Continuants with equal values, a combinatorial approach

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Abstract

A regular continuant is the denominator K of a terminating regular continued fraction, interpreted as a function of the partial quotients. We regard K as a function defined on the set of all finite words on the alphabet $1 < 2 < 3 < \dots$ with values in the positive integers. Given a word $w = w_1 \cdots w_n$ with $w_i \in \mathbb{N}$ we define its multiplicity $\mu(w)$ as the number of times the value $K(w)$ is assumed in the Abelian class $\mathcal{X}(w)$ of all permutations of the word w. We prove that there is an infinity of different lacunary alphabets of the form $\{b_1 < \cdots < b_t < l+1 < l+2 < \cdots < s\}$ with $b_j, t, l, s \in \mathbb{N}$ and s sufficiently large such that μ takes arbitrarily large values for words on these alphabets. The method of proof relies in part on a combinatorial characterisation of the word w_{max} in the class $\mathcal{X}(w)$ where K assumes its maximum.

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Introduction. Given a sequence $w = (w_1, \dots, w_n)$, of positive w_i, let $K(w)$ be the continuant of w, i.e., the denominator of the finite regular continued fraction $\frac{1}{w_1+}\frac{1}{w_2+}\cdots\frac{1}{w_{n-1}+}\frac{1}{w_n}$. We shall regard w as a *word* of length n over the alphabet $\{1 < 2 < 3 < \dots\}$ and write $w = w_1 \cdots w_n$. Since $K(w) = K(\bar{w})$, where $\bar{w} = w_n \cdots w_1$ denotes the reversal of w, we shall henceforth identify each word w with its reverse $\bar{w}$. Let $\mathcal{X}(w)$ denote the Abelian class of w consisting of all permutations of w. The following problem has attracted much attention and led to a number of applications (see e.g. [1, 4, 5, 7, 8]): Let $A = \{a_1 < \cdots < a_s\}$ be a finite ordered alphabet with $a_j \in \mathbb{N}$. Given a word $w = w_1 w_2 \cdots w_n$ with $w_i \in A$, find the arrangements $w_{\max}, w_{\min} \in \mathcal{X}(w)$ maximizing resp. minimizing the function $K(\cdot)$ on $\mathcal{X}(w)$. The first author [3] gave an explicit description of both extremal arrangements $w_{\max}$ and $w_{\min}$ and showed that in each case the arrangement is unique (up to reversal) and independent of the actual values of the positive integers a_i. He also investigated the analogous problem for the semi-regular continuant K' defined as the denominator of the semi-regular continued fraction $\frac{J'}{K'} = \frac{1}{w_1-}\frac{1}{w_2-}\cdots\frac{1}{w_{n-1}-}\frac{1}{w_n}$ with entries $w_i \in \{2, 3, ...\}$. He gave a fully combinatorial description of the minimizing arrangement $w'_{\min}$ for $K'(\cdot)$ on $\mathcal{X}(w)$ and showed that the arrangement is unique (up to reversal) and independent of the actual values of the positive integers a_i. However, the determination of the maximizing arrangement $w'_{\max}$ for the semi-regular continuant turned out to be more difficult. He showed that in the special case of a 2-digit alphabet $\{(2 \leq)\ a_1 < a_2\}$, the maximizing arrangement $w'_{\max}$ is a Sturmian word and is independent of the values of the a_i. Recently the second author together with M. Edson and A. De Luca [8] developed an algorithm for constructing $w'_{\max}$ over any ternary alphabet $\{(2 \leq)\ a_1 < a_2 < a_3\}$, and showed that the maximizing arrangement is independent of the choice of the digits. In contrast, they exhibited examples of words $w = w_1 \cdots w_n$ over a 4-digit alphabet $A = \{(2 \leq)\ a_1 < a_2 < a_3 < a_4\}$

for which the maximizing arrangement for $K'(\cdot)$ is not unique and depends on the actual values of the positive integers a_1 through a_4. In the course of these investigations the following problem came up: given an alphabet A of positive integers, we say that a word w on A has multiplicity $\mu = \mu(w)$ if the value $K(w)$ occurs precisely μ times in the multi-set $\{K(x) : x \in \mathcal{X}(w)\}$. The multiplicity $\mu'(w)$ is defined analogously for the semi-regular continuant $K'(w)$. Thus each Abelian class $\mathcal{X}(w)$ is split into subclasses of equally valued words. Question: is it true that μ can take arbitrarily large values for infinitely many alphabets and is there a combinatorial proof of this? Our aim here is to give a positive answer to this question in the case of regular continued fractions.

Theorem. Fix positive integers $1 \le t \le l < s$, $b_1 < \ldots < b_t \le l$ and let A be an ordered alphabet of the form $\{b_1 < \ldots < b_t < l+1 < \ldots < s\}$. Then for all s sufficiently large, there exists an infinite sequence of words w_k over A with multiplicities $\mu(w_k) \to \infty$ as $k \to \infty$.

It should be noted that for fixed s one obtains the largest possible alphabet $A' = \{1 < 2 < \cdots < s\}$ by choosing $b_1 = t = l = 1$ $(< s)$. Our proof makes use of the combinatorial structure of $w_{\max}$ found by the first author in [3].

Preliminaries. We introduce some notation. Let $w = w_1 \cdots w_n$ be a word of length $n \ge 2$ with $w_j \in \mathbb{N}$ $(j = 1, \ldots, n)$. The regular continuant of w has a matrix representation

$$K(w_1) = w_1 \quad \text{and} \quad K(w) = \det \begin{pmatrix} w_1 & -1 & 0 & \cdots & 0 \\ 1 & w_2 & -1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & w_{n-1} & -1 \\ 0 & \cdots & 0 & 1 & w_n \end{pmatrix}, \; n \ge 2$$

It can also be defined recursively by $K(\{\}) = 1$ ($\{\}$ = empty word), $K(w_1) = w_1$ and $K(w_1 \cdots w_j) = w_j K(w_1 \cdots w_{j-1}) + K(w_1 \cdots w_{j-2})$ for $j \ge 2$. For each $1 \le k \le m \le n$ we set $w_{k,m} := w_k \cdots w_m$ and $W := W_{1,n}$, $W_{k,m} := K(w_{k,m})$. The following fundamental formula goes back to the late 19th century and can be found in Perron [2], p.11, (4): $(W =)\ W_{1,n} = W_{1,j} W_{j+1,n} + W_{1,j-1} W_{j+2,n}$ $(j \in \{1, \ldots, n-1\})$. From this we infer the simple but useful inequality

$$W_{1,n} < 2 W_{1,j} W_{j+1,n}. \tag{1}$$

Let $A = \{a_1 < \cdots < a_s\} \subset \mathbb{N}$. We consider a word $w = w_1 \cdots w_n := a_1^{p_1} \cdots a_s^{p_s}$ of length n with Parikh vector $\mathbf{p} = (p_1, \ldots, p_s)$ with $p_1 + \cdots + p_s = n$ where

$$a^r = \underbrace{aa \cdots a}_{r\text{-times}}$$

denotes a sequence of r equal elements a. Let $\mathcal{X} = \mathcal{X}(A, \mathbf{p})$ denote the set of all permutations of w where we identify each word v with its reverse $\bar{v}$. Let $N(A, \mathbf{p})$ denote the cardinality of $\mathcal{X}$. Then, $N(A, \mathbf{p}) \ge \frac{n!}{2 p_1! \ldots p_s!}$. We put $W_{\max} = W_{\max}(A, \mathbf{p}) := \max\{K(v) : v \in \mathcal{X}\}$. It was shown in [3] (see (3), p. 190) that $W_{\max}$ is uniquely attained (up to reversal) by the arrangement

$$a_s L_{s-1} a_{s-2} L_{s-3} \cdots a_1^{p_1} \cdots a_{s-3} L_{s-2} a_{s-1} L_s \tag{2}$$

where $L_i = a_i^{p_i - 1}$. Let $P = P(A, \mathbf{p}) = \#\{K(v) : v \in \mathcal{X}\}$.

Proof of the Theorem. Our first goal is to describe how to specify the last digit s (≥ 2) in an alphabet $A : \{b_1 < \cdots < b_t < l+1 < \cdots < s\}$. We consider 'equipartitioned' words

$$w = w_1 \cdots w_n := b_1^m \cdots b_t^m (l+1)^m \cdots s^m.$$

corresponding to the Parikh vector $\mathbf{p} = (m, m, \ldots, m)$ in which each digit of A occurs precisely m-times in w. We will give a lower bound for s (see (7) below). To this end, we introduce the quantities $Q_{r,m-1} := K(r^{m-1})$ $(r \in 1, 2, ...)$. They are the elements of the r-th generalised Fibonacci sequence which is determined by the recursion $Q_{r,0} := 1$, $Q_{r,1} := K(r) = r$, $Q_{r,j+1} := rQ_{r,j} + Q_{r,j-1}$ $(j = 1, 2, \ldots)$.

Claim: $Q_{r,j-1} < (r+1)^j$ for each fixed $r \geq 1$ and all $j \geq 1$.

To prove the claim, we proceed by induction on j: This is obviously true for $j = 1$ and $j = 2$. Then by the induction hypothesis

$$\begin{aligned} Q_{r,j-1} &= rQ_{r,j-2} + Q_{r,j-3} < r(r+1)^{j-1} + (r+1)^{j-2} \\ &= (r+1)^{j-2}(r(r+1)+1) < (r+1)^{j-2}(r+1)^2 \\ &= (r+1)^j. \end{aligned}$$

In order to obtain an upper bound for the number $P(A, \mathbf{p})$, it suffices to consider words over the largest allowed s-digit alphabet $A' : \{1 < \cdots < s\}$, $b_1 = t = l = 1$ $(< s)$, with Parikh vector $\mathbf{p}' = (\underbrace{m, m, \ldots, m}_{s\text{-times}})$. Clearly

$$P(A, \mathbf{p}) \leq W_{\max}(A, \mathbf{p}) \leq W_{\max}(A', \mathbf{p}')$$

and by (2)

$$w_{\max}(A', \mathbf{p}') = s \cdot (s-1)^{m-1} \cdot (s-2) \cdots 1 \cdot 1^{m-1} \cdots (s-2)^{m-1} \cdot (s-1) \cdot s^{m-1}. \tag{3}$$

By iteration of (1) applied to the decomposition in (3) we obtain the inequalities

$$\begin{aligned} W_{\max}(A', \mathbf{p}') &= K(w_{\max}(A', \mathbf{p}')) \\ &< 2^{2s} \; s \cdot (s-1) \cdots 3 \cdot 2 \; \prod_{j=1}^{s} K(j^{m-1}) \\ &= 2^{2s} \; s! \; \prod_{j=1}^{s} Q_{j,m-1} \\ &< 2^{2s} \; s! \; \prod_{j=1}^{s} (j+1)^m \\ &= 2^{2s} \; s!((s+1)!)^m \end{aligned}$$

and hence

$$P(A, \mathbf{p}) < 2^{2s}\ s!\ ((s+1)!)^m. \tag{4}$$

For each $s \geq 2$ we define $m_0 = m_0(s)$ to be the smallest positive integer such that

$$2^{2s}\ s! \leq \left(\frac{100}{99}\right)^{m_0}.$$

Then

$$P(A, \mathbf{p}) < \left(\frac{100}{99}\ ((s+1)!)\right)^m \quad \text{for all} \ \ m \geq m_0(s). \tag{5}$$

On the other hand, we have the following lower bound for the number of different words in $\mathcal{X}(w)$:

$$N(A, \mathbf{p}) \geq \frac{((s-l+t)\,m)!}{2(m!)^{s-l+t}} \tag{6}$$

Based on the condition (7) below, we will later make a choice of $s = s'(t, l)$ depending on the parameters t, l. We apply the estimates provided by Sterling's formula to the factorial terms occurring in relations (5) and (6) to obtain

$$(P(A, \mathbf{p}))^{1/m} < \frac{100}{99}(s+1)! < \frac{100}{99}\frac{12}{11}\ \mathrm{e}^{-(s+1)}(s+1)^{s+1}\sqrt{2\pi(s+1)}.$$

$$(((s-l+t)\,m)!)^{1/m} > \mathrm{e}^{-(s-l+t)}((s-l+t)\,m)^{s-l+t}\sqrt{2\pi(s-l+t)\,m}^{\,1/m}.$$

$$\left(2\,(m!)^{s-l+t}\right)^{1/m} < \mathrm{e}^{-(s-l+t)}m^{s-l+t}\left(2\ \frac{12}{11}\ \left(\sqrt{2\pi m}\right)^{s-l+t}\right)^{1/m}.$$

When we put the right hand sides of the last two inequalities together, the terms e^{s-l+t} and m^{s-l+t} cancel out, and if we keep the parameters t, l fixed for the moment, the terms of the form $\sqrt{\cdot}^{\,1/m}$ tend to 1 as $m \to \infty$. Letting $m \to \infty$ we get

$$\begin{aligned}
\lim_{m\to\infty}\left(\frac{N(A,\mathbf{p})}{P(A,\mathbf{p})}\right)^{1/m} &\geq \frac{99}{100}\frac{11}{12}\frac{\mathrm{e}^{s+1}(s-l+t)^{s-l+t}}{\sqrt{2\pi(s+1)}(s+1)^{s+1}} = \\
&= \frac{363}{400}\frac{\mathrm{e}^{s+1}(s+1-l+t-1)^{\ s+1-l+t-1}}{\sqrt{2\pi(s+1)}(s+1)^{s+1}} \\
&= \frac{363}{400}\frac{\mathrm{e}^{s+1}}{\sqrt{2\pi(s+1)}\ (s-l+t)^{l-t+1}}\left(1-\frac{l-t+1}{s+1}\right)^{s+1}.
\end{aligned}$$

For fixed t, l $(l-t \geq 1)$ the function $f(t,l,s) = \left(1-\frac{l-t+1}{s+1}\right)^{s+1}$ in the variable s is strictly increasing on the interval $[l-t+1, \infty)$ with $f(t,l,s) \nearrow \mathrm{e}^{-(l-t)-1}$ as $s \to \infty$. We define s_0 to be the lowest integer such that $f(t,l,s_0) \geq \frac{1}{2}\ \mathrm{e}^{-(l-t)-1}$. Then

$$\lim_{m\to\infty}\left(\frac{N(A,\mathbf{p})}{P(A,\mathbf{p})}\right)^{1/m} \geq \frac{363}{400}\frac{\mathrm{e}^{s+1}}{\sqrt{2\pi(s+1)}\ (s-l+t)^{l-t+1}}\ \frac{1}{2}\ \mathrm{e}^{-(l-t)-1} \quad =: H(t,l,s)$$

for all $s \geq s_0$. Obviously there exists some sufficiently large $s' = s'(t,l) \geq s_0$ such that

$$H(t,l,s') > 1. \tag{7}$$

Therefore the right hand side of

$$\left(\frac{N(A,\mathbf{p})}{P(A,\mathbf{p})}\right) > (H(t,l,s'))^m \tag{8}$$

can be made arbitrarily large by letting $m \to \infty$. We call an $(s'-l+t)$-digit alphabet $A = \{(1 \leq)\ b_1 < \cdots < b_t < \cdots < s'\}$ *admissible* if $s' = s'(t,l)$ fulfills condition (7) We consider the word $u(A,\mathbf{p}_1) = (b_1)^{m_1} \cdots (b_t)^{m_1}(l+1)^{m_1} \cdots (s')^{m_1}$ of length $n = (s'-l+t)\,m_1$ with Parikh vector $\mathbf{p}_1 = ((m_1)^{s'-l+t})$ where we choose $m_1 \geq m_0$ such that $\left(\frac{N(A,\mathbf{p}_1)}{P(A,\mathbf{p}_1)}\right) > (H(t,l,s'))^{m_1}$. The multi-set $\mathcal{X}_1 = \mathcal{X}(A,\mathbf{p}_1)$ is made up of the $N(A,\mathbf{p}_1) = \#\mathcal{X}_1$ permuted arrangements of u. There exists at least one word $w_1 \in \mathcal{X}_1$ with multiplicity $\mu \geq 2$ because otherwise we would have $N(A,\mathbf{p}_1) = P(A,\mathbf{p}_1)$ which contradicts (8) with $m = m_1$. Let μ_1 (≥ 2) be the maximal multiplicity attained by words $w \in \mathcal{X}_1$. Next choose $m_2 > m_1(s')$ such that $H(t,l,s')^{m_2} > \mu_1$. We claim that at least one word w_2 from $\mathcal{X}_2 = \mathcal{X}(A,\mathbf{p}_2),\ \mathbf{p}_2 = ((m_2)^{s'-l+t})$ has multiplicity $\mu > \mu_1$. Otherwise we would have $N(A,\mathbf{p}_2) \leq \mu_1 P(A,\mathbf{p}_2)$ which contradicts (8) with $m = m_2$. Next let μ_2 $(\geq \mu_1)$ be the maximal multiplicity attained by words $w \in \mathcal{X}_2$. Proceeding with this construction step by step we end up with a sequence of words w_k on A with multiplicities $\mu_k \to \infty$ as $k \to \infty$. The construction can be carried out for infinitely many different admissible alphabets. This completes the proof of the Theorem. $\square$

The question remains largely unsolved in the case of semi-regular continuants though it seems certain that the behavior is quite similar to the regular case.

There is some evidence supporting the following

Conjecture. Given any ordered alphabet $A = \{a_1 < \cdots < a_s\}$ $(a_j \in \mathbb{N},\ s \geq 2)$, let $\mu \geq 2$ be a positive integer. Then there exist infinitely many words on A whose multiplicity is precisely μ. The problem appears to require a difficult investigation into the values of continuants. Most likely our theorem and the conjecture also hold for continuants of semi-regular continued fractions. Unfortunately no higher-dimensional analogue of the theorem is available at present for $s \geq 4$ due to the fact that very little is known about the maximizing arrangements $w'_{\max}$ for $s \geq 4$ (see [8]).

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