

Algebras of Sets and Coherent Sets of Gambles

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Abstract. In a recent work we have shown how to construct an information algebra of coherent sets of gambles defined on general possibility spaces. Here we analyze the connection of such an algebra with the *set algebra* of subsets of the possibility space on which gambles are defined and the *set algebra* of sets of its *atoms*. Set algebras are particularly important information algebras since they are their *prototypical* structures. Furthermore, they are the algebraic counterparts of classical propositional logic. As a consequence, this paper also details how propositional logic is naturally embedded into the theory of *imprecise probabilities*.

Keywords: Desirability · Information algebras · Order theory · Imprecise probabilities · Coherence.

1 Introduction and Overview

While analysing the compatibility problem of coherent sets of gambles, Miranda and Zaffalon [13] have recently remarked that their main results could be obtained also using the theory of information algebras [6]. This observation has been taken up and deepened in some of our recent work [12,17]: we have shown that the founding properties of desirability can in fact be abstracted into properties of information algebras. Stated differently, desirability makes up an information algebra of coherent set of gambles.

Information algebras are algebraic structures composed by ‘pieces of information’ that can be manipulated by operations of *combination*, to aggregate them, and *extraction*, to extract information regarding a specific question. From the point of view of information algebras, sets of gambles defined on a possibility space Ω are pieces of information about Ω . It is well known that coherent sets of gambles are ordered by inclusion and, in this order, there are maximal elements [4]. In the language of information algebras such elements are called *atoms*. In particular, any coherent set of gambles is contained in a maximal set (an atom) and it is the intersection (meet) of all the atoms it is contained in. An information algebra with these properties is called atomistic. Atomistic information algebras have the universal property of being embedded in a set algebra, which is an information algebra whose elements are sets. This is an important

representation theorem for information algebras, since set algebras are a special kind of algebras based on the usual set operations. Conversely, any such set algebra of subsets of Ω is embedded in the algebra of coherent sets of gambles defined on Ω . These links between set algebras and the algebra of coherent sets of gambles are the main topic of the present work.

After recalling the main concepts introduced in our previous work in Sections 2–4, in Section 5 we establish the basis to show that sets of atoms of the information algebra of coherent sets of gambles, form indeed a set algebra. In Section 6 we define the concept of *embedding* for information algebras and finally, in Section 7, we show the links between set algebras (of subsets of Ω and of sets of atoms) and the algebra of coherent sets of gambles.

Since set algebras are algebraic counterparts of classical propositional logic, the results of this paper details how the latter is formally part of the theory of imprecise probabilities [16]. We refer also to [3] for another aspect of this issue.

2 Desirability

Consider a set Ω of possible worlds. A gamble over this set is a bounded function $f : \Omega \rightarrow \mathbb{R}$. It is interpreted as an uncertain reward in a linear utility scale. A subject might desire a gamble or not, depending on the information they have about the experiment whose possible outcomes are the elements of Ω . We denote the set of all gambles on Ω by $\mathcal{L}(\Omega)$, or more simply by \mathcal{L} , when there is no possible ambiguity. We also introduce $\mathcal{L}^+(\Omega) := \{f \in \mathcal{L}(\Omega) : f \geq 0, f \neq 0\}$, or simply \mathcal{L}^+ when no ambiguity is possible, the set of non-negative non-vanishing gambles. These gambles should always be desired, since they may increase the wealth with no risk of decreasing it. As a consequence of the linearity of our utility scale, we assume also that a subject disposed to accept the transactions represented by f and g , is disposed to accept also $\lambda f + \mu g$ with $\lambda, \mu \geq 0$ not both equal to 0. More generally speaking, we consider the notion of a coherent set of gambles [16]:

Definition 1 (Coherent set of gambles). *We say that a subset \mathcal{D} of \mathcal{L} is a coherent set of gambles if and only if \mathcal{D} satisfies the following properties:*

- D1. $\mathcal{L}^+ \subseteq \mathcal{D}$ [Accepting Partial Gains];
- D2. $0 \notin \mathcal{D}$ [Avoiding Status Quo];
- D3. $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$ [Additivity];
- D4. $f \in \mathcal{D}, \lambda > 0 \Rightarrow \lambda f \in \mathcal{D}$ [Positive Homogeneity].

So, \mathcal{D} is a convex cone. Let us denote with $C(\Omega)$, or simply with C , the family of coherent sets of gambles on Ω . This leads to the concept of natural extension.

Definition 2 (Natural extension for gambles). *Given a set $\mathcal{K} \subseteq \mathcal{L}$, we call $\mathcal{E}(\mathcal{K}) := \text{posi}(\mathcal{K} \cup \mathcal{L}^+)$, where $\text{posi}(\mathcal{K}') := \left\{ \sum_{j=1}^r \lambda_j f_j : f_j \in \mathcal{K}', \lambda_j > 0, r \geq 1 \right\}$, for every set $\mathcal{K}' \subseteq \mathcal{L}$, its natural extension.*

The natural extension $\mathcal{E}(\mathcal{D})$ of a set of gambles \mathcal{D} is coherent if and only if $0 \notin \mathcal{E}(\mathcal{D})$.

In [17] we showed that $\Phi(\Omega) := C(\Omega) \cup \{\mathcal{L}(\Omega)\}$, or simply Φ when there is no possible ambiguity, is a complete lattice under inclusion [1], meet is intersection and join is defined for any family of sets $\mathcal{D}_i \in \Phi$ as

$$\bigvee_{i \in I} \mathcal{D}_i := \bigcap \left\{ \mathcal{D} \in \Phi : \bigcup_{i \in I} \mathcal{D}_i \subseteq \mathcal{D} \right\}.$$

Note that, if the family of coherent sets \mathcal{D}_i has no upper bound in C , then its join is simply \mathcal{L} . Moreover, we defined the following closure operator [1] on subsets of gambles.

$$\mathcal{C}(\mathcal{D}') := \bigcap \{ \mathcal{D} \in \Phi : \mathcal{D}' \subseteq \mathcal{D} \}. \tag{1}$$

It is possible to notice that $\mathcal{C}(\mathcal{D}) = \mathcal{E}(\mathcal{D})$ if $0 \notin \mathcal{E}(\mathcal{D})$, that is if $\mathcal{E}(\mathcal{D})$ is coherent. Otherwise we may have $\mathcal{E}(\mathcal{D}) \neq \mathcal{L}(\Omega)$.

The most informative cases of coherent sets of gambles, i.e., coherent sets that are not proper subsets of other coherent sets, are called *maximal*. The following proposition provides a characterisation of such maximal elements [4, Proposition 2].

Proposition 1 (Maximal coherent set of gambles). *A coherent set of gambles \mathcal{D} is maximal if and only if*

$$(\forall f \in \mathcal{L} \setminus \{0\}) f \notin \mathcal{D} \Rightarrow -f \in \mathcal{D}.$$

We shall denote maximal sets with M to differentiate them from the general case of coherent sets. These sets play an important role with respect to information algebras (see Section 5). Another important class is that of *strictly desirable* sets of gambles [16].³

Definition 3 (Strictly desirable set of gambles). *A coherent set of gambles D is said to be strictly desirable if and only if it satisfies*

$$(\forall f \in D \setminus \mathcal{L}^+) (\exists \delta > 0) f - \delta \in D.$$

For strictly desirable sets, we shall employ the notation D^+ .

3 Structure of Questions and Possibilities

In this section we review the main results about the structure of Ω [7,8,17]. With reference to our previous work [17], we recall that coherent sets of gambles are understood as pieces of information describing beliefs about the elements in Ω .

³ Strictly desirable sets of gambles are important because they are in a one-to-one relation with *coherent lower previsions*; these are a generalization of the usual expectation operator on gambles. Given a coherent lower prevision $\underline{P}(\cdot)$, $D^+ := \{f \in \mathcal{L} : \underline{P}(f) > 0\} \cup \mathcal{L}^+$ is a strictly desirable set of gambles [16, Section 3.8.1].

Beliefs may be originally expressed relative to different questions or variables that we identify by families of equivalence relations \equiv_x on Ω for x in some index set Q . A question $x \in Q$ has the same answer in possible worlds $\omega \in \Omega$ and $\omega' \in \Omega$, if $\omega \equiv_x \omega'$.

There is a partial order between questions capturing granularity: question y is finer than question x if $\omega \equiv_y \omega'$ implies $\omega \equiv_x \omega'$. This can be expressed also considering partitions $\mathcal{P}_x, \mathcal{P}_y$ of Ω whose blocks are respectively, the equivalence classes $[\omega]_x, [\omega]_y$ of the equivalence relations \equiv_x, \equiv_y , representing possible answers to x and y . Then $\omega \equiv_y \omega'$ implies $\omega \equiv_x \omega'$, meaning that any block $[\omega]_y$ of partition \mathcal{P}_y is contained in some block $[\omega]_x$ of partition \mathcal{P}_x . If this is the case, we say equivalently that: $x \leq y$ or $\mathcal{P}_x \leq \mathcal{P}_y$.⁴ Partitions $Part(\Omega)$ of any set Ω , form a lattice under this order [5]. In particular, the partition $\sup\{\mathcal{P}_x, \mathcal{P}_y\} := \mathcal{P}_x \vee \mathcal{P}_y$ of two partitions $\mathcal{P}_x, \mathcal{P}_y$ is, in this order, the partition obtained as the non-empty intersections of blocks of \mathcal{P}_x with blocks of \mathcal{P}_y . It can be equivalently expressed also as $\mathcal{P}_{x \vee y}$. Definition of meet $\mathcal{P}_x \wedge \mathcal{P}_y$, or equivalently $\mathcal{P}_{x \wedge y}$, is somewhat involved [5]. We usually assume that the set of questions Q analyzed, considered together with their associated partitions denoted with $\mathcal{Q} := \{\mathcal{P}_x : x \in Q\}$, is a join-sub-semilattice of $(Part(\Omega), \leq)$ [1]. In particular, we assume often that the top partition in $Part(\Omega)$, i.e. \mathcal{P}_\top (where the blocks are singleton sets $\{\omega\}$ for $\omega \in \Omega$), belongs to \mathcal{Q} . A gamble f on Ω is called *x-measurable*, iff for all $\omega \equiv_x \omega'$ we have $f(\omega) = f(\omega')$, that is, if f is constant on every block of \mathcal{P}_x . It could then also be considered as a function (a gamble) on the set of blocks of \mathcal{P}_x . We denote with $\mathcal{L}_x(\Omega)$, or more simply with \mathcal{L}_x when no ambiguity is possible, the set of all *x-measurable* gambles.

We recall also the logical independence and conditional logical independence relation between partitions [7,8].

Definition 4 (Independent Partitions). For a finite set of partitions $\mathcal{P}_1, \dots, \mathcal{P}_n \in Part(\Omega)$, $n \geq 2$, let us define

$$R(\mathcal{P}_1, \dots, \mathcal{P}_n) := \{(B_1, \dots, B_n) : B_i \in \mathcal{P}_i, \cap_{i=1}^n B_i \neq \emptyset\}.$$

We call the partitions independent, if $R(\mathcal{P}_1, \dots, \mathcal{P}_n) = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$.

Definition 5 (Conditionally Independent Partitions). Consider a finite set of partitions $\mathcal{P}_1, \dots, \mathcal{P}_n \in Part(\Omega)$, and a block B of a partition \mathcal{P} (contained or not in the list $\mathcal{P}_1, \dots, \mathcal{P}_n$), then define for $n \geq 1$,

$$R_B(\mathcal{P}_1, \dots, \mathcal{P}_n) := \{(B_1, \dots, B_n) : B_i \in \mathcal{P}_i, \cap_{i=1}^n B_i \cap B \neq \emptyset\}.$$

We call $\mathcal{P}_1, \dots, \mathcal{P}_n$ conditionally independent given \mathcal{P} , iff for all blocks B of \mathcal{P} , $R_B(\mathcal{P}_1, \dots, \mathcal{P}_n) = R_B(\mathcal{P}_1) \times \dots \times R_B(\mathcal{P}_n)$.

This relation holds if and only if $B_i \cap B \neq \emptyset$ for all $i = 1, \dots, n$, implies that $B_1 \cap \dots \cap B_n \cap B \neq \emptyset$. In this case we write $\perp\{\mathcal{P}_1, \dots, \mathcal{P}_n\}|\mathcal{P}$ or, for $n = 2$,

⁴ In the literature usually the inverse order between partitions is considered. However, this order better corresponds to our natural order of questions by granularity.

$\mathcal{P}_1 \perp \mathcal{P}_2 | \mathcal{P}$. $\mathcal{P}_x \perp \mathcal{P}_y | \mathcal{P}_z$ can be indicated also with $x \perp y | z$. We may also say that $\mathcal{P}_1 \perp \mathcal{P}_2 | \mathcal{P}$ if and only if $\omega \equiv_{\mathcal{P}} \omega'$ implies the existence of an element $\omega'' \in \Omega$ such that $\omega \equiv_{\mathcal{P}_1 \vee \mathcal{P}} \omega''$ and $\omega' \equiv_{\mathcal{P}_2 \vee \mathcal{P}} \omega''$. The three-place relation $\mathcal{P}_1 \perp \mathcal{P}_2 | \mathcal{P}$ is, in particular, a *quasi-separoid* [7], a retract of the concept of separoid [1].

Theorem 1. *Given $\mathcal{P}, \mathcal{P}', \mathcal{P}_1, \mathcal{P}_2 \in \text{Part}(\Omega)$, we have:*

- C1** $\mathcal{P}_1 \perp \mathcal{P}_2 | \mathcal{P}_2$;
- C2** $\mathcal{P}_1 \perp \mathcal{P}_2 | \mathcal{P}$ implies $\mathcal{P}_2 \perp \mathcal{P}_1 | \mathcal{P}$;
- C3** $\mathcal{P}_1 \perp \mathcal{P}_2 | \mathcal{P}$ and $\mathcal{P}' \leq \mathcal{P}_2$ imply $\mathcal{P}_1 \perp \mathcal{P}' | \mathcal{P}$;
- C4** $\mathcal{P}_1 \perp \mathcal{P}_2 | \mathcal{P}$ implies $\overline{\mathcal{P}_1} \perp \overline{\mathcal{P}_2} \vee \mathcal{P} | \mathcal{P}$.

From these properties, it follows that $\mathcal{P}_x \perp \mathcal{P}_y | \mathcal{P}_z \iff \mathcal{P}_{x \vee z} \perp \mathcal{P}_{y \vee z} | \mathcal{P}_z$, which we use very often later on.

4 Information Algebra of Coherent Sets of Gambles

In [17] we showed that Φ with the following operations:

1. Combination: $\mathcal{D}_1 \cdot \mathcal{D}_2 := \mathcal{C}(\mathcal{D}_1 \cup \mathcal{D}_2)$,
2. Extraction: $\epsilon_x(\mathcal{D}) := \mathcal{C}(\mathcal{D} \cap \mathcal{L}_x)$ for $x \in Q$,

is a *domain-free information algebra* that we call the *domain-free information algebra of coherent sets of gambles*. Combination captures aggregation of pieces of belief, and extraction describes filtering the part of information relative to a question $x \in Q$. Information algebras are particular *valuation algebras* as defined by [14] but with idempotent combination. Domain-free versions of valuation algebras have been proposed by Shafer [18]. Idempotency of combination has important consequences, such as the possibility to define an information order, atoms, approximation, and more [6,7]. It also offers—the subject of the present paper—important connections to set algebras.

Here we remind the characterizing properties of the domain-free information algebra Φ together with a system of questions Q and a family E of extraction operators $\epsilon_x : \Phi \rightarrow \Phi$ for $x \in Q$:

1. *Semigroup*: (Φ, \cdot) is a commutative semigroup with a null element $0 = \mathcal{L}(\Omega)$ and a unit $1 = \mathcal{L}^+(\Omega)$.
2. *Quasi-Separoid*: (Q, \leq) is a join semilattice and $x \perp y | z$ with $x, y, z \in Q$, a quasi-separoid.
3. *Existential Quantifier*: For any $x \in Q$, $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D} \in \Phi$:
 - (a) $\epsilon_x(0) = 0$,
 - (b) $\epsilon_x(\mathcal{D}) \cdot \mathcal{D} = \mathcal{D}$,
 - (c) $\epsilon_x(\epsilon_x(\mathcal{D}_1) \cdot \mathcal{D}_2) = \epsilon_x(\mathcal{D}_1) \cdot \epsilon_x(\mathcal{D}_2)$.
4. *Extraction*: For any $x, y, z \in Q$, $\mathcal{D} \in \Phi$, such that $x \vee z \perp y \vee z | z$ and $\epsilon_x(\mathcal{D}) = \mathcal{D}$, we have:

$$\epsilon_{y \vee z}(\mathcal{D}) = \epsilon_{y \vee z}(\epsilon_z(\mathcal{D})).$$

5. *Support*: For any $\mathcal{D} \in \Phi$ there is an $x \in Q$ so that $\epsilon_x(\mathcal{D}) = \mathcal{D}$, i.e. a *support* of \mathcal{D} [17], and for all $y \geq x$, $y \in Q$, $\epsilon_y(\mathcal{D}) = \mathcal{D}$.

When we need to specify all the constructing elements of the domain-free information algebra Φ , we can refer to it with the tuple $(\Phi, \mathcal{Q}, \leq, \perp, \cdot, \cdot, E)$, where E is the family of the extraction operators constructed starting from $x \in Q$ or, equivalently, from partitions in \mathcal{Q} .⁵ When we do not need this degree of accuracy, we can refer to it simply as Φ . Analogous considerations can be made for other domain-free information algebras. Notice that, in particular, (Φ, \cdot) is an idempotent, commutative semigroup. So, a partial order is defined by $\mathcal{D}_1 \leq \mathcal{D}_2$ if $\mathcal{D}_1 \cdot \mathcal{D}_2 = \mathcal{D}_2$. Then $\mathcal{D}_1 \leq \mathcal{D}_2$ if and only if $\mathcal{D}_1 \subseteq \mathcal{D}_2$. This order is called an *information order* [17]. This definition entails the following facts: $\epsilon_x(\mathcal{D}) \leq \mathcal{D}$ for every $\mathcal{D} \in \Phi$, $x \in Q$; given $\mathcal{D}_1, \mathcal{D}_2 \in \Phi$, if $\mathcal{D}_1 \leq \mathcal{D}_2$, then $\epsilon_x(\mathcal{D}_1) \leq \epsilon_x(\mathcal{D}_2)$ for every $x \in Q$ [6].

5 Atoms and Maximal Coherent Sets of Gambles

Maximal coherent sets M are *atoms* in the information algebra of coherent sets of gambles [17]. This is a well-known concept in (domain-free) information algebras. We remind the following elementary properties of atoms [6], immediately derivable from the definition. If M, M_1 and M_2 are atoms of Φ and $\mathcal{D} \in \Phi$, then

1. $M \cdot \mathcal{D} = M$ or $M \cdot \mathcal{D} = 0$,
2. either $\mathcal{D} \leq M$ or $M \cdot \mathcal{D} = 0$,
3. either $M_1 = M_2$ or $M_1 \cdot M_2 = 0$.

We indicate with $At(\Phi)$ the set of all atoms of Φ , and with $At(\mathcal{D})$ the set of all atoms M which dominate $\mathcal{D} \in \Phi$, that is, $At(\mathcal{D}) := \{M \in At(\Phi) : \mathcal{D} \subseteq M\}$. Furthermore, Φ is *atomic* [6], i.e. for any $\mathcal{D} \neq 0$ the set $At(\mathcal{D})$ is not empty, and *atomistic*, i.e. for any $\mathcal{D} \neq 0$, $\mathcal{D} = \bigcap At(\mathcal{D})$. It is a general result of atomistic information algebras that the subalgebras $\epsilon_x(\Phi)$ are also atomistic [11]. Moreover, in [17], we showed that $At(\epsilon_x(\Phi)) = \epsilon_x(At(\Phi)) = \{\epsilon_x(M) : M \in At(\Phi)\}$ for any $x \in Q$ and, therefore, we call $\epsilon_x(M)$ for $M \in At(\Phi)$ and $x \in Q$ *local atoms* for x . Local atoms $M_x = \epsilon_x(M)$ for x induce a partition At_x of $At(\Phi)$ with blocks $At(M_x)$. If M and M' belong to the same block, we say that $M \equiv_x M'$.

Let us indicate with $Part(At(\Phi))$ the set of these partitions. As for $Part(\Omega)$, we can introduce a partial order on $Part(At(\Phi))$ defined as: $At_x \leq At_y$ if $M \equiv_y M'$ implies $M \equiv_x M'$, for every $M, M' \in At(\Phi)$. $Part(At(\Phi))$ forms a lattice under this order where, in particular, $At_x \vee At_y$ is the partition obtained as the non-empty intersections of blocks of At_x with blocks of At_y [5]. We claim moreover that these partitions of $At(\Phi)$ mirror the partitions $\mathcal{P}_x \in \mathcal{Q}$. Before stating this main result, we need the following lemma.

Lemma 1. *Let us consider $M, M' \in At(\Phi)$ and $x \in Q$. Then*

$$M \equiv_x M' \iff \epsilon_x(M) = \epsilon_x(M') \iff M, M' \in At(\epsilon_x(M)).$$

Hence, $At_x \leq At_y$ if and only if $At(\epsilon_x(M)) \supseteq At(\epsilon_y(M))$ for every $M \in At(\Phi)$.

⁵ When we need to be explicit about partitions, we can indicate the extraction operator ϵ_x also as $\epsilon_{\mathcal{P}_x}$, where $\mathcal{P}_x \in \mathcal{Q}$ is the partition associated to the question $x \in Q$.

Proof. If $M \equiv_x M'$, there exists a local atom M_x such that $M, M' \in At(M_x)$. Therefore, $M, M' \geq M_x$ and $\epsilon_x(M), \epsilon_x(M') \geq M_x$ [17, Lemma 15, item 3]. However, $\epsilon_x(M), \epsilon_x(M')$ and M_x are all local atoms, hence $\epsilon_x(M) = \epsilon_x(M') = M_x$. The converse is obvious.

For the second part, let us suppose $At(\epsilon_y(M)) \subseteq At(\epsilon_x(M))$ for every $M \in At(\Phi)$, and consider $M', M'' \in At(\Phi)$ such that $M' \equiv_y M''$. Then $M', M'' \in At(\epsilon_y(M'))$ and hence $M', M'' \in At(\epsilon_x(M'))$, which implies $M' \equiv_x M''$. Vice versa, consider $At_x \leq At_y$ and $M' \in At(\epsilon_y(M))$ for some $M, M' \in At(\Phi)$. Then, $M \equiv_y M'$, hence $M \equiv_x M'$ and so $M' \in At(\epsilon_x(M))$.

Now we can state the main result of this section.

Theorem 2. *The map $\mathcal{P}_x \mapsto At_x$, from the lattice of partitions $(Part(\Omega), \leq)$ of Ω to the lattice partitions $(Part(At(\Phi)), \leq)$ of $At(\Phi)$, preserves order and join. Furthermore it preserves also conditional independence relations, that is, $\mathcal{P}_x \perp \mathcal{P}_y | \mathcal{P}_z$ implies $At_x \perp At_y | At_z$.*

Proof. If $x \leq y$, then $\epsilon_x(M) \leq \epsilon_y(M)$ for any atom $M \in At(\Phi)$ [17, Lemma 15, item 4]. Therefore, $At(\epsilon_x(M)) \supseteq At(\epsilon_y(M))$ for any $M \in At(\Phi)$, hence $At_x \leq At_y$. The converse is also true: indeed, if $At_x \leq At_y$, then $At(\epsilon_x(M)) \supseteq At(\epsilon_y(M))$ for any $M \in At(\Phi)$. This implies in particular that $\epsilon_x(M) = \cap At(\epsilon_x(M)) \subseteq \cap At(\epsilon_y(M)) = \epsilon_y(M)$ for any $M \in At(\Phi)$, thanks to the fact that $At(\Phi)$ is atomistic. Now, for any D coherent, consider the family $\{M_j\}_{j \in J} := At(D)$. Then we have:

$$\epsilon_x(D) = \epsilon_x(\cap_{j \in J} M_j) = \cap_{j \in J} \epsilon_x(M_j) \subseteq \cap_{j \in J} \epsilon_y(M_j) = \epsilon_y(\cap_{j \in J} M_j) = \epsilon_y(D),$$

thanks to [17, Theorem 17]. Therefore we have $\epsilon_x(D) \subseteq \epsilon_y(D)$ also for any $D \in \mathcal{C}$. Applying it to $\mathcal{D} := \mathcal{E}(\{f\})$ for every $f \in \mathcal{L}_x \setminus (\mathcal{L}_x^+ \cup \{f \in \mathcal{L}_x : f \leq 0\})$, we obtain that $\mathcal{L}_x \subseteq \mathcal{L}_y$, from which it follows that $x \leq y$ [17, Section 3]. So the map $\mathcal{P}_x \mapsto At_x$ is an order isomorphism [1, Def. 1.34], therefore it also preserves joins [1, Prop. 2.19].

For the second part, recall that $x \perp y | z$ if and only if $x \vee z \perp y \vee z | z$. Consider then local atoms $M_{x \vee z}, M_{y \vee z}$ and M_z so that

$$At(M_{x \vee z}) \cap At(M_z) \neq \emptyset, \quad At(M_{y \vee z}) \cap At(M_z) \neq \emptyset.$$

Hence, there is an atom $M' \in At(M_{x \vee z}) \cap At(M_z)$ and an atom $M'' \in At(M_{y \vee z}) \cap At(M_z)$. Therefore, $M_{x \vee z} = \epsilon_{x \vee z}(M')$, $M_{y \vee z} = \epsilon_{y \vee z}(M'')$ and $M_z = \epsilon_z(M') = \epsilon_z(M'')$. Now, thanks to the Existential Quantifier axiom, we have:

$$\epsilon_z(M_{x \vee z} \cdot M_{y \vee z} \cdot M_z) = \epsilon_z(M_{x \vee z} \cdot M_{y \vee z}) \cdot M_z.$$

Thanks to [17, Theorem 16] and [17, Lemma 15, item 6],⁶ we obtain

$$\epsilon_z(M_{x \vee z} \cdot M_{y \vee z}) \cdot M_z = \epsilon_z(\epsilon_{x \vee z}(M')) \cdot \epsilon_z(\epsilon_{y \vee z}(M'')) \cdot M_z = \epsilon_z(M') \cdot \epsilon_z(M'') \cdot M_z \neq 0.$$

⁶ [17, Theorem 16, item 2] indeed, can be rewritten also as follows: let $\mathcal{D}_1, \mathcal{D}_2 \in \Phi$ and $x, y, z \in Q$, if \mathcal{D}_1 has support $x \vee z$, \mathcal{D}_2 has support $y \vee z$ and $x \perp y | z$, then $\epsilon_z(\mathcal{D}_1 \cdot \mathcal{D}_2) = \epsilon_z(\mathcal{D}_1) \cdot \epsilon_z(\mathcal{D}_2)$.

Therefore $M_{x \vee z} \cdot M_{y \vee z} \cdot M_z \neq 0$ [17, Lemma 15, item 2] and hence, since the algebra is atomic, there is an atom $M''' \in \text{At}(M_{x \vee z} \cdot M_{y \vee z} \cdot M_z)$. Then $M_{x \vee z}, M_{y \vee z}, M_z \leq M'''$, whence $M''' \in \text{At}(M_{x \vee z}) \cap \text{At}(M_{y \vee z}) \cap \text{At}(M_z)$ and so $\text{At}_x \perp \text{At}_y | \text{At}_z$.

6 Information Algebras Homomorphisms

We are interested in homomorphisms between algebras:

Definition 6 (Domain-free information algebras homomorphism). *Let $(\Psi, \mathcal{Q}, \leq_\Psi, \perp_\Psi, \cdot_\Psi, E)$ and $(\Psi', \mathcal{Q}', \leq_{\Psi'}, \perp_{\Psi'}, \cdot_{\Psi'}, E')$ be two domain-free information algebras, where $E := \{\epsilon_{\mathcal{P}}, \mathcal{P} \in \mathcal{Q}\}$ and $E' := \{\epsilon'_{\mathcal{P}'}, \mathcal{P}' \in \mathcal{Q}'\}$ are respectively the families of the extraction operators of the two algebras. A tuple (f, h, g) of maps $f : \Psi \rightarrow \Psi'$, $h : \mathcal{Q} \rightarrow \mathcal{Q}'$ and $g : E \mapsto E'$ defined as $g : \epsilon_{\mathcal{P}} \rightarrow \epsilon'_{h(\mathcal{P})}$, is an homomorphism between $(\Psi, \mathcal{Q}, \leq_\Psi, \perp_\Psi, \cdot_\Psi, E)$ and $(\Psi', \mathcal{Q}', \leq_{\Psi'}, \perp_{\Psi'}, \cdot_{\Psi'}, E')$ if and only if:*

1. $f(\psi \cdot_\Psi \phi) = f(\psi) \cdot_{\Psi'} f(\phi)$, for every $\phi, \psi \in \Psi$;
2. $f(0_\Psi) = 0_{\Psi'}$ and $f(1_\Psi) = 1_{\Psi'}$, if we indicate with $0_\Psi, 1_\Psi$ and $0_{\Psi'}, 1_{\Psi'}$ respectively, the 0 and the 1 elements of Ψ and Ψ' ;
3. if $\mathcal{P}_1 \leq_\Psi \mathcal{P}_2$ then $h(\mathcal{P}_1) \leq_{\Psi'} h(\mathcal{P}_2)$, for every $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{Q}$;
4. $h(\mathcal{P}_1 \vee_\Psi \mathcal{P}_2) = h(\mathcal{P}_1) \vee_{\Psi'} h(\mathcal{P}_2)$ for every $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{Q}$, if we indicate with $\mathcal{P}_1 \vee_\Psi \mathcal{P}_2$, the join of $\mathcal{P}_1, \mathcal{P}_2$ with respect to \leq_Ψ and with $h(\mathcal{P}_1) \vee_{\Psi'} h(\mathcal{P}_2)$, the join of $h(\mathcal{P}_1), h(\mathcal{P}_2)$ with respect to $\leq_{\Psi'}$;
5. $\mathcal{P}_1 \perp_\Psi \mathcal{P}_2 | \mathcal{P}$ implies $h(\mathcal{P}_1) \perp_{\Psi'} h(\mathcal{P}_2) | h(\mathcal{P})$ for every $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P} \in \mathcal{Q}$;
6. $f(\epsilon_{\mathcal{P}}(\psi)) = g(\epsilon_{\mathcal{P}})(f(\psi))$, for all $\psi \in \Psi$ and $\epsilon_{\mathcal{P}} \in E$ with $\mathcal{P} \in \mathcal{Q}$.

If the maps are one-to-one, then $(\Psi, \mathcal{Q}, \leq_\Psi, \perp_\Psi, \cdot_\Psi, E)$ is said to be *embedded* into $(\Psi', \mathcal{Q}', \leq_{\Psi'}, \perp_{\Psi'}, \cdot_{\Psi'}, E')$. If they are also bijective, the homomorphism is said to be an *isomorphism* between the two algebras.

This definition is an extension of the information algebras homomorphism given in [7],⁷ for domain-free information algebras for which \mathcal{Q} is potentially different from \mathcal{Q}' . If $\mathcal{Q} = \mathcal{Q}'$, or equivalently $Q = Q'$, it collapses to the simpler definition in [7].

7 Set Algebras

Archetypes of information algebras are so-called set algebras, where the elements are subsets of some universe, combination is intersection, and extraction is related to so-called saturation operators. Starting with the set Ω of possibilities, representing possible worlds, pieces of information may be given by subsets S of Ω , meaning that the unknown world must be an element of S . As before, questions $x \in Q$ are modeled by partitions \mathcal{P}_x or equivalence relation \equiv_x , where

⁷ In [7] the set of questions Q is used in place of the set of partitions \mathcal{Q} . Here we need to be more explicit about partitions.

$\omega \equiv_x \omega'$ means that question x has the same answer in possible worlds ω and ω' . We first specify the set algebra of subsets of Ω and show then that this algebra may be embedded into the information algebra of coherent sets. Conversely, we show that the algebra \mathcal{F} of coherent sets of gambles may itself be embedded into a set algebra of its atoms, so is, in some precise sense, itself a set algebra. This is a general result for atomistic information algebras [6,11].

To any partition \mathcal{P}_x of Ω there corresponds a saturation operator defined for any subset $S \subseteq \Omega$ by

$$\sigma_x(S) := \{\omega \in \Omega : (\exists \omega' \in S) \omega \equiv_x \omega'\}. \quad (2)$$

The following are well-known properties of saturation operators.

Lemma 2. *For all $S, T \subseteq \Omega$ and any partition \mathcal{P}_x of Ω :*

1. $\sigma_x(\emptyset) = \emptyset$,
2. $S \subseteq \sigma_x(S)$,
3. $\sigma_x(\sigma_x(S) \cap T) = \sigma_x(S) \cap \sigma_x(T)$,
4. $\sigma_x(\sigma_x(S)) = \sigma_x(S)$,
5. $S \subseteq T \Rightarrow \sigma_x(S) \subseteq \sigma_x(T)$,
6. $\sigma_x(\sigma_x(S) \cap \sigma_x(T)) = \sigma_x(S) \cap \sigma_x(T)$.

Proof. Items 1, 2, 4, 5 are obvious. For item 6, consider $\omega \in \sigma_x(\sigma_x(S) \cap \sigma_x(T))$. Then there is a $\omega' \in \sigma_x(S) \cap \sigma_x(T)$ so that $\omega \equiv_x \omega'$. In particular, $\omega' \in \sigma_x(S)$, hence $\omega \in \sigma_x(\sigma_x(S)) = \sigma_x(S)$ by item 4. At the same time, $\omega' \in \sigma_x(T)$, hence $\omega \in \sigma_x(\sigma_x(T)) = \sigma_x(T)$. Then $\omega \in \sigma_x(S) \cap \sigma_x(T)$. By item 2 we must then have equality. Regarding item 3, $\sigma_x(\sigma_x(S) \cap T) \subseteq \sigma_x(S) \cap \sigma_x(T)$ by item 2, 5 and 6. So consider an element $\omega \in \sigma_x(S) \cap \sigma_x(T)$. Then, there exist $\omega' \in S$ and $\omega'' \in T$ such that $\omega \equiv_x \omega'$ and $\omega \equiv_x \omega''$. By transitivity it follows that $\omega'' \equiv_x \omega'$ so that $\omega'' \in \sigma_x(S)$. But then $\omega \equiv_x \omega'' \in \sigma_x(S) \cap T$ implies $\omega \in \sigma_x(\sigma_x(S) \cap T)$ and this proves item 3.

Note that the first three items of this theorem imply that σ_x is an existential quantifier relative to intersection as combination. This is a first step to construct a domain-free information algebra of subsets of Ω . Then we limit possible questions to the same join semilattice (Q, \leq) considered in the previous sections. Moreover, we consider on it the quasi-separoid three-place relation: $x \perp y|z$ with $x, y, z \in Q$, defined before.

Now, we want the support axiom to be satisfied. Hence, if \mathcal{P}_\top belongs to Q , then we have $\sigma_\top(S) = S$ for all $S \subseteq \Omega$. Otherwise, we must limit ourselves to the subsets of Ω for which there is a support $x \in Q$. We call these sets *saturated* with respect to some $x \in Q$, and we indicate them with $P_Q(\Omega)$ or more simply with P_Q when no ambiguity is possible. Clearly, if the top partition belongs to Q , $P_Q(\Omega) = P(\Omega)$, the power set of Ω . So in what follows we can refer more generally to sets in $P_Q(\Omega)$. Note that in particular $\Omega, \emptyset \in P_Q(\Omega)$ for every join semilattice (Q, \leq) . At this point the support axiom is satisfied. Indeed, if $x \leq y$ with $x, y \in Q$, then $\omega \equiv_y \omega'$ implies $\omega \equiv_x \omega'$, so that $\sigma_y(S) \subseteq \sigma_x(S)$. Then, if x

is a support of S , we have $S \subseteq \sigma_y(S) \subseteq \sigma_x(S) = S$, hence $\sigma_y(S) = S$. Moreover $(P_Q(\Omega), \cap)$ is a commutative semigroup with the empty set as the null element and Ω as the unit. Indeed, the only property we need to prove, is that $P_Q(\Omega)$ is closed under intersection. Then, let us consider S and T , two subsets of Ω with support $x \in Q$ and $y \in Q$ respectively. Then they have also both supports $x \vee y$ that belongs to Q , because it is a join semilattice. Therefore, thanks to Lemma 2, we have

$$\sigma_{x \vee y}(S \cap T) = \sigma_{x \vee y}(\sigma_{x \vee y}(S) \cap \sigma_{x \vee y}(T)) = \sigma_{x \vee y}(S) \cap \sigma_{x \vee y}(T) = S \cap T.$$

So, $P_Q(\Omega)$ is closed under intersection. It remains only to verify the extraction property to conclude that $P_Q(\Omega)$ forms a domain-free information algebra.

Theorem 3. *Given $x, y, z \in Q$, suppose $x \vee z \perp y \vee z | z$. Then, for any $S \in P_Q(\Omega)$,*

$$\sigma_{y \vee z}(\sigma_x(S)) = \sigma_{y \vee z}(\sigma_z(\sigma_x(S))).$$

Proof. From $\sigma_z(\sigma_x(S)) \supseteq \sigma_x(S)$ we obtain $\sigma_{y \vee z}(\sigma_z(\sigma_x(S))) \supseteq \sigma_{y \vee z}(\sigma_x(S))$. Consider therefore an element $\omega \in \sigma_{y \vee z}(\sigma_z(\sigma_x(S)))$. Then there are elements μ, μ' and ω' so that $\omega \equiv_{y \vee z} \mu \equiv_z \mu' \equiv_x \omega'$ and $\omega' \in S$. This means that ω, μ belong to some block $B_{y \vee z}$ of partition $\mathcal{P}_{y \vee z}$, μ, μ' to some block B_z of partition \mathcal{P}_z and μ', ω' to some block B_x of partition \mathcal{P}_x . It follows that $B_x \cap B_z \neq \emptyset$ and $B_{y \vee z} \cap B_z \neq \emptyset$. Then $x \vee z \perp y \vee z | z$ implies, thanks to properties of a separoid, that $x \perp y \vee z | z$. Therefore, we have $B_x \cap B_{y \vee z} \cap B_z \neq \emptyset$, and in particular, $B_x \cap B_{y \vee z} \neq \emptyset$. So there is a $\lambda \in B_x \cap B_{y \vee z}$ such that $\omega \equiv_{y \vee z} \lambda \equiv_x \omega' \in S$, hence $\omega \in \sigma_{y \vee z}(\sigma_x(S))$. So we have $\sigma_{y \vee z}(\sigma_x(S)) = \sigma_{y \vee z}(\sigma_z(\sigma_x(S)))$.

Hence, these algebras of sets, with intersection as combination and saturation as extraction, form domain-free information algebras. Such algebras will be called *set algebras*. A set algebra of subsets of Ω can be embedded in the information algebra of coherent sets of gambles defined on Ω . For any set $S \in P_Q(\Omega)$, define

$$\mathcal{D}_S := \{f \in \mathcal{L}(\Omega) : \inf_{\omega \in S} f(\omega) > 0\} \cup \mathcal{L}^+(\Omega).$$

If $S \neq \emptyset$, this is clearly a coherent set. The next theorem shows that the map $S \mapsto \mathcal{D}_S$ together with the map $\sigma_x \mapsto \epsilon_x$ is an information algebra homomorphism, according to the simpler definition given in [7]. It can be applied in fact, because in this case the set of partitions/questions analyzed by the two information algebras is the same.

Theorem 4. *Let $S, T \in P_Q(\Omega)$ and $x \in Q$. Then*

1. $\mathcal{D}_S \cdot \mathcal{D}_T = \mathcal{D}_{S \cap T}$,
2. $\mathcal{D}_\emptyset = \mathcal{L}(\Omega)$, $\mathcal{D}_\Omega = \mathcal{L}^+(\Omega)$,
3. $\epsilon_x(\mathcal{D}_S) = \mathcal{D}_{\sigma_x(S)}$.

Proof. 1. Note that $D_S = \mathcal{L}^+$ or $D_T = \mathcal{L}^+$ if and only if $S = \Omega$ or $T = \Omega$. Clearly in this case we have immediately the result. The same is true if $D_S = \mathcal{L}$ or

$D_T = \mathcal{L}$, which is equivalent to have $S = \emptyset$ or $T = \emptyset$. Now suppose $D_S, D_T \neq \mathcal{L}^+$ and $D_S, D_T \neq \mathcal{L}$. If $S \cap T = \emptyset$, then $\mathcal{D}_{S \cap T} = \mathcal{L}(\Omega)$. Consider $f \in \mathcal{D}_S$ and $g \in \mathcal{D}_T$. Since S and T are disjoint, we have $f \in \mathcal{D}_S$ and $\tilde{g} \in \mathcal{D}_T$, where \tilde{f}, \tilde{g} are defined in the following way:

$$\tilde{f}(\omega) := \begin{cases} f(\omega) & \text{for } \omega \in S, \\ -g(\omega) & \text{for } \omega \in T, \\ 0 & \text{for } \omega \in (S \cup T)^c, \end{cases} \quad \tilde{g}(\omega) := \begin{cases} -f(\omega) & \text{for } \omega \in S, \\ g(\omega) & \text{for } \omega \in T, \\ 0 & \text{for } \omega \in (S \cup T)^c. \end{cases}$$

However, $\tilde{f} + \tilde{g} = 0 \in \mathcal{E}(\mathcal{D}_S \cup \mathcal{D}_T)$, hence $\mathcal{D}_S \cdot \mathcal{D}_T = \mathcal{L}(\Omega) = \mathcal{D}_{S \cap T}$. Assume then that $S \cap T \neq \emptyset$. Note that $\mathcal{D}_S \cup \mathcal{D}_T \subseteq \mathcal{D}_{S \cap T}$ so that $\mathcal{D}_S \cdot \mathcal{D}_T$ is coherent and $\mathcal{D}_S \cdot \mathcal{D}_T \subseteq \mathcal{D}_{S \cap T}$. Consider then a gamble $f \in \mathcal{D}_{S \cap T}$. Select a $\delta > 0$ and define two functions

$$f_1(\omega) := \begin{cases} 1/2f(\omega) & \text{for } \omega \in (S \cap T), \\ \delta & \text{for } \omega \in S \setminus T, \\ f(\omega) - \delta & \text{for } \omega \in T \setminus S, \\ 1/2f(\omega) & \text{for } \omega \in (S \cup T)^c, \end{cases} \quad f_2(\omega) := \begin{cases} 1/2f(\omega) & \text{for } \omega \in (S \cap T), \\ f(\omega) - \delta & \text{for } \omega \in S \setminus T, \\ \delta & \text{for } \omega \in T \setminus S, \\ 1/2f(\omega) & \text{for } \omega \in (S \cup T)^c. \end{cases}$$

Then $f = f_1 + f_2$ and $f_1 \in \mathcal{D}_S, f_2 \in \mathcal{D}_T$. Therefore $f \in \mathcal{E}(\mathcal{D}_S \cup \mathcal{D}_T) = \mathcal{C}(\mathcal{D}_S \cup \mathcal{D}_T) =: \mathcal{D}_S \cdot \mathcal{D}_T$, hence $\mathcal{D}_S \cdot \mathcal{D}_T = \mathcal{D}_{S \cap T}$.

2. Both have been noted above.

3. First of all it can be noticed that, if $S \in P_Q(\Omega)$, then $\sigma_x(S) \in P_Q(\Omega)$. So $\mathcal{D}_{\sigma_x(S)}$ is well defined. Furthermore, if S is empty, then $\epsilon_x(\mathcal{D}_\emptyset) = \mathcal{L}(\Omega)$ so that item 3 holds in this case. Hence, assume $S \neq \emptyset$. Then we have

$$\epsilon_x(\mathcal{D}_S) := \mathcal{C}(\mathcal{D}_S \cap \mathcal{L}_x) = \text{posi}(\mathcal{L}^+(\Omega) \cup (\mathcal{D}_S \cap \mathcal{L}_x)).$$

Consider a gamble $f \in \mathcal{D}_S \cap \mathcal{L}_x$. We have $\inf_S f > 0$ and f is x -measurable. If $\omega \equiv_x \omega'$ for some $\omega' \in S$ and $\omega \in \Omega$, then $f(\omega) = f(\omega')$. Therefore $\inf_{\sigma_x(S)} f = \inf_S f > 0$, hence $f \in \mathcal{D}_{\sigma_x(S)}$. Then $\mathcal{C}(\mathcal{D}_S \cap \mathcal{L}_x) \subseteq \mathcal{C}(\mathcal{D}_{\sigma_x(S)}) = \mathcal{D}_{\sigma_x(S)}$. Conversely, consider a gamble $f \in \mathcal{D}_{\sigma_x(S)}$. $\mathcal{D}_{\sigma_x(S)}$ is a strictly desirable set of gambles.⁸ Hence, if $f \in \mathcal{D}_{\sigma_x(S)}$, $f \in \mathcal{L}^+(\Omega)$ or there is $\delta > 0$ such that $f - \delta \in \mathcal{D}_{\sigma_x(S)}$. If $f \in \mathcal{L}^+(\Omega)$, then $f \in \epsilon_x(\mathcal{D}_S)$. Otherwise, let us define for every $\omega \in \Omega$, $g(\omega) := \inf_{\omega' \equiv_x \omega} f(\omega') - \delta$. If $\omega \in S$, then $g(\omega) > 0$ since $\inf_{\sigma_x(S)} (f - \delta) > 0$. So we have $\inf_S g \geq 0$ and g is x -measurable. However, $\inf_S (g + \delta) = \inf_S g + \delta > 0$ hence $(g + \delta) \in \mathcal{D}_S \cap \mathcal{L}_x$ and $f \geq g + \delta$. Therefore $f \in \mathcal{C}(\mathcal{D}_S \cap \mathcal{L}_x)$.

Item 3. guarantees that, if $S \in P_Q(\Omega)$, then there exists an $x \in Q$ such that x is a support of D_S . Notice moreover that the two maps are one-to-one, therefore it is in particular an embedding of the set algebra $P_Q(\Omega)$ into $\Phi(\Omega)$.

Next we construct a set algebra of subsets of $At(\Phi)$. For this purpose we consider the set of partitions At_x with $x \in Q$. We denote them as $Part_Q(At(\Phi))$. Moreover, we indicate with σ_x the related saturation operators defined similarly to (2), and with $At_Q(\Phi)$ the subsets of $At(\Phi)$ saturated with respect to some $At_x \in Part_Q(At(\Phi))$. By Theorem 2 restricted to \mathcal{P}_x with $x \in Q$, it is possible to

⁸ $\underline{P}(f) := \inf_S(f)$ for every $f \in \mathcal{L}$ with $S \neq \emptyset$, is a coherent lower prevision [16].

derive that, if (Q, \leq) is a join semilattice, then $(Part_Q(At(\Phi)), \leq)$ is also a join semilattice with $At_x \perp At_y | At_z$ a quasi-separoid [7, Theorem 2.6]. So, thanks to Lemma 2 and the reasoning below, also $At_Q(\Phi)$ is a set algebra with intersection as combination and saturation relative to partitions At_x as extraction. Moreover, thanks again to Theorem 2, we know that $h : \mathcal{P}_x \mapsto At_x$, satisfies items 3, 4 and 5 of Definition 6. Therefore, we need only an analog of Theorem 4 for $f : D \mapsto At(D)$ and $g : \epsilon_x \mapsto \sigma_x$, to conclude that (f, h, g) is an information algebra homomorphism between Φ and $At_Q(\Phi)$.

Theorem 5. *For any element $\mathcal{D}_1, \mathcal{D}_2$ and \mathcal{D} of Φ and all $x \in Q$,*

1. $At(\mathcal{D}_1 \cdot \mathcal{D}_2) = At(\mathcal{D}_1) \cap At(\mathcal{D}_2)$,
2. $At(\mathcal{L}(\Omega)) = \emptyset$, $At(\mathcal{L}(\Omega)^+) = At(\Phi)$,
3. $At(\epsilon_x(\mathcal{D})) = \sigma_x(At(\mathcal{D}))$.

Proof. Item 2 is obvious. If there is a an atom $M \in At(\mathcal{D}_1 \cdot \mathcal{D}_2)$, then $M \geq \mathcal{D}_1 \cdot \mathcal{D}_2 \geq \mathcal{D}_1, \mathcal{D}_2$ and thus $M \in At(\mathcal{D}_1)$ and $M \in At(\mathcal{D}_2)$, hence $M \in At(\mathcal{D}_1) \cap At(\mathcal{D}_2)$. Conversely, if $M \in At(\mathcal{D}_1) \cap At(\mathcal{D}_2)$, then $\mathcal{D}_1, \mathcal{D}_2 \leq M$, hence $\mathcal{D}_1 \cdot \mathcal{D}_2 \leq M$ and $M \in At(\mathcal{D}_1 \cdot \mathcal{D}_2)$. This shows that $At(\mathcal{D}_1 \cdot \mathcal{D}_2) = At(\mathcal{D}_1) \cap At(\mathcal{D}_2)$.

Furthermore, if $\epsilon_x(\mathcal{D}) = 0$, then $\mathcal{D} = 0$ and $At(\mathcal{D}) = \emptyset$, hence $\sigma_x(\emptyset) = \emptyset$ and vice versa [17, Lemma 15, item 2]. Assume therefore $At(\mathcal{D}) \neq \emptyset$ and consider $M \in \sigma_x(At(\mathcal{D}))$. There is then a $M' \in At(\mathcal{D})$ so that $\epsilon_x(M) = \epsilon_x(M')$. But $\mathcal{D} \leq M'$, hence $\epsilon_x(\mathcal{D}) \leq \epsilon_x(M') = \epsilon_x(M) \leq M$. Thus $M \in At(\epsilon_x(\mathcal{D}))$. Conversely consider $M \in At(\epsilon_x(\mathcal{D}))$. We claim that $\epsilon_x(M) \cdot \mathcal{D} \neq 0$. Because otherwise $0 = \epsilon_x(\epsilon_x(M) \cdot \mathcal{D}) = \epsilon_x(M) \cdot \epsilon_x(\mathcal{D}) = \epsilon_x(M \cdot \epsilon_x(\mathcal{D}))$, which is not possible since $\epsilon_x(\mathcal{D}) \leq M$. So there is an $M' \in At(\epsilon_x(M) \cdot \mathcal{D})$ so that $\mathcal{D} \leq \epsilon_x(M) \cdot \mathcal{D} \leq M'$. We conclude that $M' \in At(\mathcal{D})$. Furthermore, $\epsilon_x(\epsilon_x(M) \cdot \mathcal{D}) = \epsilon_x(M) \cdot \epsilon_x(\mathcal{D}) \leq \epsilon_x(M')$. It follows that $\epsilon_x(M') \cdot \epsilon_x(M) \cdot \epsilon_x(\mathcal{D}) \neq 0$ and therefore $\epsilon_x(M) \cdot \epsilon_x(M') \neq 0$. But $\epsilon_x(M) \cdot \epsilon_x(M') = \epsilon_x(M \cdot \epsilon_x(M'))$ so that $M \cdot \epsilon_x(M') \neq 0$ and therefore $\epsilon_x(M') \leq M$ since M is an atom, and then $\epsilon_x(M') \leq \epsilon_x(M)$. Proceed in the same way from $\epsilon_x(M) \cdot \epsilon_x(M') = \epsilon_x(M' \cdot \epsilon_x(M))$ to obtain $\epsilon_x(M) \leq \epsilon_x(M')$. So finally $\epsilon_x(M) = \epsilon_x(M')$, which together with $M' \in At(\mathcal{D})$ tells us that $M \in \sigma_x(At(\mathcal{D}))$. This means that $At(\epsilon_x(\mathcal{D})) = \sigma_x(At(\mathcal{D}))$.

Item 3. again guarantees that if $D \in \Phi$, then $At(D) \in At_Q(\Phi)$. Moreover, since Φ is atomistic, the maps f, h, g are all one-to-one and then the homomorphism is an embedding. We can say therefore that Φ is in fact a set algebra.

8 Conclusions

This paper presents an extension of our work on information algebras related to gambles on a possibility set that is not necessarily multivariate [17]. In particular, here we analyze the relation between the domain-free version of the information algebra of coherent sets of gambles and the archetypes of information algebras, i.e., sets algebras. Specifically, we show that it is in fact a set algebra. These facts could also be expressed equivalently in the *labeled view* of information algebras, better adapted to computational purposes [6,17]. This is left for future work, along with other aspects such as the question of conditioning.

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