# On the approximation ratio of LZ-End to LZ77 

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August 17, 2021


#### Abstract

A family of Lempel-Ziv factorizations is a well-studied string structure. The LZ-End factorization is a member of the family that achieved faster extraction of any substrings (Kreft \& Navarro, TCS 2013). One of the interests for LZ-End factorizations is the possible difference between the size of LZ-End and LZ77 factorizations. They also showed families of strings where the approximation ratio of the number of LZ-End phrases to the number of LZ77 phrases asymptotically approaches 2 . However, the alphabet size of these strings is unbounded. In this paper, we analyze the LZ-End factorization of the period-doubling sequence. We also show that the approximation ratio for the period-doubling sequence asymptotically approaches 2 for the binary alphabet.


## 1 Introduction

The Lempel-Ziv 77 compression (LZ77) [33] is one of the most successful lossless compression algorithms to date. On the practical side, LZ77 and its variants have been used as a core of compression software such as zip, gzip, rar, and compressed formats such as PNG, JPEG, PDF. In addition to these real world applications, compressed self-indexing structures based on LZ77 have been proposed [10, 11, 12, 24]. An LZ77-based compressed representation of a string allowing for fast access, rank, and select queries also exists [2].

On the (more) theoretical side, the left-to-right greedy factorization in LZ77, a.k.a. the $L Z 77$ factorization, has widely been considered for decades. It parses a given input string $w$ into a sequence $p_{1}, \ldots, p_{z}$ of non-empty substrings such that $p_{1}=w[1]$ and $p_{i}$ for $i \geq 2$ is the shortest prefix of $p_{i} \cdots p_{z}$ that does not occur in $p_{1} \cdots p_{i-1}$. This implies that the prefix $p_{i}\left[1 . .\left|p_{i}\right|-1\right]$ occurs in $p_{1} \cdots p_{i-1}$, and such an occurrence is called a source of $p$.

Among many versions of LZ77 (c.f. [9, 13, 20, 21, 22, 29, 34), this paper focuses on the $L Z$-End compressor proposed by Kreft and Navarro 21. It is also based on a greedy parsing $q_{1}, \ldots, q_{z^{\prime}}$ of an input string, with a restriction that for each phrase $q_{i}$ there has to be a source which ends at the right-end of a phrase in $q_{1}, \ldots, q_{i-1}$. This constraint permits fast substring extraction without expanding the whole input string. It is known that the LZ-End compression can be computed

[^0]in linear time in the input string length [17], or in compressed space with slight slow-down on compression time [16].

One can regard LZ-End as a mix of LZ77 and LZ78 [34, since in the LZ78 factorization the source of each phrase has to begin and end at boundaries of previous phrases. Since LZ78 belongs to the class of grammar compression [6], LZ-End can be seen as a new bridge between grammar compression and LZ77.

Now, a natural question arises. How good is the compression performance of LZ-End? Practical evaluation in the literature [21] has revealed that the compression ratio of LZ-End is quite close to that of LZ77 (at most $20 \%$ worse), but very little is understood in theory. As in the literature, we measure and compare the sizes of LZ-End and LZ77 by the numbers $z^{\prime}$ and $z$ of their phrases in the factorizations, i.e., " $z^{\prime}$ versus $z$ ".

Since LZ77 is an optimal greedy unidirectional parsing, $z^{\prime} \geq z$ always holds. Thus we are concerned with the approximation ratio of LZ-End to LZ77, which is defined by $z^{\prime} / z$. Kreft and Navarro 21] presented a simple family of strings for which $z^{\prime} / z$ is asymptotically 2 over an alphabet of size $n / 3$, where $n$ is the length of the string. Kreft and Navarro [21] conjectured that the upper bound for $z^{\prime} / z$ is also 2 , but to our knowledge no non-trivial upper bound is known.

In this paper, we show that the same lower bound for $z^{\prime} / z$ can be obtained on a binary alphabet, thus significantly reducing the number of distinct characters used in the analysis from $n / 3$ to 2 . In particular, we prove that $z^{\prime} / z$ is asymptotically 2 for the period-doubling sequences, an interesting family of recursive strings. While the LZ77-factorization of the period-doubling sequences has an obvious structure (Proposition 10), the LZ-End factorization of the period-doubling sequences has a non-trivial structure and needs careful analysis (see our extensive discussions in Section 4 for detail).

Since the LZ77 factorization (without self-references) and the LZ-End factorization for the unary string $a^{n}$ are the same, our result uses a minimum possible number of distinct characters to achieve such a lower bound for $z^{\prime} / z$.

Related work. A famous variant of the LZ77 factorization, which is called the C-factorization [9] and is denoted by $w=c_{1} \cdots c_{x}$, differs from the LZ77 in that each phrase $c_{i}$ is either a fresh character or the longest prefix of $c_{i} \cdots c_{x}$ that occurs in $c_{1} \cdots c_{i-1}$. The size $x$ of the C -factorization is known to be a lower bound for the size of the smallest grammar which generates only the input string [30]. A comparison of the LZ77 factorization and the C-factorization was also considered in the literature [3, 26]. The structure of the C-factorization of the period-doubling sequences was investigated in [3]. We emphasize that our analysis of the LZ-End factorization of the perioddoubling sequences is independent and is quite different from this existing work [3].

Relative $L Z(R L Z)$ is a practical modification of LZ77 which efficiently compresses a collection of highly repetitive sequences [22]. In [20] an RLZ-based factorization of a string, called the ReLZ-factorization, was proposed. The approximation ratio of ReLZ to LZ77 was shown to be $\Omega(\log n)$ [20], where $n$ denotes the length of the input string. On the other hand, in practice ReLZ was larger than LZ77 by at most a factor of two in all the tested cases in [20].

## 2 Preliminaries

### 2.1 Strings

Let $\Sigma$ be the binary alphabet. An element of $\Sigma^{*}$ is called a string. The length of a string $w$ is denoted by $|w|$. The empty string $\varepsilon$ is the string of length 0 . Let $\Sigma^{+}$be the set of non-empty strings, i.e., $\Sigma^{+}=\Sigma^{*} \backslash\{\varepsilon\}$. For a string $w=x y z, x, y$ and $z$ are called a prefix, substring, and suffix of $w$, respectively. They are called a proper prefix, a proper substring, and a proper suffix of $w$ if $x \neq w, y \neq w$, and $z \neq w$, respectively. Further, we say that $w$ has an internal occurrence of $y$ if $y$ occurs in $w$ as a proper substring which is neither a prefix nor a suffix. The $i$-th character of a string $w$ is denoted by $w[i]$, where $1 \leq i \leq|w|$. For a string $w$ and two integers $1 \leq i \leq j \leq|w|$, let $w[i . . j]$ denote the substring of $w$ that begins at position $i$ and ends at position $j$. For convenience,
let $w[i . . j]=\varepsilon$ when $i>j$. For any $1 \leq i \leq|w|, w[i . .|w|] \cdot w[1 . . i-1]$ is called a cyclic rotation of $w$. If a cyclic rotation of $w$ is not equal to $w$, the cyclic rotation is said to be proper. For any string $w$, let $w^{1}=w$ and let $w^{k}=w w^{k-1}$ for any integer $k \geq 2$, i.e., $w^{k}$ is the $k$-times repetition of $w$. A string $w$ is said to be primitive if $w$ cannot be written as $x^{k}$ for any $x \in \Sigma^{*}$ and $k \geq 2$. Let $\bar{c}$ be the opposite character of $c$ in a binary alphabet (e.g., $\bar{a}=b, \bar{b}=a$ for alphabet $\{a, b\}$ ). For any non-empty binary string $w, \widehat{w}$ denotes the string $w[1 . .|w|-1] \cdot \overline{w[|w|}]$. We sometimes use $\mathrm{b}(x)$ and $\mathrm{e}(x)$ as the beginning position and the ending position of a substring $x$ of a given string $w$, if the occurrence of $x$ in $w$ is clear from a discussion.

### 2.2 Lempel-Ziv factorizations

We introduce the Lempel-Ziv 77 and LZ-End factorizations.
Definition 1 (LZ77 [33 ${ }^{2}$. The Lempel-Ziv 77 factorization (LZ77 factorization for short) of a string $w$ is the factorization $\mathrm{LZ} Z_{77}(w)=p_{1}, \ldots, p_{z}$ of $w$ such that $p_{i}\left[1 . .\left|p_{i}\right|-1\right]$ is the longest prefix of $p_{i} \cdots p_{z}$ which occurs in $p_{1} \cdots p_{i-1}$. As an exception, the last phrase $p_{z}$ can be a suffix of $w$ which occurs in $p_{1} \cdots p_{z-1}$.

Definition 2 (LZ-End 21). The LZ-End factorization of a string $w$ is the factorization $\mathrm{LZ}_{\text {end }}(w)=$ $q_{1}, \ldots, q_{z^{\prime}}$ of $w$ such that $q_{i}\left[1 . .\left|q_{i}\right|-1\right]$ is the longest prefix of $q_{i} \cdots q_{z^{\prime}}$ which occurs as a suffix of $q_{1} \cdots q_{j}$ for some $j<i$. As an exception, the last phrase $q_{z^{\prime}}$ can be a suffix of $w$ which occurs as a suffix of $q_{1} \cdots q_{j}$ for some $j<z^{\prime}$.

We refer to each $p_{i}$ and $q_{i}$ as an $L Z$ phrase and LZ-End phrase, respectively. For each phrase, associated longest substring is called a source of the phrase. $\mathbf{z}_{77}(w)$ and $\mathbf{z}_{\text {end }}(w)$ denote the number of the LZ phrases and the LZ-End phrases of a string $w$, respectively. For each $1 \leq i \leq \mathrm{z}_{\text {end }}(w)$, $\operatorname{LZ}_{\text {end }}(w)[i]$ denotes the $i$-th LZ-End phrase of $\mathrm{LZ}_{\text {end }}(w)$. Let $\mathrm{LZ}_{\text {end }}(w)$.last be the last LZ-End phrase of a string $w$, i.e., $L Z_{\text {end }}(w)$ last $=\operatorname{LZ} Z_{\text {end }}(w)\left[Z_{\text {end }}(w)\right]$. Fig. 1 shows examples of two factorizations.

Figure 1: The upper one shows the LZ77 factorization of $w$ and the lower one shows the LZ-End factorization of $w$, where $w=a b a a a b a b a b a a a b a a a b a a a b a b a b a a a b a b$. This $w$ is the fifth perioddoubling sequence $S_{5}$ which will be defined later.

### 2.3 Period-doubling sequence

The period-doubling sequence (cf. [1]) is one of the prominent automatic sequences. Let $S_{k}$ be the $k$-th period-doubling sequence for any $k \geq 0$. The following two definitions are equivalent:

Definition 3. $S_{0}=a$ and $S_{k}=\phi\left(S_{k-1}\right)$ for $k \geq 1$ where $\phi$ is the morphism such that $\phi(a)=$ $a b, \phi(b)=a a$.

Definition 4. $S_{0}=a$ and $S_{k}=S_{k-1} \cdot \widehat{S_{k-1}}$ for $k \geq 1$.
Let $n_{k}$ be the length of the $k$-th period-doubling sequence, i.e., $n_{k}=2^{k}$.

[^1]
## 3 Properties on period-doubling sequence

The period-doubling sequences have many good combinatorial properties (see cf. [1]). In this section, we introduce helpful properties for our results on the period-doubling sequences.

Lemma 5. For any $k \geq 0, S_{k}$ is primitive.
Proof. If $S_{k}$ is not primitive, $S_{k}$ has a period $2^{i}$ for some $i$. This implies that $S_{k}\left[n_{k} / 2\right]=S_{k}\left[n_{k}\right]$, which contradicts Definition 4 .

Lemma 6 (Proposition 8.1.5 of [25]). If a string $w$ is primitive, ww has no internal occurrence of $w$.
Lemma 7. For any $k \geq 2, S_{k}=A_{k} B_{k} A_{k} A_{k}$ where $A_{k}=S_{k-2}$ and $B_{k}=\widehat{A_{k}}$. Moreover, $A_{k}=A_{k-1} B_{k-1}$ and $B_{k}=A_{k-1} A_{k-1}$ for any $k \geq 3$.

Proof. Straightforward from Definition 3 .
Lemma 8. For any $k \geq 2, A_{k} A_{k}, A_{k} B_{k}$, and $B_{k} A_{k}$ have no internal occurrence of $A_{k}$. Hence the number of occurrences of $A_{k}$ in $S_{k}=A_{k} B_{k} A_{k} A_{k}$ is 3 .

Proof. If $k=2$, the lemma clearly holds. We assume $k \geq 3$. Since $A_{k}=S_{k-2}, A_{k}$ is primitive. By Lemma 6. $A_{k} A_{k}$ has no internal occurrence of $A_{k}$. Since $A_{k} B_{k}=\widehat{A_{k} A_{k}}, A_{k} B_{k}$ also has no internal occurrence of $A_{k}$. Similarly, $A_{k-1} A_{k-1}$ and $A_{k-1} B_{k-1}$ have no internal occurrence of $A_{k-1}$. Also, by Lemma 7, $B_{k} A_{k}$ can be written as $A_{k-1} A_{k-1} A_{k-1} B_{k-1}$. These imply that $B_{k} A_{k}$ have no internal occurrence of $A_{k}=A_{k-1} B_{k-1}$.

Lemma 9. For any $k \geq 3$ and any proper cyclic rotation $\alpha$ of $A_{k}$, the number of occurrences of $\alpha$ in $A_{k} A_{k} A_{k}, A_{k} B_{k}$, and $B_{k} A_{k}$ are 2, 1, and 0, respectively.

Proof. Since $A_{k}=S_{k-2}$ and Lemma 5. $A_{k}$ is primitive. This implies that $\alpha$ is also primitive. Thus, $A_{k} A_{k}$ has exactly one (internal) occurrences of $\alpha$. Namely, $\alpha$ occurs in $A_{k} A_{k} A_{k}$ exactly two times. Since $A_{k} B_{k}=\widehat{A_{k} A_{k}}, A_{k} B_{k}$ also has exactly one (internal) occurrence of $\alpha$. Finally, let us consider $B_{k} A_{k}=A_{k-1} A_{k-1} A_{k-1} B_{k-1}$. In a similar way of the proof of Lemma 8 we can show that both $A_{k-1} A_{k-1}$ and $A_{k-1} B_{k-1}$ have no internal occurrence of $B_{k-1}$. From this facts and Lemma 8 , $A_{k-1}$ occurs exactly three times and $B_{k-1}$ occurs exactly once in $B_{k} A_{k}$. If $\alpha=B_{k-1} A_{k-1}, \alpha$ cannot occur in $B_{k} A_{k}$. Otherwise, $\alpha$ can be written as either $x B_{k-1} y$ or $x^{\prime} A_{k-1} y^{\prime}$ where $x$ (resp. $y$ ) is a non-empty suffix (resp. prefix) of $A_{k-1}$, and $x^{\prime}$ (resp. $y^{\prime}$ ) is a non-empty suffix (resp. prefix) of $B_{k-1}$. If $\alpha=x B_{k-1} y, \alpha$ cannot occur in $B_{k} A_{k}$ due to the constraint of $B_{k-1}$. If $\alpha=x^{\prime} A_{k-1} y^{\prime}, \alpha$ cannot occur in $B_{k} A_{k}$ due to the constraint of $A_{k-1}$ and the difference between the last characters of $A_{k-1}$ and $x^{\prime}$. Therefore $\alpha$ cannot occur in $B_{k} A_{k}$ in all cases.

## 4 Factorizations of period-doubling sequence

By the definition of LZ77, the following proposition immediately holds:
Proposition 10. $\mathrm{LZ}_{77}\left(S_{k}\right)=\left(S_{0}, \widehat{S_{0}}, \widehat{S_{1}}, \ldots, \widehat{S_{k-1}}\right)$ and thus $\mathrm{z}_{77}\left(S_{k}\right)=k+1$.
In this section, we mainly discuss the LZ-End factorization of the period-doubling sequence, and give the following result.

Theorem 11. $\mathrm{z}_{\mathrm{end}}\left(S_{k}\right)=2 k-f(k)$ where $f(k)=O\left(\log ^{*} k\right)$.
By Proposition 10 and Theorem 11, we can reach our goal of this paper:
Corollary 12. There exists a family of binary strings $w$ such that the ratio $\mathrm{z}_{\mathrm{end}}(w) / \mathrm{z}_{77}(w)$ asymptotically approaches 2.

In the rest of this paper, we show Theorem 11 . The next lemma gives the LZ-End factorization of the period-doubling sequence. Notice that statement (I) in the lemma is not an immediate property for the LZ-End factorization due to the next example. Let $S=a b a a b a b a a b b a b b a a b a b a$. Then,

$$
\begin{aligned}
\mathrm{LZ}_{\mathrm{end}}(S) & =a|b| a a|b a| b a a b|b a b| b a a b a b \mid a \\
\mathrm{LZ}_{\mathrm{end}}(S a b a) & =a|b| a a|b a| b a a b|b a b| b a a b a b a a b a .
\end{aligned}
$$

Lemma 13. For any $k \geq 5$, the following statements (I)-(IV) hold.
(I) $\mathrm{LZ} \mathrm{Z}_{\text {end }}\left(S_{k}\right)[i]=\mathrm{LZ} \mathrm{Z}_{\text {end }}\left(S_{k-1}\right)[i]$ for every $1 \leq i \leq \mathbf{z}_{\text {end }}\left(S_{k-1}\right)-1$.
(II) $\mathrm{z}_{\mathrm{end}}\left(S_{k}\right) \geq \mathrm{z}_{\mathrm{end}}\left(S_{k-1}\right)+1$.

Let

$$
\begin{aligned}
w_{k} & =\mathrm{LZ} \mathrm{Z}_{\mathrm{end}}\left(S_{k}\right)\left[\mathrm{z}_{\mathrm{end}}\left(S_{k-1}\right)\right] \\
x_{k} & =\mathrm{LZ} \mathrm{Z}_{\mathrm{end}}\left(S_{k}\right)\left[\mathrm{z}_{\mathrm{end}}\left(S_{k-1}\right)+1\right] \\
y_{k} & =S_{k}\left[\mathrm{e}\left(x_{k}\right)+1 . . n_{k}\right] \text { (possibly empty). }
\end{aligned}
$$

(III) If $w_{k} \neq \mathrm{LZ}_{\text {end }}\left(S_{k-1}\right)$.last,

$$
\left|w_{k}\right|=\frac{1}{8} n_{k}+1,\left|x_{k}\right|=\frac{3}{8} n_{k},\left|y_{k}\right|=\frac{3}{16} n_{\ell(k)}-(k-\ell(k))-1,
$$

where $\ell(k)=\max \left\{i \mid i \leq k, w_{i}=\mathrm{LZ}_{\text {end }}\left(S_{i-1}\right)\right.$.last $\}$.
Otherwise (if $w_{k}=\mathrm{LZ} \mathrm{Z}_{\text {end }}\left(S_{k-1}\right)$.last),

$$
\left|w_{k}\right|=\frac{3}{16} n_{k},\left|x_{k}\right|=\frac{5}{16} n_{k}+1,\left|y_{k}\right|=\frac{3}{16} n_{k}-1
$$

(IV) If $\left|y_{k}\right| \geq 2, y_{k}\left[1 . .\left|y_{k}\right|-1\right]$ has another occurrence to the left which ends with some LZ-End phrase of $S_{k}$. Namely, $y_{k}$ is the last LZ-End phrase of $S_{k}$ if $y_{k}$ is not empty.

Proof. In this proof, we use $\mathrm{Z}^{\prime}{ }_{k}=\mathrm{LZ} Z_{\text {end }}\left(S_{k}\right)$ and $\mathrm{z}^{\prime}{ }_{k}=\mathrm{z}_{\text {end }}\left(S_{k}\right)$ for simplicity. We prove this lemma by induction on $k$.

Suppose that $k=5$. The LZ-End factorizations of $S_{4}, S_{5}$ are

$$
\begin{aligned}
\mathrm{Z}^{\prime} & =a|b| a a|a b a| b a b \mid a a a b a a \\
\mathbf{Z}_{5}^{\prime} & =a|b| a a|a b a| b a b|a a a b a a| a b a a a b a b a b a \mid a a b a b .
\end{aligned}
$$

Statements (I) and (II) clearly hold. Then, $w_{5}=a a a b a a, x_{5}=a b a a a b a b a b a, y_{5}=a a b a b$. Hence, statement (III) holds since $n_{5}=32$ and $w_{5}=\mathrm{Z}^{\prime}{ }_{4}$.last (i.e., the latter case). Statement (IV) also holds since $y_{5}[1 . .4]=a a b a$ has an occurrence which ends with the fourth phrase $a b a$.

Suppose that all the statements hold for any $k \in\left[5, k^{\prime}-1\right]$ for some $k^{\prime}>5$. We show that all the statements hold for $k^{\prime}$. Firstly, suppose on the contrary that statement (I) does not hold for $k^{\prime}$. This implies that there exists a phrase $T=S_{k^{\prime}}\left[\mathrm{b}\left(\mathrm{Z}^{\prime}{ }_{k-1}[i]\right) . . j\right]$ for some $i<\mathrm{z}^{\prime}{ }_{k^{\prime}-1}$ and $j>n_{k^{\prime}-1}$. Since $\left|x_{k^{\prime}-1} y_{k^{\prime}-1}\right| \geq \frac{3}{8} n_{k^{\prime}-1}>\frac{1}{4} n_{k^{\prime}-1}$ and $x_{k^{\prime}-1} y_{k^{\prime}-1}$ is a substring of $T, T$ has an internal occurrence of the length- $\frac{1}{4} n_{k^{\prime}-1}$ suffix $A_{k^{\prime}-1}$ of $S_{k^{\prime}-1}$. By Lemma 8 (showing the occurrences of $A_{k-1}$ in $S_{k-1}$ ), $A_{k^{\prime}-1}$ occurs exactly three times in $S_{k^{\prime}}\left[1 . . n_{k^{\prime}-1}\right]$. The first occurrence of $A_{k^{\prime}-1}$ cannot be included by a source of $T$ since $A_{k^{\prime}-1}$ is not a prefix of $T[1 . .|T|-1]$. In addition, the second occurrence of $A_{k^{\prime}-1}$ also cannot be included by a source of $T$ since the source overlaps phrase $T$. Thus, $T[1 .|T|-1]$ cannot have another occurrence to the left as a source of $T$. This contradicts that $T$ is an LZ-End phrase of $S_{k^{\prime}}$ at that position. Hence, statement (I) holds for $k^{\prime}$. Due to statement (I), $w_{k^{\prime}}$ must have $y_{k^{\prime}-1}$ as a prefix. On the other hand, $w_{k^{\prime}}$ cannot reach the end of $S_{k^{\prime}}$. Hence, statement (II) also holds. Thanks to statements (I) and (II) for $k^{\prime}$, three substrings $w_{k^{\prime}}, x_{k^{\prime}}$, and $y_{k^{\prime}}$ are well-defined (see Fig. 2 and 5 for illustrations).

Next, we show statements (III) and (IV).


Figure 2: Illustration for the LZ-End factorization when $w_{k^{\prime}} \neq \mathrm{Z}^{\prime}{ }_{k^{\prime}-1}$. last.


Figure 3: Illustration for a part of the proof. $W$ is a candidate of a source of phrase $w_{k}^{\prime}$.

- Assume that $\ell\left(k^{\prime}-1\right)=\ell\left(k^{\prime}\right)$ (i.e., $w_{k^{\prime}} \neq \mathbf{Z}^{\prime}{ }_{k^{\prime}-1}$.last). We consider a phrase $w_{k^{\prime}}$. If $\left|y_{k^{\prime}-1}\right|=$ $0, x_{k^{\prime}-1}$ is the suffix of length $\frac{3}{8} n_{k^{\prime}-1}$ of $S_{k^{\prime}-1}$, i.e., $x_{k^{\prime}-1}=B_{k^{\prime}-2} A_{k^{\prime}-1}$. From Lemma 8 , $x_{k^{\prime}-1}$ does not have other occurrences to the left. This implies that $w_{k^{\prime}}=x_{k^{\prime}-1}$. This contradicts to $w_{k^{\prime}} \neq \mathbf{Z}^{\prime}{ }_{k^{\prime}-1}$. last. Thus, $\left|y_{k^{\prime}-1}\right|>0$ holds. Namely, $x_{k^{\prime}-1}=\mathrm{Z}^{\prime}{ }_{k^{\prime}-1}\left[\mathbf{z}^{\prime}{ }_{k^{\prime}-1}-1\right]$ and $y_{k^{\prime}-1}=\mathrm{Z}^{\prime}{ }_{k^{\prime}-1}$.last (see also Fig. 2]. Let $W$ be the string of length $\frac{1}{8} n_{k^{\prime}}$ which begins at $\mathrm{b}\left(\mathrm{Z}^{\prime}{ }_{k^{\prime}-1}\right.$.last $) . \ell\left(k^{\prime}-1\right)=\ell\left(k^{\prime}\right)$ also implies that $\ell\left(k^{\prime}-1\right)<k^{\prime}$. Hence, $\left|y_{k^{\prime}-1}\right|<\frac{3}{16} n_{\ell\left(k^{\prime}-1\right)} \leq$ $\frac{3}{32} n_{k^{\prime}}<\frac{1}{8} n_{k^{\prime}}$. This fact means that $W$ is a proper cyclic rotation of $A_{k^{\prime}-1}$. By Lemma 9 , $W$ occurs twice to the left (one is in $A_{k^{\prime}-1} B_{k^{\prime}-1}$, the other is in $A_{k^{\prime}-1} A_{k^{\prime}-1}$ ). Since the second occurrence ends with phrase $Z^{\prime}{ }_{k^{\prime}}\left[\mathbf{z}^{\prime}{ }_{k^{\prime}-1}-1\right], W c_{W}$ is a candidate of phrase $w_{k^{\prime}}$ where $c_{W}$ is the character preceded by $W$. Assume on the contrary that a source of phrase $w_{k^{\prime}}$ is $W u$ for some $u \in \Sigma^{+}$(see Fig. 33). The second occurrence of $W$ cannot be the beginning position of a source of $w_{k^{\prime}}$ since $W u$ overlaps $w_{k^{\prime}}$. Hence, the only candidate of the beginning position of source $W u$ is in the first $A_{k^{\prime}-1} B_{k^{\prime}-1}$. Moreover, $W u$ cannot contain $B_{k^{\prime}-1}$ since the original $W u$ occurs in $A_{k^{\prime}-1} A_{k^{\prime}-1} \cdots$. Thus, $W u$ is a proper substring of $A_{k^{\prime}-1} A_{k^{\prime}-1}$ and $A_{k^{\prime}-1} B_{k^{\prime}-1}$. In other words, $u^{\prime} W u$ is a proper prefix of $A_{k^{\prime}-1} A_{k^{\prime}-1}$ and $A_{k^{\prime}-1} B_{k^{\prime}-1}$ for some $u^{\prime}$. Since $x_{k^{\prime}-1}$ is a proper substring of $A_{k^{\prime}-1} A_{k^{\prime}-1}, x_{k^{\prime}-1}$ also occurs in $u^{\prime} W u$. Hence, this contradicts that phrase $x_{k^{\prime}-1}$ ends with $W$ (i.e., $x_{k^{\prime}-1}$ has to be a longer phrase.), and then, $w_{k^{\prime}}=W c_{W}$. Next, we consider a phrase $x_{k^{\prime}}$. By the definition of the period-doubling sequence, there exists a clear candidate $X$ of a source which ends at e $\left(x_{k^{\prime}-1}\right)$ (see Fig. (4). Then, an equation $\left|y_{k^{\prime}-1}\right|+\frac{1}{2} n_{k^{\prime}}=\left|w_{k^{\prime}}\right|+|X|+\left|y_{k^{\prime}-1}\right|$ stands w.r.t. the length of suffix $S_{k^{\prime}}\left[\mathrm{b}\left(y_{k^{\prime}-1}\right) . . n_{k^{\prime}}\right]$. Thus, $|X|=\frac{3}{8} n_{k^{\prime}}-1$ holds since $\left|w_{k^{\prime}}\right|=\frac{1}{8} n_{k^{\prime}}+1$. This implies that $X$ has $B_{k^{\prime}-1} A_{k^{\prime}-1}$ as a substring. There does not exist a longer candidate since $B_{k^{\prime}-1} A_{k^{\prime}-1}$


Figure 4: Illustration for a part of the proof. $X$ is a candidate of a source of phrase $x_{k}^{\prime}$.


Figure 5: Illustration for the LZ-End factorization when $w_{k^{\prime}}=\mathrm{Z}^{\prime}{ }_{k^{\prime}-1}$.last.
has only one occurrence to the left. Hence, $x_{k^{\prime}}=X c_{X}$ where $c_{X}$ is the character preceded by $X$. Finally, we consider the suffix $y_{k^{\prime}}$ of $S_{k^{\prime}}$. If $\left|y_{k^{\prime}}\right| \geq 2$, from the above discussion, $y_{k^{\prime}-1}\left[2 . .\left|y_{k^{\prime}-1}\right|-1\right]=y_{k^{\prime}}\left[1 . .\left|y_{k^{\prime}}\right|-1\right]$ holds. Since $y_{k^{\prime}-1}\left[2 . .\left|y_{k^{\prime}-1}\right|-1\right]$ has an occurrence to the left which ends with some phrase ( $\because$ statement (IV) for $k^{\prime}-1$ ), $y_{k^{\prime}}\left[1 . .\left|y_{k^{\prime}}\right|-1\right]$ too. Therefore, statements (III) and (IV) also hold.

- Assume that $\ell\left(k^{\prime}-1\right) \neq \ell\left(k^{\prime}\right)$ (i.e., $w_{k^{\prime}}=\mathrm{Z}^{\prime}{ }_{k^{\prime}-1}$.last). We can show that all the statements also hold for this case in a similar way. If we assume $\left|y_{k^{\prime}-1}\right|>0$, then $\left|w_{k^{\prime}}\right|>\left|y_{k^{\prime}-1}\right|$ holds by the above discussions. This contradicts that $w_{k^{\prime}}=Z_{k^{\prime}-1}$.last, and hence, $\left|y_{k^{\prime}-1}\right|=0$ and $w_{k^{\prime}}=x_{k^{\prime}-1}$ hold (see Fig. 55. Hence, $\left|w_{k^{\prime}}\right|=\left|x_{k^{\prime}-1}\right|=\frac{3}{8} n_{k^{\prime}-1}=\frac{3}{16} n_{k^{\prime}}$. We consider a phrase $x_{k^{\prime}}$ that begins at position $\frac{1}{2} n_{k^{\prime}}+1$. Let $X^{\prime}=S_{k^{\prime}}\left[1\right.$.ee $\left.\left(w_{k^{\prime}-1}\right)\right]$ be a clear candidate of a source of $x_{k^{\prime}}$. Since $\left|X^{\prime}\right|=\frac{1}{2} n_{k^{\prime}}-\frac{3}{16} n_{k^{\prime}}=\frac{5}{16} n_{k^{\prime}}, X^{\prime}$ has $A_{k}^{\prime}$ as a prefix. From Lemma 8 , $X^{\prime}$ is the only candidate of a source, and thus $x_{k^{\prime}}=X^{\prime} c_{X^{\prime}}$ where $c_{X^{\prime}}=S_{k^{\prime}}\left[\frac{13}{16} n_{k^{\prime}}+1\right]$ is the character preceded by $X^{\prime}$. Moreover, the length of $y_{k^{\prime}}$ is $\frac{1}{2} n_{k^{\prime}}-\left(\frac{5}{16} n_{k^{\prime}}+1\right)=\frac{3}{16} n_{k^{\prime}}-1$. Since $\left|y_{k^{\prime}}\right|=\left|w_{k^{\prime}}\right|-1$ and phrase $w_{k^{\prime}}$ is a suffix of $S_{k^{\prime}-1}$, a source of $w_{k^{\prime}}$ can be also a source of $y_{k^{\prime}}$. Namely, $y_{k^{\prime}}$ is the last phrase. Thus, all the statements also hold for this case.

Therefore, this lemma holds.
We have just finished showing the form of the LZ-End factorization of $S_{k}$. Now, we will analyze the number of phrases of the factorization. Let $\mathcal{K}$ be the sequence of integers $k$ which satisfies $\ell(k)=k$. Let $k_{m}^{*}$ denotes the $m$-th smallest integer in $\mathcal{K}$. Each $k_{m}^{*}$ can be represented by the following recurrence formula:

## Lemma 14.

$$
k_{1}^{*}=5 \text { and } k_{m}^{*}=k_{m-1}^{*}+\frac{3}{16} \cdot 2^{k_{m-1}^{*}} \text { for } m \geq 2 \text {. }
$$

Proof. Let $m$ be an integer greater than one. By the discussion of the proof for the previous lemma, $\left|y_{i-1}\right|-1=\left|y_{i}\right|$ holds for any integer $i \in\left[k_{m-1}^{*}+1, k_{m}^{*}-1\right]$. In addition, $\left|y_{k_{m}^{*}-1}\right|=0$.

Hence,

$$
k_{m}^{*}=k_{m-1}^{*}+\left|y_{k_{m-1}^{*}}\right|+1=k_{m-1}^{*}+\frac{3}{16} n_{k_{m-1}^{*}}=k_{m-1}^{*}+\frac{3}{16} \cdot 2^{k_{m-1}^{*}}
$$

Lemma 15. For any $k \geq 5$,

$$
\mathrm{z}_{\mathrm{end}}\left(S_{k}\right)=2 k-f(k),
$$

where $f(k)$ is a function such that $f(k)=m+1$ if $k \in\left[k_{m}^{*}-1, k_{m+1}^{*}-2\right]$.
Proof. By Lemma 13, if $\left|y_{k}\right|=0$ (i.e., $k+1 \in \mathcal{K}$ ), then $\mathrm{z}_{\text {end }}\left(S_{k}\right)=\mathrm{z}_{\text {end }}\left(S_{k-1}\right)+1$ holds, otherwise, $\mathrm{z}_{\text {end }}\left(S_{k}\right)=\mathrm{z}_{\text {end }}\left(S_{k-1}\right)+2$ holds. Hence, for any $k \in\left[k_{m}^{*}-1, k_{m+1}^{*}-2\right]$,

$$
\mathrm{z}_{\mathrm{end}}\left(S_{k}\right)=\mathrm{z}_{\mathrm{end}}\left(S_{5}\right)+2(k-5)-(m-1)=2 k-(m+1)=2 k-f(k) .
$$

Lemma 16. $f(k)=O\left(\log ^{*} k\right)$.
Proof. By Lemma 14 ,

$$
k_{m}^{*}=O\left(2^{k_{m-1}^{*}}\right) \subseteq O\left(2^{2}\right)
$$

Thus, $m=O\left(\log ^{*} k\right)$ holds. This implies that $f(k)=O\left(\log ^{*} k\right)$ by Lemma 15 .
By Lemmas 15 and 16 . Theorem 11 holds.

## 5 Conclusions and further work

Let $z^{\prime}$ and $z$ be the number of phrases in the LZ-End and LZ77 factorizations in a string. In this paper, we proved that the approximation ratio $z^{\prime} / z$ of LZ-End to LZ77 is asymptotically 2 for the period-doubling sequences. This significantly reduces the number of distinct characters needed to achieve such a lower bound from $n / 3$ (in the existing work [21) to 2 (in this work). We believe that our work initiates analysis of theoretical performance of LZ-End compression.

A lot of interesting further work remains for LZ-End, including the following:

- Is our lower bound for the approximation ratio tight? Kreft and Navarro 21 conjectured that $z^{\prime} / z \leq 2$ holds for any string. We performed some exhaustive experiments on binary strings and the result supports their conjecture.
- Is the size $z^{\prime}$ of the LZ-End factorization a lower bound for the size $g$ of the smallest grammar generating the input string? It is known that the size of the C-factorization [9, a variant of LZ77, is a lower bound of $g$ [30, 6]. In particular case of the period-doubling sequences, there exists the following small SLP (i.e., grammar in the Chomsky normal form) generating the $k$-th period-doubling sequence: $S_{k}=S_{k-1} T_{k}, T_{k}=S_{k-2} S_{k-2}, \ldots, S_{1}=a b, S_{0}=a$. Following [30], the size of an SLP is evaluated by the number of productions and thus the above grammar is of size $2 k+1$. It is quite close to the size of the LZ-End factorization which is $2 k-O\left(\log ^{*} k\right)$ but is slightly larger.
- Interesting relationships between the size of the C-factorization and other string repetitive measures such as the size $r$ of the run-length BWT [5], the size $s$ of the smallest run-length SLP [28], the size $\ell$ of the Lyndon factorization [7, the size $b$ of the smallest bidirectional scheme [31], the size $\gamma$ of the smallest string attractor [18], the substring complexity $\delta$ [8], have been considered in the literature [4, 14, 15, 19, 23, 27, 32. Can we extend these results to the LZ-End?


## Acknowledgments

This work was supported by JSPS KAKENHI Grant Numbers JP20J11983 (TM), JP20J21147 (MF), JP18K18002 (YN), JP21K17705 (YN), JP18H04098 (MT), JP20H05964 (MT), and by JST PRESTO Grant Number JPMJPR1922 (SI).

## References

[1] J.-P. Allouche and J. Shallit. Automatic Sequences: Theory, Applications, Generalizations. Cambridge University Press, 2003.
[2] D. Belazzougui, T. Gagie, P. Gawrychowski, J. Kärkkäinen, A. O. Pereira, S. J. Puglisi, and Y. Tabei. Queries on LZ-bounded encodings. In A. Bilgin, M. W. Marcellin, J. SerraSagristà, and J. A. Storer, editors, 2015 Data Compression Conference, DCC 2015, Snowbird, UT, USA, April 7-9, 2015, pages 83-92. IEEE, 2015.
[3] J. Berstel and A. Savelli. Crochemore factorization of Sturmian and other infinite words. In R. Kralovic and P. Urzyczyn, editors, Mathematical Foundations of Computer Science 2006, 31st International Symposium, MFCS 2006, Stará Lesná, Slovakia, August 28-September 1, 2006, Proceedings, volume 4162 of Lecture Notes in Computer Science, pages 157-166. Springer, 2006.
[4] P. Bille, T. Gagie, I. L. Gørtz, and N. Prezza. A separation between RLSLPs and LZ77. J. Discrete Algorithms, 50:36-39, 2018.
[5] M. Burrows and D. Wheeler. A block-sorting lossless data compression algorithm. Technical report, DIGITAL SRC RESEARCH REPORT, 1994.
[6] M. Charikar, E. Lehman, D. Liu, R. Panigrahy, M. Prabhakaran, A. Sahai, and A. Shelat. The smallest grammar problem. IEEE Trans. Inf. Theory, 51(7):2554-2576, 2005.
[7] K. T. Chen, R. H. Fox, and R. C. Lyndon. Free differential calculus, IV. the quotient groups of the lower central series. Annals of Mathematics, 68(1):81-95, 1958.
[8] A. R. Christiansen, M. B. Ettienne, T. Kociumaka, G. Navarro, and N. Prezza. Optimal-time dictionary-compressed indexes. ACM Trans. Algorithms, 17(1):8:1-8:39, 2021.
[9] M. Crochemore. An optimal algorithm for computing the repetitions in a word. Information Processing Letters, 12(5):244-250, 1981.
[10] H. H. Do, J. Jansson, K. Sadakane, and W. Sung. Fast relative Lempel-Ziv self-index for similar sequences. Theor. Comput. Sci., 532:14-30, 2014.
[11] T. Gagie, P. Gawrychowski, J. Kärkkäinen, Y. Nekrich, and S. J. Puglisi. A faster grammarbased self-index. In A. Dediu and C. Martín-Vide, editors, Language and Automata Theory and Applications - 6th International Conference, LATA 2012, A Coruña, Spain, March 59, 2012. Proceedings, volume 7183 of Lecture Notes in Computer Science, pages 240-251. Springer, 2012.
[12] T. Gagie, P. Gawrychowski, J. Kärkkäinen, Y. Nekrich, and S. J. Puglisi. LZ77-based selfindexing with faster pattern matching. In A. Pardo and A. Viola, editors, LATIN 2014: Theoretical Informatics - 11th Latin American Symposium, Montevideo, Uruguay, March 31 - April 4, 2014. Proceedings, volume 8392 of Lecture Notes in Computer Science, pages 731742. Springer, 2014.
[13] K. Goto, H. Bannai, S. Inenaga, and M. Takeda. LZD factorization: Simple and practical online grammar compression with variable-to-fixed encoding. In F. Cicalese, E. Porat, and U. Vaccaro, editors, Combinatorial Pattern Matching - 26th Annual Symposium, CPM 2015, Ischia Island, Italy, June 29-July 1, 2015, Proceedings, volume 9133 of Lecture Notes in Computer Science, pages 219-230. Springer, 2015.
[14] J. Kärkkäinen, D. Kempa, Y. Nakashima, S. J. Puglisi, and A. M. Shur. On the size of Lempel-Ziv and Lyndon factorizations. In H. Vollmer and B. Vallée, editors, 34 th Symposium on Theoretical Aspects of Computer Science, STACS 2017, March 8-11, 2017, Hannover, Germany, volume 66 of LIPIcs, pages 45:1-45:13. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.
[15] D. Kempa and T. Kociumaka. Resolution of the Burrows-Wheeler transform conjecture. In 61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020, pages 1002-1013. IEEE, 2020.
[16] D. Kempa and D. Kosolobov. LZ-End parsing in compressed space. In A. Bilgin, M. W. Marcellin, J. Serra-Sagristà, and J. A. Storer, editors, 2017 Data Compression Conference, DCC 2017, Snowbird, UT, USA, April 4-7, 2017, pages 350-359. IEEE, 2017.
[17] D. Kempa and D. Kosolobov. LZ-End parsing in linear time. In K. Pruhs and C. Sohler, editors, 25th Annual European Symposium on Algorithms, ESA 2017, September 4-6, 2017, Vienna, Austria, volume 87 of LIPIcs, pages 53:1-53:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.
[18] D. Kempa and N. Prezza. At the roots of dictionary compression: string attractors. In I. Diakonikolas, D. Kempe, and M. Henzinger, editors, Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018, pages 827-840. ACM, 2018.
[19] T. Kociumaka, G. Navarro, and N. Prezza. Towards a definitive measure of repetitiveness. In Y. Kohayakawa and F. K. Miyazawa, editors, LATIN 2020: Theoretical Informatics 14th Latin American Symposium, São Paulo, Brazil, January 5-8, 2021, Proceedings, volume 12118 of Lecture Notes in Computer Science, pages 207-219. Springer, 2020.
[20] D. Kosolobov, D. Valenzuela, G. Navarro, and S. J. Puglisi. Lempel-Ziv-like parsing in small space. Algorithmica, 82(11):3195-3215, 2020.
[21] S. Kreft and G. Navarro. On compressing and indexing repetitive sequences. Theor. Comput. Sci., 483:115-133, 2013.
[22] S. Kuruppu, S. J. Puglisi, and J. Zobel. Relative Lempel-Ziv compression of genomes for large-scale storage and retrieval. In E. Chávez and S. Lonardi, editors, String Processing and Information Retrieval - 17th International Symposium, SPIRE 2010, Los Cabos, Mexico, October 11-13, 2010. Proceedings, volume 6393 of Lecture Notes in Computer Science, pages 201-206. Springer, 2010.
[23] K. Kutsukake, T. Matsumoto, Y. Nakashima, S. Inenaga, H. Bannai, and M. Takeda. On repetitiveness measures of Thue-Morse words. In C. Boucher and S. V. Thankachan, editors, String Processing and Information Retrieval - 27th International Symposium, SPIRE 2020, Orlando, FL, USA, October 13-15, 2020, Proceedings, volume 12303 of Lecture Notes in Computer Science, pages 213-220. Springer, 2020.
[24] J. Kärkkäinen and E. Ukkonen. Lempel-Ziv parsing and sublinear-size index structures for string matching (extended abstract). In Proc. 3rd South American Workshop on String Processing (WSP'96, pages 141-155. Carleton University Press, 1996.
[25] M. Lothaire. Applied combinatorics on words, volume 105. Cambridge University Press, 2005.
[26] S. Mitsuya, Y. Nakashima, S. Inenaga, H. Bannai, and M. Takeda. Compressed communication complexity of Hamming distance. Algorithms, 14(4):116, 2021.
[27] G. Navarro, C. Ochoa, and N. Prezza. On the approximation ratio of ordered parsings. IEEE Trans. Inf. Theory, 67(2):1008-1026, 2021.
[28] T. Nishimoto, T. I, S. Inenaga, H. Bannai, and M. Takeda. Dynamic index and LZ factorization in compressed space. Discret. Appl. Math., 274:116-129, 2020.
[29] T. Nishimoto and Y. Tabei. LZRR: LZ77 parsing with right reference. In A. Bilgin, M. W. Marcellin, J. Serra-Sagristà, and J. A. Storer, editors, Data Compression Conference, DCC 2019, Snowbird, UT, USA, March 26-29, 2019, pages 211-220. IEEE, 2019.
[30] W. Rytter. Application of Lempel-Ziv factorization to the approximation of grammar-based compression. Theor. Comput. Sci., 302(1-3):211-222, 2003.
[31] J. A. Storer and T. G. Szymanski. Data compression via textual substitution. J. ACM, 29(4):928-951, 1982.
[32] Y. Urabe, Y. Nakashima, S. Inenaga, H. Bannai, and M. Takeda. On the size of overlapping Lempel-Ziv and Lyndon factorizations. In N. Pisanti and S. P. Pissis, editors, 30th Annual Symposium on Combinatorial Pattern Matching, CPM 2019, June 18-20, 2019, Pisa, Italy, volume 128 of LIPIcs, pages 29:1-29:11. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.
[33] J. Ziv and A. Lempel. A universal algorithm for sequential data compression. IEEE Trans. Inf. Theory, 23(3):337-343, 1977.
[34] J. Ziv and A. Lempel. Compression of individual sequences via variable-rate coding. IEEE Trans. Inf. Theory, 24(5):530-536, 1978.


[^0]:    ${ }^{1}$ This version of LZ77 is often called non-overlapping LZ77 or LZ77 without self-references, since each phrase $p_{i}$ never overlaps with any of its sources.

[^1]:    ${ }^{2}$ This definition of LZ77 is different from the original one 33 (see 21 for more information).

