Weighted Prefix Normal Words: Mind the Gap

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Abstract. A prefix normal word is a binary word whose prefixes contain at least as many 1s as any of its factors of the same length. Introduced by Fici and Lipták in 2011 the notion of prefix normality is so far only defined for words over the binary alphabet. In this work we investigate a generalisation for finite words over arbitrary finite alphabets, namely weighted prefix normality. We prove that weighted prefix normality is more expressive than binary prefix normality. Furthermore, we investigate the existence of a weighted prefix normal form since weighted prefix normality comes with several new peculiarities that did not already occur in the binary case. We characterise these issues and finally present a standard technique to obtain a generalised prefix normal form for all words over arbitrary, finite alphabets.

1 Introduction

Complexity measures of words are a central topic of investigation when dealing with properties of sequences, e.g. factor complexity [4,2,6,26], binomial complexity [25,19,23,22], cyclic complexity [12]. Characterising the maximum density of a particular letter in the set of factors of a given length, hence considering an abelian setting, falls into that category (see for instance [13,24,5] and the references therein). Such characterisations inevitably prompt the search for and investigation of normal forms representing words with equivalent measures. Prefix normality is the concept considered in this paper and was first introduced by Fici and Lipták in 2011 [17] as a property describing the distribution of a designated letter within a binary word. A word over the binary alphabet $\{0, 1\}$ is prefix normal if its prefixes contain at least as many 1s as any of its factors of the same length. For example, the word 1101001 is prefix normal. Thus, prefixes of prefix normal words give an upper bound for the amount of 1s any other factor of the word may contain. For a given binary word w the maximum-1s function maps n to the maximum amount of 1s, a length-n factor of w can have. In [11] Burcsi et al. show that there exists exactly one prefix normal word (the prefix normal form) in the set of all binary words that have an identical maximum-1s function, e.g. the prefix normal form of 1001101 is 1101001.

From an application point of view this complexity measure is directly connected to the *Binary Jumbled Pattern Matching Problem (BJPM)* (see e.g. [1,7,9]

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and for the general JPM, see e.g. [21]). The BJPM problem is to determine whether a given finite binary word has factors containing given amounts of 1s and 0s. In [17] prefix normal forms were used to construct an index for the BJPM problem in O(n) time where n is the given word's length. The fastest known algorithm for this problem has a runtime of $O(n^{1.864})$ (see [14]). In [3] Balister and Gerke showed that the number of length-*n* prefix normal words is $2^{n-\Theta(\log^2(n))}$, and the class of a given prefix normal word contains at most $2^{n-O(\sqrt{n\log(n)})}$ elements. In more theoretical settings, the language of binary prefix normal words has also been extended to infinite binary words [16]. Prefix normality has been shown to be connected to other fields of research within combinatorics on words, e.g. Lyndon words [17] and bubble languages [10]. Furthermore, efforts have been made to recursively construct prefix normal words, via the notions of extension critical words (collapsing words) and prefix normal palindromes [18,10,15]. The goal therein was to learn more about the number of words with the same prefix normal form and the number of prefix normal palindromes. Very recently in [8] a Gray code for prefix normal words in amortized polylogarithmic time per word was generated. Four sequences related to prefix normal words can be found in the OEIS [20]: A194850 (number of prefix normal words of length n), A238109 (list of prefix normal words over the binary alphabet), A238110 (maximum number of binary words of length n having the same prefix normal form), and A308465 (number of prefix normal palindromes of length n).

Our contribution. In this work, we investigate a generalisation of prefix normality for finite words over arbitrary finite alphabets. We define a *weight measure*, which is a morphic function assigning a weight (an element from an arbitrary but a priori chosen monoid) to every letter of an arbitrary finite alphabet. Based on those weights we can again compare factors and prefixes of words over this alphabet w.r.t. their weight. A word is *prefix normal w.r.t. a weight measure* if no factor has a higher weight than that of the prefix of the same length. Note here, for some weight measures not every word has a unique prefix normal form. We prove basic properties of weight measures for which every word has a prefix normal form. Finally, we define a standard weight measure which only depends on the alphabetic order of the letters and a unique *weighted prefix normal form* that is not depending on the choice of a weight measure.

Structure of the paper. In Section 2, we define the basic terminology. Following that, in Section 3, we prove that weighted prefix normality is a proper generalisation of the binary case and present our results on the existence of a weighted prefix normal form. Finally, in Section 4, we present our main theorem on the standard weight measure as well as the weighted prefix normal form.

Some further insights about weighted prefix normality, e.g. a second but less powerful approach and some basic properties related to binary prefix normality, are also given in the Appendix A.

2 Preliminaries

Let \mathbb{N} denote the positive natural numbers $\{1, 2, 3, ...\}, \mathbb{Z}$ the integer numbers, and $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $i, j \in \mathbb{N}$, we define the interval $[i, j] := \{n \in \mathbb{N} \mid i \leq n \leq j\}$ and for $n \in \mathbb{N}$, we define [n] := [1, n] and $[n]_0 := [0, n]$. For two monoids A and B with operations * and \circ respectively, a function $\mu : A \to B$ is a *morphism* if $\mu(x * y) = \mu(x) \circ \mu(y)$ holds for all $x, y \in A$. Notice, if the domain A is a free monoid over some set S, a morphism from $A \to B$ is sufficiently defined by giving a mapping from S to B.

An alphabet Σ is a finite set of letters. A word is a finite sequence of letters from a given alphabet. Let Σ^* denote the set of all finite words over Σ , i.e. the free monoid over Σ . Let ε denote the *empty word* and set $\Sigma^+ := \Sigma^* \setminus \{\varepsilon\}$ as the free semigroup over Σ . We denote the length of a word $w \in \Sigma^*$ by |w|, i.e. the number of letters in w. Thus $|\varepsilon| = 0$ holds. Let w be a word of length $n \in \mathbb{N}$. Let w[i] denote the *i*th letter of w for $i \in [|w|]$, and set $w[i \dots j] = w[i] \cdots w[j]$ for $i, j \in [|w|]$ and $i \leq j$. Let $w[i \dots j] = \varepsilon$ if i > j. The number of occurrences of a letter $\mathbf{a} \in \Sigma$ in w is denoted by $|w|_{\mathbf{a}} = |\{i \in [|w|] \mid w[i] = \mathbf{a}\}|$. We say $x \in \Sigma^*$ is a factor of w if there exist $u, v \in \Sigma^*$ with w = uxv. In this case u is called a prefix of w. We denote the set of w's factors (resp. prefixes) by Fact(w) (resp. $\operatorname{Pref}(w)$ and $\operatorname{Fact}_i(w)$ ($\operatorname{Pref}_i(w)$ resp.) denotes the set of factors (prefixes) of length $i \in [|w|]$. Given a total order < over Σ let $<_{lex}$ denote the extension of < to a lexicographic order over Σ^* . Fixing a strictly totally ordered alphabet $\Sigma = {\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n}$ with $\mathbf{a}_i < \mathbf{a}_j$ for $1 \le i < j \le n$, the *Parikh vector* of a word is defined by $p: \Sigma^* \to \mathbb{N}^n : w \mapsto (|w|_{\mathbf{a}_1}, |w|_{\mathbf{a}_2}, \dots, |w|_{\mathbf{a}_n})$. For a function f set $f(A) = \{f(a) \mid a \in A\} \text{ for } A \subseteq \operatorname{dom}(f).$

Before we define the weight measures and weighted prefix normality we recall the definition for binary prefix normality as introduced by Fici and Lipták in [17].

Definition 1. ([17]) Given $w \in \{0, 1\}^*$ the maximum-ones function f_w and the prefix-ones function p_w are respectively defined by $f_w : [|w|]_0 \to \mathbb{N}_0$, $i \mapsto \max(|\operatorname{Fact}_i(w)|_1)$ and $p_w : [|w|]_0 \to \mathbb{N}_0$, $i \mapsto |\operatorname{Pref}_i(w)|_1$. The word w is called prefix normal if $f_w = p_w$ holds.

Our generalisation of binary prefix normality is based on so called *weight* measures, i.e. we apply weights represented by elements from a strictly totally ordered monoid A to every letter of the alphabet. In the following we denote the neutral element of an arbitrary monoid A by $\mathbb{1}_A$, its operation by \circ_A , and its total order by $<_A$ (in the case of existence).

Definition 2. Let A be a totally ordered monoid. A morphism $\mu : \Sigma^* \to A$ is a weight measure over the alphabet Σ w.r.t. A if $\mu(vw) = \mu(wv)$ and $\mu(w) <_A \mu(wv)$ hold for all words $w \in \Sigma^*$ and $v \in \Sigma^+$. We refer to the second property as the increasing property. We say the weights of the letters of Σ are the base weights of μ , so $\mu(\Sigma)$ is the set of all base weights.

Remark 3. Notice that if there exists a weight measure μ w.r.t. the monoid A then |A| is infinite, \circ_A is commutative on $\mu(\Sigma^*)$, and $\mu(\varepsilon) = \mathbb{1}_A$ holds. Moreover,

the increasing property of weight measures ensures that only the neutral element ε of Σ^* is mapped to the neutral element $\mathbb{1}_A$. Hence, we will see that our factorand prefix-weight functions are strictly monotonically increasing in contrast to the functions defined in [17]. However, if we allow letters from Σ to be also assigned the neutral weight $\mathbb{1}_A$, we get the known results for binary alphabets.

Remark 4. Notice, that a weight measure μ can be defined for any alphabet Σ in two steps: choose some infinite commutative monoid with a total and strict order and assign a base weight that is greater than the neutral element to each letter in Σ . Since μ is a morphism, the weight of a word $w \in \Sigma^*$ is well defined.

In the following definition we introduce seven (for us the most intuitive) special types of weight measures.

Definition 5. A weight measure μ over the alphabet Σ w.r.t. the monoid A is \diamond injective if μ is injective on Σ ,

- \diamond alphabetically ordered if $\mu(a) \leq_A \mu(b)$ holds for all a, b ∈ Σ with a ≤ b for a total order ≤ on Σ,
- \diamond binary if $|\mu(\Sigma)| = 2$ holds, and non-binary if $|\mu(\Sigma)| > 2$,
- \diamond natural if A is \mathbb{N}_0 or \mathbb{N} with \leq_A being the usual order < on integers,
- \diamond a sum weight measure if it is natural and the operation on A is +,
- \diamond a product weight measure if it is natural and the operation on A is *,
- \diamond prime if it is a product weight measure and $\mu(\Sigma) \subseteq \mathbb{P}$ holds.

Consider, for instance, the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. The weight measure μ over Σ with $\mu(\mathbf{a}) = 1$, $\mu(\mathbf{b}) = 2$, and $\mu(\mathbf{c}) = 3$ is non-binary, natural, and with the monoid $(\mathbb{N}_0, +)$ it is a sum weight measure. It cannot be a product weight measure with $(\mathbb{N}, *)$ since then $\mu(\mathbf{a}) = 1$ would violate the increasing property. However, the weight measure ν over Σ w.r.t. $(\mathbb{N}, *)$ with $\nu(\mathbf{a}) = 2$, $\nu(\mathbf{b}) = 3$, and $\nu(\mathbf{c}) = 5$ is not only a product weight measure, but also a prime weight measure (for further insights regarding prime weight measures see Subsection A.2).

Remark 6. For the binary alphabet $\Sigma = \{0, 1\}$ a sum weight measure μ with $\mu(w) = |w|_1$ for all $w \in \Sigma^*$ cannot exist since we would have $\mu(0) = 0 = \mu(\varepsilon)$ which is a contradiction to the increasing property. Later on we are going to circumvent this problem by setting $\mu(w) = |w|_1 + |w|$ for all $w \in \Sigma^*$ when implementing binary prefix normality via the usage of weight measures. Alternatively, we may relax the increasing property and allow $\mu(0) = 0$; this results exactly in the same properties as discussed in [17].

We now define the analogues to the maximum-ones and prefix-ones functions.

Definition 7. Let $w \in \Sigma^*$ and μ be a weight measure over the alphabet Σ w.r.t. the monoid A. Define the factor-weight function $f_{w,\mu}$ and prefix-weight function $p_{w,\mu}$ respectively by $f_{w,\mu}$: $[|w|]_0 \to A$, $i \mapsto \max(\mu(\operatorname{Fact}_i(w)))$ and $p_{w,\mu}$: $[|w|]_0 \to A$, $i \mapsto \mu(\operatorname{Pref}_i(w))$.

For instance, let μ be a sum weight measure with the base weights $\mu(\mathbf{a}) = 1$, $\mu(\mathbf{n}) = 2$, $\mu(\mathbf{b}) = 3$ for the alphabet $\Sigma = \{\mathbf{a}, \mathbf{n}, \mathbf{b}\}$.

Now consider the words banana and nanaba. Table 1 shows the mappings of

i	$1\ 2\ 3\ 4\ 5\ 6$
$p_{\texttt{nanaba},\mu}(i)$	$2\ 3\ 5\ 6\ 9\ 10$
$f_{\texttt{nanaba},\mu}(i)$	$3\ 4\ 6\ 7\ 9\ 10$
$p_{\text{banana},\mu}(i), f_{\text{banana},\mu}(i)$	$3\ 4\ 6\ 7\ 9\ 10$

Table 1. Comparing banana's andnanaba's prefix- and factor-weights.

their prefix- and factor-weight functions. The factor-weight function of nanaba is realised by the factors b, ab, nab, anab, nanab, nanaba.

Finally, we define a generalised approach for prefix normality, namely the weighted prefix normality for a given weight measure μ . As in the binary case,

for a prefix normal word the factor-weight function and the prefix-weight function have to be identical.

Definition 8. Let $w \in \Sigma^*$ and let μ be a weight measure over Σ . We say w is μ -prefix normal (or weighted prefix normal w.r.t. μ) if $p_{w,\mu} = f_{w,\mu}$ holds.

In the example above we see $p_{\text{banana},\mu} = f_{\text{banana},\mu}$ holds and hence banana is prefix normal w.r.t. μ . On the other hand we have $p_{\text{nanaba},\mu}(1) = 2 < 3 = f_{\text{nanaba},\mu}(1)$ and therefore nanaba is not prefix normal w.r.t. μ .

3 Weighted Prefix Normal Words and Weighted Prefix Normal Form

In this section we show that *weighted prefix normality* is a proper generalisation of binary prefix normality and further investigate the weighted prefix normal form. By examining special properties of weight measures, we intend to guide the reader from the general approach to a characterisation of special weight measures for which every word has a weighted prefix normal equivalent, namely *injective and gapfree weight measures*. (Some useful basic properties which are direct generalisations of the binary case can be found in Subsection A.1.) Before we define the analogue to the prefix-equivalence for factor weights, we show that weighted prefix normality is more general and more expressive than binary prefix normality, i.e. every statement on binary prefix normality can be expressed by weighted prefix normality but not vice versa.

Proposition 9. Binary prefix normality is expressible by weighted prefix normality, i.e. there exists a weight measure μ such that μ -prefix normality is equivalent to binary prefix normality.

Proof. We construct a sum weight measure μ over the binary alphabet $\Sigma = \{0, 1\}$. Let $\mu(1) = 2$ and $\mu(0) = 1$. Then $|w|_1 + |w| = \mu(w)$ holds for any binary word $w \in \Sigma^*$. We have $f_w(i) + i = \max(|\operatorname{Fact}_i(w)|_1) + i = \max(\mu(\operatorname{Fact}_i(w))) = f_{w,\mu}(i)$ and $p_w(i) + i = p_{w,\mu}(i)$ for all $i \in [|w|]$. Therefore, w is μ -prefix normal if and only if it is prefix normal.

With the binary sum weight measure μ over $\Sigma = \{0, 1\}$ where $\mu(1) = 2$ and $\mu(0) = 1$, we can transform any statement on binary prefix normality into an analogue in the weighted setting. For example, for w = 11001101 we have $f_w(4) = 3$ and $p_w(4) = 2$ (so w is not prefix normal) and in the weighted setting we have $f_{w,\mu}(4) = 7 = f_w(4) + 4$ and $p_{w,\mu}(4) = 6 = p_w(4) + 4$; or in general $f_w(i) = f_{w,\mu}(i) - i$ holds for all $w \in \Sigma^*$ and $i \in [|w|]$. Therefore, w is μ -prefix normal if and only if it is prefix normal.

Definition 10. Let μ be a weight measure over Σ . Two words $w, w' \in \Sigma^*$ are factor-weight equivalent w.r.t. μ (denoted by $w \sim_{\mu} w'$) if $f_{w,\mu} = f_{w',\mu}$ holds. We denote the equivalence classes by $[w]_{\sim_{\mu}} := \{w' \in \Sigma^* \mid w \sim_{\mu} w'\}$.

In the following we highlight three peculiarities about the factor-weight equivalence that do not occur in the binary case: the existence of factor-weight equivalent words with different Parikh vectors, the existence of multiple words that are weighted prefix normal and factor-weight equivalent, and the *absence* of a factor-weight equivalent word that is weighted prefix normal. The words banana and nanaba over $\Sigma = \{a, n, b\}$ with the weight measure $\mu(a) = 1$, $\mu(n) = 2$, and $\mu(b) = 3$ are factor-weight equivalent. The complete equivalence class is given by $[banana]_{\sim_{\mu}} = \{ananab, anaban, abanan, nanaba, nabana, banana\}.$ Notice that all words in the class have the same Parikh vector but only banana is μ -prefix normal. If we were to add c to Σ and expand μ by $\mu(c) = \mu(n) = 2$ then $[banana]_{\sim_{\mu}}$ contains all words previously in it but also those where some ns are substituted by c. So $[banana]_{\sim \mu}$ contains four μ -prefix normal words, namely banana, bacana, banaca, and bacaca. Lastly, consider the sum weight measure ν over the alphabet $\Sigma = \{a, n, x\}$ with the base weights $\nu(a) = 1$, $\nu(n) = 2$, $\nu(x) = 4$. Now $[xaxn]_{\sim_{\nu}}$ only contains xaxn and its reverse nxax. Interestingly none of the two words are ν -prefix normal, witnessed by $f_{xaxn,\nu} = f_{nxax,\nu} = (4, 6, 9, 11),$ $p_{xaxn,\nu} = (4, 5, 9, 11)$, and $p_{nxax,\nu} = (2, 6, 7, 11)$ (the functions are written as sequences for brevity). In order for a weighted prefix normal word to exist in the class, a letter with weight $f_{xaxn,\nu}(3) - f_{xaxn,\nu}(2) = 9 - 6 = 3$ is missing. For example with such a letter **b** in Σ with $\nu(\mathbf{b}) = 3$ the word **xnbn** is ν -prefix normal and in $[xaxn]_{\sim_{\nu}}$. These examples show that factor-weight equivalence classes can contain words with different Parikh vectors, multiple prefix normal words, and even no prefix normal words at all. We now investigate the question which weight measures cause such peculiar equivalence classes and characterise the equivalence classes that contain a single weighted prefix normal word, a normal form, as it always exists for the binary case (see [17]).

Definition 11. For $w \in \Sigma^*$ and a weight measure μ over Σ we define the μ -prefix normal subset of the factor-weight equivalence class of w by $\mathcal{P}_{\mu}(w) := \{v \in [w]_{\sim_{\mu}} \mid p_{v,\mu} = f_{v,\mu}\}.$

In the example above, multiple prefix normal words in a single class are a direct result of ambiguous base weights, i.e. non-injective weight measure: all letters with the same weights are interchangeable in any word with no effect on the weight of that word; thus there exist multiple prefix normal words for such a word. By choosing an injective weight measure we can avoid this behaviour. However, the problematic case where some equivalence classes contain no prefix normal words at all, still remains. We give a characterisation of special, so called

gapfree, weight measures and show that they guarantee the existence of a prefix normal word in every equivalence class of the factor-weight equivalence. Before we prove the just stated claims, we formally define the previous observations of gaps.

Definition 12. A weight measure μ over the alphabet Σ w.r.t. the monoid A is gapfree, if for all words $w \in \Sigma^*$ and all $i \in [|w|]$ there exists an $\mathbf{a} \in \Sigma$ such that $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(\mathbf{a})$ holds. Otherwise, if for any word $w \in \Sigma^*$ and an $i \in [|w|]$ there exists no $\mathbf{a} \in \Sigma$ such that $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(\mathbf{a})$ holds we say μ is gapful and has a gap over the word w at the index i.

Consider for example the sum weight measure over $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ with $\mu(\mathbf{a}) = 2$, $\mu(\mathbf{b}) = 4$, and $\mu(\mathbf{c}) = 6$. We show that μ is gapfree by proving the existence of letters in Σ with weight $x_i = f_{w,\mu}(i) - f_{w,\mu}(i-1) \in \mathbb{N}$ for all $w \in \Sigma^*$ and $i \in [|w|]$. Since the factor-weight function is defined as a maximum, we get $x_i \leq \mu(\mathbf{c}) = 6$. On the other hand $x_i \geq \mu(\mathbf{a}) = 2$ because the factor-weight function is strictly increasing. Since all the base weights $\mu(\Sigma) = \{2,4,6\}$ are even, the same is true for $f_{w,\mu}(i)$ and $f_{w,\mu}(i-1)$. Thus, x_i has to be even as well. This implies $x_i \in \{2,4,6\} = \mu(\Sigma)$. Hence, there exist letters in Σ with the appropriate weight to fill every possible gap, i.e. μ is gapfree. As a counter example, the sum weight measure ν over Σ with $\nu(\mathbf{a}) = 1$, $\nu(\mathbf{b}) = 3$, and $\nu(\mathbf{c}) = 4$ is gapful. Consider the word $w = \mathbf{bcac}$ then ν has a gap over w at the index 3 since $f_{w,\nu}(3) = 9$ (witnessed by the factor \mathbf{cac}) and $f_{w,\nu}(2) = 7$ (witnessed by the factor \mathbf{bc}) and there is no letter with weight 2.

Coming back to the original question of multiple prefix normal words, the following theorem characterises exactly when an equivalence class contains no, exactly one, or more than one weighted prefix normal word.

Theorem 13. Let μ be a weight measure over Σ . Then

- there exists $w \in \Sigma^*$ such that $|\mathcal{P}_{\mu}(w)| = 0$ iff μ is gapful,
- there exists $w \in \Sigma^*$ such that $|\mathcal{P}_{\mu}(w)| > 1$ iff μ is not injective, and
- for all $w \in \Sigma^*$ we have $|\mathcal{P}_{\mu}(w)| = 1$ iff μ gapfree and injective.

Proof. Let μ be a weight measure over Σ w.r.t. the monoid A. For the first equivalence consider that μ is gapful. Then there exists some word $w \in \Sigma^*$ and an index $i \in [|w|]$ such that w has a gap at i. Thus there exists no $n \in A$ for which $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A n$ holds. Now suppose there exists some word $w' \in \mathcal{P}_{\mu}(w)$. For such a word $p_{w',\mu}(i) = f_{w',\mu}(i) = f_{w,\mu}(i)$ and $p_{w',\mu}(i-1) = f_{w',\mu}(i-1) = f_{w',\mu}(i-1) \circ_A \mu(w'[i]) = f_{w',\mu}(i-1) \circ_A \mu(w'[i]) = p_{w',\mu}(i-1) \circ_A \mu(w'[i]) = p_{w',\mu}(i)$. Which is a contradiction to the gap, so $\mathcal{P}_{\mu}(w) = \emptyset$ holds. For the second direction choose $w \in \Sigma^*$ with $\mathcal{P}_{\mu}(w) = \emptyset$. Suppose μ is gapfree, so $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(\mathbf{a})$ holds for all $i \in [|w|]$ and appropriate $\mathbf{a} \in \Sigma$. Then we have a contradiction by constructing a word $w' \in \mathcal{P}_{\mu}(w)$ as follows: Choose $w'[1] \in \Sigma$ with $\mu(w'[1]) = f_{w,\mu}(1)$, which is possible according to the assumption for i = 1. And for $i \in [|w|]$ we can inductively choose $w'[i] \in \Sigma$ with $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(w'[i])$, which

is also possible according to the assumption. Now $p_{w',\mu} = f_{w',\mu}$ and $p_{w',\mu} = f_{w,\mu}$ hold by construction, so $w' \in \mathcal{P}_{\mu}(w)$ holds.

For the second claim let μ be not injective. Therefore, we have some distinct letters $\mathbf{a}, \mathbf{b} \in \Sigma$ which have the same weight $\mu(\mathbf{a}) = \mu(\mathbf{b}) = i \in A$. So $[\mathbf{a}]_{\sim \mu} = [\mathbf{b}]_{\sim \mu}$ and $p_{\mathbf{a},\mu} = f_{\mathbf{b},\mu}$ both hold directly. Consequently $\{\mathbf{a},\mathbf{b}\} \subseteq \mathcal{P}_{\mu}(\mathbf{a})$ follows. For the second direction consider $w, u, v \in \Sigma^*$ with $u \neq v$ and $\{u,v\} \subseteq \mathcal{P}_{\mu}(w)$. By the definition of $\mathcal{P}_{\mu}(w)$, the prefix-weight function of u and v are both equal to the factor-weight function of w. So $p_{u,\mu} = p_{v,\mu}$ holds, and therefore $\mu(u[j]) =$ $\mu(v[j])$ holds for all $j \in [|u|]$. On the other hand because u and v are different words, there exists some $i \in [|u|]$ with $u[i] \neq v[i]$. In other words, μ is not injective.

The third claim follows directly from the first two.

Definition 14. Let μ be a gapfree and injective weight measure over Σ and let $w \in \Sigma^*$. Then $|\mathcal{P}_{\mu}(w)| = 1$ and its element is the μ -prefix normal form of w.

Again with the alphabet $\Sigma = \{a, n, b, x\}$ and the sum weight measure μ over Σ with base weights $\mu(a) = 1$, $\mu(n) = 2$, $\mu(b) = 3$, and $\mu(x) = 4$ we have $\mathcal{P}_{\mu}(nanaba) = \{banana\}$ and $\mathcal{P}_{\mu}(xaxn) = \{xnbn\}$. So banana is the μ -prefix normal form of nanaba and xnbn is the μ -prefix normal form of xaxn. Additionally, notice xaxn is an example of a word such that its Parikh vector is different from that of its prefix normal form.

Remark 15. Let μ be a gapfree and injective weight measure over the alphabet Σ w.r.t. the monoid A and $w \in \Sigma^*$. Then the μ -prefix normal form w' of w can be constructed inductively: $w'[1] = \mathbf{a}$ if $f_{w,\mu}(1) = \mu(\mathbf{a})$ and for all $i \in [|w|]$, i > 1 set $w'[i] = \mathbf{a} \in \Sigma$ if $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(\mathbf{a})$. In contrast, for a weight measure that is gapfree but *not* injective this inductive construction can be used to non-deterministically construct all prefix normal words within the factor-weight equivalence class of a word. (A proof that the prefix normal form is indeed obtained by this construction can be found in Subsection A.4.)

4 Gapfree Weight Measures

In this section we investigate the behaviour of gapfree weight measures in more detail. In order to present a natural and gapfree *standard weight measure* for ordered alphabets that is *equivalent* to every other injective, alphabetically ordered, and gapfree weight measure (over arbitrary monoids) - and thus works as a representative, we give an alternative condition for gapfree weight measures; the so called weight measures with *stepped based weights*.

First of all, by their definition we can infer that every binary weight measure is gapfree. Consequently we consider non-binary weight measures for the rest of this section.

Lemma 16. All binary weight measures are gapfree.

Proof. Let μ be a binary weight measure over Σ w.r.t. A and with the two base weights $\mu(\Sigma) = \{x, y\}$, where $x <_A y$. W.l.o.g. let μ be injective, so Σ is binary as well. Furthermore w.l.o.g. let $\Sigma = \{0, 1\}$ and $\mu(0) = x$, $\mu(1) = y$. Now let $w \in \Sigma^*$ and $i \in [|w|]$. Then $f_{w,\mu}(i)$ is realised by some factor $u \in \text{Fact}_i(w)$ with $\mu(u) = f_{w,\mu}(i)$ and $f_{w,\mu}(i-1)$ is realised by some factor $v \in \text{Fact}_i(v)$ with $\mu(v) = f_{w,\mu}(i-1)$. Now $|v|_1 - |u|_1 \in \{0, 1\}$ holds because otherwise $\mu(v)$ or $\mu(u)$ would not be the maximum weight a factor of length i or i-1 has. In total either $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(1)$ or $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(0)$ holds. Therefore μ is gapfree.

Remark 17. By Lemma 16, we see that when modelling binary prefix normality by means of weighted prefix normality (e.g. in the proof of Theorem 9) we automatically have the existence of a unique binary prefix normal form as expected.

In the last section we saw that we have exactly one weighted prefix normal form in a factor-weight equivalence class iff the weight measure is injective and gapfree. We now give an alternative condition under which a weight measure is gapfree, which in most cases is easier to check. Later we will also see that this condition is part of a proper characterisation for gapfree weight measures.

Definition 18. Let A be a strictly totally ordered monoid. A step function is a right action of an element $s \in A$ (the step) on A, i.e. $\sigma_s : A \to A; a \mapsto a \circ_A s$. The weight measure μ over Σ w.r.t the monoid A is said to have stepped base weights if there exists a step function σ_s for some $s \in A$ such that $\mu(\Sigma) = \{\sigma_s^i(\min(\mu(\Sigma))) \mid i \in [0, |\mu(\Sigma)| - 1]\}$ holds.

In the previous example for $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, the gapfree sum weight measure μ over Σ with $\mu(\mathbf{a}) = 2$, $\mu(\mathbf{b}) = 4$, and $\mu(\mathbf{c}) = 6$ has stepped base weights with the step of 2. In contrast, the gapful sum weight measure ν over Σ with $\nu(\mathbf{a}) = 1$, $\nu(\mathbf{b}) = 3$, and $\nu(\mathbf{c}) = 4$ does not, because $\nu(\mathbf{b}) - \nu(\mathbf{a}) = 2$ but $\nu(\mathbf{c}) - \nu(\mathbf{b}) = 1$. In general, stepped base weights imply *gapfreeness* but not vice versa (see Subsection A.3 in the appendix).

Proposition 19. All weight measures with stepped base weights are gapfree.

Proof. Let μ be a non-binary weight measure over Σ w.r.t A with stepped base weights. W.l.o.g let μ be injective and let Σ be of the form $\{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1}\}$, where the letters are in ascending order of their weight, so $\mu(\mathbf{a}_i) <_A \mu(\mathbf{a}_{i+1})$ holds for all $i \in [0, n-2]$.

Assume there exists a step function σ with the step $s \in A$ such that $\mu(\Sigma)$ is of the form $\{\sigma^i(\min(\mu(\Sigma))) \mid i \in [0, |\mu(\Sigma)| - 1]\}$. Consequently the weight of every letter in Σ is $\mu(\mathbf{a}_i) = \sigma^i(\min(\mu(\Sigma)))$ for all $i \in [0, n - 1]$. In particular we have $\mu(\mathbf{a}_0) = \min(\mu(\Sigma))$. Now consider some word $w \in \Sigma^*$ and index $l \in$ [|w|], then $f_{w,\mu}(l)$ is realised by some factor $\mathbf{a}_{p_1} \dots \mathbf{a}_{p_l} \in \operatorname{Fact}_l(w)$ with some sequence $p_1, \dots, p_l \in [0, n - 1]$. Therefore $f_{w,\mu}(l)$ is of the form $\sigma^{p_1}(\mu(\mathbf{a}_0)) \circ_A$ $\cdots \circ_A \sigma^{p_l}(\mu(\mathbf{a}_0))$. And similarly $f_{w,\mu}(l - 1)$ is realised by some other factor $\mathbf{a}_{q_1} \dots \mathbf{a}_{q_{l-1}} \in \operatorname{Fact}_{l-1}(w)$ for some sequence $q_1, \dots, q_{l-1} \in [0, n - 1]$, and we have $f_{w,\mu}(l-1) = \sigma^{q_1}(\mu(\mathbf{a}_0)) \circ_A \cdots \circ_A \sigma^{l_i}(\mu(\mathbf{a}_0))$. Now let $m = \sum_{i=1}^{l} (p_i)$ and $o = \sum_{i=1}^{l-1} (q_i)$ be the number of steps in the weights $f_{w,\mu}(l)$ and $f_{w,\mu}(l-1)$. So they are the maximum number of steps any of w's factors of length l and l-1 can have in their weight.

First of all $m - o \ge 0$ holds because we know the factor-weight function is strictly increasing and otherwise m would not be the maximum for length l. We also know m - o < n holds because if $m - o \ge n$ held, o would not be the maximum number of steps in a factor of length l - 1. We now know $f_{w,\mu}(l)$ and $f_{w,\mu}(l-1)$ only differ by k := m - o steps where $0 \le k < n$ holds. Consequently $f_{w,\mu}(l) = f_{w,\mu}(l-1) \circ_A \sigma^k(\mathbf{a}_0)$ holds and because μ has stepped base weights there exists such a letter $\mathbf{a}_k \in \Sigma$ with $\mu(\mathbf{a}_k) = \sigma^k(\mathbf{a}_0)$. Thus μ is gapfree. \Box

For further investigations of gapfree weight measures we define an equivalence on weight measures based on their behaviour on words of the same length.

Definition 20. Let μ_A and μ_B be weight measures over the same alphabet Σ w.r.t. the monoids A and B. We say that μ_A and μ_B are equivalent if for all words $v, w \in \Sigma^n$, for some $n \in \mathbb{N}$, we have $\mu_A(v) <_A \mu_A(w)$ iff $\mu_B(v) <_B \mu_B(w)$.

The reasoning behind such an equivalence of weight measures lies in the fact that using different but equivalent weight measures does not change their relative behaviour. Most notably, Definition 20 and the totality of the orders imply $\mu_A(v) = \mu_A(w)$ iff $\mu_B(v) = \mu_B(w)$ and therefore, the prefix normal form remains.

For instance, considering again the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ and the gapfree sum weight measure μ over Σ with $\mu(\mathbf{a}) = 2$, $\mu(\mathbf{b}) = 4$, and $\mu(\mathbf{c}) = 6$ as well as the product weight measure ν over Σ with $\nu(\mathbf{a}) = 2$, $\nu(\mathbf{b}) = 6$, and $\nu(\mathbf{c}) = 18$. Then μ and ν are equivalent since they both are alphabetically ordered and $2 + 3\frac{\mu(w)}{2} - 1 = \nu(w)$ holds for all $w \in \Sigma^*$. Therefore, since μ is gapfree so is ν , and for instance $\mathcal{P}_{\mu}(\mathbf{bcac}) = \{\mathbf{cbbb}\} = \mathcal{P}_{\nu}(\mathbf{bcac})$ holds.

Proposition 21. The prefix normal form of any word is the same w.r.t. equivalent weight measures, i.e. $\mathcal{P}_{\mu}(w) = \mathcal{P}_{\nu}(w)$ holds for all $w \in \Sigma^*$ if μ and ν are equivalent weight measures.

Proof. Assume μ and ν are equivalent weight measures over Σ . Then for all words $w \in \Sigma^*$ and $u \in \operatorname{Fact}_i(w)$ with $i \in [|w|]$ it holds $\mu(u) = f_{w,\mu}$ iff $\nu(u) = f_{w,\nu}$. Consequently with the construction given in Remark 15 we have that the μ -prefix normal form is the same as the ν -prefix normal form. \Box

Before we present the generalised weight measure, we prove three auxiliary lemmata and give the definition of the standard weight measure.

Lemma 22. For any two equivalent weight measures, if one of them is gapfree, injective, or alphabetically ordered then so is the other.

Proof. Let μ_A, μ_B be equivalent weight measures over Σ w.r.t. monoids A, B.

1) Assume μ_A gapfree but suppose μ_B not. There exists a word $w \in \Sigma^*$ with a gap at $i \in [|w|]$ regarding μ_B . So for every $\mathbf{x} \in \Sigma$ we have $f_{w,\mu_B}(i) = \mu_B(u) \neq$

 $\mu_B(v\mathbf{x}) = f_{w,\mu_B}(i-1) \circ_B \mu_B(\mathbf{x})$ where $u \in \operatorname{Fact}_i(w)$ and $v \in \operatorname{Fact}_{i-1}(w)$. Since μ_A and μ_B are equivalent also $\mu_A(u) \neq \mu_A(v\mathbf{x})$ holds for all $\mathbf{x} \in \Sigma$. This is a contradiction since μ_A is gapfree.

2) Assume μ_A injective but suppose μ_B not. There exist letters $\mathbf{a}, \mathbf{b} \in \Sigma$ with $\mu_B(\mathbf{a}) = \mu_B(\mathbf{b})$. Since μ_A and μ_B are equivalent also $\mu_A(\mathbf{a}) = \mu_A(\mathbf{b})$ holds contradicting the assumption.

3) Follows directly by the definition of equivalent weight measures.

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Finally, we define the *standard weight measure* as an injective gapfree weight measure that is innate to any strictly totally ordered alphabet.

Definition 23. Let $\Sigma = \{a_1, a_2, ..., a_n\}$ be a strictly totally ordered alphabet, where $n \in \mathbb{N}$. We define the standard weight measure μ_{Σ} as the alphabetically ordered sum weight measure over Σ with base weights $\mu_{\Sigma}(a_i) = i$ for all $i \in [n]$.

For instance, considering again the alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ with the usual order, the standard weight measure μ_{Σ} has the base weights $\mu_{\Sigma}(\mathbf{a}) = 1$, $\mu_{\Sigma}(\mathbf{a}) = 2$, and $\mu_{\Sigma}(\mathbf{a}) = 3$. And in the following, we will see that indeed μ_{Σ} is equivalent to both μ and ν from the previous example.

Lemma 24. The standard weight measure is gapfree, injective, and alphabetically ordered.

Proof. The standard weight measure is gapfree by Proposition 19, since it is a sum weight measure with stepped base weights. It is injective and alphabetically ordered by definition.

The definition of the equivalence on weight measures raises the question whether the standard weight measure is suitable as a representative for all gapfree, injective, and alphabetically ordered weight measures. If there were other equivalence classes of such weight measures then the standard weight measure would merely represent *one* of many choices. To answer this question we first present a peculiar property every gapfree weight measure has and then present our main theorem on the equivalence class of the standard weight measure.

Lemma 25. Let μ be an injective and alphabetically ordered weight measure over Σ w.r.t. the monoid A. Let Σ be strictly totally ordered by \leq_{Σ} and let $\Sigma = \{a_1, \ldots, a_n\}$ with $n \in \mathbb{N}_{>2}$ and $a_1 \leq_{\Sigma} a_2 \leq_{\Sigma} \cdots \leq_{\Sigma} a_n$. If μ has no gap over any word of the form cacb where $a \leq_{\Sigma} b \leq_{\Sigma} c \in \Sigma$ then $\mu(a_i a_{i+x}) = \mu(a_{i+y} a_{i+x-y})$ holds for all $i, x, y \in \mathbb{N}$ with y < x and $i + x \leq n$.

Proof. By Induction over x. The case for x = 1 is trivial.

Firstly consider the case x = 2. W.l.o.g. let i = 1, so in this case we show that $\mu(\mathbf{a}_1\mathbf{a}_3) = \mu(\mathbf{a}_2\mathbf{a}_2)$ holds. Consider $u = \mathbf{a}_3\mathbf{a}_1\mathbf{a}_3\mathbf{a}_2$. Assuming μ has no gaps over words of this form and since $f_{u,\mu}(3) = \mu(\mathbf{a}_3\mathbf{a}_1\mathbf{a}_3)$ and $f_{u,\mu}(2) = \mu(\mathbf{a}_3\mathbf{a}_2)$ hold we know there exists some $z \in [n]$ such that $\mu(\mathbf{a}_1\mathbf{a}_3) = \mu(\mathbf{a}_2\mathbf{a}_2)$. Since $\mu(\mathbf{a}_i) <_A \mu(\mathbf{a}_{i+1})$ we have $\mu(\mathbf{a}_1) <_A \mu(\mathbf{a}_z) <_A \mu(\mathbf{a}_3)$. Therefore z = 2 and $\mu(\mathbf{a}_1\mathbf{a}_3) = \mu(\mathbf{a}_2\mathbf{a}_2)$ hold. Secondly consider x > 2. Let w.l.o.g $y \leq \lfloor \frac{x}{2} \rfloor$. By induction we assume the claim holds for all smaller x, e.g. (1) $\mu(a_i a_{i+x-1}) = \mu(a_{i+y} a_{i+x-y-1})$ holds and also (2) $\mu(a_j a_{j+y+1}) = \mu(a_{j+1} a_{j+y})$ where j = i + x - y - 1 holds since $y + 1 \leq \lfloor \frac{x}{2} \rfloor + 1 < x$. By (1) we have $\mu(a_i a_{i+x-1} a_{i+x}) = \mu(a_{i+y} a_{i+x-1-y} a_{i+x})$ and by (2) rewritten as $\mu(a_{i+x-y-1} a_{i+x}) = \mu(a_{i+x-y} a_{i+x-1})$ we get $\mu(a_{i+y} a_{i+x-1-y} a_{i+x}) = \mu(a_{i+y} a_{i+x-y} a_{i+x-1})$. Therefore $\mu(a_i a_{i+x}) = \mu(a_{i+y} a_{i+x-y})$ holds. \Box

Theorem 26. Let μ be a non-binary, injective, and alphabetically ordered weight measure over the alphabet Σ which is strictly ordered by $<_{\Sigma}$. The following statements are equivalent:

1. μ is gapfree.

2. μ has no gap over any word of the form cacb where $a <_{\Sigma} b <_{\Sigma} c \in \Sigma$. 3. μ is equivalent to the standard weight measure μ_{Σ} .

Proof. $(1. \Rightarrow 2.)$ Follows immediately from the definition of gapfree weight measures.

 $(2. \Rightarrow 3.)$ Let $\Sigma = \{a_1, \ldots, a_n\}$ with $n \in \mathbb{N}_{>2}$ and $\mathbf{a}_1 <_{\Sigma} \mathbf{a}_2 <_{\Sigma} \cdots <_{\Sigma} \mathbf{a}_n$. Let $k \in \mathbb{N}$ and $v = \mathbf{a}_{i_1} \ldots \mathbf{a}_{i_k} \in \Sigma^k$ and $w = \mathbf{a}_{j_1} \ldots \mathbf{a}_{j_k} \in \Sigma^k$ for all $i_{\ell}, j_{\ell} \in [n]$ and $\ell \in [k]$. Now $\mu_{\Sigma}(v) = \sum_{\ell=1}^k i_{\ell}$ and $\mu_{\Sigma}(w) = \sum_{\ell=1}^k j_{\ell}$ hold.

We show $\mu(v) <_A \mu(w) \Leftrightarrow \mu_{\Sigma}(v) < \mu_{\Sigma}(w)$ holds by induction over k:

Case k = 1: Trivial since μ and μ_{Σ} are alphabetically ordered.

For the further cases assume w.l.o.g. that v and w share no letters and let their letters be ordered increasingly, i.e. let $i_{\ell} \neq j_{\ell'}$ and $i_{\ell} \leq i_{\ell+1}$ and $j_l \leq j_{\ell+1}$ for all $\ell, \ell' \in [k-1]$, furthermore w.l.o.g. let $i_1 < j_1$.

Case k > 2: We consider two subcases dependent on i_k and j_1 . Notice $j_1 = i_k$ can not occur since v and w share no letters.

Subcase $i_k < j_1$: In this case we know $i_\ell < j_{\ell'}$ for all $\ell, \ell' \in [k]$. Therefore $\mu(v) <_A \mu(w)$ and $\mu_{\Sigma}(v) <_A \mu_{\Sigma}(w)$ both hold immediately.

Subcase $j_1 < i_k$: In this case we choose $x = i_k - i_1$ and $y = j_1 - i_1$, consequently y < x holds. By Lemma 25 we have $\mu(v) = \mu(\mathbf{a}_{i_1} \dots \mathbf{a}_{i_{k-1}} \mathbf{a}_{i_1+x}) = \mu(\mathbf{a}_{i_1+y}\mathbf{a}_{i_2} \dots \mathbf{a}_{i_{k-1}}\mathbf{a}_{i_1+x-y})$. The claim follows since $i_1 + y = j_1$ and by the induction hypotheses we have $\mu(v') <_A \mu(w') \Leftrightarrow \mu_{\Sigma}(v') < \mu_{\Sigma}(w')$ for $v' = v[2 \dots k]$ and $w' = w[2 \dots k]$.

 $(3. \Rightarrow 1.)$ Follows immediately by Proposition 22 since the standard weight measure is gapfree.

Notice, with $(1. \Leftrightarrow 2.)$ in the above we know that any gapful weight measure over $\Sigma = \{a, b, c\}$ already has a gap over bcac. For instance, consider the sum weight measure μ over Σ with $\mu(a) = 1$, $\mu(b) = 2$, and $\mu(c) = 4$. We see that μ is gapful, since it has a gap over the word w = ccabccb at index 5, witnessed by the factor-weight function $f_{w,\mu} = (4, 8, 10, 12, 15, 19, 21)$ and the fact that there is no letter with weight 15 - 12 = 3. We visualise this gap in Figure 1. However, we already have a gap within the even shorter word bcac at index 3, witnessed by $f_{bcac,\mu} = (4, 6, 9, 11)$.

On the other hand, with $(1. \Leftrightarrow 3.)$ in Theorem 26 we immediately see there only exists *one* equivalence class of gapfree, injective, and alphabetically ordered

weight measures w.r.t. the same alphabet, justifying our choice of μ_{Σ} as the standard weight measure. Also, since by transitivity all gapfree, injective, and alphabetically ordered weight measures w.r.t. to the same alphabet are equivalent, they therefore yield the same prefix normal form (by Proposition 21). In other words, assuming a strictly totally ordered alphabet, every word has exactly one weighted prefix normal form that is independent of any chosen gapfree, injective, and alphabetically ordered weight measure over the same alphabet. With that in mind, paralleling the work presented by Fici and Lipták in [17], we introduce the weighted prefix normal form of a word $w \in \Sigma^*$.

Definition 27. Let Σ be a strictly totally ordered alphabet and let $w \in \Sigma^*$. We say the μ_{Σ} -prefix normalform is the weighted prefix normal form of w or simply the prefix normal form of w.

For instance, consider the strictly totally ordered alphabet $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$, with the standard weight measure μ_{Σ} such that $\mu_{\Sigma}(\mathbf{a}) = 1, \mu_{\Sigma}(\mathbf{b}) =$ $2, \mu_{\Sigma}(\mathbf{c}) = 3$. The weighted prefix normal form of bcac is cbbb, since $\mathcal{P}_{\mu_{\Sigma}}(\text{bcac}) = \{\text{cbbb}\}$ holds as seen in previous examples. With Theorem 26 the same also holds for any other gapfree, injective, and alphabetically ordered weight measure.



Fig. 1. Visualisation of the factor-weight function's gap for w = ccabccb.

Remark 28. By Theorem 26 we immediately see

that the gapfree property of a weight measure is decidable. Since any gapful weight measure already has a gap over a word of length four using three letters, one can check whether a weight measure is gapfree in the following way: test for all $\binom{|\Sigma|}{3}$ possible enumerations of three letters $\mathbf{a} <_{\Sigma} \mathbf{b} <_{\Sigma} \mathbf{c}$ whether there exist an $\mathbf{x} \in \Sigma$ with $\mu(\mathbf{bx}) = \mu(\mathbf{ac})$. Notice, that Σ is finite and we obtain a running time of $\mathcal{O}(|\Sigma|^4)$.

5 Conclusions

In this work we presented the generalisation of prefix normality on binary alphabets as introduced by [17] to arbitrary alphabets by applying weights to the letters and comparing the weight of a factor with the weight of the prefix of the same length.

Since one of the main properties of binary prefix normality, namely the existence of a unique prefix normal form, does not hold for weighted prefix normality with arbitrary weight measures, we investigated necessary restrictions to obtain a unique prefix normal form even in the generalised setting. Here, it is worth noticing that we did not only generalise the size of the alphabet but also the weights are rather general: they belong to any (totally ordered) monoid. This is of interest because some peculiarities do not occur if \mathbb{N} or \mathbb{N}_0 are chosen. In Section 3 we proved that there always exists a unique prefix normal form if the weight measure is gapfree and injective. In Section 4 we further demonstrated that all gapfree weight measures over the same alphabet are equivalent and therefore every word has the same weighted prefix normal form w.r.t. each of them. Which led to the definition of the standard weight measure and ultimately to a unique prefix normal form in the generalised setting that exists independent of chosen weight measures. Additionally, we showed that *gapfreeness* as a property of weight measures is decidable and can easily be checked in time $\mathcal{O}(|\Sigma|^4)$.

However, the exact behaviour of the weighted prefix normal form, or generally factor-weight equivalent words, especially regarding changes in their Parikh vectors, remains an open problem. Moreover, a reconnection of weighted prefix normality to the initial problem of indexed jumbled pattern matching would be of some interest and might prove useful when investigating pattern matching problems w.r.t. a non-binary alphabet.

Finally, we like to mention that an easier, but weaker, approach to work with prefix normality on arbitrary alphabets can be achieved by considering a subset X of Σ : each letter **a** in a word is treated like a **1** if $\mathbf{a} \in X$ and **0** otherwise, which can also be expressed by weighted prefix normality (see Subsection A.5 in the Appendix).

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Α Further Insights

In this section we present results which are not necessarily important to understand the weighted prefix normality but which give a more detailed insight, e.g. in the relation of weighted prefix normality and the original prefix normality, introduced in [17]. Therefore, we start this section with the adaption of the position function into the weighted setting and present the analogous results. Afterwards, we prove that the converse of Proposition 19 does not hold true in general, i.e. not every gapfree weight measure has stepped based weights. Following that, we present some insights on how to obtain an injective weight measure from a non-injective one. We end this part with the naïve alternative approach to generalise the binary prefix normality, namely by subset prefix normality, and prove that this generalisations is already covered by weighted prefix normality.

A.1 Weighted Position Functions

In the following we define a more general version of the binary position function defined in [17]. With the binary context in mind this function is defined to give the position of the ith 1 in the word w, i.e. $pos_w(i) := min\{k \mid p_w(k) = i\}$ for all $i \in [p_w(|w|)]$ and $w \in \{0,1\}^*$. However, in the weighted context for most words not every weight corresponds to a prefix with exactly the same weight. Thus, we do not define a single exact position function, but two functions which together enclose the position within a word where a certain weight is reached. Only if both functions return the same position for some word and weight, that word's prefix up to that position has exactly that weight.

Definition 29. Let $w \in \Sigma^*$ and let μ be a weight measure over Σ w.r.t. the monoid A. We define the max-position function and min-position function respectively by $\max pos_{w,\mu} : A \to [|w|]_0, i \mapsto \max\{k \in [|w|]_0 \mid p_{w,\mu}(k) \le i\}$ and $\min pos_{w,\mu} : A \to [|w|]_0, i \mapsto \min\{k \in [|w|]_0 \mid p_{w,\mu}(k) \ge i\}.$

By this definition, we are able to prove similar statements to the binary prefix normality.

Lemma 30. Let μ be a weight measure over Σ w.r.t. a monoid $A, w \in \Sigma^*$, $j,k \in [|w|]_0$ and $x,y \in A$. Then $p_{w,\mu}$ and $f_{w,\mu}$ have the following properties:

- (1) $j < k \text{ iff } f_{w,\mu}(j) \prec f_{w,\mu}(k) \text{ iff } p_{w,\mu}(j) \prec p_{w,\mu}(k),$

- (2) $p_{w,\mu}(\max p_{w,\mu}(x)) \leq x \leq p_{w,\mu}(m) p_{w,\mu}(x)$, (3) $\max p_{w,\mu}(p_{w,\mu}(k)) = \min p_{w,\mu}(p_{w,\mu}(k)) = k$, (4) if $\max p_{w,\mu}(x) < j$ then $x \prec p_{w,\mu}(j)$ and if $j < \min p_{w,\mu}(x)$ then $p_{w,\mu}(j) \prec x,$
- (5) $\max_{w,\mu}(x) \le \min_{w,\mu}(x)$, (6) $x \prec y$ implies $\max_{w,\mu}(x) \le \max_{w,\mu}(y)$ as well as $\min_{w,\mu}(x) \le \max_{w,\mu}(x)$ $\operatorname{minpos}_{w,\mu}(y).$

Proof.

- (1) With the increasing property of weight measures the equivalences follow from the definition of the factor-weight function as the maximum over all factors and the fact that every prefix itself is a prefix of every longer prefix.
- (2) Directly follows by the definition of the max-position and min-position function as $\max pos_{w,\mu}(x) = \max\{i \in [|w|]_0 \mid p_{w,\mu}(i) \leq x\}$ and $\min pos_{w,\mu}(x) = \min\{i \in [|w|]_0 \mid x \leq p_{w,\mu}(i)\}.$
- (3) Follows by the definition of the max-position and min-position function and the fact that the prefix-weight function is strictly increasing (see (1)).
- (4) Follows by the definition of the max-position function as a maximum and the min-position function as a minimum.
- (5) Follows by (1) and (2).
- (6) Suppose otherwise, so let $x \prec y$ but $\max p_{w,\mu}(x) > \max p_{w,\mu}(y)$ holds. With (4), we then have $y \prec p_{w,\mu}(\max p_{w,\mu}(x))$ and with (2) we have $p_{w,\mu}(\max p_{w,\mu}(x)) \preceq x$. Together these are a contradiction to $x \prec y$. Now suppose $\min p_{w,\mu}(x) > \min p_{w,\mu}(y)$ holds. Again with (4), we have $p_{w,\mu}(\min p_{w,\mu}(y)) \prec x$ and with (2) we have $y \preceq p_{w,\mu}(\min p_{w,\mu}(y))$. Which is again a contradiction to $x \prec y$.

Lemma 31. For a weight measure μ over the alphabet Σ w.r.t. a monoid (A, \circ) and $w \in \Sigma^*$, $f_{w,\mu}(j) \leq f_{w,\mu}(i) \circ f_{w,\mu}(j-i)$ holds for all $i, j \in [|w|]_0$ with i < j.

Proof. Let $i, j \in [|w|]_0$ be indexes with i < j. Now suppose $f_{w,\mu}(j) \succ f_{w,\mu}(i) \circ f_{w,\mu}(j-i)$. Let $u \in \operatorname{Fact}_j(w)$ be a factor of w with $\mu(u) = f_{w,\mu}(j)$. Then by the definition of the factor-weight function, $\mu(u[1 \dots i]) \preceq f_{w,\mu}(i)$ and $\mu(u[(i+1) \dots j]) \preceq f_{w,\mu}(j-i)$ both hold. And thus $f_{w,\mu}(j) \succ \mu(u[1 \dots i]) \circ \mu(u[(i+1) \dots j]) = \mu(u)$ holds. This is a contradiction because u was chosen with $\mu(u) = f_{w,\mu}(j)$, so the original claim follows. \Box

Lemma 32. For a weight measure μ over the alphabet Σ w.r.t. a monoid (A, \circ) and $w \in \Sigma^*$ the following properties are equivalent:

- (1) w is μ -prefix normal,
- (2) $p_{w,\mu}(j) \leq p_{w,\mu}(i) \circ p_{w,\mu}(j-i)$ for all $i, j \in [|w|]_0$ with i < j,
- (3) $\operatorname{minpos}_{w,\mu}(\mu(v)) \leq |v|$ for all $v \in \operatorname{Fact}(w)$,
- (4) $\max_{w,\mu}(a) + \min_{w,\mu}(b) \leq \min_{w,\mu}(a \circ b)$ for all $a, b \in A$, with $a \circ b \leq \mu(w)$.

Proof. $(1) \Rightarrow (2)$. Follows by the second additional lemma, since for any prefix normal word the prefix- and factor-weight function are equal by definition.

 $(2) \Rightarrow (3)$. Assume we have (2) but suppose there exists $v \in Fact(w)$ with $|v| < \min_{w,\mu}(\mu(v))$. Now we write v as $v = w[i+1\dots j]$ for some $i, j \in [|w|]_0$

with i < j. Then we have $p_{w,\mu}(j) = p_{w,\mu}(i) \circ \mu(v)$. And by (8) of the first additional lemma, we know that $\mu(v) \succ p_{w,\mu}(|v|)$ holds so in total we have $p_{w,\mu}(j) \succ p_{w,\mu}(i) \circ p_{w,\mu}(|v|)$. Which is a contradiction to (2), because we have $|v| = |w[i+1\dots j]| = j-i$.

(3) \Rightarrow (1). Assume we have (3). Let $i \in [|w|]$ and let $v \in Fact(w)$ with $\mu(v) = f_{w,\mu}(i)$ then we have $|v| \ge \min p_{w,\mu}(\mu(v))$. By (2) and (4) of the first additional lemma follows $p_{w,\mu}(|v|) \succeq p_{w,\mu}(\min p_{w,\mu}(\mu(v))) \succeq \mu(v)$ from which (1) follows directly because we now have $p_{w,\mu}(i) \succeq f_{w,\mu}(i)$.

 $(3) \Rightarrow (4)$. Let $a, b \in A$ with $a \circ b \leq \mu(w)$, $m = \min pos_{w,\mu}(a \circ b)$ and $n = \max pos_{w,\mu}(a)$. Now consider w's factor $v = w[n+1\dots m]$ which has a length of m-n. So $p_{w,\mu}(n) \circ \mu(v) = p_{w,\mu}(m)$ and |v| = m-n each follow. By (3) and (4) of the first additional lemma we know $a \circ \mu(v) \succeq a \circ b$ and therefore $\mu(v) \succeq b$ holds. Again by (11) of the first additional lemma we get $\min pos_{w,\mu}(\mu(v)) \geq \min pos_{w,\mu}(b)$. So in total with (3) follows $\min pos_{w,\mu}(b) \leq \min pos_{w,\mu}(\mu(v)) \leq |v| = m - n = \min pos_{w,\mu}(a \circ b) - \max pos_{w,\mu}(a)$.

 $(4) \Rightarrow (3).$ Let $v \in \operatorname{Fact}(w)$, we write v as $v = w[i+1\ldots j]$ for some $i, j \in [|w|]_0$. So with the first additional lemma we have $\operatorname{minpos}_{w,\mu}(p_{w,\mu}(i)) = i$ and $\operatorname{minpos}_{w,\mu}(p_{w,\mu}(i) \circ \mu(v)) = \operatorname{minpos}_{w,\mu}(p_{w,\mu}(j)) = j$. By (4) we then have $\operatorname{minpos}_{w,\mu}(\mu(v)) \leq \operatorname{minpos}_{w,\mu}(p_{w,\mu}(i) \circ \mu(v)) - \operatorname{minpos}_{w,\mu}(p_{w,\mu}(i)) = j - i = |v|.$

A.2 Prime Weight Measures

In this subsection we briefly examine prime weight measures regarding a certain unique properties they inherit from the prime numbers. Notice, for injective prime weight measure the weight of any word is characteristic for its Parikh vector by the uniqueness of the prime number factorisation: any two words with the same weight under an injective prime weight measure must have exactly the same letters, i.e. the same Parikh vectors. Thus, it is not unreasonable to assume they might prove interesting regarding the *Indexed JPM* (IJPM), since in the binary case prefix normal forms always have the same Parikh vectors as their equivalent words. In our more general case, we saw that for some words the weighted prefix normal form has a different Parikh vector. However, this is prohibited by the above mentioned property of prime weight measures. Consequently, every prime weight measure has gaps over these words and is therefore gapful.

Lemma 33. All non-binary prime weight measures are gapful.

Proof. Since μ is a non-binary prime weight measure, we have $\mu(\Sigma) \subseteq \mathbb{P}$ and $|\mu(\Sigma)| > 2$. Thus, there exist letters $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 such that $\mu(\mathbf{a}_1), \mu(\mathbf{a}_2)$, and $\mu(\mathbf{a}_3)$ are pairwise different prime numbers. Assume w.l.o.g. $\mu(\mathbf{a}_1) < \mu(\mathbf{a}_2) < \mu(\mathbf{a}_3)$. Suppose that μ is gapfree, i.e. for all words $w \in \Sigma^*$ and all $i \in [|w| - 1]$ there exists an $x \in \Sigma$ with $f_{w,\mu}(i+1) = f_{w,\mu}(i) * \mu(x)$. Consider $w = \mathbf{a}_2\mathbf{a}_3\mathbf{a}_1\mathbf{a}_3$. By $\mu(\mathbf{a}_2\mathbf{a}_3\mathbf{a}_1) < \mu(\mathbf{a}_3\mathbf{a}_1\mathbf{a}_3)$ and $\mu(\mathbf{a}_1\mathbf{a}_3) = \mu(\mathbf{a}_3\mathbf{a}_1) < \mu(\mathbf{a}_2\mathbf{a}_3)$ we get $f_{w,\mu}(3) = \mu(\mathbf{a}_3\mathbf{a}_3)$.

 $\mu(\mathbf{a}_3\mathbf{a}_1\mathbf{a}_3)$ and $f_{w,\mu}(2) = \mu(\mathbf{a}_2\mathbf{a}_3)$. By the supposition, there exists an $x \in \Sigma$ with $\mu(\mathbf{a}_3\mathbf{a}_1\mathbf{a}_3) = f_{w,\mu}(3) = f_{w,\mu}(2) * \mu(x) = \mu(\mathbf{a}_2\mathbf{a}_3) * \mu(x)$. In other words $\mu(\mathbf{a}_1) * \mu(\mathbf{a}_3) = \mu(\mathbf{a}_2) * \mu(x)$ must hold. This is a direct contradiction to the uniqueness of the prime number factorisation. Therefore no non-binary prime weight measure can be gapfree, as witnessed by the word examined above. \Box

In some sense the prime weight measure even is the *most* gapful weight measure, since it has gaps between all of its base weights by the definition of prime numbers. So if there exists a weight measure that has a gap over some word, then also every prime weight measure has a gap over that word. This sentiment leads us to believe that prime weight measures might not be as helpful in solving the IJMP as initially assumed.

A.3 Gapfree and Stepped Based Weights

In this subsection, we prove that the converse of Proposition 19 does not hold in general, i.e. stepped based weights and *gapfreeness* are not equivalent. For this purpose, we define a relatively technical monoid V equipped with a weightfunction μ .

Definition 34. Let $\Sigma = \{a, b, c\}$ and let V be the strictly totally ordered monoid where $V = \{ \begin{pmatrix} a \\ b \end{pmatrix} \mid a, b \in \mathbb{N}_0 \}$, \circ_V is the usual addition on vectors, $\mathbb{1}_V = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $<_V$ is the order obtained by the lexicographical expansion of the usual less than onto vectors, e.g $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \prec \begin{pmatrix} 0 \\ 2 \end{pmatrix} \prec \begin{pmatrix} 1 \\ 1 \end{pmatrix} \prec \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ holds.

Lemma 35. The weight measure μ over Σ w.r.t. V with the base weights $\mu(\mathbf{a}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, $\mu(\mathbf{b}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\mu(\mathbf{c}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ does not have stepped base weights.

Proof. It is easy to see that there exists no $x \in V$ with $\binom{0}{2} + x = \binom{1}{1}$, since 1 < 2. So there exists no step function for μ .

Lemma 36. The weight measure μ over Σ w.r.t. V with the base weights $\mu(\mathbf{a}) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \ \mu(\mathbf{b}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ and \ \mu(\mathbf{c}) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ is gapfree.

Proof. For $i \in [|w|]$ let $u \in \operatorname{Fact}_{i+1}(w)$ be the factor determining $f_{w,\mu}(i+1)$ and $v \in \operatorname{Fact}_i(w)$ be the factor determining $f_{w,\mu}(i)$ such that i is minimal with u and v not overlapping (if they overlap, the non-overlapping parts are taken as u and v respectively). Now choose $r, s, t \in \mathbb{Z}$ with $r = |u|_{a} - |v|_{a}, s = |u|_{b} - |v|_{b}$, and $t = |u|_{c} - |v|_{c}$. Thus we have r + s + t = 1 by

$$r+s+t = |u|_{\mathbf{a}} - |v|_{\mathbf{a}} + |u|_{\mathbf{b}} - |v|_{\mathbf{b}} + |u|_{\mathbf{c}} - |v|_{\mathbf{c}} = |u| - |v| = i+1-i=1.$$

Moreover we have

$$\mu(u) = \binom{|u|_{\mathbf{b}} + 2|u|_{\mathbf{c}}}{2|u|_{\mathbf{a}} + |u|_{\mathbf{b}}} = \binom{|v|_{\mathbf{b}} + s + 2|v|_{\mathbf{c}} + 2t}{2|v|_{\mathbf{a}} + 2r + |v|_{\mathbf{b}} + s} = \mu(v) + \binom{s + 2t}{2r + s}.$$

And with $|v|_{b} = |u|_{b} - s = |u|_{b} + r + t - 1$ we get

$$\mu(u) = \begin{pmatrix} |u|_{\mathbf{b}} + 2|v|_{\mathbf{c}} + 2t \\ |u|_{\mathbf{b}} + 2|v|_{\mathbf{a}} + 2r \end{pmatrix} \text{ and } \mu(v) = \begin{pmatrix} |u|_{\mathbf{b}} + r + t - 1 + 2|v|_{\mathbf{c}} \\ 2|v|_{\mathbf{a}} + |u|_{\mathbf{b}} + r + t - 1 \end{pmatrix}.$$

By evaluating $f_{w,\mu}(i+1) = \mu(u) >_V \mu(v) = f_{w,\mu}(i)$ we get the following two cases: if $|u|_{b} + 2|v|_{c} + 2t = |u|_{b} + r + t - 1 + 2|v|_{c}$ and $|u|_{b} + 2|v|_{a} + 2r > 2r$ $2|v|_{a} + |u|_{b} + r + t - 1$ hold we get t = r - 1 and r > t - 1, thus t + 1 = r. If $|u|_{\mathbf{b}}+2|v|_{\mathbf{c}}+2t>|u|_{\mathbf{b}}+r+t-1+2|v|_{\mathbf{c}}$ holds we get t+1>r. Hence, in general we know $t+1 \ge r$ must hold. Now set u' = u[2..|u|] (the case u' = u[1..|u| - 1] is symmetric). By the assumption that v and u do not overlap we have $\mu(u') <_V$ $\mu(v)$. We now evaluate this inequality in a similar fashion but also considering the three possible letters for u[1].

case 1: u[1] = a

We have $\mu(u') = {\binom{|u'|_b + 2|u'|_c}{2|u'|_a + |u'|_b}} = {\binom{|u|_b + 2|u|_c}{2(|u|_a - 1) + |u|_b}} = {\binom{|u|_b + 2|v|_c + 2t}{2|v|_a + 2r - 2 + |u|_b}}.$ By $\mu(u') <_A \mu(v)$ we have either $|u|_b + 2|v|_c + 2t = |u|_b + r + t - 1 + 2|v|_c$ and $2|v|_{a} + 2r - 2 + |u|_{b} < 2|v|_{a} + |u|_{b} + r + t - 1 \text{ which implies } t = r - 1 \text{ and } r - 1 < t,$ which is a contradiction, or $|u|_{b} + 2|v|_{c} + 2t < |u|_{b} + r + t - 1 + 2|v|_{c}$ which implies t < r-1, which is a contradiction to $t+1 \ge r$. Hence we get $u[1] \ne a$ **case 2:** u[1] = b

We have $\mu(u') = {|u'|_{\mathfrak{b}} + 2|u'|_{\mathfrak{c}} \choose 2|u'|_{\mathfrak{a}} + |u'|_{\mathfrak{b}}} = {|u|_{\mathfrak{b}} - 1 + 2|u|_{\mathfrak{c}} \choose 2|u|_{\mathfrak{a}} + |u|_{\mathfrak{b}} - 1} = {|u|_{\mathfrak{b}} - 1 + 2|v|_{\mathfrak{c}} + 2t \choose 2|v|_{\mathfrak{a}} + 2t + |u|_{\mathfrak{b}} - 1}$. By $\mu(u') <_V$ $|u|_{b} - 1 < 2|v|_{a} + |u|_{b} + r + t - 1$ which gives again a contradiction by t = r and r < t, or $|u|_{b} - 1 + 2|v|_{c} + 2t < |u|_{b} + r + t - 1 + 2|v|_{c}$ which implies t < r. case 3: u[1] = c

We have $\mu(u') = \binom{|u'|_{\mathbf{b}}+2|u'|_{\mathbf{c}}}{2|u'|_{\mathbf{a}}+|u'|_{\mathbf{b}}} = \binom{|u|_{\mathbf{b}}+2(|u|_{\mathbf{c}}-1)}{2|u|_{\mathbf{a}}+|u|_{\mathbf{b}}} = \binom{|u|_{\mathbf{b}}+2|v|_{\mathbf{c}}+2t-2}{2|v|_{\mathbf{a}}+2r+|u|_{\mathbf{b}}}$. By $\mu(u') <_A \mu(v)$ we have here either $|u|_{\mathbf{b}} + 2|v|_{\mathbf{c}} + 2t - 2 = |u|_{\mathbf{b}} + r + t - 1 + 2|v|_{\mathbf{c}}$ and $2|v|_{a} + 2r + |u|_{b} < 2|v|_{a} + |u|_{b} + r + t - 1$ which leads to the contradiction t-1 = r and r < t-1, or $|u|_{b} + 2|v|_{c} + 2t - 2 < |u|_{b} + r + t - 1 + 2|v|_{c}$ which implies t < r + 1.

Hence, in all cases we get t < r + 1 and by t + 1 > r we know t = r - 1 or t = rmust hold. We can now prove that μ is gapfree by distinguishing these cases. **case 1:** t = r - 1

By r + s + t = 1 we get s = -2t and consequently

$$\begin{aligned} f_{w,\mu}(i+1) &= f_{w,\mu}(i) \circ_V \binom{s+2t}{2r+s} = f_{w,\mu}(i) \circ_V \binom{0}{2r-2t} \\ &= f_{w,\mu}(i) \circ_V \binom{0}{2(r-t)} = f_{w,\mu}(i) \circ_V \binom{0}{2} \\ &= f_{w,\mu}(i) \circ_V \mu(\mathbf{a}). \end{aligned}$$

case 2: t = rBy r + s + t = 1 we get s = 1 - 2t and consequently

$$\begin{split} f_{w,\mu}(i+1) &= f_{w,\mu}(i) \circ_V \binom{s+2t}{2r+s} = f_{w,\mu}(i) \circ_V \binom{1}{1} \\ &= f_{w,\mu}(i) + \mu(\mathbf{b}). \end{split}$$

Thus in both cases exists an $x \in \Sigma$ with $f_{w,\mu}(i+1) = f_{w,\mu}(i) \circ_V \mu(x)$.

A.4 Injective Weight Measures

In this section we first show that in the case of an gapfree and injective weight measure the prefix normal form can be computed deterministically, and nondeterministically if the weight measure is not injective. Afterwards, we investigate non-injective weight measures, specifically we provide a construction that can be used to transform any weight measure into an injective one. Thus, w.l.o.g. we always may assume to have an injective weight measure.

Lemma 37. Let μ be a gapfree and injective weight measure over the alphabet Σ w.r.t. the monoid A and $w \in \Sigma^*$. Then the μ -prefix normal form w' of w can be constructed inductively: $w'[1] = \mathbf{a}$ if $f_{w,\mu}(1) = \mu(\mathbf{a})$ and for all $i \in [|w|]$, i > 1 set $w'[i] = \mathbf{a} \in \Sigma$ if $f_{w,\mu}(i) = f_{w,\mu}(i-1) \circ_A \mu(\mathbf{a})$. In contrast, for a weight measure that is gapfree but not injective this inductive construction can be used to non-deterministically construct all prefix normal words within the factor-weight equivalence class of a word.

Definition 38. Let μ be a weight measure over Σ w.r.t. the monoid A. We define the μ -projected alphabet $\Sigma_{\mu} := \{[\mathbf{a}]_{\mu} \mid \mathbf{a} \in \Sigma\}$, where $[\mathbf{a}]_{\mu} = \{\mathbf{b} \in \Sigma \mid \mu(\mathbf{b}) = \mu(\mathbf{a})\}$ for $\mathbf{a} \in \Sigma$ and set μ 's projected weight measure as the weight measure $\hat{\mu}$ over Σ_{μ} w.r.t. A with the base weights $\hat{\mu}([\mathbf{a}]_{\mu}) = \mu(\mathbf{a})$. Finally for \mathbf{a} word $w \in \Sigma^*$ we construct its μ -projection $w_{\mu} \in \Sigma^*_{\mu}$ with $w_{\mu} := [w[1]]_{\mu} \dots [w[|w|]]_{\mu}$.

Lemma 39. For a weight measure μ over an alphabet Σ and a word $w \in \Sigma^*$ we have $\hat{\mu}(w_{\mu}) = \mu(w)$ and the projected weight measure $\hat{\mu}$ is injective on Σ_{μ} .

Remark 40. With this construction a word w and its μ -projection w_{μ} behave the same way under any function that is based on the weights of the letters in the words, e.g. $f_{w,\mu} = f_{w_{\mu},\hat{\mu}}, p_{w,\mu} = p_{w_{\mu},\hat{\mu}}, \max p_{w,\mu} = \max p_{w_{\mu},\hat{\mu}}$, and $\min pos_{w,\mu} = \min pos_{w_{\mu},\hat{\mu}}$ all hold. Analogously, all other statements depending on those functions hold for the μ -projection of the words as well. Thus, w_{μ} represents all words within the set $\{v \in \Sigma^* \mid v[i] \in w[i] \text{ for all } i \in [|w|]\}$.

The following theorem essentially shows that the prefix normal form of a projected word like in Definition 38 represents the set of prefix normal words that are factor-weight equivalent to the original word. In other words, for some $w \in \Sigma^*$ the sets $\mathcal{P}_{\mu}(w)$ and $\mathcal{P}_{\hat{\mu}}(w_{\mu})$ represent the same prefix normal words over Σ that are equivalent to w. Thus, also in the non-injective case we are able to obtain one prefix normal form by considering projections.

Theorem 41. Let μ be a gapfree weight measure over Σ and let $w \in \Sigma^*$. Then with $w' \in \mathcal{P}_{\hat{\mu}}(w_{\mu})$ we have $\mathcal{P}_{\mu}(w) = \{v \in \Sigma^* \mid v[i] \in w'[i] \text{ for all } i \in [|w|]\}.$

With Theorem 41 we can also accurately calculate the cardinality of $\mathcal{P}_{\mu}(w)$ for some word $w \in \Sigma^*$ and a non-injective weight measure μ .

Corollary 42. Let μ be a gapfree weight measure over the alphabet Σ , $w \in \Sigma^*$, and $w' = \mathcal{P}_{\hat{\mu}}(w_{\mu})$. Then $|\mathcal{P}_{\mu}(w)| = \prod_{i=1}^{|w|} |w'[i]|$ holds.

We conclude this section by revisiting an example w.r.t. the projected weight measure. Again consider the sum weight measure μ over $\Sigma = \{a, n, c, b\}$ with the base weights $\mu(a) = 1$, $\mu(n) = \mu(c) = 2$, and $\mu(b) = 3$. Then μ 's projected weight measure $\hat{\mu}$ is a weight measure over the alphabet $\Sigma_{\mu} = \{\{a\}, \{n, c\}, \{b\}\}$ with the base weights $\hat{\mu}(\{a\}) = 1$, $\hat{\mu}(\{n, c\}) = 2$, and $\hat{\mu}(\{b\}) = 3$ and we see that $\hat{\mu}$ is injective on Σ_{μ} . We already know that nanaba has multiple factor-weight equivalent words that are prefix normal, specifically we have $\mathcal{P}_{\mu}(\text{nanaba}) =$ $\{banana, bacana, banaca, bacaca\}$. Thus, we have the the prefix normal form $\{b\}\{a\}\{n, c\}\{a\}\{n, c\}\{a\}$ of $(\text{nanaba})_{\mu} = \{n, c\}\{a\}\{n, c\}\{a\}\{b\}\{a\}$. All factorweight equivalent and prefix normal words are represented by this word when reading it as a non-deterministic concatenation of letters, like shown in Theorem 41, i.e., we have $\mathcal{P}_{\mu}(\text{nanaba}) = \{v \in \Sigma^* \mid v[i] \in \mathcal{P}_{\hat{\mu}}((\text{nanaba})_{\mu})[i], i \in [6]\}$.

A.5 Subset Prefix Normality

In this section we briefly investigate a naïve approach to generalise binary prefix normality and prove that it is already covered by the weight measure approach. The main idea is if Σ is a finite alphabet to take a subset $X \subseteq \Sigma$ and instead of counting the amount of 1 or 0 respectively we count how many letters in a prefix or factor are contained in X. Therefore, We generalise the notation $|w|_{\mathbf{a}}$ for a letter $\mathbf{a} \in \Sigma$ to $|w|_X := |\{i \in [|w|] \mid w[i] \in X\}|$ for $X \subseteq \Sigma$, i.e. $|w|_X$ is the number of letters of w that are elements of X.

Definition 43. Let $w \in \Sigma^*$ and $X \subseteq \Sigma$. We define the prefix-X-function $p_{w,X}$ and the maximum-X-factor function $f_{w,X}$ respectively by $p_{w,X} : [|w|]_0 \to \mathbb{N}, i \mapsto$ $|\operatorname{Pref}_i(w)|_X$ and $f_{w,X} : [|w|]_0 \to \mathbb{N}, i \mapsto \max(|\operatorname{Fact}_i(w)|_X)$. We say that w is X-prefix normal (or subset prefix normal w.r.t X) if $p_{w,X} = f_{w,X}$ holds.

We now show that subset prefix normality is indeed a generalisation of binary prefix normality, and also that subset prefix normality can already be expressed by means of weighted prefix normality. However this is not possible the other way around. So in total we see that weighted prefix normality is more expressive and therefore a more useful generalisation.

Theorem 44. Binary prefix normality is expressible by subset prefix normality. (I.e. there exists $X \subseteq \{0, 1\}$ such that X-prefix normality is equivalent to binary prefix normality.)

Proof. W.l.o.g. consider just 1-prefix normality for the binary case. We choose $X \subseteq \Sigma$ with $X = \{1\}$. Then $|w|_1 = |w|_X$ holds for any binary word $w \in \{0, 1\}^*$. It follows that $f_w(i) = \max(|\operatorname{Fact}_i(w)|_1) = \max(|(\operatorname{Fact}_i(w)|_X) = f_{w,X}(i)$ and $p_w(i) = p_{w,X}(i)$ hold for all $i \in [|w|]$. Therefore, w is X-prefix normal if and only if it is 1-prefix normal. So, with such an X every statement on binary prefix normality.

In other words, in the context of the binary alphabet $\{1\}$ -prefix normality and prefix normality are the same.

Theorem 45. Subset prefix normality is expressible by weighted prefix normality. (I.e. for every $X \subseteq \Sigma$ there exists a weight measure μ such that μ -prefix normality is equivalent to X-prefix normality.)

Proof. Let Σ be an alphabet and let $X \subseteq \Sigma$. We construct a sum weight measure μ over Σ . Let $\mu(\mathbf{x}) = 2$ and $\mu(\mathbf{y}) = 1$ for every $\mathbf{x} \in X$ and $\mathbf{y} \in \Sigma \setminus X$. Then $|w|_1 + |w| = \mu(w)$ holds for any word $w \in \Sigma$. It follows that $f_{w,X}(i) + |w| = \max(|Fact_i(w)|_X) + |w| = \max(\mu(Fact_i(w))) = f_{w,\mu}(i)$ and $p_{w,X}(i) + |w| = p_{w,\mu}(i)$ hold for all $i \in [|w|]$. Therefore, w is μ -prefix normal if and only if it is X-prefix normal. So, with such a weight measure every statement on subset prefix normality.

By Theorem 45 we immediately see that subset prefix normality behaves exactly like weighted prefix normality when using a binary weight measure, which we know by Lemma 16 is gapfree.