras $\mathcal{A}_{X}=\mathbb{R}\left[\mathcal{S}_{X}\right]$ give a presheaf $\mathcal{A}$. An $\mathcal{A}$-module is a collection of modules $\mathcal{M}_{X}$ over $\mathcal{A}_{X}$, for each $X \in \mathrm{Ob} \mathbf{S}$, with an action that is "natural" in $X$. The main example is the following: for any adapted probability functor $\mathcal{Q}: \mathbf{S} \rightarrow$ Meas, one introduces a contravariant functor $\mathcal{F}=\mathcal{F}(\mathcal{Q})$ declaring that $\mathcal{F}(X)$ are the measurable functions on $\mathcal{Q}(X)$, and $\mathcal{F}(\pi)$ is precomposition with $\mathcal{Q}(\pi)$ for each morphism $\pi$ in $\mathbf{S}$. The monoid $\mathcal{S}_{X}$ acts on $\mathcal{F}(X)$ by the rule:

$$
\begin{equation*}
\forall Y \in \mathcal{S}_{X}, \forall \rho \in \mathcal{Q}(X), \quad Y . \phi(\rho)=\int_{E_{Y}} \phi\left(\left.\rho\right|_{Y=y}\right) \mathrm{d} \pi_{*}^{Y X} \rho(y) \tag{1}
\end{equation*}
$$

where $\pi_{*}^{Y X}$ stands for the marginalization $\mathcal{Q}\left(\pi^{Y X}\right)$ induced by $\pi^{Y X}: X \rightarrow Y$ in $\mathbf{S}$. This action can be extended by linearity to $\mathcal{A}_{X}$ and is natural in $X$.

In [10], the information cohomology $H^{\bullet}(\mathbf{S}, \mathcal{F})$ is defined using derived functors in the category of $\mathcal{A}$-modules, and then described explicitly, for each degree $n \geq 0$, as a quotient of $n$-cocycles by $n$-coboundaries. For $n=1$, the coboundaries vanish, so we simply have to describe the cocycles. Let $\mathcal{B}_{1}(X)$ be the $\mathcal{A}_{X}$-module freely generated by a collection of bracketed symbols $\{[Y]\}_{Y \in \mathcal{S}_{X}}$; an arrow $\pi: X \rightarrow Y$ induces an inclusion $\mathcal{B}_{1}(Y) \hookrightarrow \mathcal{B}_{1}(X)$, so $\mathcal{B}_{1}$ is a presheaf. A 1-cochain is a natural transformations $\varphi: \mathcal{B}_{1} \Rightarrow \mathcal{F}$, with components $\varphi_{X}: \mathcal{B}_{1}(X) \rightarrow \mathcal{F}(X) ;$ we use $\varphi_{X}[Y]$ as a shorthand for $\varphi_{X}([Y])$. The naturality implies that $\varphi_{X}[Z](\rho)$ equals $\varphi_{Z}[Z]\left(\pi_{*}^{Z X} \rho\right)$, a property that [2] called locality; sometimes we write $\Phi_{Z}$ instead of $\varphi_{Z}[Z]$. A 1-cochain $\varphi$ is a 1-cocycle iff

$$
\begin{equation*}
\forall X \in \mathrm{Ob} \mathbf{S}, \forall X_{1}, X_{2} \in \mathcal{S}_{X}, \quad \varphi_{X}\left[X_{1} \wedge X_{2}\right]=X_{1} \cdot \varphi_{X}\left[X_{2}\right]+\varphi_{X}\left[X_{1}\right] \tag{2}
\end{equation*}
$$

Remark that this is an equality of functions in $\mathcal{Q}(X)$.
An information structure is finite if for all $X \in \mathrm{Ob} S, E_{X}$ is finite. In this case, [10, Prop. 4.5.7] shows that, whenever an object $X$ can be written as a product $Y \wedge Z$ and $E_{X}$ is "close" to $E_{Y} \times E_{Z}$, as formalized by the definition of
nondegenerate product [10, Def. 4.5.6], then there exists $K \in \mathbb{R}$ such that for all $W \in \mathcal{S}_{X}$ and $\rho$ in $\mathcal{Q}(Z)$

$$
\begin{equation*}
\Phi_{W}(\rho)=-K \sum_{w \in E_{W}} \rho(w) \log \rho(w) \tag{3}
\end{equation*}
$$

The continuous case is of course more delicate. In the case of $\mathcal{E}$ taking values in vector spaces, and $\mathcal{Q}$ made of gaussian laws, 9 treated it as follows. We start with a vector space $E$ with Euclidean metric $M$, and a poset $\mathbf{S}$ of vector subspaces of $E$, ordered by inclusion, satisfying the hypotheses stated above; remark that $\wedge$ corresponds to intersection. Then we introduce $\mathcal{E}$ by $V \in \mathrm{Ob} \mathbf{S} \mapsto E_{V}:=E / V$, and further identify $E_{V}$ is $V^{\perp}$ using the metric (so that we only deal with vector subspaces of $E$ ). We also introduce a sheaf $\mathcal{N}$, such that $\mathcal{N}(X)$ consists of affine subspaces of $E_{X}$ and is closed under intersections; the sheaf is supposed to be closed under the projections induced by $\mathcal{E}$ and to contain the fibers of all these projections. On each affine subspace $N \in \mathcal{N}(X)$ there is a unique Lebesgue measure $\mu_{X, N}$ induced by the metric $M$. We consider a sheaf $\mathcal{Q}$ such that $\mathcal{Q}(X)$ are probabilities measures $\rho$ that are absolutely continuous with respect to $\mu_{X, N}$ for some $N \in \mathcal{N}(X)$ and have a gaussian density with respect to it. We also introduce a subfunctor $\mathcal{F}^{\prime}$ of $\mathcal{F}$ made of functions that grow moderately (i.e. at most polynomially) with respect to the mean, in such a way that the integral (1) is always convergent. Ref. 9] called a triple $(\mathbf{S}, \mathcal{E}, \mathcal{N})$ sufficiently rich when there are "enough supports", in the sense that one can perform marginalization and conditioning with respect to projections on subspaces generated by elements of at least two different bases of $E$. In this case, we showed that every 1-cocycle $\varphi$, with coefficients in $\mathcal{F}^{\prime}(\mathcal{Q})$, there are real constants $a$ and $c$ such that, for every $X \in \mathrm{Ob} \mathbf{S}$ and every gaussian law $\rho$ with support $E_{X}$ and variance $\Sigma_{\rho}$ (a nondegenerate, symmetric, positive bilinear form on $E_{X}^{*}$ ),

$$
\begin{equation*}
\Phi_{X}(\rho)=a \operatorname{det}\left(\Sigma_{\rho}\right)+c \cdot \operatorname{dim}\left(E_{X}\right) \tag{4}
\end{equation*}
$$

Moreover, $\varphi$ its completely determined by its behavior on nondegenerate laws. (The measure $\mu_{X}=\mu_{X, E_{X}}$ is enough to define the determinant $\operatorname{det}\left(\Sigma_{\rho}\right)$, 9 , Sec. 11.2.1], but the latter can also be computed w.r.t. a basis of $E_{X}$ such that $\left.M\right|_{E_{X}}$ is represented by the identity matrix.)

## 3 An extended model

### 3.1 Information structure, supports, and reference measures

In this section, we introduce a more general model, that allows us to "mix" discrete and continuous variables. It is simply the product of a structure of discrete observables and a structure of continuous ones.

Let $\left(\mathbf{S}_{d}, \mathcal{E}_{d}\right)$ be a finite information structure, such that for every $n \in \mathbb{N}$, there exist $Y_{n} \in \mathrm{Ob} \mathbf{S}_{d}$ with $\left|E_{Y_{n}}\right| \cong\{1,2, \ldots, n\}=:[n]$, and for every $X \in \mathrm{Ob} \mathbf{S}_{d}$, there
is a $Z \in \mathrm{Ob} \mathbf{S}_{d}$ that can be written as non-degenerate product and such that $Z \rightarrow X$; this implies that (3) holds for every $W \in \operatorname{Ob} \mathbf{S}_{d}$. Let $\left(\mathbf{S}_{c}, \mathcal{E}_{c}, \mathcal{N}_{c}\right)$ be a sufficiently rich triple in the sense of the previous section, associated to a vector space $E$ with metric $M$, so that (4) holds for every $X \in \mathrm{Ob} \mathbf{S}_{c}$.

Let $\left(\mathbf{S}, \mathcal{E}: \mathbf{S} \rightarrow\right.$ Meas) be the product $\left(\mathbf{S}_{c}, \mathcal{E}_{c}\right) \times\left(\mathbf{S}_{d}, \mathcal{E}_{d}\right)$ in the category of information structures, see [10, Prop. 2.2.2]. By definition, every object $X \in \mathbf{S}$ has the form $\left\langle X_{c}, X_{d}\right\rangle$ for $X_{c} \in \mathrm{Ob} \mathbf{S}_{c}$ and $X_{d} \in \mathrm{Ob} \mathbf{S}_{d}$, and $\mathcal{E}(X)=\mathcal{E}\left(X_{1}\right) \times$ $\mathcal{E}\left(X_{2}\right)$, and there is an arrow $\pi:\left\langle X_{1}, X_{2}\right\rangle \rightarrow\left\langle Y_{1}, Y_{2}\right\rangle$ in $\mathbf{S}$ if and only if there exist arrows $\pi_{1}: X_{1} \rightarrow Y_{1}$ in $\mathbf{S}_{c}$ and $\pi_{2}: X_{2} \rightarrow Y_{2}$ in $\mathbf{S}_{d}$; under the functor $\mathcal{E}$, such $\pi$ is mapped to $\mathcal{E}_{c}\left(\pi_{1}\right) \times \mathcal{E}_{d}\left(\pi_{2}\right)$. There is an embedding in the sense of information structures, see [9, Sec. 1.4], $\mathbf{S}_{c} \hookrightarrow \mathbf{S}, X \rightarrow\langle X, \mathbf{1}\rangle$; we call its image the "continuous sector" of $\mathbf{S}$; we write $X$ instead of $\langle X, \mathbf{1}\rangle$ and $\mathcal{E}(X)$ instead of $\mathcal{E}(\langle X, \mathbf{1}\rangle)=E(X) \times\{*\}$. The "discrete sector" is defined analogously.

We extend the sheaf of supports $\mathcal{N}_{c}$ to the whole $\mathbf{S}$ setting $\mathcal{N}_{d}(Y)=2^{Y} \backslash\{\emptyset\}$ when $Y \in \mathrm{Ob}_{d}$, and then $\mathcal{N}(Z)=\left\{A \times B \mid A \in N_{c}(X), B \in N_{d}(Y)\right\}$ for any $Z=\langle X, Y\rangle \in \operatorname{Ob} \mathbf{S}$. The resulting $\mathcal{N}$ is a functor on $\mathbf{S}$ closed under projections and intersections, that contains the fibers of every projection.

For every $X \in \mathrm{Ob} \mathbf{S}$ and $N \in \mathcal{N}(X)$, there is a unique reference measure $\mu_{X, N}$ compatible with $M$ : on the continuous sector it is the Lebesgue measure given by the metric $M$ on the affine subspaces of $E$, on the discrete sector it is the counting measure restricted to $N$, and for any product $A \times B \subset E_{X} \times E_{Y}$, with $X \in \mathrm{Ob} \mathbf{S}_{c}$ and $Y \in \mathrm{Ob} \mathbf{S}_{d}$, it is just $\sum_{y \in B} \mu_{A}^{y}$, where $\mu_{A}^{y}$ is the image of $\mu_{A}$ under the inclusion $A \hookrightarrow A \times B, a \mapsto(a, y)$. We write $\mu_{X}$ instead of $\mu_{X, E_{X}}$.

The disintegration of the counting measure into counting measures is trivial. The disintegration of the Lebesgue measure $\mu_{V, N}$ on a support $N \subset E_{V}$ under the projection $\pi^{W V}: E_{V} \rightarrow E_{W}$ of vector spaces is given by the Lebesgue measures on the fibers $\left(\pi^{W, V}\right)^{-1}(w)$, for $w \in E_{W}$. We recall that we are in a framework where $E_{V}$ and $E_{W}$ are identified with subspaces of $E$, which has a metric $M$; the disintegration formula is just Fubini's theorem.

To see that disintegrations exist under any arrow of the category, consider first an object $Z=\langle X, Y\rangle$ and arrows $\tau: Z \rightarrow X$ and $\tau^{\prime}: Z \rightarrow Y$, when $X \in \operatorname{Ob} \mathbf{S}_{c}$ and $Y \in \operatorname{Ob} \mathbf{S}_{d}$. By definition $E_{Z}=E_{Y} \times E_{X}=\cup_{y \in E_{Y}} E_{X} \times\{y\}$, and the canonical projections $\pi^{Y Z}: E_{Z} \rightarrow E_{Y}$ and $\pi^{X Z}: E_{Z} \rightarrow E_{X}$ are the images under $\mathcal{E}$ of $\tau$ and $\tau^{\prime}$, respectively. Set $E_{Z, y}:=\left(\pi^{Y Z}\right)^{-1}(y)=E_{X} \times\{y\}$ and $E_{Z, x}:=\left(\pi^{X Z}\right)^{-1}(x)=\{x\} \times E_{Y}$. According to the previous definitions, $E_{Z}$ has reference measure $\mu_{Z}=\sum_{y \in E_{Y}} \mu_{X}^{y}$, where $\mu_{X}^{y}$ is the image of $\mu_{X}$ under the inclusion $E_{X} \rightarrow E_{X} \times\{y\}$. Hence by definition, $\left\{\mu_{X}^{y}\right\}_{y \in E_{Y}}$ is $\left(\pi^{Y Z}, \mu_{Y}\right)$ disintegration of $\mu_{Z}$. Similarly, $\mu_{Z}$ has as $\left(\pi^{X Z}, \mu_{X}\right)$-disintegration the family of measures $\left\{\mu_{Y}^{x}\right\}$, where each $\mu_{Y}^{x}$ is the counting measure restricted on the fiber $E_{Z, x} \cong E_{Y}$.

More generally, the disintegration of reference measure $\mu_{Z, A \times B}=\sum_{y \in B} \mu_{A}^{y}$ on a support $A \times B$ of $Z=\langle X, Y\rangle$ under the arrow $\left\langle\pi_{1}, \pi_{2}\right\rangle: Z \rightarrow Z^{\prime}=\left\langle X^{\prime}, Y^{\prime}\right\rangle$ is the collection of measures $\left(\mu_{Z, A \times B, x^{\prime}, y^{\prime}}\right)_{\left(x^{\prime}, y^{\prime}\right) \in \pi_{1}(A) \times \pi_{2}(B)}$ such that

$$
\begin{equation*}
\mu_{Z, A \times B, x^{\prime}, y^{\prime}}=\sum_{y \in \pi_{2}^{-1}\left(y^{\prime}\right)} \mu_{X, A, x^{\prime}}^{y} \tag{5}
\end{equation*}
$$

where $\mu_{X, A, x^{\prime}}^{y}$ is the image measure, under the inclusion $E_{X} \rightarrow E_{X} \times\{y\}$, of the measure $\mu_{X, A, x^{\prime}}$ that comes from the $\left(\pi_{1}, \mu_{X^{\prime}, \pi_{1}(A)}\right)$-disintegration of $\mu_{X, A}$.

### 3.2 Probability laws and probabilistic functionals

Consider the subfunctor $\Pi(\mathcal{N})$ of $\Pi$ that associates to each $X \in \mathrm{ObS}$ the set $\Pi(X ; \mathcal{N})$ of probability measures on $\mathcal{E}(X)$ that are absolutely continuous with respect to the reference measure $\mu_{X, N}$ on some $N \in \mathcal{N}(X)$. We define the (affine) support or carrier of $\rho$, denoted $A(\rho)$, as the unique $A \in \mathcal{N}(X)$ such that $\rho \ll \mu_{X, A}$.

In this work, we want to restrict our attention to subfunctors $\mathcal{Q} \subset \Pi(\mathcal{N})$ of probability laws such that:

1. $\mathcal{Q}$ is adapted;
2. for each $\rho \in \mathcal{Q}(X)$, the differential entropy $S_{\mu_{A(\rho)}}(\rho):=-\int \log \frac{\mathrm{d} \rho}{\mathrm{d} \mu_{A(\rho)}} \mathrm{d} \rho$ exists i.e. it is finite;
3. when restricted to probabilities in $\mathcal{Q}(X)$ with the same carrier $A$, the differential entropy is a continuous functional in the total variation norm;
4. for each $X \in \operatorname{Ob} \mathbf{S}_{c}$ and each $N \in \mathcal{N}(X)$, the gaussian mixtures carried by $N$ are contained in $\mathcal{Q}(X)$-cf. next section.

Problem 1. The characterization of functors $\mathcal{Q}$ that satisfy properties $1 .-4$
Below we use kernel estimates, which interact nicely with the total variation norm. This norm is defined for every measure $\rho$ on $\left(E_{X}, \mathfrak{B}_{X}\right)$ by $\|\rho\|_{T V}=$ $\sup _{A \in \mathfrak{B}_{X}}|\rho(A)|$. Let $\varphi$ be a real-valued functional defined on $\Pi\left(E_{X}, \mu_{A}\right)$, and $L_{1}^{1}\left(A, \mu_{A}\right)$ the space of functions $f \in L^{1}\left(A, \mu_{A}\right)$ with total mass 1 i.e. $\int_{A} f \mathrm{~d} \mu_{A}=$ 1 ; the continuity of $\varphi$ in the total variation distance is equivalent to the continuity of $\tilde{\varphi}: L_{1}^{1}\left(A, \mu_{A}\right) \rightarrow \mathbb{R}, f \mapsto \varphi\left(f \cdot \mu_{A}\right)$ in the $L^{1}$-norm, because of Scheffe's identity [5. Thm. 45.4].

The characterization referred to in Problem 1 might involve the densities or the moments of the laws. It is the case with the main result that we found concerning convergence of the differential entropy and its continuity in total variation [6. Or it might resemble Otáhal's result [8]: if densities $\left\{f_{n}\right\}$ tend to $f$ in $L^{\alpha}(\mathbb{R}, \mathrm{d} x)$ and $L^{\beta}(\mathbb{R}, \mathrm{d} x)$, for some $0<\alpha<1<\beta$, then $-\int f_{n}(x) \log f_{n}(x) \mathrm{d} x \rightarrow$ $-\int f(x) \log f(x) \mathrm{d} x$.

For each $X \in \mathrm{Ob} \mathbf{S}$, let $\mathcal{F}(X)$ be the vector space of measurable functions of $\left(\rho, \mu_{M}\right)$, equivalently $\left(\mathrm{d} \rho / d \mu_{A(\rho)}, \mu_{M}\right)$, where $\rho$ is an element of $\mathcal{Q}(X), \mu_{M}$ is a global determination of reference measure on any affine subspace given by the metric $M$, and $\mu_{A(\rho)}$ is the corresponding reference measure on the carrier of $\rho$ under this determination.

We want to restrict our attention to functionals for which the action (11) is integrable. Of course, these depends on the answer to Problem 1.

Problem 2. What are the appropriate restrictions on the functionals $\mathcal{F}(X)$ to guarantee the convergence of (11)?

## 4 Computation of 1-cocycles

### 4.1 A formula for gaussian mixtures

Let $\mathcal{Q}$ be a probability functor satisfying conditions (1)-(4) in Subsection 3.2, and $\mathcal{G}$ a linear subfunctor of $\mathcal{F}$ such that (1) converges for laws in $\mathcal{Q}$. In this section, we compute $H^{1}(\mathbf{S}, \mathcal{G})$.

Consider a generic object $Z=\langle X, Y\rangle=\langle X, 1\rangle \wedge\langle 1, Y\rangle$ of $\mathbf{S}$; we write everywhere $X$ and $Y$ instead of $\langle X, 1\rangle$ and $\langle 1, Y\rangle$. We suppose that $E_{X}$ is an Euclidean space of dimension $d$. Remind that $E_{Y}$ is a finite set and $E_{Z}=E_{X} \times E_{Y}$. Let $\left\{G_{M_{y}, \Sigma_{y}}\right\}_{y \in E_{Y}}$ be gaussian densities on $E_{X}$ (with mean $M_{y}$ and covariance $\Sigma_{y}$ ), and $p: E_{Y} \rightarrow[0,1]$ a density on $E_{Y}$; then $\rho=\sum_{y \in E_{Y}} p(y) G_{M_{y}, \Sigma_{y}} \mu_{X}^{y}$ is a probability measure on $E_{Z}$, absolutely continuous with respect to $\mu_{Z}$, with density $r(x, y):=p(y) G_{M_{y}, \Sigma_{y}}(x)$. We have that $\pi_{*}^{Y Z} \rho$ has density $p$ with respect to the counting measure $\mu_{Y}$, whereas $\pi_{*}^{X Z} \rho$ is absolutely continuous with respect to $\mu_{X}$ (see [3, Thm. 3]) with density

$$
\begin{equation*}
\mu_{Z}^{x}(r)=\int_{E_{X}} \frac{\mathrm{~d} \rho}{\mathrm{~d} \mu_{Z}} \mathrm{~d} \mu_{Z}^{x}=\sum_{y \in E_{Y}} p(y) G_{M_{y}, \Sigma_{y}}(x) \tag{6}
\end{equation*}
$$

i.e. it is a gaussian mixture. For conciseness, we utilize here linear-functional notation for some integrals, e.g. $\mu_{Z}^{x}(r)$. The measure $\rho$ has a $\pi^{X Z}$-disintegration into probability laws $\left\{\rho_{x}\right\}_{x \in E_{X}}$, such that each $\rho_{x}$ is concentrated on $E_{Z, x} \cong E_{Y}$ and

$$
\begin{equation*}
\rho_{x}(x, y)=\frac{p(y) G_{M_{y}, \Sigma_{y}}(x)}{\sum_{y \in E_{Y}} p(y) G_{M_{y}, \Sigma_{y}}(x)} \tag{7}
\end{equation*}
$$

In virtue of the cocycle condition (remind that $\Phi_{Z}=\varphi_{Z}[Z]$ ),

$$
\begin{equation*}
\Phi_{Z}(\rho)=\varphi_{Z}[Y](\rho)+\sum_{y \in E_{Y}} p(y) \varphi_{Z}[X]\left(G_{M_{y}, \Sigma_{y}} \mu_{X}^{y}\right) \tag{8}
\end{equation*}
$$

Locality implies that $\varphi_{Z}[Y](\rho)=\Phi_{Y}\left(\pi_{*}^{Y Z} \rho\right)$, and the later equals $-b \sum_{y \in E_{Y}} p(y) \log p(y)$, for some $b \in \mathbb{R}$, in virtue of the characterization of cocycles restricted to the discrete sector. Similarly, $\varphi_{Z}[X]\left(G_{M_{y}, \Sigma_{y}} \mu_{X}^{y}\right)=\Phi_{X}\left(G_{M_{y}, \Sigma_{y}}\right)=$ $a \log \operatorname{det}\left(\Sigma_{y}\right)+c \operatorname{dim} E_{X}$ for some $a, c \in \mathbb{R}$, since our hypotheses are enough to characterize the value of any cocycle restricted to the gaussian laws on the continuous sector. Hence

$$
\begin{equation*}
\Phi_{Z}(\rho)=-b \sum_{y \in E_{Y}} p(y) \log p(y)+\sum_{y \in E_{Y}} p(y)\left(a \log \operatorname{det}\left(\Sigma_{y}\right)+c \operatorname{dim} E_{X}\right) . \tag{9}
\end{equation*}
$$

Remark that the same argument can be applied to any densities $\left\{f_{y}\right\}_{y \in Y}$ instead $\left\{G_{M_{y}, \Sigma_{y}}\right\}$. Thus, it is enough to determine $\Phi_{X}$ on general densities, for each $X \in \mathbf{S}_{c}$, to characterize completely the cocycle $\varphi$.

The cocycle condition also implies that

$$
\begin{equation*}
\Phi_{Z}(\rho)=\Phi_{X}\left(\pi^{X Z} \rho\right)+\int_{E_{X}} \varphi_{Z}[Y]\left(\rho_{x}\right) \mathrm{d} \pi_{*}^{X Z} \rho(x) \tag{10}
\end{equation*}
$$

The law $\pi_{*}^{X Z} \rho$ is a composite gaussian, and the law $\rho_{x}$ is supported on the discrete space $E_{Z, x} \cong E_{Y}$, with density

$$
\begin{equation*}
(x, y) \mapsto \rho_{x}(y)=\frac{p(y) G_{M_{y}, \Sigma_{y}}(x)}{\sum_{y^{\prime} \in E_{Y}} p\left(y^{\prime}\right) G_{M_{y}, \Sigma_{y}}(x)}=\frac{r(x, y)}{\mu_{Z}^{x}(r)} \tag{11}
\end{equation*}
$$

Using again locality and the characterization of cocycles on the discrete sector, we deduce that

$$
\begin{equation*}
\varphi_{Z}[Y]\left(\rho_{x}\right)=\Phi_{Y}\left(\pi_{*}^{Y Z} \rho_{x}\right)=-b \sum_{y \in E_{Y}} \rho_{x}(y) \log \rho_{x}(y) \tag{12}
\end{equation*}
$$

A direct computation shows that $\sum_{y \in E_{Y}} \rho_{x}(y) \log \rho_{x}(y)$ equals:

$$
\begin{equation*}
\sum_{y \in E_{Y}} p(y) \log p(y)-\sum_{y \in E_{Y}} p(y) S_{\mu_{X}}\left(G_{M_{y}, \Sigma_{y}}\right)+S_{\mu_{X}}\left(\mu_{Z}^{x}(r)\right) \tag{13}
\end{equation*}
$$

where $S_{\mu_{X}}$ is the differential entropy, defined for any $f \in L^{1}\left(E_{X}, \mu_{X}\right)$ by $S_{\mu_{X}}(f)=$ $-\int_{E_{X}} f(x) \log f(x) \mathrm{d} \mu_{X}(x)$.

It is well known that $S_{\mu_{X}}\left(G_{M, \Sigma}\right)=\frac{1}{2} \log \operatorname{det} \Sigma+\frac{\operatorname{dim} E_{X}}{2} \log (2 \pi e)$.
Equating the right hand sides of (9) and (10), we conclude that
$\left.\Phi_{X}\left(\pi^{X Z} \rho\right)=\sum_{y \in E_{Y}} p(y)\left(\left(a-\frac{b}{2}\right) \log \operatorname{det} \Sigma_{y}+c d-\frac{b d}{2} \log (2 \pi e)\right)\right)+b S_{\mu_{X}}\left(\mu_{Z}^{x}(r)\right)$.

### 4.2 Kernel estimates and main result

Equation 17 gives an explicit value to the functional $\Phi_{X} \in \mathcal{F}(X)$ evaluated on a gaussian mixture. Any density in $L^{1}\left(E_{X}, \mu_{X}\right)$ can be approximated by a random mixture of gaussians. This approximation is known as a kernel estimate.

Let $X_{1}, X_{2}, \ldots$ be a sequence of independently distributed random elements of $\mathbb{R}^{d}$, all having a common density $f$ with respect to the Lebesgue measure $\lambda_{d}$. Let $K$ be a nonnegative Borel measurable function, called kernel, such that $\int K \mathrm{~d} \lambda_{d}=1$, and $\left(h_{n}\right)_{n}$ a sequence of positive real numbers. The kernel estimate of $f$ is given by

$$
\begin{equation*}
f_{n}(x)=\frac{1}{n h_{n}^{d}} \sum_{i=1}^{n} K\left(\frac{x-X_{i}}{h_{n}}\right) \tag{15}
\end{equation*}
$$

The distance $J_{n}=\int_{E}\left|f_{n}-f\right| \mathrm{d} \lambda_{d}$ is a random variable, invariant under arbitrary automorphisms of $E$. The key result concerning these estimates [4, Ch. 3, Thm. 1] says, among other things, that $J_{n} \rightarrow 0$ in probability as $n \rightarrow \infty$ for some $f$ if and only if $J_{n} \rightarrow 0$ almost surely as $n \rightarrow \infty$ for all $f$, which holds if and only if $\lim _{n} h_{n}=0$ and $\lim _{n} n h_{n}^{d}=\infty$.

Theorem 1. Let $\varphi$ be a 1-cocycle on $\mathbf{S}$ with coefficients in $\mathcal{G}, X$ an object in $\mathbf{S}_{c}$, and $\rho$ a probability law in $\mathcal{Q}(X)$ absolutely continuous with respect to $\mu_{X}$. Then, there exist real constants $c_{1}, c_{2}$ such that

$$
\begin{equation*}
\Phi_{X}(\rho):=\varphi_{X}[X](\rho)=c_{1} S_{\mu_{A}}(\rho)+c_{2} \operatorname{dim} E_{X} \tag{16}
\end{equation*}
$$

Proof. By hypothesis. $X=\langle X, \mathbf{1}\rangle$ is an object in the continuous sector of $\mathbf{S}$. Let $f$ be any density of $\rho$ with respect to $\mu_{X},\left(X_{n}\right)_{n \in \mathbb{N}}$ an i.i.d sequence of points of $E_{X}$ with law $\rho$, and $\left(h_{n}\right)$ any sequence such that $h_{n} \rightarrow 0$ and $n h_{n}^{d} \rightarrow \infty$. Let $\left(X_{n}(\omega)\right)_{n}$ be any realization of the process such that $f_{n}$ tend to $f$ in $L^{1}$. We introduce, for each $n \in \mathbb{N}$, the kernel estimate (15) evaluated at $\left(X_{n}(\omega)\right)_{n}$, taking $K$ equal to the density of a standard gaussian. Each $f_{n}$ is the density of a composite gaussian law $\rho_{n}$ equal to $n^{-1} \sum_{i=1}^{n} G_{X_{i}(\omega), h_{n}^{2} I}$. This can be "lifted" to $Z=\left\langle X, Y_{n}\right\rangle$, this is, there exists a law $\tilde{\rho}$ on $E_{Z}:=E_{X} \times[n]$ with density $r(x, i)=p(i) G_{X_{i}(\omega), h_{n}^{2} I}(x)$, where $p:[n] \rightarrow \mathbb{R}$ is taken to be the uniform law. The arguments of Section 4 then imply that

$$
\begin{equation*}
\Phi_{X}\left(\rho_{n}\right)=2 d\left(a-\frac{b}{2}\right) \log h_{n}+c d-\frac{b d}{2} \log (2 \pi e)+b S_{\mu_{A}}\left(\rho_{n}\right) \tag{17}
\end{equation*}
$$

In virtue of the hypotheses on $\mathcal{Q}, S_{\mu_{A}}(\rho)$ is finite and $S_{\mu_{A}}\left(\rho_{n}\right) \rightarrow S_{\mu_{A}}(\rho)$. Since $\Phi_{X}$ is continuous when restricted to $\Pi\left(A, \mu_{A}\right)$ and $\Phi_{X}(f)$ is a real number, we conclude that necessarily $a=b / 2$. The statement is then just a rewriting of (17).

This is the best possible result: the dimension is an invariant associated to the reference measure, and the entropy depends on the density.

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