# Preventing Small ( $\mathbf{s}, \mathbf{t}$ )-Cuts by Protecting Edges 

## A Preprint

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#### Abstract

We introduce and study Weighted Min ( $s, t$ )-Cut Prevention, where we are given a graph $G=(V, E)$ with vertices $s$ and $t$ and an edge cost function and the aim is to choose an edge set $D$ of total cost at most $d$ such that $G$ has no $(s, t)$-edge cut of capacity at most $a$ that is disjoint from $D$. We show that Weighted Min $(s, t)$-Cut Prevention is NP-hard even on subcubcic graphs when all edges have capacity and cost one and provide a comprehensive study of the parameterized complexity of the problem. We show, for example W[1]-hardness with respect to $d$ and an FPT algorithm for $a$.


## 1 Introduction

Network interdiction is a large class of optimization problems with direct applications in operations research [8, 9, $21,22,23$ ]. In these problems one player wants to achieve a certain goal (for example finding a short path between two given vertices $s$ and $t$ ), and another player wants to modify the network to prevent this. Given the enormous importance of the maximum-flow/min-cut problem it comes as no surprise that two-player games where an attacker wants to decrease the maximum $(s, t)$-flow of a network by deleting edges have been considered [8,22]. We study an inverse problem: an attacker wants to find an $(s, t)$-cut of capacity at most $a$ and a defender wants to protect edges in order to increase the capacity of any minimum $(s, t)$-cut in $G$ to at least $a+1$. Alternatively, we may think that the defender increases the capacity of some edges to $a+1$ in such a way that the maximum $(s, t)$-flow of the resulting network exceeds the given threshold $a$. The formal problem definition reads as follows.

$$
\begin{aligned}
& \text { Weighted Min }(s, t) \text {-CUT PREVENTION (WMCP) } \\
& \text { Input: A graph } G=(V, E) \text {, two vertices } s, t \in V \text {, a cost function } c: E \rightarrow \mathbb{N} \text {, a capacity } \\
& \text { function } \omega: E \rightarrow \mathbb{N} \text {, and integers } d \text { and } a \text {. } \\
& \text { Question: Is there a set } D \subseteq E \text { with } c(D):=\sum_{e \in D} c(e) \leq d \text { such that for every }(s, t) \text {-cut } A \subseteq \\
& (E \backslash D) \text { in } G \text { we have } \omega(A):=\sum_{e \in A} \omega(e)>a \text { ? }
\end{aligned}
$$

The special case where we have only unit capacities and unit costs is referred to as Min $(s, t)$-Cut Prevention (MCP). A different problem also called Minimum st-Cut Interdiction has been studied recently [1] but in this problem the graph is directed and the interdictor may freely choose the amount of increase in edge capacities. In our formulation, the interdictor may only decide to fully protect an edge or to leave it unprocted. To the best of our knowledge, this formulation of WMCP has not been considered so far. We study the classical complexity of WMCP and its parameterized complexity with respect to $a, d$, and important structural parameterizations of the input graph $G$.

Related Work. Many interdiction problems have been studied from a (parameterized) complexity perspective: In Matching Interdiction [23], one wants to remove vertices or edges to decrease the weight of a maximum-weight matching. In the Most Vital Edges in MST problem, one aims to remove edges to decrease the weight of any maximum spanning tree. In Shortest-Path Interdiction [17], also known as Shortest Path Most Vital

[^0]Table 1: Parameter overview for WMCP and MCP. We write NP-h if the problem is NP-hard even if the corresponding parameter is a constant.

|  | $a$ | $d$ | $\Delta$ | $d+\Delta$ | vc | pw + fvs |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| WMCP | FPT | W[1]-h | NP-h | W[1]-h | weakly NP-h | weakly NP-h |
|  |  |  |  | if $\Delta=3$ | Thm. 8 | Thm. 8 |
|  |  |  |  | W[1]-h | W[1]-h |  |
|  | Thm. 5 | Lem. 4 | Thm. 1 | Thm. 1 | Thm. 9 | Thm. 9 |
| MCP | FPT | W[1]-h | NP-h | FPT | FPT | W[1]-h |
|  | Thm. 5 | Lem. 4 | Thm. 3 | Thm. 3 | Thm. 7 | Thm. 10 |

Edges [3, 13] and Minimum Length-Bounded Cut [2], one wants to remove edges to increase the length of a shortest $(s, t)$-path above a certain threshold. All of these problems are NP-hard and the study of their classical and parameterized complexity has received a lot of attention [3, 15, 13, 23].

Our Results. An overview of our results is given in Table 1. We show that WMCP and MCP are NP-hard even on subcubic graphs. This motivates a parameterized complexity study with respect to the natural parameters defender budged $d$ and attacker budget $a$ and with respect to structural parameters of the input graph $G$. Here, we consider the structural parameters treewidth and vertex cover number of $G$ as well as pathwidth and feedback vertex set number of $G$. Our main results are as follows. MCP and WMCP are $\mathrm{W}[1]$-hard with respect to the defender budget $d$ and FPT with respect to the attacker budget $a$. MCP and WMCP are W[1]-hard with respect to the combined parameter pathwidth of $G$ plus feedback vertex set number of $G$ and thus also $\mathrm{W}[1]$-hard with respect to the treewidth of $G$. The hardness for these parameters motivates a study of the vertex cover number $\mathrm{vc}(G)$. We show that MCP is FPT with respect to $\operatorname{vc}(G)$, whereas WMCP is weakly NP-hard even for $\operatorname{vc}(G)=2$ and $\mathrm{W}[1]$-hard with respect to $\operatorname{vc}(G)$ even when all capacities and costs are encoded in unary. Finally, we provide a polynomial kernel for WMCP parameterized by $\operatorname{vc}(G)+a$ and complement this result by showing that MCP and WMCP do not admit polynomial kernels with respect to the large combined parameter $d+a+\operatorname{tw}(G)+\operatorname{lp}(G)+\Delta(G)$ where $\operatorname{lp}(G)$ denotes the length of a longest path in $G$ and $\Delta(G)$ denotes the maximum degree. Overall, our results give a comprehensive complexity overview of WMCP and MCP.

## 2 Preliminaries

For integers $i$ and $j$ with $i \leq j$, we define $[i, j]:=\{k \in \mathbb{N} \mid i \leq k \leq j\}$.
An (undirected) graph $G=(V, E)$ consists of a set of vertices $V$ and a set of edges $E \subseteq\binom{V}{2}$. Throughout this work, let $n:=|V|$ and $m:=|E|$. For vertex sets $S \subseteq V$ and $T \subseteq V$ we denote with $E_{G}(S, T):=\{\{s, t\} \in E \mid s \in$ $S, t \in T\}$ the edges between $S$ and $T$. Moreover, we define $E_{G}(S):=E_{G}(S, S)$ and $E_{G}(v, S):=E_{G}(\{v\}, S)$ for $v \in V$. For a vertex set $S \subseteq V$ we denote with $G[S]:=\left(S, E_{G}(S)\right)$ the induced subgraph of $S$ in $G$. Moreover, for an edge set $D \subseteq E$ we let $G-D:=(V, E \backslash D)$ and $G[D]:=(V, D)$. For a vertex $v \in V$, we denote with $N_{G}(v):=\{w \in V \mid\{v, w\} \in E\}$ the open neighborhood of $v$ in $G$. Analogously, for a vertex set $S \subseteq V$, we define $N_{G}(S):=\bigcup_{v \in S} N_{G}(S) \backslash S$. If $G$ is clear from the context, we may omit the subscript. A sequence of distinct vertices $P=\left(v_{0}, \ldots, v_{k}\right)$ is a path or $\left(v_{0}, v_{k}\right)$-path of length $k$ in $G$ if $\left\{v_{i-1}, v_{i}\right\} \in E(G)$ for all $i \in[1, k]$. We denote with $V(P)$ the vertices of $P$ and with $E(P)$ the edges of $P$. Let $s$ and $t$ be distinct vertices of $V$. An edge set $A \subseteq E$ is an $(s, t)$ (edge)-cut in $G$ if there is no $(s, t)$-path in $G-A$. A graph $G=(V, E)$ is connected if there is an $(a, b)$-path in $G$ for each pair of distinct vertices $a, b \in V$. Moreover, we call a vertex set $S$ a connected component of $G$ if $G[S]$ is connected and if there is no $S^{\prime} \supset S$ such that $G\left[S^{\prime}\right]$ is connected.

### 2.1 Graph parameter

Let $G=(V, E)$ be a graph. Moreover, we denote with $\Delta(G):=\max \left\{\left|N_{G}(v)\right| \mid v \in V\right\}$ the maximum degree of $G$.
A set $S \subseteq V$ is a feedback vertex set for $G$ if $G-S$ is acyclic, that is, if for each pair of distinct vertices $a, b \in V \backslash S$ there is at most one $(a, b)$-path in $G-S$. The size of the smallest size feedback vertex set for $G$ is denoted by fvs $(G)$.
A path composition $\mathcal{B}$ for a graph $G=(V, E)$ is a sequence of bags $B_{1}, \ldots, B_{q}$ where $B_{j} \subseteq V$ for each $j \in[1, q]$, such that:

1. for every vertex $v \in V$, there is at least one $i \in[1, q]$ with $v \in B_{i}$,
2. for each edge $e \in E$, there is at least one $i \in[1, q]$ such that $e \subseteq B_{i}$, and
3. if $v \in B_{i} \cap B_{j}$ with $i \leq j$, then $v \in B_{k}$ for each $k \in[i, j]$.

The width of a path decomposition $\mathcal{B}$ is the size of the largest bag in $\mathcal{B}$ minus one and the pathwidth of a graph $G$ is the minimal width of any path decomposition of $G$ which is denoted by $\mathrm{pw}(G)$.
A tree decomposition of a graph $G=(V, E)$ is a pair $(\mathcal{T}, \beta)$ consisting of a directed tree $\mathcal{T}=(\mathcal{V}, \mathcal{A}, r)$ with root $r \in \mathcal{V}$ and a function $\beta: \mathcal{V} \rightarrow 2^{V}$ such that

1. for every vertex $v \in V$, there is at least one $x \in \mathcal{V}$ with $v \in \beta(x)$,
2. for each edge $\{u, v\} \in E$, there is at least one $x \in X$ such that $u \in \beta(x)$ and $v \in \beta(x)$, and
3. for each vertex $v \in V$, the subgraph $\mathcal{T}\left[\mathcal{V}_{v}\right]$ is connected, where $\mathcal{V}_{v}:=\{x \in \mathcal{V} \mid v \in \beta(x)\}$.

We call $\beta(x)$ the bag of $x$. The width of a tree decomposition is the size of the largest bag minus one and the treewidth of a graph $G$ is the minimal width of any tree decomposition of $G$ denoted by $\operatorname{tw}(G)$.
We consider tree decompositions with specific properties. A node $x \in \mathcal{V}$ is called:

1. a leaf node if $x$ has no child nodes in $\mathcal{T}$,
2. a forget node if $x$ has exactly one child node $y$ in $\mathcal{T}$ and $\beta(y)=\beta(x) \cup\{v\}$ for some $v \in V \backslash \beta(x)$,
3. an introduce node if $x$ has exactly one child node $y$ in $\mathcal{T}$ and $\beta(y)=\beta(x) \backslash\{v\}$ for some $v \in V \backslash \beta(y)$, or
4. a join node if $x$ has exactly two child nodes $y$ and $z$ in $\mathcal{T}$ and $\beta(x)=\beta(y)=\beta(z)$.

A tree decomposition $(\mathcal{T}=(\mathcal{V}, \mathcal{A}, r), \beta)$ is called nice if every node $x \in \mathcal{V}$ is either a leaf node, a forget node, an introduce node, or a join node.

For a node $x \in \mathcal{V}$, we define with $V_{x}$ the union of all bags $\beta(y)$, where $y$ is reachable from $x$ in $\mathcal{T}$. Moreover, we set $G_{x}:=G\left[V_{x}\right]$ and $E_{x}:=E_{G}\left(V_{x}\right)$.
The tree-depth $\operatorname{td}(G)$ is the smallest height of any directed tree $T=(V(G), A)$ with the property that for each edge $\{u, w\} \in E(G)$ either $u$ is an ancestor of $w$ in $T$ or vice versa.
Two instances $I$ and $I^{\prime}$ of the same decision problem $L$ are equivalent if $I$ is a yes-instance of $L$ if and only if $I^{\prime}$ is a yes-instance of $L$. A reduction rule for a decision problem $L$ is an algorithm $A$ that transforms any instance $I$ of $L$ into another instance $A(I)$ of $L$. We call $A$ safe, if for each instance $I$ of $L, I$ and $A(I)$ are equivalent instances of $L$. A reduction rule $A$ is exhaustively applied for an instance $I$ if $A(I)=I$.
For details on parameterized complexity, we refer to the standard monograph [10].
The two further variants of WMCP that we study are defined as follows.
Zero-Weight Min $(s, t)$-Cut Prevention (ZWMCP)
Input: A graph $G=(V, E)$, two vertices $s, t \in V$, a cost function $c: E \rightarrow \mathbb{N}$, a capacity function $\omega: E \rightarrow \mathbb{N} \cup\{0\}$, and integers $d$ and $a$.
Question: Is there a set $D \subseteq E$ with $c(D):=\sum_{e \in D} c(e) \leq d$ such that for every $(s, t)$-cut $A \subseteq$ $(E \backslash D)$ in $G$ it holds that $\omega(A):=\sum_{e \in A} \omega(e)>a$ ?

Min $(s, t)$-Cut Prevention (MCP)
Input: A graph $G=(V, E)$, two vertices $s, t \in V$, and integers $d$ and $a$.
Question: Is there an edge set $D \subseteq E$ of size at most $d$ such that every disjoint $(s, t)$-cut $A \subseteq$ ( $E \backslash D$ ) in $G$ has size more than $a$ ?

Informally, we search for a cheap set of edges $S$ such that every disjoint $(s, t)$-cut $M$ is expensive.
Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of any of the above problems (in the case of MCP, $c(e):=$ $\omega(e):=1$ for all $e \in E$ ). We call an edge set $D \subseteq E$ a solution of $I$ if every $(s, t)$-cut $A \subseteq E \backslash D$ has capacity at least $a+1$ according to $\omega$. A solution $D$ of $I$ is called a minimum solution of $I$, if there is no solution $D^{\prime}$ of $I$ with $c\left(D^{\prime}\right)<c(D)$.

### 2.2 Basic Observations

We assume without loss of generality that $G$ is connected and that $c(e) \leq d+1$ and $\omega(e) \leq a+1$ for each edge $e \in E(G)$, as otherwise we can decrease these weights accordingly. Furthermore, we can assume that $d \leq c(E)$ where $c(E)$ denotes the total sum of edge-costs. Analogously, we can assume that $a \leq \omega(E)$.

Fact 1. Let $G=(V, E)$ be a graph, let $\omega: E \rightarrow \mathbb{N}$ be a capacity function, and let $D \subseteq E$. Then, in $n^{\mathcal{O}(1)}$ time we can compute an $(s, t)$-cut $A \subseteq E \backslash D$ with $\omega(A) \leq a$ or report that no such $(s, t)$-cut exists.

Proof. We define a capacity function $\omega^{\prime}: E \rightarrow \mathbb{N}$ by $\omega^{\prime}(e):=a+1$ if $e \in D$ and $\omega^{\prime}(e):=\omega(e)$ otherwise. We then compute a $\min (s, t)$-cut $A$ in $G$ with respect to the new capacity function $\omega^{\prime}$ in $n^{\mathcal{O}(1)}$ time. If $\omega^{\prime}(A) \leq a$, then we return $A$. Otherwise, we report that no such $(s, t)$-cut exists.

Observe that if $\omega^{\prime}(A) \leq a$, then $A \subseteq E \backslash D$, since $\omega^{\prime}(e)=a+1$ for every $e \in D$. Otherwise, if $\omega^{\prime}(A)>a$, then every $(s, t)$-cut in $G$ either contains an edge from $D$ or has capacity bigger than $a$. Thus, the algorithm is correct.

Lemma 1. Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of ZWMCP and let $e^{*}=\left\{v^{*}, w^{*}\right\} \in E$ with $c\left(e^{*}\right)>$ 1. Moreover, let $I^{\prime}:=\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), s, t, c^{\prime}, \omega^{\prime}, d, a\right)$ be the instance of ZWMCP obtained by replacing $e^{*}$ by an $\left(v^{*}, w^{*}\right)$-path with $c\left(e^{*}\right)$ edges of cost 1 and capacity $\omega\left(e^{*}\right)$. Then, $I$ and $I^{\prime}$ are equivalent instances of ZWMCP and $I^{\prime}$ can be computed in $\mathcal{O}\left(c\left(e^{*}\right) \cdot|I|\right)$ time.

Proof. The running time bound follows immediately by the construction. It remains to prove the correctness. Let $E^{*}:=E^{\prime} \backslash E$ be the edges of the $\left(v^{*}, w^{*}\right)$-path in $G^{\prime}$ that replaces the edge $e^{*}$.

We show that $I$ is a yes-instances of ZWMCP if and only if $I^{\prime}$ is a yes-instances of ZWMCP.
$(\Rightarrow)$ Let $D$ be a solution of $I$ of cost at most $d$.
Case 1: $e^{*} \in D$. We set $D^{\prime}:=D \backslash\left\{e^{*}\right\} \cup E^{*}$. Note that $c^{\prime}\left(D^{\prime}\right) \leq d$. We show that $D^{\prime}$ is a solution of $I^{\prime}$. Assume towards a contradiction that there is an $(s, t)$-cut $A^{\prime} \subseteq E^{\prime} \backslash D^{\prime}$ in $G^{\prime}$ with $\omega\left(A^{\prime}\right) \leq a$. By definition of $D^{\prime}$ it follows that $A^{\prime} \subseteq E \backslash\left\{e^{*}\right\}$ with $\omega\left(A^{\prime}\right) \leq a$. Moreover, since we obtained $G^{\prime}$ from $G$ by replacing $e^{*}$ with a path consisting of the edges $E^{*}$ and $A^{\prime}$ is disjoint to $E^{*}$, it follows that $A^{\prime}$ is an $(s, t)$-cut in $G$, a contradiction.

Case 2: $e^{*} \notin D$. Note that $D \subseteq E^{\prime}$ and that $c^{\prime}(D) \leq d$. We show that $D$ is a solution of $I^{\prime}$. Assume towards a contradiction that there is an $(s, t)$-cut $A^{\prime} \subseteq E^{\prime} \backslash D$ in $G^{\prime}$ with $\omega\left(A^{\prime}\right) \leq a$.
If $A^{\prime} \cap E^{*}=\emptyset$, then $A^{\prime}$ is also an $(s, t)$-cut disjoint to $D$ in $G$ with $\omega\left(A^{\prime}\right) \leq a$. A contradiction. Otherwise, $A^{\prime} \cap E^{*} \neq$ $\emptyset$. Hence, $A:=A^{\prime} \backslash E^{*} \cup\left\{e^{*}\right\}$ is an $(s, t)$-cut disjoint to $D$ in $G$ with $\omega(A) \leq a$, a contradiction.
$(\Leftarrow)$ Let $D^{\prime}$ be a solution of $I^{\prime}$.
Case 1: $E^{*} \subseteq D^{\prime}$. We set $D:=D^{\prime} \backslash E^{*} \cup\left\{e^{*}\right\}$. Note that $c(D) \leq d$. We show that $D$ is a solution of $I$. Assume towards a contradiction that there is an $(s, t)$-cut $A \subseteq E \backslash A$ in $G$ with $\omega(A) \leq a$. By definition of $D$, it holds that $A \subseteq E^{\prime} \backslash D^{\prime}$. Moreover, since we obtained $G^{\prime}$ from $G$ by replacing $e^{*}$ with a path consisting of the edges $E^{*}$ and $A$ is disjoint to $E^{*}$, it follows that $A$ is an $(s, t)$-cut in $G^{\prime}$ with $\omega^{\prime}(A) \leq a$, a contradiction.

Case 2: $E^{*} \nsubseteq D^{\prime}$. We set $D:=D^{\prime} \backslash E^{*}$. Note that $c(D) \leq d$. We show that $D$ is a solution of $I$. Assume towards a contradiction that there is an $(s, t)$-cut $A \subseteq E \backslash D$ in $G$ with $\omega(A) \leq a$.
If $e^{*} \notin A$, then $A$ is also an $(s, t)$-cut disjoint to $D^{\prime}$ in $G^{\prime}$ with $\omega^{\prime}(A) \leq a$. A contradiction. Otherwise, $e^{*} \in A$. Hence, $A^{\prime}:=A \backslash\left\{e^{*}\right\} \cup\left\{e^{\prime}\right\}$ for some $e^{\prime} \in E^{*} \backslash D^{\prime}$ is an $(s, t)$-cut disjoint to $D^{\prime}$ in $G^{\prime}$ with $\omega^{\prime}\left(A^{\prime}\right) \leq a$, a contradiction.

Lemma 2. Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of $Z W M C P$ and let $e^{*}=\left\{v^{*}, w^{*}\right\} \in E$ with $\omega\left(e^{*}\right)>$ 1. Moreover, let $I^{\prime}:=\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), s, t, c^{\prime}, \omega^{\prime}, d, a\right)$ be the instance of ZWMCP obtained by updating the capacity of $e^{*}$ to 1 and by adding $\omega\left(e^{*}\right)-1$ many $\left(v^{*}, w^{*}\right)$-paths with two edges of cost $c\left(e^{*}\right)$ and capacity 1 each. Then, I and $I^{\prime}$ are equivalent instances of ZWMCP and $I^{\prime}$ can be computed in $\mathcal{O}\left(\omega\left(e^{*}\right) \cdot|I|\right)$ time.

Proof. The running time bound follows immediately by the construction. It remains to prove the correctness. By $V^{*}:=V^{\prime} \backslash V$ we denote the vertices and by $E^{*}:=E^{\prime} \backslash E$ we denote the edges added to $G$ to obtain the graph $G^{\prime}$. We prove that $I$ is a yes-instance of ZWMCP if and only if $I^{\prime}$ is a yes-instance for ZWMCP.
$(\Rightarrow)$ Let $D \subseteq E$ be a solution of $I$ of cost at most $d$.
Case 1: $e^{*} \in D$. We set $D^{\prime}:=D$. Clearly, $c\left(D^{\prime}\right) \leq d$. We show that $D^{\prime}$ is a solution of $I^{\prime}$. Assume towards a contradiction that there is an $(s, t)$-cut $A^{\prime} \subseteq E^{\prime} \backslash D^{\prime}$ in $G^{\prime}$ with $\omega\left(A^{\prime}\right) \leq a$. Recall that each edge in $E^{*}$ is on a path between $v^{*}$ and $w^{*}$. Since $e^{*} \in D^{\prime}$, we conclude that $A^{\prime} \cap E^{*}=\emptyset$. Hence, $A^{\prime}$ is also an $(s, t)$-cut of capacity at most $a$ in $G$, a contradiction.

Case 2: $e^{*} \notin D$. Observe that $D \subseteq E^{\prime}$ and that $c^{\prime}(D) \leq d$. We show that $D$ is a solution of $I^{\prime}$. Assume towards a contradiction that there is an inclusion-minimal $(s, t)$-cut $A^{\prime} \subseteq E^{\prime} \backslash D$ in $G^{\prime}$ with $\omega\left(A^{\prime}\right) \leq a$.

If $A^{\prime} \cap E^{*}=\emptyset$, then $A^{\prime}$ is also an $(s, t)$-cut disjoint to $D$ in $G$ with $\omega\left(A^{\prime}\right) \leq a$, a contradiction. Otherwise, $A^{\prime} \cap E^{*} \neq \emptyset$. Recall that all edges in $E^{*}$ are on paths with two edges between $v^{*}$ and $w^{*}$. Thus, $A^{\prime}$ is also an $\left(v^{*}, w^{*}\right)$-cut in $G^{\prime}$. Hence, $\left\{v^{*}, w^{*}\right\} \in A^{\prime}$ and for each vertex $z \in V^{*}$ at least one adjacent edge is contained in $A^{\prime}$. Since $\left|V^{*}\right|=$ $\omega\left(e^{*}\right)-1$, we conclude that $\left|A^{\prime} \cap E^{*}\right| \geq \omega\left(e^{*}\right)-1$. Thus, $A^{\prime} \backslash E^{*}$ is an $(s, t)$-cut of cost at most $a$ in $G$, a contradiction.
$(\Leftarrow)$ Let $D^{\prime}$ be a solution of $I^{\prime}$ of cost at most $d$. By $P_{z}$ we denote the path $\left(v^{*}, z, w^{*}\right)$ for some vertex $z \in V^{*}$.
Case 1: $E\left(P_{z}\right) \subseteq D^{\prime}$ for some $z \in V^{*}$ or $e^{*} \in D^{\prime}$. We set $D:=D^{\prime} \backslash E^{*} \cup\left\{e^{*}\right\}$. Note that since $c(e)=c\left(e^{*}\right)$ for each edge $e \in E^{*}$ we obtain $c(D) \leq d$. We show that $D^{\prime}$ is a solution of $I$. Assume towards a contradiction that there is an $(s, t)$-cut $A \subseteq E \backslash D$ in $G$ with $\omega(A) \leq a$. By definition of $D$, we observe that $A \subseteq E^{\prime} \backslash D^{\prime}$. Moreover, since we obtained $G^{\prime}$ from $G$ by adding $\omega\left(e^{*}\right)-1$ paths consisting of the edges $E^{*}$, and the fact that $A$ is disjoint to $E^{*}$, we conclude that $A$ is an $(s, t)$-cut in $G^{\prime}$ with $\omega^{\prime}(A) \leq a$, a contradiction.
Case 2: $E\left(P_{z}\right) \nsubseteq D^{\prime}$ for each $z \in V^{*}$ and $e^{*} \notin D^{\prime}$. We set $D:=D^{\prime} \backslash E^{*}$. Note that $c(D) \leq d$. We show that $D$ is a solution of $I$. Assume towards a contradiction that there is an $(s, t)$-cut $A \subseteq E \backslash D$ in $G$ with $\omega(A) \leq a$.
If $e^{*} \notin A$, then $A$ is also an $(s, t)$-cut disjoint to $D^{\prime}$ in $G^{\prime}$ with $\omega^{\prime}(A) \leq a$, a contradiction. Otherwise, $e^{*} \in A$. Let $A^{\prime}:=A \backslash\left\{e^{*}\right\} \cup\left\{\left\{v^{*}, z\right\} \mid z \in V^{*}\right\}$. Note that for each $e \in E^{*} \cup\left\{e^{*}\right\}$ we have $\omega(e)=1$ and that $\left|V^{*}\right|=\omega\left(e^{*}\right)-1$. Hence, $A^{\prime}$ is an $(s, t)$-cut disjoint to $D^{\prime}$ in $G^{\prime}$ with $\omega^{\prime}\left(A^{\prime}\right) \leq a$, a contradiction.

Recall that we can assume $c(e) \leq d+1$ and $\omega(e) \leq a+1$ for each edge $e \in E$. Hence, the subsequent application of Lemmas 1 and 2 leads to the following.
Corollary 1. Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of WMCP. Then, one can compute in $(n+a+$ $d)^{\mathcal{O}(1)}$ time an equivalent instance $I^{\prime}=\left(G^{\prime}, s^{\prime}, t^{\prime}, d, a\right)$ of MCP .

The next definition will be a useful tool in several proofs in this work.
Definition 1. Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of WMCP , and let $e=\{u, w\} \in E$. The merge of $u$ and $w$ in $I$ is the instance $I^{\prime}$ obtained from $I$ by removing $u$ and $w$ from $G$ and adding a new vertex $v_{\{u, w\}}$ which is adjacent to $N(\{u, w\})$. The cost and capacity for each edge in $E \cap E^{\prime}$ are set to the corresponding cost and capacity in $E$, and for each $x \in N(\{u, w\})$,

$$
\begin{aligned}
& \text { - } c^{\prime}\left(\left\{v_{\{u, w\}}, x\right\}\right)=\min \left\{c\left(e^{\prime}\right) \mid e^{\prime} \in E(x,\{u, w\})\right\} \text {, and } \\
& \text { - } \omega^{\prime}\left(\left\{v_{\{u, w\}}, x\right\}\right)=\sum_{e^{\prime} \in E(x,\{u, w\})} \omega\left(e^{\prime}\right) \text {. }
\end{aligned}
$$

Rule 1. If $G$ contains an edge $e^{*}=\left\{u^{*}, w^{*}\right\} \in E$ which is not contained in any inclusion-minimal ( $s, t$ )-cut of capacity at most a in $G$, then merge $u^{*}$ and $w^{*}$.
Lemma 3. Rule 1 is safe and can be applied exhaustively in polynomial time.
Proof. Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of WMCP and let $I^{\prime}=\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), s^{\prime}, t^{\prime}, c^{\prime}, \omega^{\prime}, d, a\right)$ be the merge of $u^{*}$ and $w^{*}$ in $I$. We show that $I$ and $I^{\prime}$ are equivalent instances of WMCP.
$(\Rightarrow)$ Let $D \subseteq E$ be a solution of $I$ of cost at most $d$.
Claim 1. The set $D^{*}:=D \backslash\left\{e^{*}\right\}$ is a solution of $I$.
Proof. Assume towards a contradiction that $D^{*}$ is not a solution of $I$. Then, there is an inclusion-minimal $(s, t)$ cut $A \subseteq E \backslash D^{*}$ of capacity at most $a$ in $G$. By the condition of Rule 1 , it holds that $e^{*} \notin A$. Note that $A$ avoids $D^{*}$. This contradicts the fact that $D^{*}$ is a solution.

Due to Claim 1 we can assume that $e^{*} \notin D$. We set $D^{\prime}:=\left(D \cap E^{\prime}\right) \cup\left\{\left\{v_{e^{*}}, x\right\} \mid\left\{u^{*}, x\right\} \in D\right.$ or $\left.\left\{w^{*}, x\right\} \in D\right\}$. By definition of $c^{\prime}$ it follows that $D^{\prime}$ has cost at most $c(D)$. Hence, it remains to show that $D^{\prime}$ is a solution of $I^{\prime}$.

Assume towards a contradiction that $D^{\prime}$ is not a solution of $I^{\prime}$. Then, there is an $\left(s^{\prime}, t^{\prime}\right)$-cut $A^{\prime} \subseteq E^{\prime} \backslash D^{\prime}$ of capacity at most $a$ in $G^{\prime}$. We set $A:=\left(A^{\prime} \cap E\right) \cup\left\{e \in E\left(x, e^{*}\right) \mid\left\{v_{e^{*}}, x\right\} \in A^{\prime}\right\}$. Note that $A \subseteq E \backslash D$. By definition of $\omega^{\prime}$, we obtain that $\omega(A)=\omega^{\prime}\left(A^{\prime}\right) \leq a$. Since $\left\{x, v_{e^{*}}\right\} \in A^{\prime}$ if and only if $E\left(x, e^{*}\right) \subseteq A$, and $A$ and $A^{\prime}$ agree on $E \cap E^{\prime}$, we obtain that $A$ is an $(s, t)$-cut in $G$ which contradicts the fact that $D$ is a solution of $I$. Consequently, $I$ is a yes-instance of WMCP.
$(\Leftarrow)$ Let $D^{\prime} \subseteq E^{\prime}$ be a solution of $I^{\prime}$ of cost at most $d$. We set $D:=\left(D^{\prime} \cap E\right) \cup\left\{e_{x} \mid\left\{v_{e^{*}}, x\right\} \in D^{\prime}\right\}$, where $e_{x}$ is an edge in $E\left(x, e^{*}\right)$ with minimal cost. By definition of $c^{\prime}$ it follows that $c(D) \leq c^{\prime}\left(D^{\prime}\right)$. It remains to show that $D$ is a solution of $I$.

Assume towards a contradiction that $D$ is not a solution of $I$. Then, there is an $(s, t)$-cut $A^{*} \subseteq E \backslash D$ of capacity at most $a$ in $G$. Since $e^{*}$ is not contained in any inclusion-minimal ( $s, t$ )-cut of capacity at most $a$, there is an inclusionminimal $(s, t)$-cut $A \subseteq A^{*} \backslash\left\{e^{*}\right\}$ of capacity at most $a$ in $G$. We set $A^{\prime}:=\left(E^{\prime} \cap A\right) \cup\left\{\left\{v_{e^{*}}, x\right\} \in E^{\prime} \mid E\left(x, e^{*}\right) \subseteq A\right\}$. By definition of $\omega^{\prime}$, we obtain $\omega^{\prime}\left(A^{\prime}\right) \leq \omega(A) \leq a$. Since $\left\{x, v_{e^{*}}\right\} \in A^{\prime}$ if and only if $E\left(x, e^{*}\right) \subseteq A$ and $A$ and $A^{\prime}$ agree on $E \cap E^{\prime}$, we obtain that $A^{\prime}$ is an $\left(s^{\prime}, t^{\prime}\right)$-cut in $G$ which contradicts the fact that $D^{\prime}$ is a solution of $I^{\prime}$. Consequently, $I$ is a yes-instance of WMCP.
It remains to bound the running time. Each application of Rule 1 reduces the number of vertices by one, and each such application can be performed in polynomial time, we obtain that, Rule 1 can be exhaustively applied in polynomial time.

## 3 NP-hardness and Parameterization by the Defender Budget $d$

In this section we prove that MCP is NP-hard and we analyze parameterization by $d$ and $\Delta(G)$. In particular, we provide a complexity dichotomy for $\Delta(G)$.
Lemma 4. WMCP is NP-complete and W[1]-hard when parameterized by d even if $G$ is bipartite, $\omega(e)=1$, and $c(e) \in \mathcal{O}(|G|)$ for all $e \in E$.

Proof. We describe a parameterized reduction from a variant of Independent SET which is known to be W[1]-hard when parameterized by $k[10,11]$.

## Regular-Independent Set

Input: An $r$-regular graph $G=(V, E)$ for some integer $r$ and an integer $k$.
Question: Is there an independent set $S \subseteq V$ of size at least $k$ in $G$ ?
Let $I:=(G=(V, E), k)$ be an instance of $r$-REGULAR-Independent SET. We describe how to construct an instance $I^{\prime}:=\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), s, t, c, \omega, d, a\right)$ of WMCP in polynomial time such that $I$ is a yes-instance of REGULARIndependent Set if and only if $I^{\prime}$ is a yes-instance of WMCP.
We start with an empty graph $G^{\prime}$ and add all vertices of $V$ to $G^{\prime}$. For each vertex $v \in V$ we also add an additional vertex $v^{\prime}$. Furthermore, for each edge $e \in E$ we add a vertex $w_{e}$, and two new vertices $s$ and $t$ to $G^{\prime}$. Moreover, we add the edges $\{s, v\},\left\{v, v^{\prime}\right\}$ and $\left\{v^{\prime}, t\right\}$ to $G^{\prime}$ for each vertex $v \in V$. Next, we add the edges $\left\{u, w_{e}\right\},\left\{v, w_{e}\right\}$, and $\left\{w_{e}, t\right\}$ to $G^{\prime}$ for each edge $e=\{u, v\} \in E$. Now, we set $\omega\left(e^{\prime}\right):=1$ for all $e^{\prime} \in E^{\prime}$. Furthermore, for each $e^{\prime} \in E^{\prime}$, we set $c\left(e^{\prime}\right):=1$ if $s \in e^{\prime}$ and $c\left(e^{\prime}\right):=k+1$ otherwise. Finally, we set $d:=k$ and $a:=n+k r-1$ where $n:=|V|$. This completes the construction of $I^{\prime}$. Observe that $G^{\prime}$ is bipartite with one partite set being $\{t\} \cup V$. Note that only the edges incident with $s$ can be protected, since all other edges have cost exactly $d+1$.

Next, we show that $I$ is a yes-instance of REGULAR-INDEPENDENT SET if and only if $I^{\prime}$ is a yes-instance of WMCP.
$(\Rightarrow)$ Let $S \subseteq V$ be an independent set of $G$ of size exactly $k=d$. We set $D^{\prime}:=\{\{s, v\} \mid v \in S\}$. Note that $D^{\prime}$ has cost exactly $d$. It remains to show that $D^{\prime}$ is a solution of $I^{\prime}$. To this end, we provide $a+1$ many paths whose edge sets may only intersect in $D^{\prime}$.
Note that for each vertex $v \in V \backslash S$ we have a path $\left(s, v, v^{\prime}, t\right)$. These are $n-k$ many. Next, consider a vertex $v \in S$. Observe that $\left(s, v, v^{\prime}, t\right)$ and $\left\{\left(s, v, w_{e}, t\right) \mid e \in E, v \in e\right\}$ are $r+1$ paths only sharing the edge $\{s, v\} \in D^{\prime}$. Since $|S|=k$ and $G$ is $r$-regular, these are $k r+k$ many paths. Moreover, since $S$ is an independent set no two vertices $u, v \in S$ have a common neighbor $w_{e}$ in $G^{\prime}$ for $e=\{u, v\}$. Hence, there are $n-k+k r+k=n+k r=a+1$ many $(s, t)$-paths in $G^{\prime}$ whose edge sets only intersect in $D^{\prime}$.
$(\Leftarrow)$ Suppose that $I^{\prime}$ is a yes-instance of WMCP. Let $D^{\prime}$ be a solution with cost at most $d$ of $I^{\prime}$. Recall that $c(e)=d+1$ for each edge $e^{\prime} \in E^{\prime}$ with $s \notin e^{\prime}$. Hence, $D^{\prime} \subseteq\{\{s, v\} \mid v \in V\}$. If $\left|D^{\prime}\right|<d$, then we add exactly $d-\left|D^{\prime}\right|$ many edges of the form $\{s, v\}$ which are not already contained in $D^{\prime}$ to $D^{\prime}$. Note that $D^{\prime}$ remains a solution of $I^{\prime}$. Thus, in the following we can assume that $\left|D^{\prime}\right|=d=k$. Let $S:=\left\{v \mid\{s, v\} \in D^{\prime}\right\}$. We prove that $S$ is an independent set in $G$.

Assume towards a contradiction that $S$ is no independent set in $G$ and let $e^{*}$ be an edge of $G[S]$. In the following, we construct an $(s, t)$-cut $A \subseteq\left(E^{\prime} \backslash D^{\prime}\right)$ in $G^{\prime}$ of size at most $a$. Let $A_{V \backslash S}:=\{\{s, v\} \mid v \notin S\}, A_{S}:=\left\{\left\{v, v^{\prime}\right\} \mid v \in S\right\}$, and $A_{E}:=\left\{\left\{v, w_{e}\right\} \in E^{\prime} \mid v \in S, e \neq e^{*}\right\}$. We show that $A:=A_{V \backslash S} \cup A_{S} \cup A_{E} \cup\left\{\left\{w_{e^{*}}, t\right\}\right\}$ is an $(s, t)$-cut of size at most $a$ in $G^{\prime}$. Note that $\left|A_{V \backslash S}\right|+\left|A_{S}\right|=n$. Moreover, since $|S|=k$ and each vertex $v \in V$ has degree exactly $r$ in $G,\left|A_{E}\right| \leq k r-2$. Hence, $A$ has capacity at most $n+k r-1=a$ since $\omega\left(e^{\prime}\right)=1$ for each $e^{\prime} \in E^{\prime}$. It remains to show that $A$ is an $(s, t)$-cut in $G^{\prime}$. Let $G^{*}:=G^{\prime}-A$. Note that $N_{G^{*}}(s)=S$ and $N_{G^{*}}(v)=\{s\}$ for each $v \in S \backslash e^{*}$.

Moreover, note that $N_{G^{*}}(v)=\left\{s, w_{e^{*}}\right\}$ for each $v \in e^{*}$ and $N_{G^{*}}\left(w_{e^{*}}\right)=e^{*}$. Hence, $A$ is an $(s, t)$-cut in $G^{\prime}$ with capacity at most $a$. A contradiction.
Consequently, $S$ is an independent set of size $k$ in $G$ and, therefore, $I$ is a yes-instance of REGULAR-Independent SET.

By applying Corollary 1, we can extend the hardness results to MCP. Note that if $k$ is odd, Corollary 1 replaces an edge with costs $k+1$ by a path of even length and thus the resulting instance of MCP is not bipartite. Hence, to obtain $\mathrm{W}[1]$-hardness in case of odd $k$, we set $c(e)=k+2$ for edges not containing $s$.

Corollary 2. MCP is NP-complete and $\mathrm{W}[1]$-hard when parameterized by d, even on bipartite graphs.
Next, we provide a complexity dichotomy for the classical complexity with respect to the maximum degree of the graph.
Lemma 5. ZWMCP can be solved in polynomial time on graphs of maximum degree two.

Proof. Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of ZWMCP where $G$ has degree at most two. Recall that we can assume without loss of generality that $G$ is connected. Observe that since $G$ has degree at most two, $G$ is either a path or a cycle.
$G$ is a path. Let $P$ be the unique $(s, t)$-path in $G$ and let $E_{A}:=\left\{e_{i} \in E(P) \mid \omega\left(e_{i}\right) \leq a\right\}$ be the set of edges of capacity at most $a$. Since $\left\{e_{i}\right\}$ is an $(s, t)$-cut of capacity at most $a$ in $G$ for every $e_{i} \in E_{A}$, we conclude that $E_{A}$ is a subset of every solution of $I$. Consequently, $I$ is a yes-instance of ZWMCP if and only if $d \geq c\left(E_{A}\right)$, since every $(s, t)$-cut $M \subseteq E \backslash E_{A}$ has capacity larger than $a$.
$G$ is a cycle. Let $P_{1}$ and $P_{2}$ be the unique $(s, t)$-paths in $G$. Moreover, let $E_{A}:=\left\{\left\{e_{i}^{1}, e_{j}^{2}\right\} \mid e_{i}^{1} \in E\left(P_{1}\right), e_{j}^{2} \in\right.$ $\left.E\left(P_{2}\right), \omega\left(e_{i}^{1}\right)+\omega\left(e_{j}^{2}\right) \leq a\right\}$ be the set of minimal $(s, t)$-cuts of capacity at most $a$ in $G$. Note that every other $(s, t)$-cut of capacity at most $a$ is a superset of any $(s, t)$-cut in $E_{A}$. Hence, $I$ is a yes-instance of ZWMCP if and only if there is a set $S \subseteq E\left(P_{1}\right) \cup E\left(P_{2}\right)$ with $c(S) \leq d$ such that $S \cap \mathbf{e} \neq \emptyset$ for all $\mathbf{e} \in E_{A}$. This is equivalent to the question if the graph $G^{\prime}$ with bipartition $\left(E\left(P_{1}\right), E\left(P_{2}\right)\right)$ and edges $E_{A}$ has a vertex cover of capacity at most $d$ with $c$ as the capacity function. This can be done in polynomial time.

Consequently, ZWMCP can be solved in polynomial time on graphs of degree at most two.
Lemma 6. WMCP is NP-hard and $\mathrm{W}[1]$-hard when parameterized by d even on subcubic graphs and even if $c(e)=1$ and $\omega(e) \in \mathcal{O}(|G|)$ for all $e \in E$.

Proof. We reduce from MCP which is W[1]-hard when parameterized by $d$ due to Corollary 2 . Let $I=$ $(G=(V, E), s, t, d, a)$ be an instance of MCP. Next, we construct an equivalent instance $I^{\prime}=\left(G^{\prime}=\right.$ $\left.\left(V^{\prime}, E^{\prime}\right), s^{\prime}, t^{\prime}, c^{\prime}, \omega^{\prime}, d^{\prime}, a^{\prime}\right)$ of WMCP as follows.

For each vertex $v \in V$ we add a path $P_{u}$ consisting of $|N(u)|$ vertices to $G^{\prime}$. We denote the vertices of $P_{u}$ by $p_{u}^{1}, \ldots p_{u}^{|N(u)|}$. In the following, we assume an arbitrary but fixed ordering on $N(u)$. Thus, the $i$-th-vertex of $N(u)$ is associated with vertex $p_{u}^{i} \in P_{u}$. Furthermore, if $v$ is the $i$-th neighbor of $u$ we also write $p_{u}^{v}$ instead of $p_{u}^{i}$ to access neighbor $v$ more conveniently. We set $c^{\prime}(e)=1$ and $\omega^{\prime}(e)=a+1$ for each edge $e \in E\left(P_{u}\right)$. Furthermore, for each edge $\{u, v\} \in E(G)$ we add the edge $\left\{p_{u}^{v}, p_{v}^{u}\right\}$ to $G^{\prime}$ with cost and capacity equal to one. Next, we set $s^{\prime}:=p_{s}^{1}$ and $t^{\prime}:=p_{t}^{1}$. Finally, we set $a^{\prime}:=a$ and $d^{\prime}:=d$.

Since each vertex in $P_{u}$ has exactly one neighbor which is not in $P_{u}$, the graph $G^{\prime}$ is subcubic. Next, we prove that $I$ is a yes-instance of MCP if and only if $I^{\prime}$ is a yes-instance of WMCP.
Let $v \in V$ and let $e$ be an edge of $P_{v}$. By the fact that $\omega^{\prime}(e)=a+1, e$ is not contained in any (inclusionminimal) $\left(s^{\prime}, t^{\prime}\right)$-cut of capacity at most $a$. Thus, by merging the endpoints of $e$, we obtain an equivalent instance of WMCP due to Lemma 3.

Let $I^{*}=\left(G^{*}=\left(V^{*}, E^{*}\right), s^{*}, t^{*}, c^{*}, \omega^{*}, d^{\prime}, a^{\prime}\right)$ be the instance of WMCP we obtain after merging the endpoints of all edges contained in any path $P_{v}$. Note that by Definition 1 it follows that $G^{*}$ is isomorphic to $G$ and $\omega^{*}(e)=c^{*}(e)=1$ for each $e \in E^{*}$. Thus, $I$ is a yes-instance of MCP if and only if $I^{*}$ is a yes-instance of WMCP.

By Lemma 5 and Lemma 6 we obtain the following.
Theorem 1. WMCP can be solved in polynomial time on graphs of maximum degree two. WMCP is NP-hard and $\mathrm{W}[1]$-hard when parameterized by $d$ even on subcubic graphs and even if $c(e)=1$ and $\omega(e) \in \mathcal{O}(|G|)$ for all $e \in E$.

Next, we strengthen the NP-hardness of WMCP on subcubic graphs to MCP.
Lemma 7. MCP is NP-complete even on subcubic graph.

Proof. We reduce from MCP. Let $I=(G=(V, E), s, t, d, a)$ be an instance of MCP. We first prove the statement for WMCP where $\omega(e)=1$ and $c(e) \in n^{\mathcal{O}(1)}$. We do this intermediate step to emphasize the main idea of the reduction. Second, we apply Corollary 1 to each edge in the instance of WMCP to obtain an equivalent instance of MCP. Note that since Corollary 1 replaces an edge by a path, the resulting instance of MCP is also subcubic. Hence, it remains to prove the statement for the restricted version of WMCP.

Next, we construct an equivalent instance $I^{\prime}=\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), s^{\prime}, t^{\prime}, c^{\prime}, \omega^{\prime}, d^{\prime}, a^{\prime}\right)$ of WMCP as follows. For each vertex $v \in V$ we add a path $P_{u}$ consisting of $N(u)$ vertices to $G^{\prime}$. We denote the vertices of $P_{u}$ by $p_{u}^{1}, \ldots p_{u}^{|N(u)|}$. In the following, we assume an arbitrary but fixed ordering on $N(u)$. Thus, the $i$-th-vertex of $N(u)$ is associated with vertex $p_{u}^{i} \in P_{u}$. Furthermore, if $v$ is the $i$-th neighbor of $u$ we also write $p_{u}^{v}$ instead of $p_{u}^{i}$ to access neighbor $v$ more convenient. We set $c^{\prime}(e):=\omega^{\prime}(e):=1$ for each edge $e \in E\left(P_{u}\right)$. Furthermore, for each edge $\{u, v\} \in E(G)$ we add the edge $\left\{p_{u}^{v}, p_{v}^{u}\right\}$ to $G^{\prime}$ and set its costs to $n^{2}$ and its capacity to one. Next, we set $s^{\prime}:=p_{s}^{1}$ and $t^{\prime}:=p_{t}^{1}$. Finally, we set $a^{\prime}:=a$ and $d^{\prime}:=d n^{2}+n(n-1)$.
Since each vertex in $P_{u}$ has exactly one neighbor which is not in $P_{u}$, the graph $G^{\prime}$ is subcubic. Next, we prove that $I$ is a yes-instance of MCP if and only if $I^{\prime}$ is a yes-instance of WMCP.
$(\Rightarrow)$ Let $D \subseteq E$ be a solution with cost at most $d$ of $I$. In the following, we construct a solution $D^{\prime} \subseteq E^{\prime}$ with cost at most $d^{\prime}$ of $I^{\prime}$.

For each edge $\{u, v\} \in D$ we add the corresponding edge $\left\{p_{u}^{v}, p_{v}^{u}\right\}$ in $G^{\prime}$ to $D^{\prime}$. Since each of these edges has cost $n^{2}$, and $|D| \leq s$, these edges contribute at most $d n^{2}$ to to cost of $D^{\prime}$. Furthermore, we add each edge in $P_{u}$ for each $u \in V$ to $D^{\prime}$. Since $P_{u}$ has at most $n-1$ edges and each edge in $P_{u}$ has cost one, all these edges contribute at most $n(n-1)$ to the total costs. Hence, $\left|D^{\prime}\right| \leq d^{\prime}$. Assume towards a contradiction that $G^{\prime}$ has an $\left(s^{\prime}, t^{\prime}\right)$-cut $A^{\prime} \subseteq E^{\prime} \backslash D^{\prime}$ with $\omega^{\prime}\left(A^{\prime}\right) \leq a^{\prime}$. Since $E\left(P_{u}\right) \subseteq D^{\prime}$ for each $u \in V$, the $(s, t)$-cut $A^{\prime}$ contains only edges of the form $\left\{p_{u}^{v}, p_{v}^{u}\right\}$ between two different paths. We define the set $A$ as the set of corresponding edges of $A^{\prime}$ in $G$. Since $\left|A^{\prime}\right| \leq a$ we obtain $|A| \leq a$. Since there is no $(s, t)$-cut of capacity at most $a$ in $G \backslash D$ and $A \cap D=\emptyset$, we conclude that there exists an $(s, t)$-path $\left(s, w_{1}, \ldots, w_{\ell}, t\right)$ in $G \backslash A$. Observe that $\left(p_{s}^{1}, \ldots, p_{s}^{w_{1}}, p_{w_{1}}^{s}, \ldots, p_{w_{1}}^{w_{2}}, p_{w_{2}}^{w_{1}}, \ldots, p_{t}^{w_{\ell}}, p_{t}^{1}\right)$ is an $\left(s^{\prime}, t^{\prime}\right)$-path in $G^{\prime}-A^{\prime}$, a contradiction to the assumption that $A^{\prime}$ is an $\left(s^{\prime}, t^{\prime}\right)$-cut in $G^{\prime}$.
$(\Leftarrow)$ Let $D^{\prime} \subseteq E^{\prime}$ be a solution with cost at most $d^{\prime}$ of $I^{\prime}$. In the following, we construct a solution $D \subseteq E$ with cost at most $d$ of $I$.

Since $d^{\prime}=d n^{2}+n(n-1), c^{\prime}(e)=n^{2}$ for each edge $e \notin E\left(P_{u}\right)$ and each $u \in V$ in $G^{\prime}$, and $c(e)=1$ for each edge $e \in E\left(P_{u}\right)$ for some $u \in V$ in $G^{\prime}$, we can assume without loss of generality that $E\left(P_{u}\right) \subseteq D^{\prime}$ for each $u \in V$. We start with an empty set $D$. For an edge $\left\{p_{u}^{v}, p_{v}^{u}\right\} \in D^{\prime}$ between two different paths, we add the edge $\{u, v\}$ to $D$. Assume towards a contradiction that $G$ has an $(s, t)$-cut $A \subseteq E \backslash D$ with $\omega(A) \leq a$. We define the set $A^{\prime}$ as the set of corresponding edges of $A$ in $G^{\prime}$. Note that since $\omega(e)=1$ for each edge $e \in E^{\prime}$ we have $\left|A^{\prime}\right| \leq a=a^{\prime}$. Since there is no ( $\left.s^{\prime}, t^{\prime}\right)$-cut of capacity at most $a$ in $G^{\prime} \backslash D^{\prime}$ and $A^{\prime} \cap D^{\prime}=\emptyset$, we conclude that there exists an $\left(s^{\prime}, t^{\prime}\right)$-path $\left(p_{s}^{1}, \ldots, p_{s}^{w_{1}}, p_{w_{1}}^{s}, \ldots, p_{w_{1}}^{w_{2}}, p_{w_{2}}^{w_{1}}, \ldots, p_{t}^{w_{\ell}}, p_{t}^{1}\right)$ in $G^{\prime}$. Thus, $\left(s, w_{1}, \ldots, w_{\ell}, t\right)$ is an $(s, t)$ path in $G \backslash A$, a contradiction to the assumption that $A$ is an $(s, t)$-cut in $G$.

Theorem 2. WMCP can be solved in $a^{d} \cdot n^{\mathcal{O}(1)}$ time.

Proof. Let $J:=(G, s, t, c, \omega, d, a)$ be an instance of WMCP. We prove the theorem by describing a simple search tree algorithm, that we initially call with $D:=\emptyset$, where $D$ represents the choice of the defender:
If $c(D)>d$, then return no. Otherwise, use the algorithm behind Lemma 1 to compute an $(s, t)$-cut $A=$ $\left\{e_{1}, \ldots, e_{z}\right\} \subseteq E \backslash D$ for some $z \leq|A|$ with $\omega(A) \leq a$. If no such $(s, t)$-cut exists, then return yes. Otherwise, we branch into the cases where $D:=D \cup\left\{e_{i}\right\}$ for each $i \in[1, z]$.
The correctness of the algorithm follows from the fact that for every $(s, t)$-cut $A \subseteq E \backslash D$ with $\omega(A) \leq a$, at least one of the edges of $A$ must be contained in any solution of $J$. It remains to consider the running time of the algorithm. We have $\omega(e) \geq 1$ for every edge $e$ in any WMCP instance. Hence, $|A| \leq a$ and therefore, the search tree algorithm branches into at most $a$ cases. Furthermore, after every branching step, $c(D)$ increases by at least 1 , since we add one additional edge to $D$ and we have $c(e) \geq 1$ for every edge $e$. Thus, the depth of the search tree is at most $d$. Together with the running time from Lemma 1 , we obtain a total running time of $a^{d} n^{\mathcal{O}(1)}$.

Lemma 8. MCP can be solved in $((d / 2+1) \cdot \Delta(G))^{d} \cdot n^{\mathcal{O}(1)}$ time, where $\Delta(G)$ denotes the maximum degree of the input graph.

Proof. Let $J:=(G, s, t, d, a)$ be an instance of MCP. We prove the theorem by showing that $a \leq d \cdot \Delta$ in non-trivial instances of MCP. Together with Theorem 2, we then obtain fixed-parameter tractability for $d+\Delta$.
If $G$ contains an $(s, t)$-path with at most $d$ edges, then $J$ is a trivial yes-instance. Thus, we may assume that for every $D \subseteq E$ with $|D| \leq d$, there is no ( $s, t$ )-path in $G$ that contains only edges from $D$. We use this assumption to prove the following claim.
Claim 2. If $a \geq(d / 2+1) \cdot \Delta$, then $J$ is $a$ no-instance.
Proof. Let $D \subseteq E$ with $|D| \leq d$. We prove that there exists an $(s, t)$-cut $A \subseteq E$ of size at most $a$. To this end, consider the graph $G_{D}:=(V, D)$ consisting only of the edges in $D$. Since $|D| \leq d$ we know that there is no $(s, t)$-path in $G_{D}$. Thus, $s$ and $t$ are in distinct connected components $C_{s} \subseteq V$ and $C_{t} \subseteq V$ in $G_{D}$. Furthermore, observe that in at least one of the induced graphs $G_{D}\left[C_{s}\right]$ or $G_{D}\left[C_{t}\right]$, there are at most $d / 2$ edges. Without loss of generality, assume that this is the case for $G_{D}\left[C_{s}\right]$. Then, $\left|C_{D}\right| \leq d / 2+1$. We define $A:=\bigcup_{v \in C_{D}} X(v)$, where $X(v) \subseteq E \backslash D$ is the set of all edges in $E \backslash D$ that are incident with $v$ in $G$. Note that $A \subseteq E \backslash D$ and that $|A| \leq\left|C_{D}\right| \cdot \Delta=(d / 2+1) \cdot \Delta \leq a$. Moreover, $A$ is an $(s, t)$-cut in $G$ since $t \notin C_{s}$.

By Claim 2, we conclude that for every non-trivial instnace of MCP, we have $a \leq d \cdot \Delta$. Together with Theorem 2, we obtain that MCP can be solved in $((d / 2+1) \cdot \Delta)^{d} \cdot n^{\mathcal{O}(1)}$ time.

By Lemma 7 and Lemma 8 we obtain the following.
Theorem 3. MCP is NP-complete even on subcubic graphs. Furthermore, MCP can be solved in $((d / 2+1) \cdot \Delta(G))^{d}$. $n^{\mathcal{O}(1)}$ time.

## 4 Parameterization by the Attacker Budget

In this section, we show that WMCP admits an FPT-algorithm for the parameter $a$. To this end, we first provide an algorithm with a running time of $a^{f(\operatorname{tw}(G))} \cdot n$ for some computable function $f$, where $\operatorname{tw}(G)$ denotes the treewidth of the graph. Afterwards, we show that for every instance of WMCP we can obtain an equivalent instance $I^{\prime}$ of WMCP in polynomial time, where every edge is contained in an inclusion-minimal $(s, t)$-cut of size at most $a$ in $I^{\prime}$. Due to previous results [16,20], the graph of $I^{\prime}$ then has treewidth at most $g(a)$ for some computable function $g$. In combination with the algorithm for $a$ and $\operatorname{tw}(G)$, we thus obtain the stated FPT-algorithm for the parameter $a$.
The algorithm with a running time of $a^{f(\operatorname{tw}(G))} \cdot n$ relies on dynamic programming over a tree decomposition. Essentially, what the attacker can achieve in the current subgraph is to disconnect specific parts of the bag and thus obtain a cheap partition. Roughly speaking, the algorithm computes the minimum cost for an edge set $D$ such that each choice of the attacker to obtain any partition disjoint from $D$ is expensive. Hence, before we describe the algorithm, we first introduce some notations for partitions.
Let $X$ be a set. We denote with $B(X)$ the collection of all partitions of $X$. Let $P \in B(X)$ be a partition of $X$ and let $v \in X$. Then, we define with $P-v:=\{R \backslash\{v\} \mid R \in P\} \backslash\{\emptyset\}$ the partition of $X \backslash\{v\}$ after removing $v$ from $P$. Analogously, for every $w \notin X$ we define $P+w:=\left\{P^{\prime} \in B(X \cup\{w\}) \mid P^{\prime}-w=P\right\}$. Note that $B(X \backslash\{v\})=\{P-v \mid P \in B(X)\}$ and $B(X \cup\{w\})=\{P+w \mid P \in B(X)\}$. Moreover, we denote with $P(v)$ the unique set of $P$ containing $v$ for a partition $P$ of $X$ and an element $v \in X$.
Let $(\mathcal{T}:=(\mathcal{V}, \mathcal{A}, r), \beta)$ be a tree decomposition of a graph $G$. Recall that fora node $x \in \mathcal{V}$, we define with $V_{x}$ the union of all bags $\beta(y)$, where $y$ is reachable from $x$ in $\mathcal{T}, G_{x}:=G\left[V_{x}\right]$, and $E_{x}:=E_{G}\left(V_{x}\right)$.
Let $P$ be a partition of $\beta(x)$, then we call an edge set $A \subseteq E_{x}$ a partition-cut for $P$ in $G_{x}$ if $v$ and $w$ are in different connected components in $G_{x}-A$ for every pair of distinct vertices $\{v, w\}$ of $\beta(x)$ with $P(v) \neq P(w)$. Note that all edges between distinct sets of $P$ are contained in every partition-cut for $P$ in $G_{x}$.
Theorem 4. Let $\operatorname{tw}(G)$ denote the treewidth of $G$. Then, ZWMCP can be solved in $a^{\operatorname{tw}(G)} \mathcal{O}(\operatorname{tw}(G)) \quad n+m$ time.
Proof. Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of ZWMCP. In the following, we assume that there is no edge $\{s, t\} \in E$ since if $c(\{s, t\}) \leq d$, then $\{\{s, t\}\}$ is a valid solution with cost at most $d$ and, thus, $I$ is a trivial yes-instance of ZWMCP. Otherwise, this edge is contained in every $(s, t)$-cut and, thus, we can simply remove the edge from the graph and reduce $a$ by $\omega(\{s, t\})$.

We describe a dynamic programming algorithm on a tree decomposition. First, we compute a nice tree decomposition $\left(\mathcal{T}=(\mathcal{V}, \mathcal{A}, r), \beta^{\prime}\right)$ of $G-\{s, t\}$ with $|\mathcal{V}| \leq 4 n$ such that the bag of the root and the bag of each leaf is the empty set in $\mathrm{tw} \mathcal{O}\left(\mathrm{tw}^{3}\right) \cdot n+m$ time [19, 4]. Next, we set $\beta(x):=\beta^{\prime}(x) \cup\{s, t\}$ for each $x \in \mathcal{V}$. Note that $(\mathcal{T}, \beta)$ is a tree decomposition of width at most tw +2 for $G$. Recall that for a node $x \in \mathcal{V}$, the vertex set $V_{x}$ is the union of all bags $\beta(y)$, where $y$ is reachable from $x$ in $\mathcal{T}, G_{x}:=G\left[V_{x}\right]$, and $E_{x}:=E_{G}\left(V_{x}\right)$.

The dynamic programming table $T$ has entries of type $T\left[x, f_{x}, D_{x}\right]$ with $x \in \mathcal{V}, f_{x}: B(\beta(x)) \rightarrow[0, a+1]$, and $D_{x} \subseteq$ $E(\beta(x))$. Each entry stores the minimal cost of an edge set $D \subseteq E_{x}$ with $D_{x}:=D \cap E(\beta(x))$ such that for every $P \in B(\beta(x))$ the capacity of every partition-cut $A \subseteq E_{x} \backslash D$ of $P$ in $G_{x}$ is at least $f_{x}(P)$.
For each entry of $T$, we will sketch the proof of the correctness of its recurrence. The formal correctness proof is then direct and thus omitted.
We start to fill the table $T$ by setting for each leaf node $\ell$ of $\mathcal{T}$ :

$$
T\left[\ell, f_{\ell}, \emptyset\right]:= \begin{cases}0 & \text { if } f_{\ell}(\{\{s\},\{t\}\})=f_{\ell}(\{\{s, t\}\})=0 \\ \infty & \text { otherwise }\end{cases}
$$

Recall that $\beta(\ell)=\{s, t\}$ and that we assumed that there is no edge between $s$ and $t$ in $G$. Hence, $G_{\ell}$ contains no edges and, thus, the empty set is a partition-cut for both $\{\{s\},\{t\}\}$ and $\{\{s, t\}\}$, and has capacity zero.
To compute the remaining entries $T\left[x, f_{x}, D_{x}\right]$, we distinguish between the three types of non-leaf nodes.
Forget node: Let $x$ be a forget node with child node $y$ and let $v$ be the unique vertex in $\beta(y) \backslash \beta(x)$. Then we compute the table entries for $x$ by:

$$
T\left[x, f_{x}, D_{x}\right]:=\min _{E_{v} \subseteq E(v, \beta(x))} T\left[y, f_{y}, D_{x} \cup E_{v}\right]
$$

where $f_{y}(P):=f_{x}(P-v)$ for each $P \in B(\beta(y))$.
The idea behind the definition of $f_{y}(P)$ is that every partition cut for $P$ in $G_{y}$ must be as expensive as the partition cut of the unique partition of $\beta(x)$ that agrees with $P$ on $\beta(x)$. By the fact that $G_{x}=G_{y}$, it follows that for each partition $P \in B(\beta(x))$, an edge set $A \subseteq E_{x}$ is a partition-cut for $P$ in $G_{x}$ if and only if $A$ is also a partition-cut for some $P^{\prime} \in P+v$ in $G_{x}$. Since we are looking for the minimal costs of an edge set $D \subseteq E_{x}$ such that every partition-cut disjoint from $D$ for $P-v$ in $G_{x}$ has capacity at least $f_{x}(P-v)$, it is thus necessary and sufficient that every partition-cut for $P$ in $G_{y}$ has capacity at least $f_{x}(P-v)$.
Introduce node: Let $x$ be an introduce node with child node $y$ and let $v$ be the unique vertex in $\beta(x) \backslash \beta(y)$. Then we compute the table entries for $x$ by:

$$
T\left[x, f_{x}, D_{x}\right]:=T\left[y, f_{y}, D_{x} \cap E(\beta(y))\right]+c\left(D_{x} \backslash E(\beta(y))\right)
$$

where $f_{y}(P):=\max \left(\{0\} \cup\left\{f_{x}\left(P^{\prime}\right)-\omega\left(A_{P^{\prime}}\right) \mid P^{\prime} \in(P+v), D_{x} \cap A_{P^{\prime}}=\emptyset\right\}\right)$ for each $P \in B(\beta(y))$ and $A_{P^{\prime}}:=$ $E\left(v, \beta(y) \backslash P^{\prime}(v)\right)$.
The idea behind the definition of $f_{y}(P)$ is that, since every partition in $P+v$ agrees with $P$ in $\beta(y)$, every partition cut for $P$ in $G_{y}$ must be sufficiently large to ensure that every partition cut for any partition in $P+v$ is as least as expensive as desired. Since we are looking for the minimum cost of an edge set $D \subseteq E_{x}$ which intersects with $E(\beta(x))$ in exactly the set $D_{x}$, the cost of $D$ is exactly $c(D \cap E(\beta(y)))+c\left(D_{x} \backslash E(\beta(y))\right)$. Let $P^{\prime} \in B(\beta(x))$. Note that $A_{P^{\prime}}$ is a subset of every partition-cut for $P^{\prime}$ in $G_{x}$. Hence, if $D_{x} \cap A_{P^{\prime}}=\emptyset$, then $f_{y}\left(P^{\prime}-v\right)$ has to be at least $f_{x}\left(P^{\prime}\right)-\omega\left(A_{P^{\prime}}\right)$. Otherwise, if $D_{x} \cap A_{P^{\prime}} \neq \emptyset$, then there is no partition-cut for $P^{\prime}$ in $G_{x}$ disjoint from $D$.
Join node: Let $x$ be a join node with child nodes $y$ and $z$. Then we compute the table entries for $x$ by:

$$
T\left[x, f_{x}, D_{x}\right]:=\min _{f_{y}: B(\beta(y)) \rightarrow[0, a+1]} T\left[y, f_{y}, D_{x}\right]+T\left[z, f_{z}, D_{x}\right]-c\left(D_{x}\right)
$$

where the mapping $f_{z}$ is given by

$$
f_{z}(P):=\max \left(0, \min \left(a+1, f_{x}(P)-f_{y}(P)+\omega(E(\beta(x)) \backslash E(P))\right)\right)
$$

with $E(P):=\cup_{R \in P} E(R)$ for each $P \in B(\beta(z))$.
The idea behind the definition of $f_{z}(P)$ is that the no partition cut for $P$ in $G_{z}$ is more expensive than the sum of any combination of partition cuts for $P$ in $G_{y}$ and $G_{z}$ minus the capacity of the cut-edges in the current bag. Recall that we are looking for the minimum cost of an edge set $D \subseteq E_{x}$ such that for each partition $P \in B(\beta(x))$, every partition-cut for $P$ in $G_{x}$ disjoint from $D$ has capacity at least $f_{x}(P)$. Since $E_{y} \cap E_{z}=E(\beta(x))$ it follows that the cost of $D$ is $c\left(S_{y}\right)+c\left(S_{z}\right)-c\left(D_{x}\right)$, where $S_{y}:=E_{y} \cap D$ and $S_{z}:=E_{z} \cap D$. Moreover, note that for every partition $P \in B(\beta(x))$,
every partition-cut $A_{\alpha} \subseteq E_{\alpha}$ for $P$ in $G_{\alpha}$ has to contain all edges of $E(\beta(x)) \backslash E(P)$, where $\alpha \in\{x, y, z\}$. Thus, we have to guarantee that $f_{y}(P)+f_{z}(P)-\omega(E(\beta(x)) \backslash E(P)) \geq f_{x}(P), f_{y}(P)>f_{x}(P)$, or $f_{z}(P)>f_{x}(P)$.
Then, there is a solution $D$ of cost at most $d$ of $I$ if and only if $T\left[r, f_{r}, \emptyset\right] \leq d$, where $r$ is the root of $\mathcal{T}, f_{r}(\{\{s, t\}\})=0$ and $f_{r}(\{\{s\},\{t\}\})=a+1$. Moreover, the corresponding set $D$ can be found via traceback. It remains to show the running time.
For every node $x$ of $\mathcal{T}$, there are $(a+2)^{|B(\beta(x))|} \cdot 2^{|\beta(x)|^{2}}$ entries. Since $(\mathcal{T}, \beta)$ has at most $4 n$ bags, each bag contains at most $k:=\mathrm{tw}+3$ vertices, and $|B(X)| \leq|X|^{|X|}$, the dynamic programming table contains at most $4 n \cdot(a+2)^{k^{k}} \cdot 2^{k^{2}}$ entries. Now, we bound the running times of the four types of bags.

- An entry for a leaf node can be computed in $\mathcal{O}(1)$ time.
- For a forget node, we can compute the function $f_{y}$ in $k^{k} \cdot k^{\mathcal{O}(1)}$ time and iterate over all possible choices for $E_{v}$ in $2^{k}$ time. Thus, an entry in $k^{k} \cdot 2^{k} \cdot k^{\mathcal{O}(1)}$ time.
- For an introduce node, we can compute the function $f_{y}$ in $k^{k} \cdot k^{\mathcal{O}(1)}$ time and thus the entry in the same running time.
- For a join node, we have $(a+2)^{k^{k}}$ possibilities for $f_{y}$ and for each of them, we can compute $f_{z}$ in $k^{k} \cdot k^{\mathcal{O}(1)}$ time. Hence, for a join node, we can compute an entry in $(a+2)^{k^{k}} \cdot k^{k} \cdot k^{\mathcal{O}(1)}$ time.

The join nodes have the worst running time for any entry. Thus, we can compute all entries of $T$ in $(a+2)^{2(\mathrm{tw}+3)^{\mathrm{tw}+3}}$. $(\mathrm{tw}+3)^{\mathrm{tw}+3} \cdot 2^{(\mathrm{tw}+3)^{2}} \cdot \mathrm{tw}^{\mathcal{O}(1)} \cdot n$ time and obtain the stated running time.

Next, we show that we can use Theorem 4 to obtain an FPT-algorithm for WMCP when parameterized by $a$. To this end, we first obtain the following corollary which follows from a result of Gutin et al. [16, Lemma 12].
Corollary 3. Let $G=(V, E)$ be a graph, let $s$ and $t$ be distinct vertices of $G$, and let a be an integer. If every edge $e \in E$ is contained in an inclusion-minimal $(s, t)$-cut of size at most $a$, then $\operatorname{tw}(G) \leq g(a)$ for some computable function $g$.

Hence, to obtain an FPT-algorithm for WMCP with the parameter $a$, we only have to find an equivalent instance in polynomial time where each edge is contained in some inclusion-minimal $(s, t)$-cut of size at most $a$. Since each edge in an instance of WMCP has capacity at least one, by applying Rule 1 exhaustively we obtain an equivalent instance of WMCP where each edge is contained in some inclusion-minimal $(s, t)$-cut of size at most $a$. Hence, we obtain the following by combining Lemma 3, Corollary 3, and Theorem 4.
Theorem 5. WMCP is FPT when parameterized by $a$.
Note that this is not possible for ZWMCP due to Theorem 6.
Theorem 6. ZWMCP is $\mathrm{W}[1]$-hard when parameterized by $d+a$ even if $\omega(e) \in\{0,1\}$ for all $e \in E$.
Proof. We describe a parameterized reduction from BICLIQUE which is known to be $\mathrm{W}[1]$-hard when parameterized by $k$ [10].

## Biclique

Input: A bipartite graph $G=(X \cup Y, E)$ with partite sets $X$ and $Y$ and an integer $k$.
Question: Does $G$ contain a $(k, k)$-biclique?
Let $I=(G=(X \cup Y, E), k)$ be an instance of BICLIQUE. Now, we describe how to construct an instance $I^{\prime}=$ $\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), s, t, c, \omega, d, a\right)$ of ZWMCP in polynomial time such that $I$ is a yes-instance of BICLIQUE if and only if $I^{\prime}$ is a yes-instance of ZWMCP.
The graph $G^{\prime}$ contains the graph $G$ as a copy together with two new vertices $s$ and $t$ and edges $F:=\{\{s, x\} \mid x \in$ $X\} \cup\{\{y, t\} \mid y \in Y\}$. Furthermore, each edge $\{x, y\} \in E$ is subdivided by a new vertex $w_{x y}$ in $G^{\prime}$. Hence, the graph $G^{\prime}$ contains the edges $\left\{x, w_{x y}\right\}$ and $\left\{w_{x y}, y\right\}$ instead of the edge $\{x, y\}$. We define $E_{X}:=\left\{\left\{x, w_{x y}\right\} \mid x \in X\right\}$ and $E_{Y}:=\left\{\left\{w_{x y}, y\right\} \mid y \in Y\right\}$. We set $d:=(2 k+1) k^{2}+2 k=2 k^{3}+k^{2}+2 k$ and for each edge $\{x, y\} \in E$ we set $c\left(\left\{x, w_{x y}\right\}\right):=d+1, \omega\left(\left\{x, w_{x y}\right\}\right):=1, c\left(\left\{w_{x y}, y\right\}\right):=2 k+1$, and $\omega\left(\left\{w_{x y}, y\right\}\right):=0$. Furthermore, for each edge $e \in F$ we set $c(e):=1$, and $\omega(e):=0$. Finally, we set $a:=k^{2}-1$ which completes the construction of $I^{\prime}$.
$(\Rightarrow)$ Suppose that $I$ is a yes-instance of Biclique. Then there exist sets $S_{X} \subseteq X$ and $S_{Y} \subseteq Y$ of size $k$ each, such that $\{x, y\} \in E$ for all $x \in S_{X}$ and $y \in S_{Y}$. We set $D:=\left\{\left\{w_{x y}, y\right\} \mid x \in S_{X}, y \in S_{Y}\right\} \cup\{\{s, x\} \mid x \in$ $\left.S_{X}\right\} \cup\left\{\{y, t\} \mid y \in S_{Y}\right\}$. Observe that $|D|=(2 k+1) k^{2}+2 k=d$. Next, we show that $D$ is a solution of $I^{\prime}$.

Consider for each $x \in S_{X}$ and for each $y \in S_{Y}$ the $(s, t)$-path $\left(s, x, w_{x y}, y, t\right)$. Since $\{s, x\} \in D,\left\{w_{x y}, y\right\} \in D$, and $\{y, t\} \in D$, each $(s, t)$-cut $A$ has to contain the edge $\left\{x, w_{x y}\right\}$. By the fact that $\omega\left(\left\{x, w_{x y}\right\}\right)=1$ for each $x \in S_{X}$ and each $y \in S_{Y}$ we observe that $\omega(A) \geq\left|S_{X} \times S_{Y}\right|=k^{2}=a+1$. Hence, $I^{\prime}$ is a yes-instance of ZWMCP.
$(\Leftarrow)$ Let $D$ be a solution with cost at most $d$ of $I^{\prime}$. Observe that for each edge $e \in E_{X}$ we have $d(e)=d+1$. Thus, $E_{X} \cap D=\emptyset$. Furthermore, observe that for each edge $e \in E_{Y}$ we have $d(e)=2 k+1$. Hence, for the set $E_{D}:=D \cap E_{Y}$ we conclude that $\left|E_{D}\right| \leq k^{2}$. Next, we define the vertex sets $S_{X}:=\{x \mid\{s, x\} \in D\}$, the set of endpoints of edges in $D$ incident with $s$, and $S_{Y}:=\{y \mid\{y, t\} \in D\}$, the set of endpoints of edges in $D$ incident with $t$. In the following, we describe an $(s, t)$-cut $A$ for $G^{\prime}$ that avoids $D$. We partition $A$ into two sets $A_{0}$ and $A_{1}$, where $A_{0}:=\left(E_{Y} \cup F\right) \backslash D$ and $A_{1}:=\left\{\left\{x, w_{x y}\right\} \mid\left\{w_{x y}, y\right\} \in E_{D}\right\}$. Next, we show that $A$ is an $(s, t)$-cut for $G^{\prime}$ :
Observe that every $(s, t)$-path contains at least one subpath $\left(x, w_{x y}, y\right)$ for a vertex $x \in X$ and a vertex $y \in Y$ as an induced subgraph. If $\left\{w_{x y}, y\right\} \in E_{D}$ then $\left\{x, w_{x y}\right\} \in A_{1}$, and otherwise if $\left\{w_{x y}, y\right\} \notin D_{S}$ then $\left\{w_{x y}, y\right\} \in A_{0}$. Furthermore, observe that the $(s, t)$-cut $A$ has capacity $\omega\left(A_{0}\right)+\omega\left(A_{1}\right)=\omega\left(A_{1}\right)=\left|E_{D}\right| \leq k^{2}$. Since $D$ is a solution of $I^{\prime}$, we conclude that $A$ has capacity at least $a+1=k^{2}$ and, thus, $\left|E_{D}\right|=\omega(A)=k^{2}$. Thus, $|D \cap F| \leq 2 k$.

Next, assume towards a contradiction that the set $E_{D}$ contains an edge $\left\{w_{x y}, y\right\}$ such that $x \notin S_{X}$ or $y \notin S_{Y}$. Without loss of generality assume that $y \notin S_{Y}$. We set $A^{*}:=A \backslash\left\{\left\{x, w_{x y}\right\}\right\}$ and show that $A^{*}$ is an $(s, t)$-cut in $G^{\prime}$. Since $\{y, t\} \in A^{*}$ and for each $x^{\prime} \in N_{G}(y) \backslash\{x\}$ either $\left\{x^{\prime}, w_{x^{\prime} y}\right\} \in A^{*}$ or $\left\{w_{x^{\prime} y}, y\right\} \in A^{*}$, we obtain that $A^{*}$ is an $(s, t)$-cut of capacity $\omega(A)-\omega\left(\left\{x, w_{x y}\right\}\right)=k^{2}-1=a$. A contradiction. Hence, for each edge $\left\{w_{x y}, y\right\} \in D$ we have $\{s, x\} \in D$ and $\{y, t\} \in D$. Since $\left|E_{D}\right|=k^{2}$, we conclude that $\left|S_{X}\right|=k=\left|S_{Y}\right|$. Consequently, $\left(S_{X}, S_{Y}\right)$ is a $(k, k)$-biclique in $G$ and, thus, $I$ is a yes-instance of BICLIQUE.

Together with Corollary 1 we obtain the following.
Corollary 4. ZWMCP is $\mathrm{W}[1]$-hard when parameterized by $d+a$ even if $c(e)=1$ and $\omega(e) \in\{0,1\}$ for all $e \in E$.

## 5 Parameterization by Vertex Cover Number

We investigate the parameterization by the vertex cover number $\operatorname{vc}(G)$. Observing that for MCP the number of protected edges $d$ is at most $2 \mathrm{vc}(G)$ in nontrivial instances, eventually leads to the following FPT result.
Theorem 7. MCP can be solved in $2^{\mathcal{O}\left(\mathrm{vc}(G)^{2}\right)} \cdot n^{\mathcal{O}(1)}$ time.

Proof. Let $J:=(G, s, t, d, a)$ be an instance of MCP, and let vc be the size of a minimum vertex cover of $G$. The algorithm that we describe here is based on two observations which we formalize in two claims. The first claim states that the defender budget $d$ is upper bounded by $2 \cdot \mathrm{vc}$.

Claim 3. If $d \geq 2 \cdot \mathrm{vc}$, then $J$ is a yes-instance.
Proof. Let $S$ be a minimum vertex cover in $G$ and let $P$ be a shortest $(s, t)$-path in $G$. Then, for every pair $v, w$ of consecutive vertices on $P$ at least one of $v$ and $w$ is contained in $S$. Consequently, there are at most $2 \cdot \mathrm{vc}(G)$ edges on $P$. If $d \geq 2 \cdot \operatorname{vc}(G)$, then the set of edges on $P$ is a solution of $J$ of size at most $d$. Thus, $J$ is a yes-instance.

Let $S$ be a minimum vertex cover in $G$, and let $I:=V \backslash S$ be the remaining independent set. With the next claim we state that only a bounded number of vertices in $I$ is needed to find a minimal solution of $J$. To this end we introduce some notation: Given a subset $X \subseteq S$, we let $I_{X}:=\left\{u \in I \mid N_{G}(u)=X\right\} \subseteq I$ denote the neighborhood class of $X$. Moreover, we let $E_{X}$ denote the set of edges between $X$ and $I_{X}$.
Claim 4. There exists a minimum solution $D$ of $J$ such that $\left|D \cap E_{X}\right| \leq|X|$ for every $X \subseteq D$.
Proof. Let $D$ be a solution of $J$. If $\left|D \cap E_{X}\right| \leq|X|$ for every $X \subseteq S$, nothing more needs to be shown. Thus, consider some $X \subseteq S$ such that $\left|D \cap E_{X}\right|>|X|$, and let $u \in I_{X}$. We then define $D^{\prime}:=\left(D \backslash E_{X}\right) \cup E(u, X)$. It then holds that $\left|D^{\prime} \cap E_{X}\right|=|X|$. Moreover, observe that $\left|D^{\prime}\right|<|D|$ and $D^{\prime} \backslash E_{X}=D \backslash E_{X}$.
We next show that $D^{\prime}$ is a solution of $J$. Let $A \subseteq E \backslash D^{\prime}$ be an $(s, t)$-cut in $G$ that is minimum among all $(s, t)$-cuts that avoid $D^{\prime}$. We first prove that $A \cap E_{X}=\emptyset$. Obviously, $E(u, X) \cap A=\emptyset$ since $E(u, X) \subseteq D^{\prime}$. Consider $u^{\prime} \in I_{X} \backslash\{u\}$. Then, since $N\left(u^{\prime}\right)=X$, for every $(s, t)$-path $P^{\prime}$ containing $u$, there are two consecutive edges $\left\{x_{1}, u^{\prime}\right\}$ and $\left\{u^{\prime}, x_{2}\right\}$
with $x_{1}, x_{2} \in X$ on $P^{\prime}$. Since $N\left(u^{\prime}\right)=N(u)$, replacing $u^{\prime}$ with $u$ defines another $(s, t)$-path $P$ in $G^{\prime}$. Then, since $\left\{x_{1}, u\right\}$ and $\left\{u, x_{2}\right\}$ are not contained in $A$, there exists another edge on $P$ that is an element of $A$. Consequently, on every $(s, t)$-path $P^{\prime}$ containing $u^{\prime}$, there exists an edge in $A$ that is not an element of $E\left(u^{\prime}, X\right)$. Then, the fact that $A$ is a minimum $(s, t)$-cut among all $(s, t)$-cuts that avoid $D^{\prime}$ implies $E\left(u^{\prime}, X\right) \cap A=\emptyset$. Therefore, $A \cap E_{X}=\emptyset$.
Then, since $A \cap E_{X}=\emptyset$ and $D^{\prime} \backslash E_{X}=D \backslash E_{X}$ we have $A \subseteq E \backslash D$. Consequently, $|A|>a$ since $D$ is a solution of $J$.

Since $D^{\prime} \backslash E_{X}=D \backslash E_{X}$, the modification of $D$ described above can be applied on all neighborhood classes $I_{X}$ independently. Therefore, there exists a minimum solution of $J$ that has the described property.

Let $X \subseteq S$ and $I_{X}:=\left\{v_{1}, \ldots, v_{\left|I_{X}\right|}\right\}$. We define $I_{X}^{\prime}$ by $I_{X}^{\prime}:=I_{X}$ if $\left|I_{X}\right| \leq|X|$ and $I_{X}^{\prime}:=\left\{v_{1}, \ldots, v_{|X|}\right\}$, otherwise. Due to Claim 4, there exists a minimum solution such that at most $|X|$ vertices in $I_{X}$ are endpoints of edges in $S$. Without loss of generality we may assume that all of these endpoints are from $I_{X}^{\prime}$. Thus, we can assume that there is a minimum solution $D \subseteq E(S) \cup \bigcup_{X \subseteq S} E\left(X, I_{X}^{\prime}\right)$. We use this assumption for the algorithm that we describe as follows.

1. If $d \geq 2 \cdot \operatorname{vc}(G)$, then return yes.
2. Otherwise, we compute a minimum vertex cover $S$. Iterate over every possible edge-set $D \subseteq E(S \cup$ $\bigcup_{X \subseteq S} E\left(X, I_{X}^{\prime}\right)$ with $|D| \leq d$, and check with the algorithm behind Lemma 1 that every ( $s, t$ )-cut $A \subseteq E \backslash S$ in $G$ has size bigger than $a$. If this is the case, then return yes.
3. If for none of the choices of $D$ the answer yes was returned in Step 2, then return no.

The correctness of the algorithm is implied by Claims 3 and 4. It remains to analyze the running time. Obviously, Step 1 and Step 3 can be performed in linear time. Consider Step 2. A minimum vertex cover can be computed in $\mathcal{O}\left(1.28^{\mathrm{vc}}+n \cdot \mathrm{vc}\right)$ time [7]. Next, observe that

$$
\begin{aligned}
\left|E\left(S \cup \bigcup_{X \subseteq S} I_{X}^{\prime}\right)\right| & \leq \mathrm{vc}^{2}+\sum_{i=0}^{\mathrm{vc}}\binom{\mathrm{vc}}{i} i^{2} \\
& \leq \mathrm{vc}^{2}\left(1+2^{\mathrm{vc}-1}\right) .
\end{aligned}
$$

Since $d<2 \cdot \mathrm{vc}$, there are less than $\left(\mathrm{vc}^{2}\left(1+2^{\mathrm{vc}-1}\right)\right)^{2 \mathrm{vc}}$ possible subsets $D \subseteq E\left(S \cup \bigcup_{X \subseteq S} I_{X}^{\prime}\right)$ with $|D| \leq d$. Together with the running time from Lemma 1, Step 2 can be performed in $\left(\mathrm{vc}^{2}\left(1+2^{\mathrm{vc}-1}\right)\right)^{2 \mathrm{vc}} \cdot n^{\mathcal{O}(1)}$ time. Altogether, the algorithm runs within the claimed running time.

Theorem 4 implies that WMCP can be solved in pseudopolynomial time on graphs with a constant treewidth and therefore on graphs with a constant vertex cover number. With the next two theorems we show that significant improvements of this result are presumably impossible.
Theorem 8. WMCP is weakly NP-hard on graphs with a vertex cover of size two .
Proof. We describe a polynomial time reduction from KNAPSACK which is known to be weakly NP-hard [14].

## KnAPSACK

Input: A set $U$, a size function $f: U \rightarrow \mathbb{N}$, a value function $g: U \rightarrow \mathbb{N}$, and two budgets $B, C \in$ $\mathbb{N}$.
Question: Is there a set of items $S \subseteq U$ such that $f(S):=\sum_{u \in S} f(u) \leq B$ and $g(S):=$ $\sum_{u \in S} g(u) \geq C$ ?

Let $I:=(U, f, g, B, C)$ be an instance of KNAPSACK. We describe how to construct an equivalent instance $I^{\prime}:=$ ( $G=(V, E), s, t, c, \omega, d, a)$ of WMCP where $G$ has a vertex cover of size two in polynomial time.
We set $V:=U \cup\{s, t\}$ and $E:=\{\{s, u\},\{u, t\} \mid u \in U\}$. Note that $\{s, t\}$ is a vertex cover of size two in $G$. Next, we set $d:=B$ and $a:=|U|+C-1$. Finally, for every $u \in U$ we set $c(\{s, u\}):=f(u), \omega(\{s, u\}):=1$, $c(\{u, t\}):=d+1$, and $\omega(\{u, t\}):=g(u)+1$.
Next, we show that $I$ is a yes-instance of KnAPSACK if and only if $I^{\prime}$ is a yes-instance of WMCP.
$(\Rightarrow)$ Suppose that $I$ is a yes-instance of KnAPSACK. Then, there is a set $S_{U} \subseteq U$ such that $f\left(S_{U}\right) \leq B=d$ and $g\left(S_{U}\right) \geq C$. We set $D:=\left\{\{s, u\} \mid u \in S_{U}\right\}$. By construction, we obtain that $c(D)=f\left(S_{U}\right) \leq d$. Let $\bar{A} \subseteq E \backslash D$ be an $(s, t)$-cut. We show that $A$ has capacity larger than $a$.

Since $\{s, u\} \in D$ for all $u \in S_{U}$ it holds that $T:=\left\{\{u, t\} \mid u \in S_{U}\right\} \subseteq A$. Note that $\omega(T)=\sum_{u \in S_{U}}(g(u)+1)=$ $g\left(S_{U}\right)+\left|S_{U}\right|$. Moreover, because of the path $(s, u, t)$ for every $u \in U \backslash S_{U}$ we obtain that $\{s, u\} \in A$ or $\{u, t\} \in A$. Since both of these edges have capacity at least one, we obtain $\omega(A) \geq \omega(T)+\left|U \backslash S_{U}\right|=g\left(S_{U}\right)+|U|=a+1$. Consequently, $I^{\prime}$ is a yes-instance of WMCP.
$(\Leftarrow)$ Suppose that $I^{\prime}$ is a yes-instance of WMCP. Then, there is a solution $D \subseteq E$ with $c(D) \leq d$. By the fact that $c(\{u, t\})=d+1$ for all $u \in U$, it follows that $D \subseteq\{\{s, u\} \mid u \in U\}$.
Let $S_{U}:=\{u \in U \mid\{s, u\} \in D\}$. By construction, $f\left(S_{U}\right)=c(D) \leq d=B$. We show that $g\left(S_{U}\right) \geq C$. Let $A \subseteq E \backslash D$ be an $(s, t)$-cut of minimum capacity. Recall that $\omega(A) \geq a+1=|U|+C$. Since $A$ is an $(s, t)$-cut in $G$ and disjoint to $D$, we know that $\{u, t\} \in A$ for all $u \in S_{U}$. Moreover, since $\omega(\{s, u\})=1 \leq \omega(\{u, t\})$ for all $u \in U$, we can assume without loss of generality, that $\{s, u\} \in A$ for all $u \in U \backslash S_{U}$. Hence, $a+1=|U|+C \leq$ $\omega(A)=\left|U \backslash S_{U}\right|+\sum_{u \in S_{U}}(g(u)+1)=g\left(S_{U}\right)+|U|$. Thus, $C \leq g\left(S_{U}\right)$. Consequently, $I$ is a yes-instance of KnAPSACK.

Theorem 9. WMCP is $\mathrm{W}[1]$-hard when parameterized by the vertex cover number $\mathrm{vc}(G)$ even if $c(e)+\omega(e) \in n^{\mathcal{O}(1)}$ and the graph is a biclique.

Proof. We describe a parameterized reduction from Bin PACKING which is W[1]-hard when parameterized by $k$ even if the size of each item is polynomial in the input size [18].

## Bin Packing

Input: A set $U$ of items, a size-function $f: U \rightarrow \mathbb{N}$, and integers $B$ and $k$.
Question: Is there a $k$-partition $\left(U_{1}, \ldots, U_{k}\right)$ of $U$ with $\sum_{u \in U_{i}} f(u)=B$ for all $i \in[1, k]$ ?
Let $I:=(U, f, B, k)$ be an instance of BIN PACKING where the size of each item is polynomial in the input size. We can assume without loss of generality that $\sum_{u \in U} f(u)=B k$, as, otherwise, $I$ is a trivial no-instance of Bin Packing. We construct an equivalent instance $I^{\prime}:=(G=(V, E), s, t, c, \omega, d, a)$ of WMCP where $G$ has a vertex cover of size $k+1$. The graph $G$ is a biclique with bipartition $(\{s\} \cup \mathcal{B},\{t\} \cup U)$ where $\mathcal{B}:=\left\{b_{1}, \ldots, b_{k}\right\}$. We set $d:=|U|$, and

$$
c(e):= \begin{cases}1 & \text { if } e \in\{\{u, b\} \mid u \in U, b \in \mathcal{B}\}, \text { and } \\ d+1 & \text { otherwise }\end{cases}
$$

Let $\lambda:=2 B \cdot|U|$, we set

$$
\omega(e):= \begin{cases}\lambda \cdot f(u) & \text { if } e=\{s, u\} \text { with } u \in U \\ \lambda \cdot B & \text { if } e=\{t, b\} \text { with } b \in \mathcal{B}, \text { and } \\ 1 & \text { otherwise }\end{cases}
$$

Finally, we set $a:=|U| \cdot k+\lambda(B k-1)$. This completes the construction of $I^{\prime}$. Figure 1 shows an example of the construction. Note that $\{s\} \cup \mathcal{B}$ is a vertex cover of $G$ of size $k+1$. It remains to show that $I$ is a yes-instance of Bin PACKING if and only if $I^{\prime}$ is a yes-instance of WMCP.
$(\Rightarrow)$ Suppose that $I$ is a yes-instance of Bin Packing. Then, there is a $k$-partition $\left(U_{1}, \ldots, U_{k}\right)$ of $U$, such that $\sum_{u \in U_{i}} f(u)=B$ for all $i \in[1, k]$. We set $D:=\left\{\left\{u, b_{i}\right\} \mid i \in[1, k], u \in U_{i}\right\}$. Note that $c(D)=d$. We next show that $D$ is a solution.
Let $A \subseteq E \backslash D$ be an $(s, t)$-cut in $G$ and let $i \in[1, k]$. Since for each $u \in U_{i}, D$ contains the edge $\left\{u, b_{i}\right\}$, the $(s, t)$-path $P_{u}:=\left(s, u, b_{i}, t\right)$ can only be cut if $\{s, u\} \in A$ or $\left\{b_{i}, t\right\} \in A$. Consequently, $\left\{\{s, u\} \mid u \in U_{i}\right\} \subseteq A$ or $\left\{b_{i}, t\right\} \in A$. Recall that $\sum_{u \in U_{i}} f(u)=B$. Hence, $\sum_{u \in U_{i}} \omega(\{s, u\})=\sum_{u \in U_{i}} \lambda f(u)=\lambda B=\omega\left(\left\{b_{i}, t\right\}\right)$. Since $P_{u}$ and $P_{w}$ are edge-disjoint if $u$ and $w$ are in distinct parts of the $k$-partition, we obtain that $\omega(A) \geq k \lambda B>a$ and, thus, $I^{\prime}$ is a yes-instance of WMCP.
$(\Leftarrow)$ Suppose that $I^{\prime}$ is a yes-instance of WMCP. Then, there is a solution $D \subseteq E$ with $c(D) \leq d$. By construction, $D \subseteq E(U, \mathcal{B})$, since all other edges have cost $d+1$.
Note that for each $u \in U$, there is some $b \in \mathcal{B}$, such that $\{u, b\} \in D$, as, otherwise $A:=\{\{s, t\}\} \cup\left\{\left\{s, u^{\prime}\right\} \mid u^{\prime} \in\right.$ $\left.U \backslash\{u\}\} \cup\left\{\left\{u, b^{\prime}\right\} \mid b^{\prime} \in \mathcal{B}\right\}\right\}$ is an $(s, t)$-cut in $G$ with capacity $\lambda(B k-f(u))+k+1<a$. Since $|D| \leq d$, we obtain that for each $u \in U$, there is exactly one $b \in \mathcal{B}$, such that $\{u, b\} \in D$.


Figure 1: An example of the construction from the proof of Theorem 9 for a Bin PaCKING instance with $f\left(u_{1}\right)=$ $f\left(u_{4}\right)=4, f\left(u_{2}\right)=1, f\left(u_{3}\right)=3, B=4$, and $k=3$. The thick edges represent a minimum solution $D$. The edge-labels represent all edge capacities that are bigger than one. Observe that every ( $s, t$ )-cut avoiding $D$ contains dashed edges that have a capacity sum of at least $12 \lambda$.

We set $U_{i}:=\left\{u \in U \mid\left\{u, b_{i}\right\} \in D\right\}$ for all $i \in[1, k]$. By the above, we obtain that $\left(U_{1}, \ldots, U_{k}\right)$ is a $k$-partition of $U$. We show that $\sum_{u \in U_{i}} f(u)=B$ for all $i \in[1, k]$.

Assume towards a contradiction that $\sum_{u \in U_{i}} f(u) \neq B$ for some $i \in[1, k]$. This is the case if and only if there is some $j \in[1, k]$ with $\sum_{u \in U_{j}} f(u)<B$. We set $A:=\{\{s, t\}\} \cup\left\{\{s, u\} \mid u \in U_{j}\right\} \cup\left\{\{b, t\} \mid b \in \mathcal{B} \backslash\left\{b_{j}\right\}\right\} \cup(E(U, \mathcal{B}) \backslash$ $D)$. Note that $\omega(A)=1+\lambda\left(\sum_{u \in U_{j}} f(u)\right)+\lambda B(k-1)+|U| \cdot(k-1) \leq \lambda(B-1)+\lambda B(k-1)+|U| \cdot k=$ $\lambda(B k-1)+|U| \cdot k=a$, since $\sum_{u \in U_{j}} f(u)<B$. It remains to show that $A$ is an $(s, t)$-cut in $G$. Observe that $N_{G-A}(t)=b_{j}$. Since $N_{G-A}\left(b_{j}\right)=\{t\} \cup U_{j}$ and $N_{G-A}(u)=\left\{b_{j}\right\}$ for each $u \in U_{j}$, we conclude that $A$ is indeed an $(s, t)$-cut in $G$. This contradicts the fact that there is no $(s, t)$-cut disjoint to $D$ in $G$ of capacity at most $a$. As a consequence, $\sum_{u \in U_{i}} f(u)=B$ for all $i \in[1, k]$ and, thus, $I$ is a yes-instance of Bin Packing.

We use Theorem 9 to extend the $\mathrm{W}[1]$-hardness to $\mathrm{pw}(G)+\mathrm{fvs}(G)$ and thus $\mathrm{vc}(G)$ in the running time stated in Theorem 7 can presumably not be replaced by $\operatorname{pw}(G)+\operatorname{fvs}(G)$. Hence, MCP is also W[1]-hard parameterized by $\operatorname{tw}(G)$.
Theorem 10. MCP is $\mathrm{W}[1]$-hard when parameterized by $\operatorname{pw}(G)+\operatorname{fvs}(G)$, where $\operatorname{pw}(G)$ denotes the pathwidth of the input graph and $\operatorname{fvs}(G)$ denotes the size of the smallest feedback vertex set of the input graph.

Proof. We reduce from WMCP which, due to Theorem 9, is W[1]-hard when parameterized by the vertex cover number $\operatorname{vc}(G)$ even if $c(e)+\omega(e) \in n^{\mathcal{O}(1)}$ and the graph is a biclique.

Let $I=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of WMCP where $c(e)+\omega(e) \in n^{\mathcal{O}(1)}$ and the graph is a biclique. Moreover, let $I^{\prime}=\left(G^{\prime}=\left(V^{\prime}, E^{\prime}\right), s, t, d, a\right)$ be the equivalent instance of MCP we obtain obtain in polynomial time by applying the construction leading to Corollary 1 . We show that both the size of the smallest feedback vertex set and the pathwidth of $G^{\prime}$ are upper-bounded by a function only depending on $\operatorname{vc}(G)$.
Let $(X, Y)$ be the bipartition of $G$ and let $X$ be the smaller part. Thus, vc $(G)=|X|$. Moreover, let $x_{1}, \ldots, x_{|X|}$ be the elements of $X$ and let $y_{1}, \ldots, y_{|Y|}$ be the elements of $Y$. Recall that we obtain $I^{\prime}$ by replacing every edge $e=$ $\{u, v\} \in E$ by a subgraph $G_{e}=\left(V_{e}, E_{e}\right)$ which consists of vertex disjoint $(u, v)$-paths (besides $u$ and $v$ ).
Note that $G_{e}$ has a path decomposition $\mathcal{B}_{e}$ of width at most three where each bag contains both endpoints of $e$. For every $y \in Y$ we set $\mathcal{B}_{y}:=\mathcal{B}_{\left\{x_{1}, y\right\}} \cdot \ldots \cdot \mathcal{B}_{\left\{x_{|X|}, y\right\}}$. Note that $\mathcal{B}_{y}$ is a path decomposition of width at most three for $G_{y}=\left(\bigcup_{x \in X} V_{\{x, y\}}, \bigcup_{x \in X} E_{\{x, y\}}\right)$.
Finally, let $\mathcal{B}:=\mathcal{B}_{y_{1}} \cdots \cdots \mathcal{B}_{y_{|Y|}}$ and let $\mathcal{B}^{\prime}$ be the sequence of bags we obtain from $\mathcal{B}$ by adding all vertices of $X$ to each of the bags of $\mathcal{B}$. By construction, $\mathcal{B}^{\prime}$ is a path decomposition of width at most $|X|+3$ for $G^{\prime}$. Hence, $\mathrm{pw}\left(G^{\prime}\right) \leq$ $|X|+3=\operatorname{vc}(G)+3$.
It remains to show that $\operatorname{fvs}\left(G^{\prime}\right) \leq \operatorname{vc}(G)$. Note that $G_{\{x, y\}}-\{x\}$ is acyclic for each $x \in X$ and $y \in Y$. Hence, $G^{\prime}-X$ is acyclic since $Y$ is an independent set in $G$ and for each pair of distinct edges $e_{1}, e_{2} \in E$ it holds that $V_{e_{1}} \cap V_{e_{2}}=$ $e_{1} \cap e_{2}$. Consequently, $\mathrm{fvs}\left(G^{\prime}\right) \leq \operatorname{vc}(G)$ and, thus, MCP is $\mathrm{W}[1]$-hard when parameterized by $\mathrm{pw}\left(G^{\prime}\right)+\mathrm{fvs}\left(G^{\prime}\right)$.

## 6 On Problem Kernelization

### 6.1 A Polynomial Kernel for $\mathrm{vc}+a$

On the positive side, we show that WMCP admits a polynomial kernel when parameterized by vc $+a$. The main tool for this kernelization is the merge of vertices according to Definition 1.

Let $J:=(G=(V, E), s, t, c, \omega, d, a)$ be an instance of WMCP. We first provide two simple reduction rules that remove degree-one vertices.
Rule 2. If $s$ has exactly one neighbor $w$ and $\omega(\{s, w\}) \leq a$, then delete $s$, set $s:=w$, and decrease $d$ by $c(\{s, w\})$. Analogously, if t has exactly on neighbor $v$ and $\omega(\{t, v\}) \leq a$, then delete $t$, set $t:=v$, and decrease $d$ by $c(\{t, v\})$.

The safeness of Rule 2 follows by the observation that, if $s$ (or $t$, respectively) is incident with a unique edge $e$ with $\omega(e) \leq a$, this edge must be part of every solution, since $M:=\{e\}$ is an $(s, t)$-cut of capacity at most $a$.
Rule 3. If there exists a degree-one vertex $v \notin\{s, t\}$, then delete $v$.
It is easy to see that Rule 3 is safe. Since $v \neq s$ and $v \neq t$, the single edge incident with $v$ is not contained in any inclusion-minimal $(s, t)$-cut and therefore not part of any minimal solution. The next reduction rule is the main idea behind the problem kernelization.
Rule 4. If there are vertices $u, v \in V$ such that a minimum $(u, v)$-cut has capacity at least $a+1$, then merge $u$ and $v$.
Lemma 9. Rule 4 is safe.
Proof. Recall that due to Lemma 3 we can safely merge edges that are not contained in any inclusion-minimal ( $s, t$ )cut of capacity at most $a$. We show the safeness of Rule 4 by applying Lemma 3 on the vertex pair $\{u, v\}$. Note that $u$ and $v$ are not necessarily adjacent in $G$. Thus, we first transform $J$ into an instance $J^{\prime}$ by adding an edge $\{u, v\}$ with cost $d+1$ and capacity $a+1$. Let $c^{\prime}$ and $\omega^{\prime}$ be the cost functions and capacity functions of $J^{\prime}$. We next show that $J$ is a yes-instance if and only if $J^{\prime}$ is a yes-instance.
$(\Rightarrow)$ Let $S$ be a solution of $J$ with $c(S) \leq d$. Since adding an edge might only increase the size of a cut, $S$ is a solution of $J^{\prime}$.
$(\Leftarrow)$ Let $S^{\prime}$ be a solution of $J^{\prime}$ with $c^{\prime}(S) \leq d$. Then, $\{u, v\} \notin S^{\prime}$ since $c^{\prime}(\{u, v\})=d+1$. We show that $S^{\prime}$ is a solution of $J$. Let $M \subseteq E \backslash S^{\prime}$ be an inclusion-minimal $(s, t)$-cut in $G$. We consider the corresponding partition $(A, B)$ of $V$. If $u \in A$ and $v \in B$ or vice versa, then $M$ is an $(u, v)$-cut in $G$ and therefore $\omega(M) \geq a+1$ by the condition of Rule 4. Otherwise, if $u$ and $v$ belong to the same partite set, then $M$ is an $(s, t)$-cut in $G^{\prime}$. Since $S$ is a solution of $J^{\prime}$ we conclude $\omega(M) \geq a+1$.

Thus, the instances $J$ and $J^{\prime}$ are equivalent. Note that $\omega^{\prime}(\{u, v\})=a+1$ implies that $\{u, v\}$ is not contained in any inclusion-minimal $(s, t)$-cut of capacity at most $a$. Then, Lemma 3 implies that $u$ and $v$ can safely be merged, which proves the safeness of Rule 4.

We now assume that $J$ is reduced regarding Rules 2-4. Before we show that the number of edges in $G$ is at most 2 . $\operatorname{vc}(G) \cdot a$, we observe that there is no degree-one vertex in $G$ : Since $J$ is reduced regarding Rule 3, every vertex in $V \backslash\{s, t\}$ has degree at least two. Furthermore, since $J$ is reduced regarding Rule 2 the vertices $s$ and $t$ are not incident with a unique edge of capacity at most $a$. Finally, since $J$ is reduced regarding Rule 4, the vertices $s$ and $t$ are not incident with a unique edge $\{s, u\}$ (or $\{t, v\}$, respectively) of capacity at least $a+1$ since the vertices $u$ and $v$ (or $t$ and $v$ ) would have been merged by Rule 4 .

To show that the number of edges in $J$ is at most $2 \cdot \mathrm{vc}(G) \cdot a$, we introduce cut trees which are special binary trees. Throughout this section, given an inner vertex $x$ of a binary tree, we let $x_{\ell}$ denote its left child and $x_{r}$ denote its right child.
Definition 2. Let $G=(V, E)$ be a graph with a capacity function $\omega: E \rightarrow \mathbb{N}$, let $S \subseteq V$ be a vertex cover of $G$. Let $T=(\mathcal{V}, \mathcal{E})$ be a binary tree with root vertex $r \in \mathcal{V}$ and $\psi: \mathcal{V} \rightarrow 2^{V}$. Then, $(T, \psi)$ is a cut tree of $G$ with respect to $S$ if

1. $\psi(r)=V$,
2. for every vertex $x \in \mathcal{V}$ with $|\psi(x) \cap S| \geq 2$, there exist vertices $u, v \in \psi(x) \cap S$ and a minimum $(u, v)$-cut $M$ in $G[\psi(x)]$ with partitions $(A, B)$ such that $\psi\left(x_{\ell}\right)=A$ and $\psi\left(x_{r}\right)=B$, and
3. every vertex $x \in \mathcal{V}$ with $|\psi(x) \cap S|=1$ is a leaf.

Recall that we consider a reduced instance $J$ with input graph $G$. In the following, let $S$ be a minimum vertex cover of $G$. We consider a cut tree $(T, \psi)$ of $G$ with respect to $S$. Observe that there is no inner vertex of $T$ that has exactly one child, and that $\{\psi(x) \mid x$ is a leaf of $T\}$ is a partition of $V$, where each set of the partition contains exactly one vertex from $S$. Thus, if $S$ is a minimum vertex cover, then T consists of at most $\mathrm{vc}(G)$ inner vertices and $\mathrm{vc}(G)$ leaves. Furthermore, note that for each inner vertex $x$, the tuple $\left(\psi\left(x_{\ell}\right), \psi\left(x_{r}\right)\right)$ is a partition of $\psi(x)$.
To give a bound on the number of edges of $G$, we associate an edge-set $E_{x}$ with every $x \in \mathcal{V}$. If $x$ is an inner vertex in $T$, then we define $E_{x}:=E_{G}\left(\psi\left(x_{\ell}\right), \psi\left(x_{r}\right)\right)$. Otherwise, if $x$ is a leaf, then we define $E_{x}:=E_{G}(\psi(x))$. Observe that for every inner vertex $x$ the edge-set $E_{x}$ is a minimum $(u, v)$-cut in $G$ for a pair of vertices $u, v \in \psi(x)$. The size bound of the number of edges mainly relies on the following lemma.
Lemma 10. Let $(T=(\mathcal{V}, \mathcal{E}), \psi)$ be a cut tree of $G=(V, E)$. Then, $E=\bigcup_{x \in \mathcal{V}} E_{x}$.
Proof. It clearly holds that $\bigcup_{x \in \mathcal{V}} E_{x} \subseteq E$ since each $E_{x} \subseteq E$. It remains to prove $E \subseteq \bigcup_{x \in \mathcal{V}} E_{x}$.
Let $e=\{u, v\} \in E$. If $e \in E_{x}$ for some leaf vertex $x$, nothing more needs to be shown. Otherwise, consider the leaf vertices $x$ and $y$ with $u \in \psi(x)$ and $v \in \psi(y)$ and let $z$ be the first common ancestor of $x$ and $y$. Then, $u \in \psi(z \ell)$ and $v \in \psi\left(z_{r}\right)$ or vice versa. Consequently, $e \in E_{G}\left(\psi\left(z_{\ell}\right), \psi\left(z_{r}\right)\right)=E_{z}$.

We now prove the main result of this subsection.
Theorem 11. There is an algorithm that, given an instance of WMCP computes an equivalent instance in polynomial time, such that the graph consists of at most $2 \mathrm{vc}(G) \cdot$ a edges.

Proof. The algorithm is simply described as follows: Apply the Rules 2-4 exhaustively. Obviously, a single application of one rule can be done in polynomial time. Then, since after every application of one of the rules the number of vertices is decreased by one, Rules 2-4 can be applied exhaustively in polynomial time.

Let $J$ be an instance of WMCP that is reduced regarding Rules $2-4$. We next use Lemma 10 to prove that the input graph $G$ consists of at most $2 \cdot \mathrm{vc} \cdot a$ edges. Recall that for every pair $(u, v)$ of vertices in $G$, there exists a $(u, v)$-cut of size at most $a$ since $J$ is reduced regarding Rule 4
Let $(T=(\mathcal{V}, \mathcal{E}), \psi)$ be a cut tree of $G$ with respect to a minimum vertex cover $S$. Let $I:=V \backslash S$ be the remaining independent set. Furthermore, let $\mathcal{L} \subseteq \mathcal{V}$ be the set of leaves of $T$ and let $\mathcal{I} \subseteq \mathcal{V}$ be the set of inner vertices of $T$. Lemma 10 then implies

$$
|E| \leq\left|\bigcup_{x \in \mathcal{I}} E_{x}\right|+\left|\bigcup_{x \in \mathcal{L}} E_{x}\right|
$$

Since every $(u, v)$-cut in $G$ has size at most $a$ and $\omega(e) \geq 1$ for every edge $e$ we conclude that $\left|E_{x}\right| \leq a$ for every $x \in \mathcal{I}$. Thus, since $T$ has at most $\operatorname{vc}(G)$ inner vertices, we have $\left|\bigcup_{x \in \mathcal{I}} E_{x}\right| \leq \operatorname{vc}(G) \cdot a$.
We next define an injective mapping $p: \bigcup_{x \in \mathcal{L}} E_{x} \rightarrow \bigcup_{x \in \mathcal{I}} E_{x}$. Observe that the existence of such a mapping implies $\left|\bigcup_{x \in \mathcal{L}} E_{x}\right| \leq\left|\bigcup_{x \in \mathcal{I}} E_{x}\right|$ and thus $|E| \leq 2 \cdot \operatorname{vc}(G) \cdot a$.
Let $\{u, v\} \in E_{x}$ for some leaf vertex $x$. Without loss of generality assume that $v \in S$ and $u \in I$. Since $J$ is reduced regarding Rules 2-4, there are no degree-one vertices in $G$ and thus, $u$ has a neighbor $w \in S \backslash\{v\}$. We then define $p(\{u, v\}):=\{u, w\}$. Note that $p(\{u, v\}) \in E_{G}(S, I)$, and that both edges $\{u, v\}$ and $p(\{u, v\})$ are incident with the same vertex $u \in I$.

We first show that $p$ is well-defined. That is, that $p(\{u, v\}) \in \bigcup_{x \in \mathcal{I}} E_{x}$ for every $\{u, v\} \in \bigcup_{x \in \mathcal{L}} E_{x}$. Since $|\psi(x) \cap S|=1$, there exists another leaf vertex $y$ with $w \in \psi(y)$. Let $z$ be the first common ancestor of $x$ and $y$. Then, $\{u, w\} \in E_{z}$. Since $z$ is an inner vertex, we conclude that $p$ is well-defined.
Next, we show that $p$ is injective. Let $e:=\{u, v\}$ and $e^{\prime}:=\left\{u^{\prime}, v^{\prime}\right\}$ be edges in $\bigcup_{x \in \mathcal{L}} E_{x}$. Let $p(e)=p\left(e^{\prime}\right)$. We show that $e=e^{\prime}$. Without loss of generality assume that $v, v^{\prime} \in S$ and $u, u^{\prime} \in I$. Then, all four edges $e, e^{\prime}, p(e)$, and $p\left(e^{\prime}\right)$ are incident with the same vertex of $I$ and thus $u=u^{\prime}$. Then, since $\{\psi(x) \mid x \in \mathcal{L}\}$ is a partition of $V$ we conclude that $e$ and $e^{\prime}$ are element of the same set $E_{x}$ for some $x \in \mathcal{L}$. Then, $|\psi(x) \cap S|=1$ implies $v=v^{\prime}$ and thus $e=e^{\prime}$. Therefore, $p$ is injective, which then implies $|E| \leq 2 \cdot \mathrm{vc}(G) \cdot a$.

Technically, the instance from Theorem 11 is not a kernel since the encoding of $d$ and the values of $c(e)$ might not be bounded by some polynomial in $a$ and vc. We use the following lemma to show that Theorem 11 implies a polynomial kernel for WMCP.

Lemma 11 ([12]). There is an algorithm that, given a vector $w \in \mathbb{Q}^{r}$ and some $W \in \mathbb{Q}$ computes in polynomial time a vector $\bar{w}=\left(w_{1}, \ldots, w_{r}\right) \in \mathbb{Z}^{r}$ where $\max _{i \in\{1, \ldots, r\}}\left|w_{i}\right| \in 2^{\mathcal{O}\left(r^{3}\right)}$ and an integer $\bar{W} \in \mathbb{Z}$ with total encoding length $\mathcal{O}\left(r^{4}\right)$ such that $w \cdot x \leq W$ if and only if $\bar{w} \cdot x \leq \bar{W}$ for every $x \in\{0,1\}^{r}$.
Corollary 5. WMCP admits a polynomial problem kernel when parameterized by $\mathrm{vc}+a$.
Proof. Let $J:=(G=(V, E), s, t, c, \omega, d, a)$ be the reduced instance from Theorem 11. Observe that both, the number of vertices $n$ and the number of edges $m$ of $G$ are polynomially bounded in vc $+a$. We define $r:=m$ and $w$ to be the $r$-dimensional vector where the entries are the values $c(e)$ for each $e \in E$. Furthermore, let $W:=d$. Applying the algorithm behind Lemma 11 computes a vector $\bar{w}$ with the property stated in the lemma and an integer $\bar{W}$ that has encoding length $\mathcal{O}\left(m^{4}\right)$.

Substituting all values $c(e)$ with the corresponding entry in $\bar{w}$ and substituting $d$ by $\bar{W}$ then converts $J$ into an equivalent instance which has a size that is polynomially bounded in $\mathrm{vc}+a$.

The algorithm behind Theorem 11 also implies a polynomial kernel for the unweighted problem MCP: We transform the unweighted instance into a weighted instance where all capacities and costs are one. Afterwards, we apply the algorithm from Theorem 11 to compute a reduced instance $J^{\prime}$. In $J^{\prime}$ all costs are one, and the capacities are at most $a+1$. We then use Corollary 1 to transform the reduced instance $J^{\prime}$ into an instance $J$ of MCP. Due to the structure of $J^{\prime}$, the number of new vertices introduced in $J$ is at most $m \cdot(a+1)$, where $m$ denotes the number of edges in $J^{\prime}$. Since $m \leq 2 \mathrm{vc}(G) \cdot a$, we obtain the following corollary.
Corollary 6. MCP admits a polynomial problemkernel with $4 \mathrm{vc}(G) \cdot a^{2}$ edges.

### 6.2 Limits of Problem Kernelization

Let $B_{q}$ be a full binary tree of height $q$. We denote the vertices on level $\ell$ as $b_{\ell, 1}^{q}, \ldots, b_{\ell, 2^{\ell}}^{q}$ for each $\ell \in[0, q]$. Hence, vertex $b_{\ell, i}^{q}$ for some $\ell \in[0, q-1]$ and some $i \in\left[1,2^{\ell}\right]$ has the neighbors $b_{\ell+1,2 i-1}^{q}$ and $b_{\ell+1,2 i}^{q}$ in the next level. The full binary tree $R_{q}$ of height $q$ with the vertices $r_{\ell, 1}^{q}, \ldots, r_{\ell, 2^{\ell}}^{q}$ on level $\ell \in[0, q]$ is defined analogously. A mirror fully binary tree $M_{q}$ is the graph obtained after merging the vertices $b_{q, i}^{q}$ and $r_{q, i}^{q}$ for each $i \in\left[1,2^{q}\right]$. By $\operatorname{lp}(G)$ we denote the length of a longest path in $G$.
Lemma 12. Let $q \geq 3$, then the longest path of a mirror fully binary tree $M_{q}$ is $2 q^{2}$.
Proof. By $L_{q}$ we denote the length of each longest path with one endpoint being $b_{0,1}^{q}$ which does not contain vertex $r_{0,1}^{q}$. We prove the following statements inductively for $q \geq 3$.

1. $L_{q}=L_{q-1}+2 q-1$.
2. $\left|V\left(P_{q}\right) \cap\left\{b_{0,1}^{q}, r_{0,1}^{q}\right\}\right|=1$ for each longest path $P_{q}$ of $M_{q}$ and $\operatorname{lp}\left(M_{q}\right)=2 L_{q}$.

Solving the recurrence implied by 1 . and 2. leads to $\operatorname{lp}\left(M_{q}\right)=2 q^{2}$. Hence, it remains to prove the two statements.
Base Case $q=3$ : By considering all possible longest paths in $M_{3}$ we show that the length of a longest path with one endpoint being $b_{0,1}^{3}$ and not containing $r_{0,1}^{3}$ is nine and that $\operatorname{lp}\left(M_{3}\right)=18$.

Inductive step $j-1 \mapsto j$ :

1. Let $Z_{j}$ be a longest path starting at $b_{0,1}^{j}$ not containing vertex $r_{0,1}^{j}$. Without loss of generality, assume that $b_{1,1}^{j} \in Z_{j}$.

First, consider the case that $r_{1,1}^{j} \notin Z_{j}$. Then, the length of $Z_{j}$ is at most the length of a longest path starting in vertex $b_{1,1}^{j}$ and not containing vertex $r_{1,1}^{j}$ in the mirror fully binary tree of height $j-1$ rooted in vertex $b_{1,1}^{j}$ plus one for the edge $\left\{b_{1,1}^{j}, b_{0,1}^{j}\right\}$. By inductive hypothesis we obtain that $Z_{j}$ has length at most $L_{j-1}+1$.

Second, consider the case that $r_{1,1}^{j} \in Z_{j}$. Since $b_{1,1}^{j}, r_{1,1}^{j} \in Z_{j}$ there exists a path connecting these two vertices. Without loss of generality, assume that vertices $b_{2,1}^{j}$ and $r_{2,1}^{j}$ are on this path of length exactly $2 j-2$. Note that $b_{1,1}^{j}$ has the neighbors $b_{0,1}^{j}$ and $b_{2,1}^{j}$ in the path and thus $b_{2,2}^{j}$ is not a neighbor of $b_{1,1}^{j}$. Thus, we can now use the inductive hypothesis. The length of each longest path starting at vertex $r_{1,1}^{j}$ and not containing
vertex $b_{1,1}^{j}$ in the mirror fully binary tree of height $j-1$ rooted in $r_{1,1}^{j}$ is at most $L_{j-1}$. Hence, the length of $Z_{j}$ is at most $L_{j-1}+2 j-1$. Thus, we obtain $L_{j}=L_{j-1}+2 j-1$.
2. Consider the case that $\left|V\left(P_{j}\right) \cap\left\{b_{0,1}^{j}, r_{0,1}^{j}\right\}\right|=1$. Then, by 1 . we can construct a path $Z_{j}$ of length $2 L_{j}=$ $2 L_{j-1}+4 j-2$ by joining two paths where one endpoint is $b_{0,1}^{j}$ which both do not contain vertex $r_{0,1}^{j}$. Thus, $\operatorname{lp}\left(M_{j}\right) \geq 2 L_{j}$. Next, assume towards a contradiction that $\left|V\left(P_{j}\right) \cap\left\{b_{0,1}^{j}, r_{0,1}^{j}\right\}\right| \neq 1$.
First, consider that case $\left|V\left(P_{j}\right) \cap\left\{b_{0,1}^{j}, r_{0,1}^{j}\right\}\right|=0$. Then, the path $P_{j}$ has length at most $\operatorname{lp}\left(M_{j-1}\right)=$ $2 L_{j-1}<2 L_{j-1}+4 j-2=2 L_{j}$, a contradiction to the existence of the path $Z_{j}$.
Second, consider the case $\left|V\left(P_{j}\right) \cap\left\{b_{0,1}^{j}, r_{0,1}^{j}\right\}\right|=2$. Let $Q_{j}$ be the unique subpath of $P_{j}$ with endpoints $b_{0,1}^{j}$ and $r_{0,1}^{j}$. Without loss of generality, $b_{1,1}^{j} \in V\left(Q_{j}\right)$ (and thus also $r_{1,1}^{j} \in V\left(Q_{j}\right)$ ). Since $M_{j}$ is a mirror fully binary tree the subpath $Q_{j}$ has length exactly $2 j$. Let $M^{\prime}$ be the mirror fully binary tree rooted at vertex $b_{1,1}^{j}$. Then $M^{\prime} \cap V\left(P_{j}\right)=V\left(Q_{j}\right) \backslash\left\{b_{0,1}^{j}, r_{0,1}^{j}\right\}$. Furthermore, $P_{j}$ can contain the edges $\left\{b_{0,1}^{j}, b_{1,2}^{j}\right\}$ and $\left\{r_{0,1}^{j}, r_{1,2}^{j}\right\}$, and a longest path in the mirror fully binary tree rooted in $b_{1,2}^{j}$ of height $j-1$. Thus, the length of $P_{j}$ is at most $\operatorname{lp}\left(M_{j-1}\right)+2 j+2=2 L_{j-1}+2 j+2<2 L_{j-1}+4 j-2=2 L_{j}$, a contradiction to the existence of the path $Z_{j}$.
Hence, $\operatorname{lp}\left(M_{j}\right)=2 L_{j}$.

On the negative side, we provide an OR-composition to exclude a polynomial kernel for the combination of almost all considered parameters with the exception of $\mathrm{vc}(G)$ and fvs $(G)$.
Theorem 12. None of the problems MCP, WMCP, and ZWMCP admits a polynomial kernel when parameterized by $d+a+\operatorname{lp}(G)+\Delta(G)+\operatorname{td}(G)$, unless NP $\subseteq$ coNP/poly, where $\operatorname{td}(G)$ denotes the treedepth of $G$.

Proof. Our strategy is as follows: First, we provide an OR-composition [5, 6] of $2^{q}$ instances of MCP to WMCP where $\omega(e)=1$ and $c(e) \in(d+q)^{\mathcal{O}(1)}$ for each edge $e$. Second, we apply Lemma 1 exhaustively to transform the constructed instance of WMCP to an equivalent instance of MCP. Clearly, the budgets $a$ and $d$ do not change. In this transformation, each edge $e$ with $c(e) \geq 2$ is replaced by a path with $c(e)$ edges. Hence, the maximum degree does not increase. Furthermore, since $c(e) \in(d+q)^{\mathcal{O}(1)}$, the length of the longest path does only increase by a factor of $(d+q)^{\mathcal{O}(1)}$ and the tree-depth is only increased by $\mathcal{O}(\log (d+q))$. Thus, this transformation preserves all five parameters. It remains to show the statement for WMCP.
Now, we prove the no polynomial kernel result for WMCP by presenting an OR-composition from MCP. Let $I_{1}, I_{2}, \ldots, I_{2^{q}}$ be instances of MCP with the same budgets $d$ and $a$, the same maximum degree $\Delta(G)$, the same length $\operatorname{lp}(G)$ of the longest path, and the same tree-depth $\operatorname{td}(G)$ for some integer $q \geq 3$. Moreover, let $I_{j}:=\left(G_{j}=\left(V_{j}, E_{j}\right), s_{j}, t_{j}, d, a\right)$.
We describe how to construct an instance $I^{*}=\left(G^{*}, s^{*}, t^{*}, c^{*}, \omega^{*}, d^{*}, a^{*}\right)$ of WMCP in polynomial time, where $d^{*}+$ $a^{*}+\operatorname{lp}\left(G^{*}\right)+\Delta\left(G^{*}\right)+\operatorname{td}\left(G^{*}\right) \in(d+a+\operatorname{lp}(G)+\Delta(G)+\operatorname{td}(G)+q)^{\mathcal{O}(1)}$ such that $I^{*}$ is a yes-instance of WMCP if and only if $I_{j}$ is a yes-instance of MCP for at least one $j \in\left[1,2^{q}\right]$.
We add the following vertices and edges to the graph $G^{*}$ :

- We add a copy of the graph $G_{j}$ for each $j \in\left[1,2^{q}\right]$ to $G^{*}$.
- Furthermore, we add $6 q$ new vertices $Z_{j}:=\left\{z_{j}^{1}, \ldots, z_{j}^{6 q}\right\}$ and we add the the edges $\left\{s_{j}, z_{j}^{i}\right\}$ and $\left\{z_{j}^{i}, t_{j}\right\}$ for each $j \in\left[1,2^{q}\right]$ and each $i \in[1,6 q]$ to $G^{*}$.
- Next, we add a full binary tree $B$ with height $q$ to $G^{*}$. We denote the vertices on level $\ell$ as $b_{\ell}^{1}, \ldots, b_{\ell}^{2}$ for each $\ell \in[0, q]$. Hence, vertex $b_{\ell}^{i}$ for some $\ell \in[0, q-1]$ and some $i \in\left[1,2^{\ell}\right]$ has the neighbors $b_{\ell+1}^{2 i-1}$ and $b_{\ell+1}^{2 i}$ in the next level. Now, we identify vertex $s_{j}$ with the leaf $b_{q}^{j}$ for each $j \in\left[1,2^{q}\right]$ and we identify vertex $s^{*}$ with the root $b_{0}^{1}$.
- Analogously, we add a full binary tree $R$ of height $q$ with the vertices $r_{\ell}^{1}, \ldots, r_{\ell}^{2^{\ell}}$ on level $\ell \in[0, q]$. Similarly, we identify vertex $t^{*}$ with the root $r_{0}^{1}$ and identify vertex $t_{j}$ with the leaf $r_{q}^{j}$ for each $j \in\left[1,2^{q}\right]$.

Next, we set $d^{*}:=2 q(d+1)+d$ and $a^{*}:=7 q+a$. Afterwards, we set $\omega^{*}(e):=1$ for each edge $e \in E\left(G^{*}\right)$. We define the costs of each edge in $E\left(G^{*}\right)$ as follows:

- For each $e \in E_{j}$ for some $j \in\left[1,2^{q}\right]$ we set $c(e):=1$.
- We set $c\left(\left\{s_{j}, z_{j}^{i}\right\}\right):=d^{*}+1$ and $c\left(\left\{z_{j}^{i}, t_{j}\right\}\right):=d^{*}+1$ for each $j \in\left[1,2^{q}\right]$ and each $i \in[1,6 q]$.
- For each edge $e$ in one of the full binary trees we set $c(e):=d+1$.

This completes the construction of $I^{*}$. Now, we prove that the parameters $a^{*}, d^{*}, \Delta\left(G^{*}\right)$, and $\operatorname{lp}\left(G^{*}\right)$ are bounded by $(a+d+\Delta(G)+\operatorname{lp}(G)+q)^{\mathcal{O}(1)}$.

- Since $d^{*}=2 q(d+1)$ and $a^{*}=7 q+a$, the statement is clear for $a^{*}$ and $d^{*}$.
- Each vertex in $B \cup R$ except the leaves have degree at most three. Furthermore, each vertex $z_{j}^{i}$ for some $j \in$ $\left[1,2^{q}\right]$ and some $i \in[1,6 q]$ as degree two and each vertex in $V_{j} \backslash\left\{s_{j}, t_{j}\right\}$ has degree at most $\Delta(G)$. Note that vertex $s_{j}$ and $t_{j}$ for each $j \in\left[1,2^{q}\right]$ has degree at most $6 q+1+\Delta(G)$. Hence, the statement is true for $\Delta\left(G^{*}\right)$.
- Observe that the graph obtained from contracting all vertices in $W_{j}:=V\left(G_{j}\right) \cup\left\{z_{j}^{1}, \ldots, z_{j}^{6 q}\right\}$ in the graph $G^{*}$ into one vertex for each $j \in\left[1,2^{q}\right]$ is a mirror fully binary tree of height $q$. Observe that by construction, $\operatorname{lp}\left(G^{*}\left(W_{j}\right)\right) \in \mathcal{O}(\operatorname{lp}(G))$ for each $j \in\left[1,2^{q}\right]$. Thus, by Lemma 12 we obtain that $\operatorname{lp}\left(G^{*}\right) \in$ $\mathcal{O}\left(\operatorname{lp}(G) \cdot q^{2}\right)$. Hence, the statement is true for $\operatorname{lp}\left(G^{*}\right)$.
- Since $\operatorname{td}\left(G_{j}\right)=\operatorname{td}(G)$ for each $j \in\left[1,2^{q}\right]$, the tree-depth of $G_{j}^{\prime}=G^{*}\left[V_{j} \cup Z_{j}\right]$ is at most $\operatorname{td}(G)+2$. Hence, there is a directed tree $T_{j}\left(V_{j} \cup Z_{j}, A_{j}\right)$ of depth at most $\operatorname{td}(G)+2$ and root $s_{j}$, such that for each edge $\{u, w\} \in E\left(G_{j}^{\prime}\right)$ either $u$ is an ancestor of $w$ in $T_{j}$ or vice versa. We define a directed tree $T^{*}=$ $\left(V\left(G^{*}\right), A^{*}\right)$ as follows. The tree $T_{j}$ is a subtree of $T^{*}$ for each $j \in\left[1,2^{q}\right]$. The vertex $b_{0}^{1}$ is the root of $T^{*}$ and for each $\ell \in[0, q-1]$ and each $i \in\left[1,2^{\ell}\right], A^{*}$ contains the $\operatorname{arcs}\left(b_{\ell}^{i}, r_{\ell}^{i}\right),\left(r_{\ell}^{i}, b_{\ell+1}^{2 i}\right)$, and $\left(r_{\ell}^{i}, b_{\ell+1}^{2 i+1}\right)$. Recall that $b_{q}^{j}=s_{j}$. Since $T_{j}$ is a subtree of $T^{*}$ it follows that for each edge $\{u, w\} \in E\left(G^{*}\right)$ either $u$ is an ancestor of $w$ in $T^{*}$ or vice versa. Moreover, since $T^{*}$ has depth at most $2 q+\operatorname{td}(G)$ we obtain the stated bound on the tree-depth of $G^{*}$.

Next, we prove the correctness. That is, we show that at least one instance $I_{j}$ has a solution $D_{j}$ with cost at most $d$ for some $j \in\left[1,2^{q}\right]$ if and only if $I^{*}$ has a solution $D$ with cost at most $d^{*}$.
$(\Rightarrow)$ Let $D_{j}$ be a solution of $I_{j}$ with cost at most $d$.
Let $P_{j}^{s}$ be the unique $\left(s^{*}, s_{j}\right)$-path in $B$ and let $P_{j}^{t}$ be the unique $\left(t_{j}, t^{*}\right)$-path in $R$. We set $D:=D_{j} \cup E\left(P_{j}^{s}\right) \cup E\left(P_{j}^{t}\right)$.
First, we show that $c^{*}(D) \leq d^{*}$. Since $B$ and $R$ are full binary trees of height $q$, both paths $P_{j}^{s}$ and $P_{j}^{t}$ consist of exactly $q$ edges. Recall that each edge in both $B$ and $R$ has cost $d+1$. Since $\left|D_{j}\right| \leq d$ and $c(e)=1$ for each edge $e \in E_{j}$, we obtain $c^{*}(D) \leq 2 q(d+1)+d$.
Second, we prove that there is no $\left(s^{*}, t^{*}\right)$-cut $A \subseteq E\left(G^{*}\right) \backslash D$ in $G^{*}$ with $\omega^{*}(A) \leq a^{*}$. To this end, we present $a^{*}+1$ many $\left(s^{*}, t^{*}\right)$ paths in $G^{*}$ whose edge sets may only intersect in $D$. Together with $\omega^{*}(e)=1$ for each edge $e \in$ $E\left(G^{*}\right)$ we then conclude that $\omega^{*}(M) \geq a^{*}+1$. We use the following notation: For two paths $P_{1}=\left(v_{1}, \ldots, v_{k}\right)$ and $P_{2}=\left(w_{1}, \ldots, w_{r}\right)$ in $G^{*}$ where $w_{r}=v_{1}$, we let $P_{1} \multimap P_{2}:=\left(v_{1}, \ldots, v_{k}=w_{1}, \ldots, w_{r}\right)$ denote the merge of $P_{1}$ and $P_{2}$.

- Let $P_{j}^{s, \ell}$ be the subpath of $P_{j}^{s}$ until level $\ell \in[0, q-1]$. Since $B$ is a full binary tree, let $b_{\ell+1}^{i}$ be the child of the endpoint of $P_{j}^{s, \ell}$ which is not contained in $P_{j}^{s}$. The subpath $P_{j}^{\ell, t}$ and the vertex $r_{\ell+1}^{i}$ are defined similarly. We consider the $\left(s^{*}, t^{*}\right)$-path $P_{j}^{s, \ell} \cdot\left(b_{\ell+1}^{i}, b_{\ell+2}^{2 i}, \ldots, b_{q}^{2^{q-\ell \cdot i}}=s_{2^{q-\ell \cdot i}}, z_{2^{q-\ell \cdot i}}^{1}, t_{2^{q-\ell \cdot i}}=r_{q}^{2^{q-\ell} \cdot i}, \ldots, r_{\ell+2}^{2 i}, r_{\ell+1}^{i}\right)$. $P_{j}^{\ell, t}$. Since $\ell \in[0, q-1]$, these are $q$ paths in total.
- Observe that $P_{j}^{s} \multimap\left(s_{j}, z_{j}^{i}, t_{j}\right) \multimap P_{j}^{t}$ is an $\left(s^{*}, t^{*}\right)$-path for each $i \in[1,6 q]$. Hence, these are $6 q$ paths in total.
- Since $D_{j}$ is a solution of $I_{j}$, there are $a+1$ many $\left(s_{j}, t_{j}\right)$-paths $P_{1}, \ldots, P_{a+1}$ in $G_{j}$ whose edge set may only intersect in $D_{j}$. Since $D_{j} \subseteq D, P_{j}^{s} \multimap P_{i} \multimap P_{j}^{t}$ is an $\left(s^{*}, t^{*}\right)$-path for each $i \in[1, a+1]$ such that $P_{i} \subseteq E_{j}$. Hence, these are $a+1$ paths in total.

Thus, $G^{*}$ contains at least $a+1$ many $\left(s^{*}, t^{*}\right)$-paths whose edge set may only intersect in $D$ and hence $I^{*}$ is a yes-instance of WMCP.
$(\Leftarrow)$ Conversely, let $D$ be a solution with cost at most $d^{*}$ of $I^{*}$. At the beginning, we prove the following statement.
Claim 5. For each solution $D$ of $I^{*}$ with cost at most $d^{*}$, there exists a $j \in\left[1,2^{q}\right]$ such that $E\left(P_{j}^{s}\right) \subseteq D$ for the unique path $P_{j}^{s}$ from $s^{*}$ to $s_{j}$ and $E\left(P_{j}^{t}\right) \subseteq D$ for the unique path $P_{j}^{t}$ from $t^{*}$ to $t_{j}$.

Proof. Assume towards a contradiction that this is not the case. We define $B_{s}:=E(B) \cap D$ and $R_{t}:=E(R) \cap D$ as the set of protected edges in the binary trees $B$ and $R$. Note that since $c^{*}(e)=d+1$ for each edge $e$ in the binary trees $B$ and $R$ and $d^{*}=2 q(d+1)+d$, we have $\left|B_{s}\right|+\left|R_{t}\right| \leq 2 q$. By $Z_{s}$ we denote the connected component of $G^{*}\left[B_{s}\right]$ containing vertex $s^{*}$. Since $\left|B_{s}\right| \leq 2 q$, we conclude that $Z_{s}$ contains at most $2 q+1$ vertices. Since $B$ is a binary tree, each vertex in $B$ has degree at most three. Recall that only vertices in level $q$ of $B$ have neighbors outside of $B$. We set $X:=E_{B}\left(Z_{s}, N\left(Z_{s}\right)\right)$. Note that $|X| \leq 3 \cdot(2 q+1) \leq 6 q+3$.
First, we consider the case that $G^{*}\left[B_{s}\right]$ does not contain the path $P_{j}^{s}$ as an induced subgraph for any $j \in\left[1,2^{q}\right]$. We set $A:=X$ and show that $M$ is an $\left(s^{*}, t^{*}\right)$-cut in $G^{*}$. Note that $|M| \leq 6 q+3 \leq a^{*}$. Observe that $A$ avoids $D$ since $Z_{s}$ is a connected component in $G^{*}[D]$ and $A$ contains only adjacent edges of $Z_{s}$. Thus, $A$ is an $\left(s^{*}, t^{*}\right)$-cut in $G^{*}$ that avoids $D$ with $\omega^{*}(A) \leq a^{*}$, a contradiction. Analogously, we can prove that $G^{*}\left[R_{t}\right]$ is a path $P_{j^{\prime}}^{t}$ for some $j^{\prime} \in\left[1,2^{q}\right]$.

Second, we consider the case that $G^{*}\left[B_{s}\right]$ is a path $P_{j}^{s}$ for some $j \in\left[1,2^{q}\right]$ and that $G^{*}\left[R_{t}\right]$ is a path $P_{j^{\prime}}^{t}$ for some $j^{\prime} \in$ $\left[1,2^{q}\right]$ and $j \neq j^{\prime}$. Recall that $c(e)=d+1$ for each edge $e$ in the binary trees $B$ and $R$ and that $d^{*}=2 q(d+1)+d$. Furthermore, note that $\left|B_{s}\right|=q=\left|R_{t}\right|$ and that $Z_{s}$ contains $q+1$ vertices. Thus, $D$ contains no other edges of $R$ than $R_{t}$. In particular, $e^{*}:=\left\{r_{q}^{j}, r_{q-1}^{\lceil j / 2\rceil}\right\} \notin D$. Recall that $t_{j}=r_{q}^{j}$. We define $A:=\left\{e^{*}\right\} \cup X$. Clearly, $A$ avoids $D$. Since $q \geq 2$, we conclude that $|A| \leq 1+6 q+1 \leq 7 q$. It remains to show that $A$ is an $\left(s^{*}, t^{*}\right)$-cut in $G^{*}$. Since $X \subseteq A$, every $\left(s^{*}, t^{*}\right)$-path $P^{*}$ in $G^{*}-X$ starts with $P_{j}^{s}$ followed by an $\left(s_{j}, t_{j}\right)$-path. Moreover, $P^{*}$ has to contains the edge $e^{*}$. Since $e^{*} \in A, A$ is indeed an $\left(s^{*}, t^{*}\right)$-cut in $G^{*}$ with $|A| \leq a^{*}$ that avoids $D$, a contradiction.

By Claim 5, let $j \in\left[1,2^{q}\right]$ such that $E\left(P_{j}^{s}\right) \cup E\left(P_{j}^{t}\right) \subseteq D$. We define $D_{j}:=D \cap E_{j}$. Observe that since $c(e)=d+1$ for each edge in both binary trees $B$ and $R$, each edge incident with vertex $z_{j^{\prime}}^{i}$ for some $j^{\prime} \in\left[1,2^{q}\right]$ and some $i \in[1,6 q]$ has cost $d^{*}+1$, and each edge in the copy of $G_{j^{\prime}}$ for some $j^{\prime} \in\left[1,2^{q}\right]$ has costs one, we conclude that $\left|D_{j}\right| \leq d$. In the following, we show that $D_{j}$ is a solution of $I_{j}$.
Assume towards a contradiction that there is an $\left(s_{j}, t_{j}\right)$-cut $A_{j} \subseteq E_{j}$ of size at most $a$ in $G_{j}$ that avoids $D_{j}$. We set $A:=A_{j} \cup\left\{\left\{s_{j}, z_{j}^{i}\right\} \mid i \in[1,6 q]\right\} \cup X$, where $X:=E_{B}\left(V\left(P_{j}^{s}\right), N\left(V\left(P_{j}^{s}\right)\right)\right)$. Note that $A$ avoids $D$. By the fact that $B$ is a binary tree of depth $q$, it follows that $|M| \leq\left|M_{j}\right|+6 q+q \leq a+7 q=a^{*}$.
Since $X \subseteq A$, every $\left(s^{*}, t^{*}\right)$-path $P^{*}$ in $G^{*}-X$ starts with $P_{j}^{s}$ followed by an $\left(s_{j}, t_{j}\right)$-path. Moreover, since $A_{j}$ is an $\left(s_{j}, t_{j}\right)$-cut in $G_{j}$ and $\left\{\left\{s_{j}, z_{j}^{i}\right\} \mid i \in[1,6 q]\right\} \subseteq A, A$ is an $\left(s^{*}, t^{*}\right)$-cut of capacity at most $a^{*}$ in $G^{*}$, a contradiction. Hence, $D_{j}$ is a solution with cost at most $d$ of $I_{j}$ and, thus, $I_{j}$ is a yes-instance of WMCP.

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