Computational Complexity of Computing a Quasi-Proper Equilibrium

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Abstract

We study the computational complexity of computing or approximating a quasi-proper equilibrium for a given finite extensive form game of perfect recall. We show that the task of computing a symbolic quasi-proper equilibrium is PPAD-complete for two-player games. For the case of zero-sum games we obtain a polynomial time algorithm based on Linear Programming. For general n-player games we show that computing an approximation of a quasi-proper equilibrium is FIXP_a-complete. Towards our results for two-player games we devise a new perturbation of the strategy space of an extensive form game which in particular gives a new proof of existence of quasi-proper equilibria for general n-player games.

1 Introduction

A large amount of research has gone into defining [22, 23, 18, 13, 25] and computing [26, 17, 10, 6, 9, 8] various refinements of Nash equilibria [19]. The motivation for introducing these refinements has been to eliminate undesirable equilibria, e.g., those relying on playing dominated strategies.

The quasi-proper equilibrium, introduced by van Damme [25], is one of the more refined solution concepts for extensive form games. Any quasi-proper equilibrium is quasi-perfect, and therefore also sequential, and also trembling hand perfect in the associated normal form game. Beyond being a further refinement of the quasi-perfect equilibrium [25], it is also conceptually related in that it addresses a deficiency of the direct translation of a normal form solution concept to extensive form games. One of the most well known refinements is Selten's trembling hand perfect equilibrium, originally defined [22] for normal form games, and the solution concept is usually referred to as normal-form perfect. This can be translated to extensive form games, by applying the trembling hand definition to each information set of each player, which yields what is now known as extensive-form perfect equilibria [23]. However, this translation introduces undesirable properties, first pointed out by Mertens [16]. Specifically, Mertens

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presents a certain two-player voting game where all extensive-form perfect equilibria have weakly dominated strategies in their support. That is, extensive-form perfection is in general inconsistent with *admissibility*. Mertens argues that quasi-perfection is conceptually superior to Selten's notion of extensive-form perfection, as it avoids the cause of the problem in Mertens' example. It achieves this with a subtle modification of the definition of extensive-form perfect equilibria, which in effect allows each player to plan as if they themselves were unable to make any future mistakes. Further discussion of quasi-perfection can be found in the survey of Hillas and Kohlberg [11].

One of the most restrictive equilibrium refinements of normal-form games is that of Myerson's normal-form proper equilibrum [18], which is a refinement of Selten's normal-form perfect equilibrium. Myerson's definition can similarly be translated to extensive form, again by applying the definition to each information set of each player, which yields the extensive-form proper equilibria. Not surprisingly, all extensive-form proper equilibria are also extensive-form perfect. Unfortunately, this also means that Merten's critique applies equally well to extensive-form proper equilibria. Again, the definition can be subtely modified to sidestep Merten's example, which then gives the definition for quasi-proper-equilibria [25]. It is exactly this solution concept that is the focus of this paper.

1.1 Contributions

The main novel idea of the paper is a new perturbation of the strategy space of an extensive form game of perfect recall, in which a Nash equilibrium is an ε-quasi-proper equilibrium of the original game. This construction works for *any* number of players and in particular directly gives a new proof of existence of quasi-proper equilibria for general n-player games.

From a computational perspective we can, in the important case of two-player games, exploit the new pertubation in conjunction with the sequence form of extensive form games [12] to compute a symbolic quasi-proper equilibrium by solving a Linear Complementarity Problem. This immediately implies PPAD-membership for the task of computing a symbolic quasi-proper equilibrium. For the case of zero-sum games a quasi-proper equilibrium can be computed by solving just a Linear program which in turn gives a polynomial time algorithm.

For games with more than two players there is, from the viewpoint of computational complexity, no particular advantage in working with the sequence form. Instead we work directly with behavior strategies and go via so-called δ-almost ε-quasi-proper equilibrium, which is a relaxation of ε-quasi-proper equilibrium. We show FIXP_a-membership for the task of computing an approximation of a quasi-proper equilibrium. We leave the question of FIXP-membership as an open problem similarly to previous results about computing Nash equilibrium refinements in games with more than two players [4, 3, 9].

Since we work with refinements of Nash equilibrium, PPAD-hardness for two-player games and FIXP_a-hardness for n-player games, with $n \geq 3$ follow directly. This combined with our membership results for PPAD and FIXP_a implies PPAD-completeness and FIXP_a-completeness, respectively.

1.2 Relation to previous work

Any strategic form game may be written as an extensive form game of comparable size, and any quasi-proper equilibrium of the extensive form representation is a proper equilibrium of the strategic form game. Hence our results fully generalize previous results for computing [24] or approximating [9] a proper equilibrium. The generalization is surprisingly clean, in the sense that if a bimatrix game is translated into an extensive form game, the strategy constraints introduced in this paper will end up being identical to those defined in [24] for the given bimatrix game. This is surprising, since a lot of details have to align for this structure to survive through a translation to a different game model. Likewise, if a strategic form game with more than two players is translated into an extensive form game, the fixed point problem we construct in this paper is identical to that for strategic form games [9].

The quasi-proper equilibria are a subset of the quasi-perfect equilibria, so our positive computational results also generalize the previous results for quasi-perfect equilibria [17]. Again, the generalization is clean; if all choices in the game are binary, then quasi-perfect and quasi-proper coincide, and the constraints introduced in this paper work out to be exactly the same as those for computing a quasi-perfect equilibrium. The present paper thus manages to cleanly generalize two different constructions in two different game models.

2 Preliminaries

2.1 Extensive Form Games

A game Γ in *extensive form* of imperfect information with n players is given as follows. The structure of Γ is determined by a finite tree T. For a non-leaf node v, let $S(v)$ denote the set of immediate successor nodes. Let Z denote the set of leaf nodes of T. In a leaf-node $z \in Z$, player i receives utility $u_i(z)$. Non-leaf nodes are either chance-nodes or decision-nodes belonging to one of the players. To every chance node v is associated a probability distribution on $S(v)$. The set P_i of decision-nodes for Player i is partitioned into information sets. Let H_i denote the information sets of Player i. To every decision node v is associated a set of $|S(v)|$ actions and these label the edges between v and $S(v)$. Every decision node belonging to a given information set h shares the same set C_h of actions. Define $m_h = |C_h|$ to be the number of actions of every decision node of h. The game Γ is of *perfect recall* if every node v belonging to an information set h of Player i share the same sequence of actions and information sets of Player i that are observed on the path from the root of T to v. We shall only be concerned with games of perfect recall [14].

A local strategy for Player i at information set $h \in H_i$ is a probability distribution b_{ih} on C_h assigning a behavior probability to each action in C_h and in turn induces a probability distribution on $S(v)$ for every $v \in h$. A local strategy b_{ih} for every information set $h \in H_i$ defines a behavior strategy b_i for Player i. The behavior strategy b_i is fully mixed if $b_{ih}(c) > 0$ for every $h \in H_i$ and every $c \in C_h$. Given a local strategy b'_{ih} denote by b_i/b'_{ih} the result of replacing b_{ih} by b'_{ih}. In particular if $c \in C_h$ we let b_i/c prescribe action c with probability 1 in h. For another behavior strategy b'_i and an information set h for Player i we let $b_i/_h b'_i$ denote the behavior strategy that chooses actions according to b_i until h is reached after which actions are chosen according to b'_i. We shall also write $b_i/_h b'_i/c = b_i/_h(b'_i/c)$. A behavior strategy profile $b = (b_1, \ldots, b_n)$ consists of a behavior strategy for each player. Let B be the set of all behavior strategy profiles of Γ. We let $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ and $(b_{-i}; b'_i) = b/b'_i = (b_1, \ldots, b_{i-1}, b'_i, b_{i+1}, \ldots, b_n)$. Furthermore, for simplicity of notation, we define $b/_h b'_i = b/(b_i/_h b'_i)$, and $b/_h b'_i/c = b/(b_i/_h b'_i/c)$.

A behavior strategy profile $b = (b_1, \ldots, b_n)$ gives together with the probability distributions of chance-nodes a probability distribution on the set of paths from the root-node to a leaf-node of T. We let $\rho_b(v)$ be the probability that v is reached by this path and for an information set h we let $\rho_b(h) = \sum_{v \in h} \rho_b(v)$ be the total probability of reaching a node of h. Note that we define $\rho_b(v)$ for all nodes v of T. When $\rho_b(h) > 0$ we let $\rho_b(v \mid h)$ be the conditional probability that node v is reached given that h is reached. The *realization weight* $\rho_{b_i}(h)$ for Player i of an information set $h \in H_i$ is the product of behavior probabilities given by b_i on any path from the root to h. Note that this is well-defined due to the assumption of perfect recall.

Given a behavior strategy profile $b = (b_1, \ldots, b_n)$, the payoff to Player i is $U_i(b) = \sum_{z \in Z} u_i(z)\rho_b(z)$. When $\rho_b(h) > 0$ the conditional payoff to Player i given that h is reached is then $U_{ih}(b) = \sum_{z \in Z} u_i(z)\rho_b(z \mid h)$.

Realization weights are also defined on actions, to correspond to Player i's weight assigned to the given action:

$$\forall h \in H_i, c \in C_h : \quad \rho_{b_i}(c) = \rho_{b_i}(h) b_i(c) \tag{1}$$

We note that the realization weight of an information set is equal to that of the most recent action by the same player, or is equal to 1, if no such action exists.

A realization plan for Player i is a strategy specified by its realization weights for that player. As shown by Koller et al. [12], the set of valid realization weights for Player i can be expressed by the following set of linear constraints

$$\forall h \in H_i : \quad \rho_{b_i}(h) = \sum_{c \in C_h} \rho_{b_i}(c) \quad \wedge \quad \forall c \in C_h : \rho_{b_i}(c) \geq 0 \tag{2}$$

in the variables $\rho_{b_i}(c)$ letting $\rho_{b_i}(h)$ refer to the realization weight of the most recent action of Player i before reaching information set h or to the constant 1 if h is the first time Player i moves. This formulation is known as the sequence form [12], and has the advantage that for two-player games, the utility of each player is bilinear, i.e., linear in the realization weights of each player. As shown by Koller et al. this allows the equilibria to be characterized by the solutions to a Linear Complementarity Problem for general sum games, and as solutions to a Linear Program for zero-sum games. We will build on this insight for computing quasi-proper equilibria of two-player games.

Given a behavior strategy for a player, the corresponding realization plan can be derived by multiplying the behavior probability of an action with the realization weight of its information set. However, it is not always the case that the reverse is possible. The behavior probability of an action is the ratio of the realization weight of an action to the realization weight of its information set, but if any of the preceeding actions by the player have probability 0, the ratio works out to $\frac{0}{0}$. In the present paper, the restriction on the strategy space ensures that no realization weight is zero, until we have retrieved the behavior probabilities.

A strategy profile b is a Nash equilibrium if for every i and every behavior strategy profile b'_i of Player i we have $U_i(b) \geq U_i(b/b'_i)$. Our object of study is quasi-proper equilibrium defined by van Damme [25] refining the Nash equilibrium. We first introduce a convenient notation for quantities used in the definition. Let b be a behavior strategy profile, h an information set of Player i such that $\rho_b(h) > 0$, and $c \in C_h$. We then define

$$K_i^{h,c}(b) = \max_{b'_i} U_{ih}(b/_h b'_i/c) \ . \tag{3}$$

When b'_i is a pure behavior strategy we say that $b/_h b'_i$ is a h-local *purification* of b. We note that $U_{ih}(b/_h b'_i/c)$ always assumes its maximum for a pure behavior strategy b'_i.

Definition 1 (Quasi-proper equilibrium)**.** *Given $\varepsilon > 0$, a behavior strategy profile b is an ε-quasi-proper equilibrium if b is fully mixed and satisfies for every i, every information set h of Player i, and every $c, c' \in C_h$, that $b_{ih}(c) \leq \varepsilon b_{ih}(c')$ whenever $K_i^{h,c}(b) < K_i^{h,c'}(b)$.*

A behavior strategy profile b is a quasi-proper equilibrium if and only if it is a limit point of a sequence of ε-quasi-proper equilibria with $\varepsilon \to^+ 0$.

We shall also consider a relaxation of quasi-proper equilibrium in analogy to relaxations of other equilibrium refinements due to Etessami [3].

Definition 2. *Given $\varepsilon > 0$ and $\delta > 0$, a behavior strategy profile b is a δ-almost ε-quasi-proper equilibrium if b is fully mixed and satisfies for every Player i, every information set h of Player i, and every $c, c' \in C_h$ that $b_{ih}(c) \leq \varepsilon b_{ih}(c')$ whenever $K_i^{h,c}(b) + \delta \leq K_i^{h,c'}(b)$.*

2.2 Strategic Form Games

A game Γ in *strategic form* with n players is given as follows. Player i has a set S_i of *pure strategies*. To a pure strategy profile $a = (a_1, \ldots, a_n)$ Player i is given utility $u_i(a)$. A mixed strategy x_i for Player i is a probability distribution on S_i. We identify a pure strategy with the mixed strategy that selects the pure strategy with probability 1. A strategy profile $x = (x_1, \ldots, x_n)$ consists of a mixed strategy for each player. To a strategy profile x Player i is given utility $U_i(x) = \sum_{a \sim x} u_i(a) \prod_j x_j(a_j)$. A strategy profile x is fully mixed if $x_i(a_i) > 0$ for all i and all $a_i \in S_i$. We let $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Given a strategy x'_i for Player i we define $(x_{-i}; x'_i) = x/x'_i = (x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)$.

A strategy profile x is a Nash equilibrium if for every i and every strategy x'_i of Player i we have $U_i(x/x'_i) \leq U_i(x)$. Myerson defined the notion of proper equilibrium [18] refining the Nash equilibrium.

Definition 3 (Proper equilibrium)**.** *Given $\varepsilon > 0$, a strategy profile x is an ε-proper equilibrium if x is fully mixed and satisfies for every i and every $c, c' \in S_i$ that $x_i(c) \leq \varepsilon x_i(c')$ whenever $U_i(x_{-i}; c) < U_i(x_{-i}; c')$.*

A strategy profile x is a proper equilibrium if and only if it is a limit point of a sequence of ε-proper equilibria with $\varepsilon \to^+ 0$.

For proper equilibrium we also consider a relaxation as suggested by Etessami [3].

Definition 4. *Given $\varepsilon > 0$ and $\delta > 0$, a strategy profile x is a δ-almost ε-proper equilibrium if x is fully mixed and satisfies for every i and every $c, c' \in S_i$ that $x_i(c) \leq \varepsilon x_i(c')$ whenever $U_i(x_{-i}; c) + \delta \leq U_i(x_{-i}; c')$.*

2.3 Complexity Classes

We give here only a brief description of the classes PPAD and FIXP and refer to Papadimitriou [20] and Etessami and Yannakakis [5] for detailed definitions and discussion of the two classes.

PPAD is a class of discrete total search problems, whose totality is guaranteed based on a parity argument on a directed graph. More formally PPAD is defined by a canonical complete problem EndOfTheLine. Here a directed graph is given implicitly by predecessor and successor circuits, and the search problem is to find a degree 1 node different from a given degree 1 node. We do not make direct use of the definition of PPAD but instead prove PPAD-membership indirectly via Lemke's algorithm [15] for solving a Linear Complementarity Problem (LCP).

FIXP is the class of real-valued total search problems that can be cast as Brouwer fixed points of functions represented by $\{+, -, *, /, \max, \min\}$-circuits computing a function mapping a convex polytope described by a set of linear inequalities to itself. The class FIXP_a is the class of *discrete* total search problems that reduce in polynomial time to *approximate* Brouwer fixed points. We will prove FIXP_a membership directly by constructing an appropriate circuit.

3 Two-Player Games

In this section, we prove that computing a single quasi-proper equilibrium of a two-player game Γ can be done in PPAD, and in the case of zero-sum games, it can be computed in P. We are using the same overall approach as has been used for computing quasi-perfect equilibria of extensive form games [17], proper equilibria of two-player games [24], and proper equilibria of poly-matrix games [9].

The main idea is to construct a new game Γ_ε, where the strategy space is slightly restricted for both players, in such a way that equilibria of the new game are ε-quasi-proper equilibria of the original game. This construction also provides a new proof of existence for quasi-proper equilibria of n-player games, since there is nothing in neither the construction nor the proof that requires the game to have only two players. However, for two players, the strategy constraints can be enforced using a symbolic infinitesimal ε, which can be part of the solution output, thereby providing a witness of the quasi-properness of the computed strategy.

We will first describe the strategy constraints. At a glance, the construction consists of fitting the strategy constraints for ε-proper equilibria [24] into the strategy constraints of each of the information sets of the sequence form [12], discussed in the preliminaries section, equation (2).

The constraints for ε-proper equilibria [24] restricts the strategy space of each player to be an ε-permutahedron. Before the technical description of this, we define the necessary generalization of the permutahedron. A permutahedron is traditionally over the vector $(1, \ldots, n)$, but it generalizes directly to any other set as well.

Definition 5 (Permutahedron)**.** *Let $\alpha \in \mathbb{R}^m$ with all coordinates being distinct. A permutation $\pi \in S_m$ acts on α by permuting the coordinates of α, i.e. $(\pi(\alpha))_i = \alpha_{\pi(i)}$. We define the permutahedron $Perm(\alpha)$ over α to be the convex hull of the set $\{\pi(\alpha) \mid \pi \in S_m\}$ of the $m!$ permutations of the coordinates of α.*

A very useful description of the permutahedron is by its $2^m - 2$ facets.

Proposition 1 (Rado [21])**.** *Suppose $\alpha_1 > \alpha_2 > \cdots > \alpha_m$. Then*

$$Perm(\alpha) = \left\{ x \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} \alpha_i \wedge \forall S \notin \{\emptyset, [m]\} : \sum_{c \in S} x_c \geq \sum_{i=1}^{|S|} \alpha_{m-i+1} \right\} .$$

As each inequality of Proposition 1 define a facet of the permutahedron, any direct formulation of the permutahedron over n elements requires $2^n - 2$ inequalities. Goemans [7] gave an asymptotically optimal extended formulation for the permutahedron, using $O(n \log n)$ additional constraints and variables. This allows a compact representation, which allows us to use ε-permutahedra [24] as building blocks for our strategy constraints.

The ε-permutahedron defined in [24] is a permutahedron over the vector $(1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{m-1})$, normalized to sum to 1. We need to generalize this, so that it can sum to any value ρ, and in a way that does not require normalization. In the following, we will abuse notation slightly, and use ρ without subscript as a real number, since it will shortly be replaced by a realization weight for each specific information set.

Definition 6 (ε-Permutahedron)**.** *For real $\rho > 0$, integers $k \geq 0$ and $m \geq 1$, and $\varepsilon > 0$ such that $\rho \geq \varepsilon^k$, define the vector $p_\varepsilon(\rho, k, m) \in \mathbb{R}^m$ by*

$$(p_\varepsilon(\rho, k, m))_i = \begin{cases} \rho - (\varepsilon^{k+1} + \cdots + \varepsilon^{k+m-1}) & , i = 1 \\ \varepsilon^{k+i-1} & , i > 1 \end{cases} ,$$

and define the ε-permutahedron $\Pi_\varepsilon(\rho, k, m) = Perm(p_\varepsilon(\rho, k, m)) \subseteq \mathbb{R}^m$.

We shall be viewing ε as a variable. Note that, by definition, $\|p_\varepsilon(\rho, k, m)\|_1 = \rho$.

Lemma 1. *Assume $0 < \varepsilon \leq 1/3$ and $\rho \geq \varepsilon^k$, for a given integer $k \geq 0$. Then for every $1 \leq i < m$ we have $(p_\varepsilon(\rho, k, m))_i \geq (p_\varepsilon(\rho, k, m))_{i+1}/(2\varepsilon)$.*

Proof. The statement clearly holds for $i > 1$. Next we see that $(p_\varepsilon(\rho, k, m))_1 = \rho - \varepsilon^{k+1}(1 - \varepsilon^{m-1})/(1 - \varepsilon) \geq \varepsilon^k - \varepsilon^{k+1}/(1 - \varepsilon) = (1/\varepsilon - 1/(1 - \varepsilon))\varepsilon^{k+1} \geq \varepsilon^{k+1}/(2\varepsilon) = (p_\varepsilon(\rho, k, m))_2/(2\varepsilon)$. $\square$

We are now ready to define the perturbed game Γ_ε.

Definition 7 (Strategy constraints)**.** *For each player i, and each information set $h \in H_i$, let $k_h = \sum_{h' < h} m_{h'}$ be the sum of the sizes of the action sets at information sets visited by Player i before reaching information set h. Now, in the perturbed game Γ_ε, restrict $(\rho_{b_i}(c_1), \rho_{b_i}(c_2), \ldots, \rho_{b_i}(c_{m_h}))$ to be in $\Pi_\varepsilon(\rho_{b_i}(h), k_h, m_h)$.*

Notice that the strategy constraints for the first information set a player visits is identical to the strategy constraints for proper equilibria of bimatrix games.

The next three lemmas describe several ways we may modify coordinates of points of $\Pi_\varepsilon(\rho, k, m)$ while staying within $\Pi_\varepsilon(\rho', k, m)$ for appropriate ρ'. These are needed for the proof of our main technical result, Proposition 2, below.

Lemma 2. *Let $0 < \varepsilon < 1/3$, $\rho \geq \varepsilon^k$, and $x \in \Pi_\varepsilon(\rho, k, m)$. Suppose for distinct c and c' we have $x_c > 2\varepsilon x_{c'}$. Then there exists $\delta > 0$ such that $x + \delta(e_{c'} - e_c) \in \Pi_\varepsilon(\rho, k, m)$ (here as usual e_i denotes the i-unit vector).*

Proof. By definition of $\Pi_\varepsilon(\rho, k, m)$ we may write x as a convex combination of the corner points of $\Pi_\varepsilon(\rho, k, m)$, $x = \sum_{\pi \in S_m} w_\pi \pi(p_\varepsilon(\rho, k, m))$, where $w_\pi \geq 0$ and $\sum_{\pi \in S_m} w_\pi = 1$. There must exist a permutation π such that $w_\pi > 0$ and $\pi^{-1}(c) < \pi^{-1}(c')$, since otherwise $x_c \leq 2\varepsilon x_{c'}$ by Lemma 1. Let $\pi' \in S_m$ such that $\pi'(\pi^{-1}(c)) = c'$, $\pi'(\pi^{-1}(c')) = c$, and $\pi'(i) = \pi(i)$ when $\pi(i) \notin \{c, c'\}$. We then have that

$$x' = x + w_\pi(\pi'(p_\varepsilon(\rho, k, m)) - \pi(p_\varepsilon(\rho, k, m))) \in \Pi_\varepsilon(\rho, k, m) \ .$$

Note now that $\pi'(p_\varepsilon(\rho, k, m)) - \pi(p_\varepsilon(\rho, k, m))$ is equal to

$$((p_\varepsilon(\rho, k, m))_{\pi^{-1}(c)} - (p_\varepsilon(\rho, k, m))_{\pi^{-1}(c')})(e_{c'} - e_c) \ .$$

Since $(p_\varepsilon(\rho, k, m))_{\pi^{-1}(c)} > (p_\varepsilon(\rho, k, m))_{\pi^{-1}(c')}$, the statement follows. □

Lemma 3. *Let $x \in \Pi_\varepsilon(\rho, k, m)$ where $\rho \geq \varepsilon^k$. Then $x + \delta e_c \in \Pi(\rho + \delta, k, m)$ for any $\delta > 0$ and c.*

Proof. This follows immediately from Proposition 1 since the inequalities defining the facets of $\Pi_\varepsilon(\rho, k, m)$ and $\Pi_\varepsilon(\rho + \delta, k, m)$ are exactly the same. □

Lemma 4. *Let $x \in \Pi_\varepsilon(\rho, k, m)$ where $0 < \varepsilon \leq 1/2$ and $\rho > \max(\varepsilon^k, 2m\varepsilon^{k+1})$. Let c be such that $x_c \geq x_{c'}$ for all c'. Then $x - \delta e_c \in \Pi_\varepsilon(\rho - \delta, k, m)$ for any $\delta \leq \min(\rho - \varepsilon^k, \rho/m - 2\varepsilon^{k+1})$.*

Proof. Since $\delta \leq \rho - \varepsilon^k$ we have $\rho - \delta \geq \varepsilon^k$, thereby satisfying the definition of $\Pi_\varepsilon(\rho - \delta, k, m)$. By the choice of c we have that $x_c \geq \rho/m$. Since we also have $\delta \leq \rho/m - 2\varepsilon^{k+1}$ it follows that $x_c - \delta \geq 2\varepsilon^{k+1}$. Thus $x_c - \delta \geq \varepsilon^{k+1} + \cdots + \varepsilon^{k+m-1}$. It then follows immediately from Proposition 1 that $x - \delta e_c \in \Pi_\varepsilon(\rho - \delta, k, m)$, since any inequality given by S with $c \in S$ is trivially satisfied, and any inequality with $c \notin S$ is unchanged from $\Pi_\varepsilon(\rho, k, m)$. □

We are now in position to prove correctness of our approach.

Proposition 2. *Any Nash equilibrium of Γ_ε is a 2ε-quasi-proper equilibrium of Γ, for any sufficiently small $\varepsilon > 0$.*

Proof. Let b be a Nash equilibrium of Γ_ε. Consider Player i for any i, any information set $h \in H_i$, and let $c, c' \in h$ be such that $b_{ih}(c) > 2\varepsilon b_{ih}(c')$. We are then to show that $K_i^{h,c}(b) \geq K_i^{h,c'}(b)$, when $\varepsilon > 0$ is sufficiently small. Let b'_i be such that $U_{ih}(b/_h b'_i/c') = K_i^{h,c'}(b)$. We may assume that b'_i is a pure behavior strategy thereby making b/b'_i a h-local purification. Let $H_{i,c'}$ be the set of those information sets of Player i that follow after h when taking action c' in h. Similarly, let $H_{i,c}$ be the set of those information sets of Player i that follow after taking action c in h. Note that by perfect recall of Γ we have that $H_{i,c'} \cap H_{i,c} = \emptyset$. Let b_i^* be any pure behavior strategy of Player i choosing $c_h^* \in C_h$ maximizing $b_{ih}(c_h^*)$, for all $h \in H_i$. We claim that $U_{ih}(b/_h b_i^*/c) \geq K_i^{h,c'}(b)$ for all sufficiently small $\varepsilon > 0$.

Let x_i be the realization plan given by b_i, let x' be the realization plan given by $b_i/_h b_i'/c'$, and let x_i^* be the realization plan given by $b_i/_h b_i^*/c$. We shall next apply Lemma 2 to h, Lemma 3 to all $h' \in H_{i,c'}$, and Lemma 4 to all $h^* \in H_{i,c}$ assigned positive realization weight by $b_i/_h b_i^*/c$, to obtain that for all sufficiently small $\varepsilon > 0$ there is $\delta > 0$ such that $\widetilde{x}_i = x_i + \delta(x_i' - x_i^*)$ is a valid realization plan of Γ_ε.

Lemma 3 can be applied whenever $\varepsilon > 0$ is sufficiently small, whereas Lemma 2 in addition makes use of the assumption that $b_{ih}(c) > 2\varepsilon b_{ih}(c')$. To apply Lemma 4, we need to prove that the player's realization weight is sufficiently large for the relevant information sets, specifically $\rho_{h'} > \varepsilon^{k_{h'}}$ for each relevant information set h'. Since b_i^* is pure, Player i's realization weight, $\rho_{h'}$ for each information set h' in $H_{i,c}$ is either 0 or ρ_c. Since $b_{ih}(c) > 2\varepsilon b_{ih}(c')$, we have that $\rho_c > \varepsilon^{k_h + |C_h| - 1} \geq \varepsilon^{k_{h'}}$ as needed.

Thus, consider $\varepsilon > 0$ and $\delta > 0$ such that $\widetilde{x}_i$ is a valid realization plan and let $\widetilde{b}_i$ be the corresponding behavior strategy. Since b is a Nash equilibrium we have $U_i(b/\widetilde{b}_i) \leq U_i(b)$. But $U_i(b/\widetilde{b}_i) = U_i(b) + \delta(U_i(b/_h b_i'/c') - U_i(b/_h b_i^*/c))$. It follows that $\delta(U_i(b/_h b_i'/c') - U_i(b/_h b_i^*/c)) \leq 0$, and since $\delta > 0$ we have $U_i(b/_h b_i^*/c) \geq U_i(b/_h b_i'/c')$. Equivalently, $U_{ih}(b/_h b_i^*/c) \geq U_{ih}(b/_h b_i'/c')$, which was to be proved. Since i and $h \in H_i$ were arbitrary, it follows that b is a 2ε-quasi-proper equilibrium in Γ, for any sufficient small $\varepsilon > 0$. □

Theorem 1. *A symbolic ε-quasi-proper equilibrium for a given two-player extensive form game with perfect recall can be computed by applying Lemke's algorithm to an LCP of polynomial size, and can be computed in* PPAD.

Proof. Given an extensive form game Γ, construct the game Γ_ε. The strategy constraints (Definition 7) are all expressed directly in terms of the realization weights of each player. Using Goemans' [7] extended formulation, the strategy constraints require only $O(\sum_h |C_h| \log |C_h|)$ additional constraints and variables, which is linearithmic in the size of the game. Furthermore, all occurrences of ε are on the right-hand side of the linear constraints. These constraints fully replace the strategy constraints of the sequence form [12]. In the sequence form, there is a single equality per information set, ensuring conservation of the realization weight. In our case, this conservation is ensured by the permutahedron constraint for each information set.

In the case of two-player games, the equilibria can be captured by an LCP of polynomial size, which can be solved using Lemke's algorithm [15], if the strategy constraints are sufficiently well behaved. Since the added strategy constraints is a collection of constraints derived from Goemans' extended formulation, the proof that the constraints are well behaved is identical to the proofs of [24, Theorem 5.1 and 5.4], which we will therefore omit here. Following the approach of [17] the solution to the LCP can be made to contain the symbolic ε, with the probabilities of the strategies being formal polynomials in the variable ε.

By Proposition 2, equilibria of Γ_ε are ε-quasi-proper equilibria of Γ. All realization weights of the computed realization plans are formal polynomials in ε. Finally, from this we may express the ε-quasi-proper equilibrium in behavior strategies, where all probabilities are rational functions in ε. □

Having computed a symbolic ε-quasi-proper equilibrium for Γ it is easy to compute the limit for $\varepsilon \to 0$, thereby giving a quasi-proper equilibrium of Γ. It is crucial here that we first convert into behavior strategies before computing the limit. In the case of zero-sum games, the same construction can be used to construct a linear program of polynomial size, whose solution would provide quasi-proper equilibria of the given game. This is again analogous to the approach of [17] and further details are hence omitted.

Theorem 2. *A symbolic ε-quasi-proper equilibrium for a given two-player extensive form zero-sum game with perfect recall can be computed in polynomial time.*

4 Multi-Player Games

In this section we show that approximating a quasi-proper equilibrium for a finite extensive-form game Γ with $n \geq 3$ players is FIXP_a-complete. As for two-player games, by Proposition 2 an ε-quasi-proper equilibrium for Γ could be obtained by computing an equilibrium of the perturbed game Γ_ε. But for more than two players we do not know how to make efficient use of this connection. Indeed, from the viewpoint of computational complexity there is no advantage in doing so. Our construction instead works by directly combining the approach and ideas of the proof of FIXP_a-completeness for quasi-perfect equilibrium in extensive form games by Etessami [3] and of the proof of FIXP_a-completeness for proper equilibrium in strategic form games by Hansen and Lund [9]. We explain below how these are modified and combined to obtain the result. The approach obtains FIXP_a membership, leaving FIXP-membership as an open problem. A quasi-proper equilibrium is defined as a limit point of a sequence of ε-quasi-proper equilibria, whose existence was obtained by the Kakutani fixed point theorem by Myerson [18]. This limit point operation in itself poses a challenge for FIXP membership. The use of the Kakutani fixed point theorem presents a further challenge. However, as we show below analougous to the case of proper equilibria [9], these may be approximated by δ-almost ε-quasi-proper equilibria, which in turn can be expressed as a set of Brouwer fixed points. In fact we show that the corresponding search problem is in FIXP.

To see how to adapt the result of Hansen and Lund [9] for strategic form games to the setting of extensive form games, it is helpful to compare the definitions of ε-proper equilibrium and δ-almost ε-proper equilibrium in strategic form games to the corresponding definitions of ε-quasi-proper equilibrium and δ-almost ε-proper equilibrium in extensive form games.

In a strategic form game, Player i is concerned with the payoffs $U_i(x_{-i}, c)$, which we may think of as *valuations* of all pure strategies $c \in S_i$. The relationship between these valuations in turn place constraints on the strategy x_i chosen by Player i in an ε-proper equilibrium or a δ-almost ε-proper equilibrium. In an extensive form game, Player i is in a given information set h considering the payoffs $K_i^{h,c}$, which we may similarly think of as *valuations* of all actions $c \in C_h$. The relationship between these valuations place constraints on the local strategy b_{ih} chosen by Player i in a ε-quasi-proper equilibrium or a δ-almost ε-proper equilibrium. These constraints are completely analogous to those placed on the strategies in strategic form games. This fact will allow us to adapt the constructions of Hansen and Lund by essentially just changing the way the valuations are computed. Etessami [3] observed that these may be computed using dynamic programming and gave a construction of formulas computing them.

Lemma 5 (cf. [3, Lemma 7]). *Given an extensive form game of perfect recall Γ, a player i, an information set h of Player i, and $c \in C_h$ there is a polynomial size $\{+,-,*,/,\max\}$-formula $V_i^{h,c}$ computable in polynomial time satisfying that for any fully mixed behavior strategy profile b it holds that $V_i^{h,c}(b) = K_i^{h,c}(b)$.*

We now state our result for multi-player games.

Theorem 3. *Given as input a finite extensive form game of perfect recall Γ with n players and a rational $\gamma > 0$, the problem of computing a behavior strategy profile b' such that there is a quasi-proper equilibrium b of Γ with $\|b' - b\|_\infty < \gamma$ is FIXP_a-complete.*

Before presenting the proof of Theorem 3 we describe the changes needed to adapt the results of Hansen and Lund [9] to extensive-form games in more details.

The first step of the construction is to establish that to compute an approximation to a quasi-proper equilibrium it is sufficient to compute (an approximation to) an ε-quasi-proper equilibrium, for a sufficiently small $\varepsilon > 0$, and further to compute an approximation to an ε-quasi-proper equilibrium it is sufficient to compute a δ-almost ε-quasi-proper equilibrium, for a sufficiently small $\delta > 0$. Both statements are obtained by invoking the "almost implies near" paradigm of Anderson [1]. The first statement generalizes essentially verbatim from the case of proper equilibrium in strategic form games [9, Lemma 4.2] and we omit the proof.

Lemma 6. *For any fixed extensive form game of perfect recall Γ, and any $\gamma > 0$, there is an $\varepsilon > 0$ so that any ε-quasi-proper equilibrium of Γ has ℓ_∞-distance at most γ to some quasi-proper equilibrium of Γ.*

We now define a *perturbed* version of Γ, restricting the domain of local behavior strategies. For $\varepsilon > 0$ and a positive integer m define $\eta_m(\varepsilon) = \varepsilon^m/m$. The η-perturbed game Γ_η restricts a local behavior strategy in every information set h to use behavior probabilities at least $\eta_{m_h}(\varepsilon)$. Let B_η be the set of such restricted behavior strategy profiles of Γ. The proof of the second statement following below very closely follows that of [9, Lemma 4.3]. For completeness we give the proof.

Lemma 7. *For any fixed extensive form game of perfect recall Γ, any $\varepsilon > 0$ and any $\gamma > 0$, there is a $\delta > 0$ so that any δ-almost ε-quasi-proper equilibrium of Γ in B_η has ℓ_∞-distance at most γ to some ε-quasi-proper equilibrium of Γ in B_η.*

Proof. Suppose to the contrary there is a game Γ, $\varepsilon > 0$, and $\gamma > 0$ so that for all $\delta > 0$ there is a δ-almost ε-quasi-proper equilibrium b_δ of Γ in B_η so that there is no ε-quasi-proper equilibrium in B_η in a γ-neighborhood (with respect to the ℓ_∞ norm) of b_δ. Consider the sequence $(b_{1/n})_{n\in\mathbb{N}}$. Since this is a sequence in a compact space, by the Bolzano-Weierstrass Theorem it has a convergent subsequence $(b_{1/n_r})_{r\in\mathbb{N}}$. Let $b^* = \lim_{r\to\infty} b_{1/n_r}$. We now claim that b^* is an ε-quasi-proper equilibrium, which will contradict the statement that there is no ε-quasi-proper equilibrium in a γ-neighborhood of any of the behavior strategy profiles $b_{1/n}$.

First, since $b_{1/n_r} \in B_\eta$ for all n_r we also have $b^* \in B_\eta$. In particular, b^* is fully mixed. The functions $K_i^{h,c}$ are well defined on B_η. Define $\nu > 0$ by

$$\nu = \min_{i,h,c,c'} \left\{ K_i^{h,c'}(b^*) - K_i^{h,c}(b^*) \mid K_i^{h,c}(b^*) < K_i^{h,c'}(b^*) \right\} \quad .$$

By continuity of the functions $K_i^{h,c}$ we have $\lim_{r\to\infty} K_i^{h,c}(b_{1/n_r}) = K_i^{h,c}(b^*)$, for all i, h, and c. Thus let N be an integer such that $\left| K_i^{h,c}(b_{1/n_r}) - K_i^{h,c}(b^*) \right| \leq \nu/3$ and such that $1/n_r \leq \nu/3$, for all i,h,c, and $r \geq N$.

Consider now an information set h of Player i and $c, c' \in C_h$ such that $K_i^{h,c}(b^*) < K_i^{h,c'}(b^*)$. By construction, for any $r \geq N$ we also have $K_i^{h,c}(b_{1/n_r}) + 1/n_r \leq K_i^{h,c'}(b_{1/n_r})$. Since b_{1/n_r} is a $(1/n_r)$-almost ε-quasi-proper equilibrium it follows that $(b_{1/n_r})_{ih}(c) \leq \varepsilon (b_{1/n_r})_{ih}(c')$. Taking the limit $r \to \infty$ we also have $b^*_{ih}(c) \leq b^*_{ih}(c')$, which shows that b^* is an ε-quasi-proper equilibrium. □

The second step is to show that given Γ, $\delta > 0$, and $\varepsilon > 0$, the task of computing a δ-almost ε-quasi-proper equilibrium of Γ belongs to FIXP. We outline the details of this below, using a slightly different notation compared to [9].

Definition 8 (cf. [9, Definition 4.4])**.** *Let $v \in \mathbb{R}^m$, $x \in \mathbb{R}^m_+$, $\delta > 0$, and $\varepsilon > 0$. We say that x satisfies the δ-almost ε-proper property with respect to valuation v if and only if $x_c \leq \varepsilon x_{c'}$ whenever $v_c + \delta \leq v_{c'}$, for all c, c'.*

Hansen and Lund [9, Definition 4.6] define a function $P_{m,\delta,\varepsilon} : \mathbb{R}^m_+ \times \mathbb{R}^m \to \mathbb{R}^m_+$ as a main ingredient of computing δ-almost ε-proper equilibrium. This is given by

$$(P_{m,\delta,\varepsilon}(x, v))_c = \min_{c'} \{\mathrm{Sel}_\delta(x_c, \varepsilon x_{c'}, v_{c'} - v_c)\} \quad ,$$

where

$$\mathrm{Sel}_\delta(x, y, z) = \begin{cases} x & \text{if } z \leq 0 \\ (1 - z/\delta)x + (z/\delta)y & \text{if } 0 \leq z \leq \delta \\ y & \text{if } \delta \leq z \end{cases}$$

is the the δ-approximate selection function, for $\delta > 0$.

The function function $P_{m,\delta,\varepsilon}$ then induces an operator $P^v_{m,\delta,\varepsilon} : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ by letting $P^v_{m,\delta,\varepsilon}(x) = P_{m,\delta,\varepsilon}(x, v)$. Define $\Delta_m = \{y \in \mathbb{R}^m \mid \|y\|_1 = 1; \forall j : y_j \geq 0\}$ and for $\eta > 0$ define $\Delta^\eta_m = \{y \in \mathbb{R}^m \mid \|y\|_1 = 1; \forall j : y_j \geq \eta\}$. We may identify the points of Δ_m and Δ^η_m with probability distributions on a set of m elements. Let $\tau_m \in \Delta_m$ be the uniform distribution on m elements. Note that $\tau_m \in \Delta^{1/m}_m$. We need the following properties of $P^v_{m,\delta,\varepsilon}$ proved by Hansen and Lund. We let $(P^v_{m,\delta,\varepsilon})^{\circ j}$ denote the j-th iteration of the operator $P^v_{m,\delta,\varepsilon}$.

Lemma 8 (cf. [9, Lemma 4.10])**.** *If $x \in \mathbb{R}^m_+$ is a fixed point of $P^v_{m,\delta,\varepsilon}$ then x satisfies the δ-almost ε-proper property with respect to v.*

Proposition 3 (cf. [9, Lemma 4.11 and Proposition 4.15])**.** *Suppose* $\varepsilon \leq 1/m$. *Then* $(P^v_{m,\delta,\varepsilon})^{\circ j}(\tau_m)$ *in contained in* $\Delta_m^{\eta_m}$ *and satisfy the* δ*-almost* $\sqrt{\varepsilon}$*-proper property with respect to* v *for all* $j \geq 2m^2$.

We now have everything needed for defining the fixed point problem. We define a function $F_{\varepsilon,\delta} : B^{\eta(\varepsilon^2)} \to B^{\eta(\varepsilon^2)}$ as follows. For $b \in B^{\eta(\varepsilon^2)}$, define the following for every i and every information set h of Player i: First we let $v_{ih} \in \mathbb{R}_{m_h}$ be given by $(v_{ih})_c = V_i^{h,c}(b)$. We then let $y_{ih} = (P^{v_i h}_{m_h,\delta,\varepsilon})^{\circ 2m_h^2}(\tau_{m_h})$ and $b'_{ih} = y_{ih}/\|y_{ih}\|_1$. Finally define $F_{\varepsilon,\delta}(b) = b'$.

Proposition 4. *Let* $\delta > 0$ *and* $0 < \varepsilon < 1$. *Then every fixed point* $b \in B^{\eta(\varepsilon^2)}$ *of* $F_{\varepsilon,\delta}$ *is a* δ*-almost* ε*-quasi-proper equilibrium of* Γ.

Proof. Suppose that $b \in B^{\eta(\varepsilon^2)}$ is a fixed point of $F_{\varepsilon,\delta}$. For every i and every information set h of Player i follows that $b_{ih} = y_{ih}/\|y_{ih}\|_1$. By Proposition 3 y_{ih} satisfies the δ-almost ε-proper property with respect to valuation v_{ih}. This implies that b_{ih} satisfies the δ-almost ε-proper property with respect to valuation v_{ih} as well. Since this holds for all i and h, we can conclude that b is a δ-almost ε-quasi-proper equilibrium. □

By Lemma 5 the valuations v_{ih} may be computed by a polynomial size $\{+,-,*,/,\max\}$-formula. Likewise, as seen from their definition, the functions $P_{m,\delta,\varepsilon}$ may be computed by polynomial size $\{+,-,*,/,\max,\min\}$-formulas. All these formulas may furthermore be constructed in polynomial time. The function $F_{\varepsilon,\delta}$ is given by combining polynomially many such formulas into a *circuit*. In conclusion we obtain the following result, analogously to [9, Theorem 4.17].

Theorem 4. *There exists a function* $F_{\varepsilon,\delta} : B^{\eta(\varepsilon^2)} \to B^{\eta(\varepsilon^2)}$ *that is given by a* $\{+,-,*,/,\max,\min\}$*-circuit computable in polynomial time from* Γ*, with the circuit having inputs* b*,* $\varepsilon > 0$*, and* $\delta > 0$*, such that for all fixed* $0 < \varepsilon < 1$ *and* $\delta > 0$*, every fixed point of* $F_{\varepsilon,\delta}$ *is a* δ*-almost* ε*-quasi-proper equilibrium of* Γ*. In particular, the problem of computing a* δ*-almost* ε*-quasi-proper equilibrium of a finite extensive form game of perfect recall* Γ *is in* FIXP.

The third step is to quantify how small $\varepsilon > 0$ and $\delta > 0$ need to be in order to guarantee that Lemma 6 and Lemma 7 apply. Such bounds can be obtained in a completely generic way using the general machinery of real algebraic geometry, cf. Basu, Pollack, and Roy [2], and was applied for the same purpose in previous works [4, 3, 9]. The approach involves formalizing the statements of Lemma 6 and Lemma 7 in the first order theory of the reals. More precisely, doing this for Lemma 6 results in a formula depending on Γ and γ with a free variable ε. This formula is built from the definition of a ε-quasi-proper equilibrium as well as the formula of Lemma 5. Applying quantifier elimination to that formula and employing known bounds on the result of this we obtain the following statement, analogously to [9, Lemma 4.18].

Lemma 9. *There exists a polynomial* q_1 *such that for any finite extensive form game* Γ *of perfect recall and any* $0 < \gamma < 1/2$*, whenever* $0 < \varepsilon < \gamma^{2^{q_1(|\Gamma|)}}$ *any* ε*-quasi-proper equilibrium of* Γ *has* L_∞*-distance at most* γ *to some quasi-proper equilibrium of* Γ.

Similarly for Lemma 7 we construct a formula depending on Γ, γ, and ε with a free variable δ. Again applying quantifier elimination to that formula and employing known bounds on the result of this we obtain the following statement, analogously to [9, Lemma 4.19].

Lemma 10. *There exists a polynomial q_2 such that for any finite extensive form game Γ of perfect recall, any $0 < \gamma < 1/2$, and any $\varepsilon > 0$, whenever $0 < \min(\delta, \varepsilon)^{2^{q_2(|\Gamma|)}}$ any δ-almost ε-quasi-proper equilibrium of Γ has L_∞-distance at most γ to some ε-quasi-proper equilibrium of Γ.*

We can now complete the proof of Theorem 3. As done for the case of approximating proper equilibrium [9], the idea is to construct two virtual infinitesimals $\delta \ll \varepsilon$, given Γ and $\gamma > 0$, by means of repeated squaring, according to Lemma 9 and Lemma 10.

Proof of Theorem 3. Given an extensive form game of perfect recall Γ and a rational $\gamma > 0$ we shall in polynomial time construct a $\{+, -, *, /, \max, \min\}$-circuit C computing a function $F : B \to B$ such that any fixed point of F is γ-close to a quasi-proper equilibrium of Γ. This is sufficient to establish FIXP_a-membership.

The circuit C will first compute $\varepsilon > 0$ satisfying the condition of Lemma 9 by repeated squaring of $\gamma/2$ exactly $q_1(|\Gamma|)$ times. Then C computes $\delta > 0$ satisfying the condition of Lemma 10 by repeated squaring of $\min(\gamma/2, \varepsilon)$ exactly $q_2(|\Gamma|)$ times. Next we need to restrict the input to $B^{\eta(\varepsilon^2)}$ before we can apply the function $F_{\varepsilon,\delta}$ of Theorem 4. For this we need to map the input $x \in B$ into $B^{\eta(\varepsilon^2)}$ by a mapping that is the identity function on $B^{\eta(\varepsilon^2)}$. One way to achieve this (cf. [9]) is to compute for every i and h a number t_{ih} such that $\sum_{c \in C_h} \max(b_{ih}(c) - t_{ih}, \eta_{m_h}(\varepsilon^2)) = 1$ using a sorting network as done by Etessami and Yannakakis [5] and then map each $b_{ih}(c)$ to $\max(b_{ih}(c) - t_{ih}, \eta_{m_h}(\varepsilon^2))$. Finally $F_{\varepsilon,\delta}$ is applied to the output of this together with the constructed ε and δ. By Theorem 4 any fixed-point of F is then a δ-almost ε-quasi-proper equilibrium of Γ. By Lemma 9 this is $\gamma/2$-close to a ε-quasi-proper equilibrium which in turn by Lemma 10 is $\gamma/2$-close to a quasi-proper equilibrium of Γ. The proof is then concluded by the triangle inequality. $\square$

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