Truth Tables Without Truth Values: On 4.27 and 4.42 of Wittgenstein's *Tractatus*

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Abstract. In 4.27 and 4.42 of his *Tractatus* Wittgenstein introduces quite complicated formulas, which are equivalent to 2^n and 2^{2^n}. This paper shows, however, that the formulas Wittgenstein presents fit particularly well with the way he thinks about truth values, logical connectives, tautologies, and contradictions. Furthermore, it will be shown how Wittgenstein could have avoided truth values even more radically. In this way it is demonstrated that the reference to truth values can indeed be substituted by talking of existing and non-existing facts.

1 Introduction

In his *Tractatus*, Wittgenstein presents a very useful diagrammatic device for logic: truth tables. Unlike Frege, however, Wittgenstein does not assume the independent objects "truth" and "falsehood." This paper aims to show how this philosophical position is reflected in Wittgenstein's formulas, which calculate the size and number of such tables, and in his truth tables themselves.

In 4.27, Wittgenstein presents a formula whose purpose is to calculate the possible combinations "with regard to the existence of n atomic facts"[1] ("*Bezüglich des Bestehens und Nichtbestehens von n Sachverhalten.*" Note, however, that the word "existence," which occurs in both English translations[2] of the *Tractatus*, is not used in the German version):

$$K_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \tag{1}$$

This formula is much more complicated than 2^n, which is equivalent to Wittgenstein's formula K_n.[3]

[1] Here, and henceforward, I quote the translation by Odgen [11], if not otherwise specified.

[2] Pears/McGuinness [9] translate the explanation of 4.27 as the "possibilities of existence and non-existence" for "*n* state of affairs.".

[3] This equivalence can easily be checked with the binomial theorem: $(x + y)^n = \sum_{k=0}^{k=n} \binom{n}{k} x^{n-k} y^k$. In the special case we now have $x = y = 1$, which yields: $2^n = (1 + 1)^n = \sum_{k=0}^{k=n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{k=n} \binom{n}{k} = K_n$.

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As a reminder:

$$\sum_{\nu=0}^{n} \binom{n}{\nu} = \frac{n!}{0!(n-0)!} + \frac{n!}{1!(n-1)!} + \cdots + \frac{n!}{n!(n-n)!} \tag{2}$$

Similarly, Wittgenstein calculates the number of possibilities "with regard to the agreement and disagreement of a proposition with the truth possibilities of n elementary propositions," which he later also calls "truth-conditions" [11, 4.431], in 4.42 with:

$$L_n = \sum_{\kappa=0}^{K_n} \binom{K_n}{\kappa} \tag{3}$$

L_n is equivalent to 2^{2^n}.

In his *Companion to Wittgenstein's Tractatus*, Max Black only points out the equivalence of the formulas K_n and L_n with 2^n and 2^{2^n}, respectively ([1, p. 215 and p. 222]; see also: [2, 198] and [3, 116]). However, I will argue that Wittgenstein actually has good reasons to present K_n instead of 2^n, and L_n instead of 2^{2^n}: Firstly, in his formulas, Wittgenstein does not assume truth values, which he argues are not independent objects, but he only assumes atomic facts. Furthermore, in L_n, Wittgenstein distinguishes between the different numbers of possible truth conditions, thereby separating out the cases in which no or all possible truth conditions are chosen: contradiction and tautology.

In what follows, I will firstly explain the combinatorial approach which underlies the different formulas and thereby show that truth values are not considered in Wittgenstein's formulas. Secondly, I will show how L_n fits with Wittgenstein's attitude towards tautology and contradiction. Thirdly, I will argue that Wittgenstein also presents the truth table in 4.31 and 4.442 in the way he does because of his rejection of truth values as independent objects. Finally, I will follow Wittgenstein's approach to its logical end by introducing pure tables of atomic facts ("*Sachverhaltstabellen*"), or truth tables without truth values.

2 The Formulas Explained from a Combinatorial Point of View

$\binom{n}{\nu}$, a fragment of K_n, and k^n, the generalized version of 2^n, are the standard formulas for calculating two of the four standard tasks in combinatorics:[4]

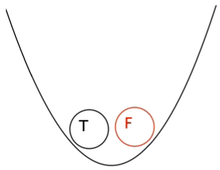
permutation with repetition	permutation without repetition
combination with repetition	combination without repetition

k^n is the formula used to calculate the number of possible permutations with repetition. $\binom{n}{\nu}$ is the formula for calculating the number of possible combinations without repetition. Hence, with K_n and 2^n one calculates the number of possible

[4] In a combination, in contrast to a permutation, the order of selection does not matter.

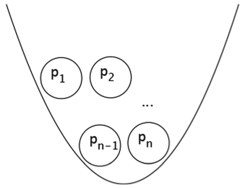
combinations of existing atomic facts or, as it is usually expressed, truth values, in quite different ways.

With 2^n one calculates the number of possibilities to choose arbitrarily n times one of two truth values. This can be illustrated as an urn problem. Consider an urn of (black and red) truth-values balls. One now asks how many ways there are to take out (and put back immediately) a truth-value ball from the urn n times. In this situation, the order is important, because it matters which truth value is assigned to which elementary proposition.



Thus, in our urn problem, one asks for the number of possibilities for putting truth-value balls at places denoted by elementary propositions. In the following picture, for example, p_1 is true and all other elementary propositions are false.

$\binom{n}{\nu}$, which occurs in K_n, instead indicates the number of possibilities for choosing ν of n elementary propositions. Thus, the balls in our urn do not represent truth values but rather elementary propositions. From this urn, ν balls are chosen without repetition. When a ball representing an elementary proposition is chosen, this means that the elementary proposition is true or, as Wittgenstein puts it, that it expresses an atomic fact. When a ball representing an elementary proposition is not chosen, this means that the elementary proposition is false or, to put it in other words, that the atomic fact does not exist.



For $\nu = 1$, for example, there are n different possibilities. Choosing the ball labeled with p_1, for example, means that only p_1 is true and any other proposition is false. For $\nu = n$, however, there is only one possibility, namely to choose every ball. The balls taken from the urn do not have to be placed in a specific order. It is only significant which balls are chosen.

With K_n one calculates the sum of the possibilities for all ν from 0 to n.[5] Thus, truth values are not considered in K_n at all, but only the existence and non-existence of atomic facts.

That an atomic fact exists is, according to Wittgenstein, the meaning of the expression “a proposition is true”:

> If the elementary proposition is true, the atomic fact exists; if it is false the atomic fact does not exist. [11, 4.25]

In 4.28, the paragraph just after 4.27, in which he presents his formula, Wittgenstein explains:

> To these combinations [of atomic facts to exist, and the others not to exist] correspond the same number of possibilities of the truth and falsehood of n elementary propositions. [11, 4.28]

However, Wittgenstein does not use the words “true” and “false” in order to denote independent objects:

> It is clear that to the complex of the signs “F” and “T” no object (or complex of objects) correspond. [11, 4.441]

Thus, instead of “the fact exists” one can say “the proposition is true,” but not that “the proposition denotes the truth.”[6]

2^{2^n} underlies the same combinatorial approach as 2^n, and L_n the same as K_n. 2^{2^n} can be illustrated as an urn problem as follows: take a ball representing a truth value and put it in a specific place, which is now not simply denoted by an elementary proposition anymore but by one of the 2^n truth value assignments. The picture on the next page, for example, illustrates that only one truth-value assignment is true, namely, that all elementary propositions are true. Thus, it represents an n-ary conjunction.

$\binom{K_n}{\kappa}$, which is part of L_n, indicates the number of possibilities for choosing κ of K_n possible combinations of elementary propositions. Thus, the balls in the

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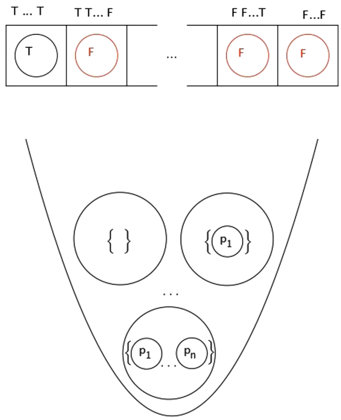
[5] A similar explanation of K_n is made by Morris [4], endnote 6 to chapter 5. Morris, however, does not contrast this to 2^n. Zalabardo [12, 187–188] points out that 2^n is the number of subsets of a set of n state of affairs. This is correct, but in this explanation it is hard to see why Wittgenstein did not choose 2^n as a formula in the first place.

[6] It is central to Wittgenstein's philosophy of logic that there are no logical objects or concepts. This sits in sharp contrast especially to Frege and also to Russell. (See [5, 59–60], [4, 205–206], [6, 52–62 and 86–93] and [3, 97–100]).

Frege first introduced the idea that propositions denote truth values, which are objects, and names of logical connectives denote functions, just like names of other concepts do. This idea is fundamental to Frege's logicism. Since for Frege mathematical propositions do have a content, and parts of mathematics are logical in nature, logic also must have a content (see [8], chapter 1). Wittgenstein in contrast already points out in his Notebook [10] in an entry dated 25 December 1914 that his “fundamental idea” is “that the logical constants are not proxies”. This is also expressed in 4.0312 of the *Tractatus*.

urn do not represent elementary propositions, as in $\binom{n}{\nu}$, but rather combinations of the elementary propositions. The choice of a ball representing a combination of elementary propositions represents the agreement with the existence of the facts described by the elementary propositions occurring in the combination, and with the non-existence of the facts described by the elementary propositions not occurring in the combination. That a ball is not chosen represents the respective disagreement. Since, according to sentence 1 and 2 of the *Tractatus*, the world is defined as all facts that exist, if there were n elementary propositions, one of the balls would represent the world.



3 Tautology and Contradiction

Besides the different combination approaches, there is another important distinction between 2^{2^n} and L_n: while one calculates all possibilities at once with 2^{2^n}, with L_n, one calculates *separately* the possibilities for every number ("κ") of combinations of elementary propositions chosen. In this way, Wittgenstein already separates the "two extreme cases" [11, 4.46]: to choose no or all elementary propositions. He calls these cases "tautology" and "contradiction." They are considered in L_n by the last and the first summand, $\binom{K_n}{K_n}$ and $\binom{K_n}{0}$. Both summands are 1 for any result of K_n.

Hence, Wittgenstein's perspective of considering "the range which is left to the facts by the proposition" [11, 4.463] is already opened up by L_n. This angle leads him to the conclusion that tautology and contradiction cannot "in any way determine reality" [11, 4.463], because "[t]autology leaves to reality the whole infinite logical space; contradiction fills the whole logical space and leaves no point to reality" [11, 4.463]. In other words, to choose no or any combination of elementary propositions does not specify our world, which is the totality of existing facts.

4 Wittgenstein's Truth Tables

With the explanation of of formula L_n in mind it is possible to understand why the truth conditions are ordered the way they are in Wittgenstein's truth tables. In particular, the truth table for three elementary propositions in 4.31 seems quite chaotic or at least less reasonable than the standard order:

p	q	r
T	T	T
F	T	T
T	F	T
T	T	F
F	F	T
F	T	F
F	F	T
F	F	F

However, if one counts the occurrences of "T" in the table, one understands the idea behind Wittgenstein's order: Wittgenstein first lists the case that three elementary propositions are chosen, then the cases that two of three are chosen, then that one of three is chosen and, at the end, that no proposition is chosen. In the normal form this is mixed up.

The way Wittgenstein presents the truth table in 4.442 also fits with the way Wittgenstein presents his formulas. In this truth table, Wittgenstein only marks the possible combinations of atomic facts with "T" and leaves a blank for the impossible combination.

p	q	
T	T	T
F	T	T
T	F	
F	F	T

This seems quite strange. Taking into account that Wittgenstein eschews the assumption of truth values as independent objects, this notation becomes less puzzling: Wittgenstein does not use "T" and "F" to assign truth values. In the left columns, he uses "T" to mark the elementary propositions that describe a fact that exists, and "F" to mark the elementary propositions that describe a fact that does not exist. In the last column he uses "T" and "F" to mark the agreement and disagreement with the existence and non-existence of the atomic facts described by the elementary propositions. Since there are no other options besides agreement and disagreement, one can only use "T" in the last column to mark the agreement and stipulate that a blank in the last column expresses disagreement [11, 4.43]. This is just what Wittgenstein does when he only uses "T"s in the last column of the table in 4.422.

5 Truth Tables Without Truth Values

Wittgenstein could even have avoided using "T" and "F," which are usually considered to denote truth values, more consistently. In 4.31 he presents truth possibilities, which he had defined just before in 4.3 as "the possibilities of the existence and non-existence of the atomic facts."

Wittgenstein could have also just listed the different combinations of atomic facts. Thus, instead of the truth table in 4.31, he could have simply presented the following table:[7]

p	q
	q
p	

This is a pure table of atomic facts ("*Sachverhaltstabelle*") or, to put it in other words, a truth table without truth values.

Avoiding signs for truth and falsehood in a truth table for a particular logical connective, such as the one introduced in 4.442, is more complicated. However, Wittgenstein introduces an alternative notation "if the sequence of the truth-possibilities is once for all determined" [11, 4.442], namely:

(TT–T) (p,q) or (TTFT) (p,q)

Such a linear notation can be easily transformed into one without truth values. One could only give a list of the possible combinations of atomic facts:

((p,q), (q), ())

In this notation, the sign for the conditional is completely substituted by brackets and commas. This illustrates Wittgenstein's remark that "[l]ogical operation signs are punctuations" [11, 5.4611].

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[7] Black [1, 217] suggests that Wittgenstein's table in 4.31 could be read as follows:

p and	q
not-p and	q
p and	not-q
not-p and	not-q

This would be in accordance with the "same-level interpretation," according to which "T and F serve merely as indicators of the positive and negative quality of the proposition signs to which they are attached" [1, 217]. Black assumes that this interpretation is correct [1, 218]. He contrasts it with the "new level" interpretation according to which "the T's and F's in the truth-table must [...] be understood to stand for truth and falsehood respectively" [1, 216]. Ricketts [7] also stresses that truth tables are "object language expressions" and not "metalinguistic devices."

To propose that p does not hold with "not-p," as Black suggests, is nevertheless kind of circular: One uses logical connectives within the truth table, and truth tables themselves, to express a logical connection.

If we now want to transform this into a truth table, we could just insert this possible combination into a third column and – just as in 4.442 – leave a blank for the impossible combinations. The truth table for the conditional would then look like this:

p	q	(p, q)
p		(p,)
	q	
		(,)

This looks a bit artificial. It shows, however, that Wittgenstein's attempt to substitute talk of "true propositions" with talk of "existing facts" ("*bestehenden Sachverhalten*") could be realized even more consistently. In combination with the reading of his formulas K_n and L_n presented in this paper this shows that "T" and "F" can in fact be perceived as mere abbreviations and not as names for objects *sui generis*. Thus, Wittgenstein managed to introduce truth tables without presupposing truth values.

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References

1. Black, M.: A Companion to Wittgenstein's Tractatus, 5th edn. Cornell University Press, Ithaca (1992)
2. McGinn, M.: Elucidating the Tractatus. Wittgenstein's Early Philosophy of Logic and Language. Clarendon Press, Oxford (2006)
3. Milne, P.: Tractatus 5.4611: 'Signs for logical operations are punctuation marks'. In: Sullivan, P., Potter, M. (eds.) Wittgenstein's Tractatus. History and Interpretation, pp. 97–124. Oxford University Press, Oxford (2013)
4. Morris, M.: Wittgenstein and the Tractatus. Routledge, New York (2008)
5. Peterson, D.: Wittgenstein's Early Philosophy. Three Sides of the Mirror. University of Toronto Press, Toronto (1990)
6. Potter, M.: Wittgenstein's Notes on Logic. Oxford University Press, Oxford (2009)
7. Ricketts, T.: Pictures, logic and the limits of sense in Wittgenstein's Tractatus. In: Sluga, H., Stern, D. (eds.) The Cambridge Companion to Wittgenstein, pp. 59–99. Cambridge University Press, Cambridge (1996)
8. Rohr, T.: Freges Begriff der Logik. Mentis, Paderborn (2020)
9. Wittgenstein, L.: Tractatus logico-philosophicus. Routledge, London (1995). Translated by D. F. Pears and B. F. McGuinness
10. Wittgenstein, L.: Notebooks 1914–1916. Ed. by G. H. Wright with an English translation by G. E. M. Anscombe. Blackwell, Oxford (2004)
11. Wittgenstein, L.: Tractatus logico-philosophicus. Routledge, London (2005). Translated by C. K. Odgen
12. Zalabardo, J.: Representation and Reality in Wittgenstein's Tractatus. Oxford University Press, Oxford (2019)