On the approximation ratio of LZ-End to LZ77

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Abstract

A family of Lempel-Ziv factorizations is a well-studied string structure. The LZ-End factorization is a member of the family that achieved faster extraction of any substrings (Kreft & Navarro, TCS 2013). One of the interests for LZ-End factorizations is the possible difference between the size of LZ-End and LZ77 factorizations. They also showed families of strings where the approximation ratio of the number of LZ-End phrases to the number of LZ77 phrases asymptotically approaches 2. However, the alphabet size of these strings is unbounded. In this paper, we analyze the LZ-End factorization of the period-doubling sequence. We also show that the approximation ratio for the period-doubling sequence asymptotically approaches 2 for the binary alphabet.

1 Introduction

The Lempel-Ziv $\gamma\gamma$ compression (LZ $\gamma\gamma$) [33] is one of the most successful lossless compression algorithms to date. On the practical side, LZ $\gamma\gamma$ and its variants have been used as a core of compression software such as zip, gzip, rar, and compressed formats such as PNG, JPEG, PDF. In addition to these real world applications, compressed self-indexing structures based on LZ $\gamma\gamma$ have been proposed [10, 11, 12, 24]. An LZ $\gamma\gamma$ -based compressed representation of a string allowing for fast access, rank, and select queries also exists [2].

On the (more) theoretical side, the left-to-right greedy factorization in LZ77, a.k.a. the LZ77factorization, has widely been considered for decades. It parses a given input string w into a sequence p_1, \ldots, p_z of non-empty substrings such that $p_1 = w[1]$ and p_i for $i \ge 2$ is the shortest prefix of $p_i \cdots p_z$ that does not occur in $p_1 \cdots p_{i-1}$. This implies that the prefix $p_i[1..|p_i|-1]$ occurs in $p_1 \cdots p_{i-1}$, and such an occurrence is called a *source* of p_i^{-1} .

Among many versions of LZ77 (c.f. [9, 13, 20, 21, 22, 29, 34]), this paper focuses on the *LZ-End* compressor proposed by Kreft and Navarro [21]. It is also based on a greedy parsing $q_1, \ldots, q_{z'}$ of an input string, with a restriction that for each phrase q_i there has to be a source which ends at the right-end of a phrase in q_1, \ldots, q_{i-1} . This constraint permits fast substring extraction without expanding the whole input string. It is known that the LZ-End compression can be computed

¹This version of LZ77 is often called *non-overlapping LZ77* or *LZ77 without self-references*, since each phrase p_i never overlaps with any of its sources.

in linear time in the input string length [17], or in compressed space with slight slow-down on compression time [16].

One can regard LZ-End as a mix of LZ77 and LZ78 [34], since in the LZ78 factorization the source of each phrase has to begin and end at boundaries of previous phrases. Since LZ78 belongs to the class of grammar compression [6], LZ-End can be seen as a new bridge between grammar compression and LZ77.

Now, a natural question arises. How good is the compression performance of LZ-End? Practical evaluation in the literature [21] has revealed that the compression ratio of LZ-End is quite close to that of LZ77 (at most 20% worse), but very little is understood in theory. As in the literature, we measure and compare the sizes of LZ-End and LZ77 by the numbers z' and z of their phrases in the factorizations, i.e., "z' versus z".

Since LZ77 is an optimal greedy unidirectional parsing, $z' \ge z$ always holds. Thus we are concerned with the approximation ratio of LZ-End to LZ77, which is defined by z'/z. Kreft and Navarro [21] presented a simple family of strings for which z'/z is asymptotically 2 over an alphabet of size n/3, where n is the length of the string. Kreft and Navarro [21] conjectured that the upper bound for z'/z is also 2, but to our knowledge no non-trivial upper bound is known.

In this paper, we show that the same lower bound for z'/z can be obtained on a binary alphabet, thus significantly reducing the number of distinct characters used in the analysis from n/3 to 2. In particular, we prove that z'/z is asymptotically 2 for the *period-doubling sequences*, an interesting family of recursive strings. While the LZ77-factorization of the period-doubling sequences has an obvious structure (Proposition 10), the LZ-End factorization of the period-doubling sequences has a non-trivial structure and needs careful analysis (see our extensive discussions in Section 4 for detail).

Since the LZ77 factorization (without self-references) and the LZ-End factorization for the unary string a^n are the same, our result uses a minimum possible number of distinct characters to achieve such a lower bound for z'/z.

Related work. A famous variant of the LZ77 factorization, which is called the *C*-factorization [9] and is denoted by $w = c_1 \cdots c_x$, differs from the LZ77 in that each phrase c_i is either a fresh character or the longest prefix of $c_i \cdots c_x$ that occurs in $c_1 \cdots c_{i-1}$. The size x of the C-factorization is known to be a lower bound for the size of the smallest grammar which generates only the input string [30]. A comparison of the LZ77 factorization and the C-factorization was also considered in the literature [3, 26]. The structure of the C-factorization of the period-doubling sequences was investigated in [3]. We emphasize that our analysis of the LZ-End factorization of the period-doubling sequences is independent and is quite different from this existing work [3].

Relative LZ (RLZ) is a practical modification of LZ77 which efficiently compresses a collection of highly repetitive sequences [22]. In [20] an RLZ-based factorization of a string, called the *ReLZ-factorization*, was proposed. The approximation ratio of ReLZ to LZ77 was shown to be $\Omega(\log n)$ [20], where *n* denotes the length of the input string. On the other hand, in practice ReLZ was larger than LZ77 by at most a factor of two in all the tested cases in [20].

2 Preliminaries

2.1 Strings

Let Σ be the binary *alphabet*. An element of Σ^* is called a *string*. The length of a string w is denoted by |w|. The empty string ε is the string of length 0. Let Σ^+ be the set of non-empty strings, i.e., $\Sigma^+ = \Sigma^* \setminus \{\varepsilon\}$. For a string w = xyz, x, y and z are called a *prefix*, *substring*, and *suffix* of w, respectively. They are called a *proper prefix*, a *proper substring*, and a *proper suffix* of w if $x \neq w$, $y \neq w$, and $z \neq w$, respectively. Further, we say that w has an *internal occurrence* of y if y occurs in w as a proper substring which is neither a prefix nor a suffix. The *i*-th character of a string w is denoted by w[i], where $1 \leq i \leq |w|$. For a string w and two integers $1 \leq i \leq j \leq |w|$, let w[i..j] denote the substring of w that begins at position i and ends at position j. For convenience,

let $w[i..j] = \varepsilon$ when i > j. For any $1 \le i \le |w|$, $w[i..|w|] \cdot w[1..i-1]$ is called a *cyclic rotation* of w. If a cyclic rotation of w is not equal to w, the cyclic rotation is said to be proper. For any string w, let $w^1 = w$ and let $w^k = ww^{k-1}$ for any integer $k \ge 2$, i.e., w^k is the k-times repetition of w. A string w is said to be primitive if w cannot be written as x^k for any $x \in \Sigma^*$ and $k \ge 2$. Let \overline{c} be the opposite character of c in a binary alphabet (e.g., $\overline{a} = b, \overline{b} = a$ for alphabet $\{a, b\}$). For any non-empty binary string w, \widehat{w} denotes the string $w[1..|w|-1] \cdot \overline{w[|w|]}$. We sometimes use b(x) and e(x) as the beginning position and the ending position of a substring x of a given string w, if the occurrence of x in w is clear from a discussion.

2.2 Lempel-Ziv factorizations

We introduce the Lempel-Ziv 77 and LZ-End factorizations.

Definition 1 (LZ77 [33]²). The Lempel-Ziv 77 factorization (LZ77 factorization for short) of a string w is the factorization $LZ_{77}(w) = p_1, \ldots, p_z$ of w such that $p_i[1..|p_i| - 1]$ is the longest prefix of $p_i \cdots p_z$ which occurs in $p_1 \cdots p_{i-1}$. As an exception, the last phrase p_z can be a suffix of w which occurs in $p_1 \cdots p_{z-1}$.

Definition 2 (LZ-End [21]). The LZ-End factorization of a string w is the factorization $\mathsf{LZ}_{\mathsf{end}}(w) = q_1, \ldots, q_{z'}$ of w such that $q_i[1..|q_i| - 1]$ is the longest prefix of $q_i \cdots q_{z'}$ which occurs as a suffix of $q_1 \cdots q_j$ for some j < i. As an exception, the last phrase $q_{z'}$ can be a suffix of w which occurs as a suffix of a suffix of $q_1 \cdots q_j$ for some j < z'.

We refer to each p_i and q_i as an *LZ phrase* and *LZ-End phrase*, respectively. For each phrase, associated longest substring is called a *source* of the phrase. $z_{77}(w)$ and $z_{end}(w)$ denote the number of the LZ phrases and the LZ-End phrases of a string w, respectively. For each $1 \le i \le z_{end}(w)$, $LZ_{end}(w)[i]$ denotes the *i*-th LZ-End phrase of $LZ_{end}(w)$. Let $LZ_{end}(w)$.last be the last LZ-End phrase of a string w, i.e., $LZ_{end}(w)$.last = $LZ_{end}(w)[z_{end}(w)]$. Fig. 1 shows examples of two factorizations.

2.3 Period-doubling sequence

The *period-doubling sequence* (cf. [1]) is one of the prominent automatic sequences. Let S_k be the k-th period-doubling sequence for any $k \ge 0$. The following two definitions are equivalent:

Definition 3. $S_0 = a$ and $S_k = \phi(S_{k-1})$ for $k \ge 1$ where ϕ is the morphism such that $\phi(a) = ab, \phi(b) = aa$.

Definition 4. $S_0 = a$ and $S_k = S_{k-1} \cdot \widehat{S_{k-1}}$ for $k \ge 1$.

Let n_k be the length of the k-th period-doubling sequence, i.e., $n_k = 2^k$.

 $^{^{2}}$ This definition of LZ77 is different from the original one [33] (see [21] for more information).

3 Properties on period-doubling sequence

The period-doubling sequences have many good combinatorial properties (see cf. [1]). In this section, we introduce helpful properties for our results on the period-doubling sequences.

Lemma 5. For any $k \ge 0$, S_k is primitive.

Proof. If S_k is not primitive, S_k has a period 2^i for some *i*. This implies that $S_k[n_k/2] = S_k[n_k]$, which contradicts Definition 4.

Lemma 6 (Proposition 8.1.5 of [25]). If a string w is primitive, ww has no internal occurrence of w.

Lemma 7. For any $k \geq 2$, $S_k = A_k B_k A_k A_k$ where $A_k = S_{k-2}$ and $B_k = \widehat{A_k}$. Moreover, $A_k = A_{k-1} B_{k-1}$ and $B_k = A_{k-1} A_{k-1}$ for any $k \geq 3$.

Proof. Straightforward from Definition 3.

Lemma 8. For any $k \ge 2$, $A_k A_k$, $A_k B_k$, and $B_k A_k$ have no internal occurrence of A_k . Hence the number of occurrences of A_k in $S_k = A_k B_k A_k A_k$ is 3.

Proof. If k = 2, the lemma clearly holds. We assume $k \ge 3$. Since $A_k = S_{k-2}$, A_k is primitive. By Lemma 6, A_kA_k has no internal occurrence of A_k . Since $A_kB_k = \widehat{A_kA_k}$, A_kB_k also has no internal occurrence of A_k . Similarly, $A_{k-1}A_{k-1}$ and $A_{k-1}B_{k-1}$ have no internal occurrence of A_{k-1} . Also, by Lemma 7, B_kA_k can be written as $A_{k-1}A_{k-1}A_{k-1}B_{k-1}$. These imply that B_kA_k have no internal occurrence of $A_k = A_{k-1}B_{k-1}$.

Lemma 9. For any $k \ge 3$ and any proper cyclic rotation α of A_k , the number of occurrences of α in $A_kA_kA_k$, A_kB_k , and B_kA_k are 2, 1, and 0, respectively.

Proof. Since $A_k = S_{k-2}$ and Lemma 5, A_k is primitive. This implies that α is also primitive. Thus, A_kA_k has exactly one (internal) occurrences of α . Namely, α occurs in $A_kA_kA_k$ exactly two times. Since $A_kB_k = \widehat{A_kA_k}$, A_kB_k also has exactly one (internal) occurrence of α . Finally, let us consider $B_kA_k = A_{k-1}A_{k-1}A_{k-1}B_{k-1}$. In a similar way of the proof of Lemma 8, we can show that both $A_{k-1}A_{k-1}$ and $A_{k-1}B_{k-1}$ have no internal occurrence of B_{k-1} . From this facts and Lemma 8, A_{k-1} occurs exactly three times and B_{k-1} occurs exactly once in B_kA_k . If $\alpha = B_{k-1}A_{k-1}$, α cannot occur in B_kA_k . Otherwise, α can be written as either $xB_{k-1}y$ or $x'A_{k-1}y'$ where x (resp. y) is a non-empty suffix (resp. prefix) of A_{k-1} , and x' (resp. y') is a non-empty suffix (resp. prefix) of B_{k-1} . If $\alpha = xB_{k-1}y$, α cannot occur in B_kA_k due to the constraint of B_{k-1} . If $\alpha = x'A_{k-1}y'$, α cannot occur in B_kA_k due to the constraint of A_{k-1} and the difference between the last characters of A_{k-1} and x'. Therefore α cannot occur in B_kA_k in all cases.

4 Factorizations of period-doubling sequence

By the definition of LZ77, the following proposition immediately holds:

Proposition 10. $\mathsf{LZ}_{77}(S_k) = (S_0, \widehat{S_0}, \widehat{S_1}, \dots, \widehat{S_{k-1}})$ and thus $\mathsf{z}_{77}(S_k) = k+1$.

In this section, we mainly discuss the LZ-End factorization of the period-doubling sequence, and give the following result.

Theorem 11. $z_{end}(S_k) = 2k - f(k)$ where $f(k) = O(\log^* k)$.

By Proposition 10 and Theorem 11, we can reach our goal of this paper:

Corollary 12. There exists a family of binary strings w such that the ratio $z_{end}(w)/z_{77}(w)$ asymptotically approaches 2.

In the rest of this paper, we show Theorem 11. The next lemma gives the LZ-End factorization of the period-doubling sequence. Notice that statement (I) in the lemma is not an immediate property for the LZ-End factorization due to the next example. Let S = abaababaabbaababaababaa. Then,

$$\mathsf{LZ}_{\mathsf{end}}(S) = a|b|aa|ba|baab|bab|baabab|a, \mathsf{LZ}_{\mathsf{end}}(Saba) = a|b|aa|ba|baab|bab|baababaabaa.$$

Lemma 13. For any $k \ge 5$, the following statements (I)-(IV) hold.

(I) $\mathsf{LZ}_{\mathsf{end}}(S_k)[i] = \mathsf{LZ}_{\mathsf{end}}(S_{k-1})[i]$ for every $1 \le i \le \mathsf{z}_{\mathsf{end}}(S_{k-1}) - 1$.

 $(II) \ \mathsf{z}_{\mathsf{end}}(S_k) \ge \mathsf{z}_{\mathsf{end}}(S_{k-1}) + 1.$

Let

$$\begin{aligned} w_k &= \mathsf{LZ}_{\mathsf{end}}(S_k)[\mathsf{z}_{\mathsf{end}}(S_{k-1})], \\ x_k &= \mathsf{LZ}_{\mathsf{end}}(S_k)[\mathsf{z}_{\mathsf{end}}(S_{k-1})+1], \\ y_k &= S_k[\mathsf{e}(x_k)+1..n_k] \ (possibly \ empty) \end{aligned}$$

(III) If $w_k \neq \mathsf{LZ}_{\mathsf{end}}(S_{k-1})$.last,

$$|w_k| = \frac{1}{8}n_k + 1, |x_k| = \frac{3}{8}n_k, |y_k| = \frac{3}{16}n_{\ell(k)} - (k - \ell(k)) - 1,$$

where $\ell(k) = \max\{i \mid i \leq k, w_i = \mathsf{LZ}_{\mathsf{end}}(S_{i-1}).\mathsf{last}\}.$ Otherwise (if $w_k = \mathsf{LZ}_{\mathsf{end}}(S_{k-1}).\mathsf{last})$,

$$|w_k| = \frac{3}{16}n_k, |x_k| = \frac{5}{16}n_k + 1, |y_k| = \frac{3}{16}n_k - 1.$$

(IV) If $|y_k| \ge 2$, $y_k[1..|y_k| - 1]$ has another occurrence to the left which ends with some LZ-End phrase of S_k . Namely, y_k is the last LZ-End phrase of S_k if y_k is not empty.

Proof. In this proof, we use $Z'_k = LZ_{end}(S_k)$ and $z'_k = z_{end}(S_k)$ for simplicity. We prove this lemma by induction on k.

Suppose that k = 5. The LZ-End factorizations of S_4, S_5 are

$$Z'_4 = a|b|aa|aba|bab|aaabaa,$$

 $Z'_5 = a|b|aa|aba|bab|aaabaa|abaaabababa|aabab.$

Statements (I) and (II) clearly hold. Then, $w_5 = aaabaa, x_5 = abaaabababaa, y_5 = aabab$. Hence, statement (III) holds since $n_5 = 32$ and $w_5 = Z'_4$.last (i.e., the latter case). Statement (IV) also holds since $y_5[1..4] = aaba$ has an occurrence which ends with the fourth phrase aba.

Suppose that all the statements hold for any $k \in [5, k'-1]$ for some k' > 5. We show that all the statements hold for k'. Firstly, suppose on the contrary that statement (I) does not hold for k'. This implies that there exists a phrase $T = S_{k'}[b(Z'_{k-1}[i])..j]$ for some $i < z'_{k'-1}$ and $j > n_{k'-1}$. Since $|x_{k'-1}y_{k'-1}| \ge \frac{3}{8}n_{k'-1} > \frac{1}{4}n_{k'-1}$ and $x_{k'-1}y_{k'-1}$ is a substring of T, T has an internal occurrence of the length $-\frac{1}{4}n_{k'-1}$ suffix $A_{k'-1}$ of $S_{k'-1}$. By Lemma 8 (showing the occurrences of A_{k-1} in S_{k-1}), $A_{k'-1}$ occurs exactly three times in $S_{k'}[1..n_{k'-1}]$. The first occurrence of $A_{k'-1}$ cannot be included by a source of T since $A_{k'-1}$ is not a prefix of T[1..|T|-1]. In addition, the second occurrence of $A_{k'-1}$ also cannot be included by a source of T since the source overlaps phrase T. Thus, T[1..|T|-1] cannot have another occurrence to the left as a source of T. This contradicts that T is an LZ-End phrase of $S_{k'}$ at that position. Hence, statement (I) holds for k'. Due to statement (I), $w_{k'}$ must have $y_{k'-1}$ as a prefix. On the other hand, $w_{k'}$ cannot reach the end of $S_{k'}$. Hence, statement (II) also holds. Thanks to statements (I) and (II) for k', three substrings $w_{k'}, x_{k'}$, and $y_{k'}$ are well-defined (see Fig. 2 and 5 for illustrations).

Next, we show statements (III) and (IV).

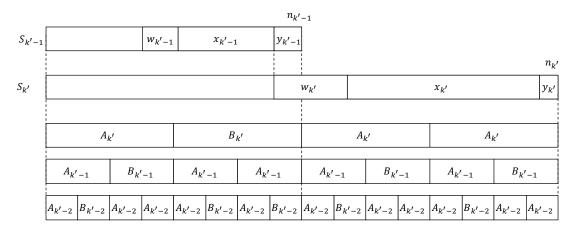


Figure 2: Illustration for the LZ-End factorization when $w_{k'} \neq \mathsf{Z'}_{k'-1}$.last.

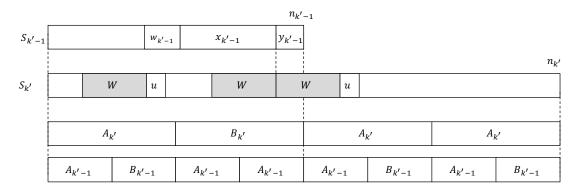


Figure 3: Illustration for a part of the proof. W is a candidate of a source of phrase w'_k .

• Assume that $\ell(k'-1) = \ell(k')$ (i.e., $w_{k'} \neq \mathsf{Z'}_{k'-1}$.last). We consider a phrase $w_{k'}$. If $|y_{k'-1}| =$ 0, $x_{k'-1}$ is the suffix of length $\frac{3}{8}n_{k'-1}$ of $S_{k'-1}$, i.e., $x_{k'-1} = B_{k'-2}A_{k'-1}$. From Lemma 8, $x_{k'-1}$ does not have other occurrences to the left. This implies that $w_{k'} = x_{k'-1}$. This contradicts to $w_{k'} \neq \mathsf{Z'}_{k'-1}$.last. Thus, $|y_{k'-1}| > 0$ holds. Namely, $x_{k'-1} = \mathsf{Z'}_{k'-1}[\mathsf{z'}_{k'-1}-1]$ and $y_{k'-1} = \mathsf{Z'}_{k'-1}$.last (see also Fig. 2). Let W be the string of length $\frac{1}{8}n_{k'}$ which begins at $b(Z'_{k'-1})$. last). $\ell(k'-1) = \ell(k')$ also implies that $\ell(k'-1) < k'$. Hence, $|y_{k'-1}| < \frac{3}{16}n_{\ell(k'-1)} \le \frac{3}{16}n_{\ell($ $\frac{3}{32}n_{k'} < \frac{1}{8}n_{k'}$. This fact means that W is a proper cyclic rotation of $A_{k'-1}$. By Lemma 9, W occurs twice to the left (one is in $A_{k'-1}B_{k'-1}$, the other is in $A_{k'-1}A_{k'-1}$). Since the second occurrence ends with phrase $\mathsf{Z}'_{k'}[\mathsf{z}'_{k'-1}-1], Wc_W$ is a candidate of phrase $w_{k'}$ where c_W is the character preceded by W. Assume on the contrary that a source of phrase $w_{k'}$ is Wu for some $u \in \Sigma^+$ (see Fig. 3). The second occurrence of W cannot be the beginning position of a source of $w_{k'}$ since Wu overlaps $w_{k'}$. Hence, the only candidate of the beginning position of source Wu is in the first $A_{k'-1}B_{k'-1}$. Moreover, Wu cannot contain $B_{k'-1}$ since the original Wu occurs in $A_{k'-1}A_{k'-1}\cdots$. Thus, Wu is a proper substring of $A_{k'-1}A_{k'-1}$ and $A_{k'-1}B_{k'-1}$. In other words, u'Wu is a proper prefix of $A_{k'-1}A_{k'-1}$ and $A_{k'-1}B_{k'-1}$ for some u'. Since $x_{k'-1}$ is a proper substring of $A_{k'-1}A_{k'-1}$, $x_{k'-1}$ also occurs in u'Wu. Hence, this contradicts that phrase $x_{k'-1}$ ends with W (i.e., $x_{k'-1}$ has to be a longer phrase.), and then, $w_{k'} = Wc_W$. Next, we consider a phrase $x_{k'}$. By the definition of the period-doubling sequence, there exists a clear candidate X of a source which ends at $e(x_{k'-1})$ (see Fig. 4). Then, an equation $|y_{k'-1}| + \frac{1}{2}n_{k'} = |w_{k'}| + |X| + |y_{k'-1}|$ stands w.r.t. the length of suffix $S_{k'}[\mathbf{b}(y_{k'-1})..n_{k'}]$. Thus, $|X| = \frac{3}{8}n_{k'} - 1$ holds since $|w_{k'}| = \frac{1}{8}n_{k'} + 1$. This implies that X has $B_{k'-1}A_{k'-1}$ as a substring. There does not exist a longer candidate since $B_{k'-1}A_{k'-1}$



Figure 4: Illustration for a part of the proof. X is a candidate of a source of phrase x'_k .

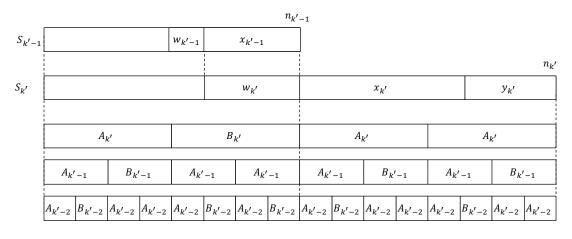


Figure 5: Illustration for the LZ-End factorization when $w_{k'} = \mathsf{Z'}_{k'-1}$.last.

has only one occurrence to the left. Hence, $x_{k'} = Xc_X$ where c_X is the character preceded by X. Finally, we consider the suffix $y_{k'}$ of $S_{k'}$. If $|y_{k'}| \ge 2$, from the above discussion, $y_{k'-1}[2..|y_{k'-1}| - 1] = y_{k'}[1..|y_{k'}| - 1]$ holds. Since $y_{k'-1}[2..|y_{k'-1}| - 1]$ has an occurrence to the left which ends with some phrase (\because statement (IV) for k' - 1), $y_{k'}[1..|y_{k'}| - 1]$ too. Therefore, statements (III) and (IV) also hold.

• Assume that $\ell(k'-1) \neq \ell(k')$ (i.e., $w_{k'} = \mathbf{Z'}_{k'-1}$.last). We can show that all the statements also hold for this case in a similar way. If we assume $|y_{k'-1}| > 0$, then $|w_{k'}| > |y_{k'-1}|$ holds by the above discussions. This contradicts that $w_{k'} = \mathbf{Z'}_{k'-1}$.last, and hence, $|y_{k'-1}| = 0$ and $w_{k'} = x_{k'-1}$ hold (see Fig. 5). Hence, $|w_{k'}| = |x_{k'-1}| = \frac{3}{8}n_{k'-1} = \frac{3}{16}n_{k'}$. We consider a phrase $x_{k'}$ that begins at position $\frac{1}{2}n_{k'} + 1$. Let $X' = S_{k'}[1.e(w_{k'-1})]$ be a clear candidate of a source of $x_{k'}$. Since $|X'| = \frac{1}{2}n_{k'} - \frac{3}{16}n_{k'} = \frac{5}{16}n_{k'}$, X' has A'_k as a prefix. From Lemma 8, X' is the only candidate of a source, and thus $x_{k'} = X'c_{X'}$ where $c_{X'} = S_{k'}[\frac{13}{16}n_{k'} + 1]$ is the character preceded by X'. Moreover, the length of $y_{k'}$ is $\frac{1}{2}n_{k'} - (\frac{5}{16}n_{k'} + 1) = \frac{3}{16}n_{k'} - 1$. Since $|y_{k'}| = |w_{k'}| - 1$ and phrase $w_{k'}$ is a suffix of $S_{k'-1}$, a source of $w_{k'}$ can be also a source of $y_{k'}$. Namely, $y_{k'}$ is the last phrase. Thus, all the statements also hold for this case.

Therefore, this lemma holds.

We have just finished showing the form of the LZ-End factorization of S_k . Now, we will analyze the number of phrases of the factorization. Let \mathcal{K} be the sequence of integers k which satisfies $\ell(k) = k$. Let k_m^* denotes the *m*-th smallest integer in \mathcal{K} . Each k_m^* can be represented by the following recurrence formula:

Lemma 14.

$$k_1^* = 5 \text{ and } k_m^* = k_{m-1}^* + \frac{3}{16} \cdot 2^{k_{m-1}^*} \text{ for } m \ge 2.$$

Proof. Let m be an integer greater than one. By the discussion of the proof for the previous lemma, $|y_{i-1}| - 1 = |y_i|$ holds for any integer $i \in [k_{m-1}^* + 1, k_m^* - 1]$. In addition, $|y_{k_m^*-1}| = 0$.

Hence,

$$k_m^* = k_{m-1}^* + |y_{k_{m-1}^*}| + 1 = k_{m-1}^* + \frac{3}{16}n_{k_{m-1}^*} = k_{m-1}^* + \frac{3}{16} \cdot 2^{k_{m-1}^*}.$$

Lemma 15. For any $k \geq 5$,

$$\mathsf{z}_{\mathsf{end}}(S_k) = 2k - f(k)$$

where f(k) is a function such that f(k) = m + 1 if $k \in [k_m^* - 1, k_{m+1}^* - 2]$.

Proof. By Lemma 13, if $|y_k| = 0$ (i.e., $k + 1 \in \mathcal{K}$), then $\mathsf{z}_{\mathsf{end}}(S_k) = \mathsf{z}_{\mathsf{end}}(S_{k-1}) + 1$ holds, otherwise, $\mathsf{z}_{\mathsf{end}}(S_k) = \mathsf{z}_{\mathsf{end}}(S_{k-1}) + 2$ holds. Hence, for any $k \in [k_m^* - 1, k_{m+1}^* - 2]$,

$$z_{end}(S_k) = z_{end}(S_5) + 2(k-5) - (m-1) = 2k - (m+1) = 2k - f(k).$$

Lemma 16. $f(k) = O(\log^* k)$.

Proof. By Lemma 14,

$$k_m^* = O(2^{k_{m-1}^*}) \subseteq O\left(2^{2^{\cdots}}\right)$$

Thus, $m = O(\log^* k)$ holds. This implies that $f(k) = O(\log^* k)$ by Lemma 15.

By Lemmas 15 and 16, Theorem 11 holds.

5 Conclusions and further work

Let z' and z be the number of phrases in the LZ-End and LZ77 factorizations in a string. In this paper, we proved that the approximation ratio z'/z of LZ-End to LZ77 is asymptotically 2 for the period-doubling sequences. This significantly reduces the number of distinct characters needed to achieve such a lower bound from n/3 (in the existing work [21]) to 2 (in this work). We believe that our work initiates analysis of theoretical performance of LZ-End compression.

A lot of interesting further work remains for LZ-End, including the following:

- Is our lower bound for the approximation ratio tight? Kreft and Navarro [21] conjectured that $z'/z \leq 2$ holds for any string. We performed some exhaustive experiments on binary strings and the result supports their conjecture.
- Is the size z' of the LZ-End factorization a lower bound for the size g of the smallest grammar generating the input string? It is known that the size of the C-factorization [9], a variant of LZ77, is a lower bound of g [30, 6]. In particular case of the period-doubling sequences, there exists the following small SLP (i.e., grammar in the Chomsky normal form) generating the k-th period-doubling sequence: $S_k = S_{k-1}T_k$, $T_k = S_{k-2}S_{k-2}$, ..., $S_1 = ab$, $S_0 = a$. Following [30], the size of an SLP is evaluated by the number of productions and thus the above grammar is of size 2k + 1. It is quite close to the size of the LZ-End factorization which is $2k O(\log^* k)$ but is slightly larger.
- Interesting relationships between the size of the C-factorization and other string repetitive measures such as the size r of the run-length BWT [5], the size s of the smallest run-length SLP [28], the size ℓ of the Lyndon factorization [7], the size b of the smallest bidirectional scheme [31], the size γ of the smallest string attractor [18], the substring complexity δ [8], have been considered in the literature [4, 14, 15, 19, 23, 27, 32]. Can we extend these results to the LZ-End?

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