PREVENTING SMALL (s, t)-CUTS by Protecting Edges

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ABSTRACT

We introduce and study WEIGHTED MIN (s,t)-CUT PREVENTION, where we are given a graph G = (V, E) with vertices s and t and an edge cost function and the aim is to choose an edge set D of total cost at most d such that G has no (s,t)-edge cut of capacity at most a that is disjoint from D. We show that WEIGHTED MIN (s,t)-CUT PREVENTION is NP-hard even on subcubcic graphs when all edges have capacity and cost one and provide a comprehensive study of the parameterized complexity of the problem. We show, for example W[1]-hardness with respect to d and an FPT algorithm for a.

1 Introduction

Network interdiction is a large class of optimization problems with direct applications in operations research [8, 9, 21, 22, 23]. In these problems one player wants to achieve a certain goal (for example finding a short path between two given vertices s and t), and another player wants to modify the network to prevent this. Given the enormous importance of the maximum-flow/min-cut problem it comes as no surprise that two-player games where an attacker wants to decrease the maximum (s, t)-flow of a network by deleting edges have been considered [8, 22]. We study an inverse problem: an attacker wants to find an (s, t)-cut of capacity at most a and a defender wants to protect edges in order to increase the capacity of any minimum (s, t)-cut in G to at least a + 1. Alternatively, we may think that the defender increases the capacity of some edges to a + 1 in such a way that the maximum (s, t)-flow of the resulting network exceeds the given threshold a. The formal problem definition reads as follows.

WEIGHTED MIN (s,t)-CUT PREVENTION (WMCP) **Input:** A graph G = (V, E), two vertices $s, t \in V$, a cost function $c : E \to \mathbb{N}$, a capacity function $\omega : E \to \mathbb{N}$, and integers d and a. **Question:** Is there a set $D \subseteq E$ with $c(D) := \sum_{e \in D} c(e) \leq d$ such that for every (s, t)-cut $A \subseteq (E \setminus D)$ in G we have $\omega(A) := \sum_{e \in A} \omega(e) > a$?

The special case where we have only unit capacities and unit costs is referred to as MIN (s, t)-CUT PREVENTION (MCP). A different problem also called MINIMUM *st*-CUT INTERDICTION has been studied recently [1] but in this problem the graph is directed and the interdictor may freely choose the amount of increase in edge capacities. In our formulation, the interdictor may only decide to fully protect an edge or to leave it unprocted. To the best of our knowledge, this formulation of WMCP has not been considered so far. We study the classical complexity of WMCP and its parameterized complexity with respect to *a*, *d*, and important structural parameterizations of the input graph *G*.

Related Work. Many interdiction problems have been studied from a (parameterized) complexity perspective: In MATCHING INTERDICTION [23], one wants to remove vertices or edges to decrease the weight of a maximum-weight matching. In the MOST VITAL EDGES IN MST problem, one aims to remove edges to decrease the weight of any maximum spanning tree. In SHORTEST-PATH INTERDICTION [17], also known as SHORTEST PATH MOST VITAL

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	a	d	Δ	$d + \Delta$	vc	pw + fvs
WMCP	FPT	W[1]-h	NP-h	W[1]-h	weakly NP-h	weakly NP-h
				if $\Delta = 3$	Thm. <mark>8</mark>	Thm. 8
					W[1]-h	W[1]-h
	Thm. <mark>5</mark>	Lem. 4	Thm. 1	Thm. 1	Thm. 9	Thm. 9
MCP	FPT	W[1]-h	NP-h	FPT	FPT	W[1]-h
	Thm. <mark>5</mark>	Lem. 4	Thm. 3	Thm. 3	Thm. 7	Thm. 10

Table 1: Parameter overview for WMCP and MCP. We write NP-h if the problem is NP-hard even if the corresponding parameter is a constant.

EDGES [3, 13] and MINIMUM LENGTH-BOUNDED CUT [2], one wants to remove edges to increase the length of a shortest (s, t)-path above a certain threshold. All of these problems are NP-hard and the study of their classical and parameterized complexity has received a lot of attention [3, 15, 13, 23].

Our Results. An overview of our results is given in Table 1. We show that WMCP and MCP are NP-hard even on subcubic graphs. This motivates a parameterized complexity study with respect to the natural parameters defender budged d and attacker budget a and with respect to structural parameters of the input graph G. Here, we consider the structural parameters treewidth and vertex cover number of G as well as pathwidth and feedback vertex set number of G. Our main results are as follows. MCP and WMCP are W[1]-hard with respect to the defender budget d and FPT with respect to the attacker budget a. MCP and WMCP are W[1]-hard with respect to the combined parameter pathwidth of G plus feedback vertex set number of G and thus also W[1]-hard with respect to the treewidth of G. The hardness for these parameters motivates a study of the vertex cover number vc(G). We show that MCP is FPT with respect to vc(G), whereas WMCP is weakly NP-hard even for vc(G) = 2 and W[1]-hard with respect to vc(G) even when all capacities and costs are encoded in unary. Finally, we provide a polynomial kernel for WMCP parameterized by vc(G) + a and complement this result by showing that MCP and WMCP do not admit polynomial kernels with respect to the large combined parameter $d + a + tw(G) + lp(G) + \Delta(G)$ where lp(G) denotes the length of a longest path in G and $\Delta(G)$ denotes the maximum degree. Overall, our results give a comprehensive complexity overview of WMCP and MCP.

2 Preliminaries

For integers *i* and *j* with $i \leq j$, we define $[i, j] := \{k \in \mathbb{N} \mid i \leq k \leq j\}$.

An (undirected) graph G = (V, E) consists of a set of vertices V and a set of edges $E \subseteq {\binom{V}{2}}$. Throughout this work, let n := |V| and m := |E|. For vertex sets $S \subseteq V$ and $T \subseteq V$ we denote with $E_G(S,T) := \{\{s,t\} \in E \mid s \in S, t \in T\}$ the edges between S and T. Moreover, we define $E_G(S) := E_G(S,S)$ and $E_G(v,S) := E_G(\{v\},S)$ for $v \in V$. For a vertex set $S \subseteq V$ we denote with $G[S] := (S, E_G(S))$ the *induced subgraph of* S *in* G. Moreover, for an edge set $D \subseteq E$ we let $G - D := (V, E \setminus D)$ and G[D] := (V, D). For a vertex $v \in V$, we denote with $N_G(v) := \{w \in V \mid \{v, w\} \in E\}$ the *open neighborhood* of v in G. Analogously, for a vertex set $S \subseteq V$, we define $N_G(S) := \bigcup_{v \in S} N_G(S) \setminus S$. If G is clear from the context, we may omit the subscript. A sequence of distinct vertices $P = (v_0, \ldots, v_k)$ is a *path* or (v_0, v_k) -*path* of length k in G if $\{v_{i-1}, v_i\} \in E(G)$ for all $i \in [1, k]$. We denote with V(P) the vertices of P and with E(P) the edges of P. Let s and t be distinct vertices of V. An edge set $A \subseteq E$ is an (s, t) (edge)-cut in G if there is no (s, t)-path in G - A. A graph G = (V, E) is *connected* if there is an (a, b)-path in G for each pair of distinct vertices $a, b \in V$. Moreover, we call a vertex set S a *connected component* of G if G[S] is connected and if there is no $S' \supset S$ such that G[S'] is connected.

2.1 Graph parameter

Let G = (V, E) be a graph. Moreover, we denote with $\Delta(G) := \max\{|N_G(v)| \mid v \in V\}$ the maximum degree of G.

A set $S \subseteq V$ is a *feedback vertex set* for G if G - S is acyclic, that is, if for each pair of distinct vertices $a, b \in V \setminus S$ there is at most one (a, b)-path in G - S. The size of the smallest size feedback vertex set for G is denoted by fvs(G).

A path composition \mathcal{B} for a graph G = (V, E) is a sequence of bags B_1, \ldots, B_q where $B_j \subseteq V$ for each $j \in [1, q]$, such that:

- 1. for every vertex $v \in V$, there is at least one $i \in [1, q]$ with $v \in B_i$,
- 2. for each edge $e \in E$, there is at least one $i \in [1, q]$ such that $e \subseteq B_i$, and

3. if $v \in B_i \cap B_j$ with $i \leq j$, then $v \in B_k$ for each $k \in [i, j]$.

The width of a path decomposition \mathcal{B} is the size of the largest bag in \mathcal{B} minus one and the pathwidth of a graph G is the minimal width of any path decomposition of G which is denoted by pw(G).

A tree decomposition of a graph G = (V, E) is a pair (\mathcal{T}, β) consisting of a directed tree $\mathcal{T} = (\mathcal{V}, \mathcal{A}, r)$ with root $r \in \mathcal{V}$ and a function $\beta : \mathcal{V} \to 2^V$ such that

- 1. for every vertex $v \in V$, there is at least one $x \in V$ with $v \in \beta(x)$,
- 2. for each edge $\{u, v\} \in E$, there is at least one $x \in X$ such that $u \in \beta(x)$ and $v \in \beta(x)$, and
- 3. for each vertex $v \in V$, the subgraph $\mathcal{T}[\mathcal{V}_v]$ is connected, where $\mathcal{V}_v := \{x \in \mathcal{V} \mid v \in \beta(x)\}$.

We call $\beta(x)$ the *bag* of x. The *width of a tree decomposition* is the size of the largest bag minus one and the *treewidth* of a graph G is the minimal width of any tree decomposition of G denoted by tw(G).

We consider tree decompositions with specific properties. A node $x \in \mathcal{V}$ is called:

- 1. a *leaf node* if x has no child nodes in \mathcal{T} ,
- 2. a forget node if x has exactly one child node y in \mathcal{T} and $\beta(y) = \beta(x) \cup \{v\}$ for some $v \in V \setminus \beta(x)$,
- 3. an *introduce node* if x has exactly one child node y in \mathcal{T} and $\beta(y) = \beta(x) \setminus \{v\}$ for some $v \in V \setminus \beta(y)$, or
- 4. a *join node* if x has exactly two child nodes y and z in \mathcal{T} and $\beta(x) = \beta(y) = \beta(z)$.

A tree decomposition ($\mathcal{T} = (\mathcal{V}, \mathcal{A}, r), \beta$) is called *nice* if every node $x \in \mathcal{V}$ is either a leaf node, a forget node, an introduce node, or a join node.

For a node $x \in \mathcal{V}$, we define with V_x the union of all bags $\beta(y)$, where y is reachable from x in \mathcal{T} . Moreover, we set $G_x := G[V_x]$ and $E_x := E_G(V_x)$.

The tree-depth td(G) is the smallest height of any directed tree T = (V(G), A) with the property that for each edge $\{u, w\} \in E(G)$ either u is an ancestor of w in T or vice versa.

Two instances I and I' of the same decision problem L are equivalent if I is a yes-instance of L if and only if I' is a yes-instance of L. A *reduction rule* for a decision problem L is an algorithm A that transforms any instance I of L into another instance A(I) of L. We call A safe, if for each instance I of L, I and A(I) are equivalent instances of L. A reduction rule A is exhaustively applied for an instance I if A(I) = I.

For details on parameterized complexity, we refer to the standard monograph [10].

The two further variants of WMCP that we study are defined as follows.

ZERO-WEIGHT MIN (s,t)-CUT PREVENTION (ZWMCP) **Input:** A graph G = (V, E), two vertices $s, t \in V$, a cost function $c : E \to \mathbb{N}$, a capacity function $\omega : E \to \mathbb{N} \cup \{0\}$, and integers d and a. **Question:** Is there a set $D \subseteq E$ with $c(D) := \sum_{e \in D} c(e) \leq d$ such that for every (s, t)-cut $A \subseteq (E \setminus D)$ in G it holds that $\omega(A) := \sum_{e \in A} \omega(e) > a$?

MIN (s, t)-CUT PREVENTION (MCP) **Input:** A graph G = (V, E), two vertices $s, t \in V$, and integers d and a. **Question:** Is there an edge set $D \subseteq E$ of size at most d such that every disjoint (s, t)-cut $A \subseteq (E \setminus D)$ in G has size more than a?

Informally, we search for a cheap set of edges S such that every disjoint (s, t)-cut M is expensive.

Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of any of the above problems (in the case of MCP, $c(e) := \omega(e) := 1$ for all $e \in E$). We call an edge set $D \subseteq E$ a *solution* of I if every (s, t)-cut $A \subseteq E \setminus D$ has capacity at least a + 1 according to ω . A solution D of I is called a *minimum solution* of I, if there is no solution D' of I with c(D') < c(D).

2.2 Basic Observations

We assume without loss of generality that G is connected and that $c(e) \leq d+1$ and $\omega(e) \leq a+1$ for each edge $e \in E(G)$, as otherwise we can decrease these weights accordingly. Furthermore, we can assume that $d \leq c(E)$ where c(E) denotes the total sum of edge-costs. Analogously, we can assume that $a \leq \omega(E)$.

Fact 1. Let G = (V, E) be a graph, let $\omega : E \to \mathbb{N}$ be a capacity function, and let $D \subseteq E$. Then, in $n^{\mathcal{O}(1)}$ time we can compute an (s,t)-cut $A \subseteq E \setminus D$ with $\omega(A) \leq a$ or report that no such (s,t)-cut exists.

Proof. We define a capacity function $\omega' : E \to \mathbb{N}$ by $\omega'(e) := a + 1$ if $e \in D$ and $\omega'(e) := \omega(e)$ otherwise. We then compute a min (s, t)-cut A in G with respect to the new capacity function ω' in $n^{\mathcal{O}(1)}$ time. If $\omega'(A) \leq a$, then we return A. Otherwise, we report that no such (s, t)-cut exists.

Observe that if $\omega'(A) \leq a$, then $A \subseteq E \setminus D$, since $\omega'(e) = a + 1$ for every $e \in D$. Otherwise, if $\omega'(A) > a$, then every (s, t)-cut in G either contains an edge from D or has capacity bigger than a. Thus, the algorithm is correct. \Box

Lemma 1. Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of ZWMCP and let $e^* = \{v^*, w^*\} \in E$ with $c(e^*) > 1$. Moreover, let $I' := (G' = (V', E'), s, t, c', \omega', d, a)$ be the instance of ZWMCP obtained by replacing e^* by an (v^*, w^*) -path with $c(e^*)$ edges of cost 1 and capacity $\omega(e^*)$. Then, I and I' are equivalent instances of ZWMCP and I' can be computed in $\mathcal{O}(c(e^*) \cdot |I|)$ time.

Proof. The running time bound follows immediately by the construction. It remains to prove the correctness. Let $E^* := E' \setminus E$ be the edges of the (v^*, w^*) -path in G' that replaces the edge e^* .

We show that I is a yes-instances of ZWMCP if and only if I' is a yes-instances of ZWMCP.

 (\Rightarrow) Let D be a solution of I of cost at most d.

Case 1: $e^* \in D$. We set $D' := D \setminus \{e^*\} \cup E^*$. Note that $c'(D') \leq d$. We show that D' is a solution of I'. Assume towards a contradiction that there is an (s, t)-cut $A' \subseteq E' \setminus D'$ in G' with $\omega(A') \leq a$. By definition of D' it follows that $A' \subseteq E \setminus \{e^*\}$ with $\omega(A') \leq a$. Moreover, since we obtained G' from G by replacing e^* with a path consisting of the edges E^* and A' is disjoint to E^* , it follows that A' is an (s, t)-cut in G, a contradiction.

Case 2: $e^* \notin D$. Note that $D \subseteq E'$ and that $c'(D) \leq d$. We show that D is a solution of I'. Assume towards a contradiction that there is an (s, t)-cut $A' \subseteq E' \setminus D$ in G' with $\omega(A') \leq a$.

If $A' \cap E^* = \emptyset$, then A' is also an (s, t)-cut disjoint to D in G with $\omega(A') \leq a$. A contradiction. Otherwise, $A' \cap E^* \neq \emptyset$. Hence, $A := A' \setminus E^* \cup \{e^*\}$ is an (s, t)-cut disjoint to D in G with $\omega(A) \leq a$, a contradiction.

 (\Leftarrow) Let D' be a solution of I'.

Case 1: $E^* \subseteq D'$. We set $D := D' \setminus E^* \cup \{e^*\}$. Note that $c(D) \leq d$. We show that D is a solution of I. Assume towards a contradiction that there is an (s,t)-cut $A \subseteq E \setminus A$ in G with $\omega(A) \leq a$. By definition of D, it holds that $A \subseteq E' \setminus D'$. Moreover, since we obtained G' from G by replacing e^* with a path consisting of the edges E^* and A is disjoint to E^* , it follows that A is an (s,t)-cut in G' with $\omega'(A) \leq a$, a contradiction.

Case 2: $E^* \not\subseteq D'$. We set $D := D' \setminus E^*$. Note that $c(D) \leq d$. We show that D is a solution of I. Assume towards a contradiction that there is an (s, t)-cut $A \subseteq E \setminus D$ in G with $\omega(A) \leq a$.

If $e^* \notin A$, then A is also an (s,t)-cut disjoint to D' in G' with $\omega'(A) \leq a$. A contradiction. Otherwise, $e^* \in A$. Hence, $A' := A \setminus \{e^*\} \cup \{e'\}$ for some $e' \in E^* \setminus D'$ is an (s,t)-cut disjoint to D' in G' with $\omega'(A') \leq a$, a contradiction.

Lemma 2. Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of ZWMCP and let $e^* = \{v^*, w^*\} \in E$ with $\omega(e^*) > 1$. Moreover, let $I' := (G' = (V', E'), s, t, c', \omega', d, a)$ be the instance of ZWMCP obtained by updating the capacity of e^* to 1 and by adding $\omega(e^*) - 1$ many (v^*, w^*) -paths with two edges of cost $c(e^*)$ and capacity 1 each. Then, I and I' are equivalent instances of ZWMCP and I' can be computed in $\mathcal{O}(\omega(e^*) \cdot |I|)$ time.

Proof. The running time bound follows immediately by the construction. It remains to prove the correctness. By $V^* := V' \setminus V$ we denote the vertices and by $E^* := E' \setminus E$ we denote the edges added to G to obtain the graph G'. We prove that I is a yes-instance of ZWMCP if and only if I' is a yes-instance for ZWMCP.

 (\Rightarrow) Let $D \subseteq E$ be a solution of I of cost at most d.

Case 1: $e^* \in D$. We set D' := D. Clearly, $c(D') \leq d$. We show that D' is a solution of I'. Assume towards a contradiction that there is an (s,t)-cut $A' \subseteq E' \setminus D'$ in G' with $\omega(A') \leq a$. Recall that each edge in E^* is on a path between v^* and w^* . Since $e^* \in D'$, we conclude that $A' \cap E^* = \emptyset$. Hence, A' is also an (s,t)-cut of capacity at most a in G, a contradiction.

Case 2: $e^* \notin D$. Observe that $D \subseteq E'$ and that $c'(D) \leq d$. We show that D is a solution of I'. Assume towards a contradiction that there is an inclusion-minimal (s, t)-cut $A' \subseteq E' \setminus D$ in G' with $\omega(A') \leq a$.

If $A' \cap E^* = \emptyset$, then A' is also an (s, t)-cut disjoint to D in G with $\omega(A') \leq a$, a contradiction. Otherwise, $A' \cap E^* \neq \emptyset$. Recall that all edges in E^* are on paths with two edges between v^* and w^* . Thus, A' is also an (v^*, w^*) -cut in G'. Hence, $\{v^*, w^*\} \in A'$ and for each vertex $z \in V^*$ at least one adjacent edge is contained in A'. Since $|V^*| = \omega(e^*) - 1$, we conclude that $|A' \cap E^*| \geq \omega(e^*) - 1$. Thus, $A' \setminus E^*$ is an (s, t)-cut of cost at most a in G, a contradiction.

 (\Leftarrow) Let D' be a solution of I' of cost at most d. By P_z we denote the path (v^*, z, w^*) for some vertex $z \in V^*$.

Case 1: $E(P_z) \subseteq D'$ for some $z \in V^*$ or $e^* \in D'$. We set $D := D' \setminus E^* \cup \{e^*\}$. Note that since $c(e) = c(e^*)$ for each edge $e \in E^*$ we obtain $c(D) \leq d$. We show that D' is a solution of I. Assume towards a contradiction that there is an (s, t)-cut $A \subseteq E \setminus D$ in G with $\omega(A) \leq a$. By definition of D, we observe that $A \subseteq E' \setminus D'$. Moreover, since we obtained G' from G by adding $\omega(e^*) - 1$ paths consisting of the edges E^* , and the fact that A is disjoint to E^* , we conclude that A is an (s, t)-cut in G' with $\omega'(A) \leq a$, a contradiction.

Case 2: $E(P_z) \notin D'$ for each $z \in V^*$ and $e^* \notin D'$. We set $D := D' \setminus E^*$. Note that $c(D) \leq d$. We show that D is a solution of I. Assume towards a contradiction that there is an (s, t)-cut $A \subseteq E \setminus D$ in G with $\omega(A) \leq a$.

If $e^* \notin A$, then A is also an (s, t)-cut disjoint to D' in G' with $\omega'(A) \leq a$, a contradiction. Otherwise, $e^* \in A$. Let $A' := A \setminus \{e^*\} \cup \{\{v^*, z\} \mid z \in V^*\}$. Note that for each $e \in E^* \cup \{e^*\}$ we have $\omega(e) = 1$ and that $|V^*| = \omega(e^*) - 1$. Hence, A' is an (s, t)-cut disjoint to D' in G' with $\omega'(A') \leq a$, a contradiction.

Recall that we can assume $c(e) \le d + 1$ and $\omega(e) \le a + 1$ for each edge $e \in E$. Hence, the subsequent application of Lemmas 1 and 2 leads to the following.

Corollary 1. Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of WMCP. Then, one can compute in $(n + a + d)^{\mathcal{O}(1)}$ time an equivalent instance I' = (G', s', t', d, a) of MCP.

The next definition will be a useful tool in several proofs in this work.

Definition 1. Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of WMCP, and let $e = \{u, w\} \in E$. The merge of u and w in I is the instance I' obtained from I by removing u and w from G and adding a new vertex $v_{\{u,w\}}$ which is adjacent to $N(\{u,w\})$. The cost and capacity for each edge in $E \cap E'$ are set to the corresponding cost and capacity in E, and for each $x \in N(\{u,w\})$,

• $c'(\{v_{\{u,w\}},x\}) = \min\{c(e') \mid e' \in E(x,\{u,w\})\}, and$

•
$$\omega'(\{v_{\{u,w\}},x\}) = \sum_{e' \in E(x,\{u,w\})} \omega(e')$$

Rule 1. If G contains an edge $e^* = \{u^*, w^*\} \in E$ which is not contained in any inclusion-minimal (s, t)-cut of capacity at most a in G, then merge u^* and w^* .

Lemma 3. *Rule 1 is safe and can be applied exhaustively in polynomial time.*

Proof. Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of WMCP and let $I' = (G' = (V', E'), s', t', c', \omega', d, a)$ be the merge of u^* and w^* in I. We show that I and I' are equivalent instances of WMCP.

 (\Rightarrow) Let $D \subseteq E$ be a solution of I of cost at most d.

Claim 1. The set $D^* := D \setminus \{e^*\}$ is a solution of I.

Proof. Assume towards a contradiction that D^* is not a solution of I. Then, there is an inclusion-minimal (s,t)-cut $A \subseteq E \setminus D^*$ of capacity at most a in G. By the condition of Rule 1, it holds that $e^* \notin A$. Note that A avoids D^* . This contradicts the fact that D^* is a solution.

Due to Claim 1 we can assume that $e^* \notin D$. We set $D' := (D \cap E') \cup \{\{v_{e^*}, x\} \mid \{u^*, x\} \in D \text{ or } \{w^*, x\} \in D\}$. By definition of c' it follows that D' has cost at most c(D). Hence, it remains to show that D' is a solution of I'.

Assume towards a contradiction that D' is not a solution of I'. Then, there is an (s', t')-cut $A' \subseteq E' \setminus D'$ of capacity at most a in G'. We set $A := (A' \cap E) \cup \{e \in E(x, e^*) \mid \{v_{e^*}, x\} \in A'\}$. Note that $A \subseteq E \setminus D$. By definition of ω' , we obtain that $\omega(A) = \omega'(A') \leq a$. Since $\{x, v_{e^*}\} \in A'$ if and only if $E(x, e^*) \subseteq A$, and A and A' agree on $E \cap E'$, we obtain that A is an (s, t)-cut in G which contradicts the fact that D is a solution of I. Consequently, Iis a yes-instance of WMCP.

 (\Leftarrow) Let $D' \subseteq E'$ be a solution of I' of cost at most d. We set $D := (D' \cap E) \cup \{e_x \mid \{v_{e^*}, x\} \in D'\}$, where e_x is an edge in $E(x, e^*)$ with minimal cost. By definition of c' it follows that $c(D) \leq c'(D')$. It remains to show that D is a solution of I.

Assume towards a contradiction that D is not a solution of I. Then, there is an (s,t)-cut $A^* \subseteq E \setminus D$ of capacity at most a in G. Since e^* is not contained in any inclusion-minimal (s,t)-cut of capacity at most a, there is an inclusion-minimal (s,t)-cut $A \subseteq A^* \setminus \{e^*\}$ of capacity at most a in G. We set $A' := (E' \cap A) \cup \{\{v_{e^*}, x\} \in E' \mid E(x, e^*) \subseteq A\}$. By definition of ω' , we obtain $\omega'(A') \leq \omega(A) \leq a$. Since $\{x, v_{e^*}\} \in A'$ if and only if $E(x, e^*) \subseteq A$ and A and A' agree on $E \cap E'$, we obtain that A' is an (s', t')-cut in G which contradicts the fact that D' is a solution of I'. Consequently, I is a yes-instance of WMCP.

It remains to bound the running time. Each application of Rule 1 reduces the number of vertices by one, and each such application can be performed in polynomial time, we obtain that, Rule 1 can be exhaustively applied in polynomial time. \Box

3 NP-hardness and Parameterization by the Defender Budget d

In this section we prove that MCP is NP-hard and we analyze parameterization by d and $\Delta(G)$. In particular, we provide a complexity dichotomy for $\Delta(G)$.

Lemma 4. WMCP is NP-complete and W[1]-hard when parameterized by d even if G is bipartite, $\omega(e) = 1$, and $c(e) \in \mathcal{O}(|G|)$ for all $e \in E$.

Proof. We describe a parameterized reduction from a variant of INDEPENDENT SET which is known to be W[1]-hard when parameterized by k [10, 11].

REGULAR-INDEPENDENT SET **Input:** An *r*-regular graph G = (V, E) for some integer *r* and an integer *k*. **Question:** Is there an independent set $S \subseteq V$ of size at least *k* in *G*?

Let I := (G = (V, E), k) be an instance of *r*-REGULAR-INDEPENDENT SET. We describe how to construct an instance $I' := (G' = (V', E'), s, t, c, \omega, d, a)$ of WMCP in polynomial time such that I is a yes-instance of REGULAR-INDEPENDENT SET if and only if I' is a yes-instance of WMCP.

We start with an empty graph G' and add all vertices of V to G'. For each vertex $v \in V$ we also add an additional vertex v'. Furthermore, for each edge $e \in E$ we add a vertex w_e , and two new vertices s and t to G'. Moreover, we add the edges $\{s, v\}, \{v, v'\}$ and $\{v', t\}$ to G' for each vertex $v \in V$. Next, we add the edges $\{u, w_e\}, \{v, w_e\}$, and $\{w_e, t\}$ to G' for each edge $e = \{u, v\} \in E$. Now, we set $\omega(e') := 1$ for all $e' \in E'$. Furthermore, for each $e' \in E'$, we set c(e') := 1 if $s \in e'$ and c(e') := k + 1 otherwise. Finally, we set d := k and a := n + kr - 1 where n := |V|. This completes the construction of I'. Observe that G' is bipartite with one partite set being $\{t\} \cup V$. Note that only the edges incident with s can be protected, since all other edges have cost exactly d + 1.

Next, we show that I is a yes-instance of REGULAR-INDEPENDENT SET if and only if I' is a yes-instance of WMCP.

 (\Rightarrow) Let $S \subseteq V$ be an independent set of G of size exactly k = d. We set $D' := \{\{s, v\} \mid v \in S\}$. Note that D' has cost exactly d. It remains to show that D' is a solution of I'. To this end, we provide a + 1 many paths whose edge sets may only intersect in D'.

Note that for each vertex $v \in V \setminus S$ we have a path (s, v, v', t). These are n - k many. Next, consider a vertex $v \in S$. Observe that (s, v, v', t) and $\{(s, v, w_e, t) \mid e \in E, v \in e\}$ are r + 1 paths only sharing the edge $\{s, v\} \in D'$. Since |S| = k and G is r-regular, these are kr + k many paths. Moreover, since S is an independent set no two vertices $u, v \in S$ have a common neighbor w_e in G' for $e = \{u, v\}$. Hence, there are n - k + kr + k = n + kr = a + 1many (s, t)-paths in G' whose edge sets only intersect in D'.

 (\Leftarrow) Suppose that I' is a yes-instance of WMCP. Let D' be a solution with cost at most d of I'. Recall that c(e) = d+1 for each edge $e' \in E'$ with $s \notin e'$. Hence, $D' \subseteq \{\{s, v\} \mid v \in V\}$. If |D'| < d, then we add exactly d - |D'| many edges of the form $\{s, v\}$ which are not already contained in D' to D'. Note that D' remains a solution of I'. Thus, in the following we can assume that |D'| = d = k. Let $S := \{v \mid \{s, v\} \in D'\}$. We prove that S is an independent set in G.

Assume towards a contradiction that S is no independent set in G and let e^* be an edge of G[S]. In the following, we construct an (s,t)-cut $A \subseteq (E' \setminus D')$ in G' of size at most a. Let $A_{V \setminus S} := \{\{s,v\} \mid v \notin S\}, A_S := \{\{v,v'\} \mid v \in S\},$ and $A_E := \{\{v, w_e\} \in E' \mid v \in S, e \neq e^*\}$. We show that $A := A_{V \setminus S} \cup A_S \cup A_E \cup \{\{w_{e^*}, t\}\}$ is an (s, t)-cut of size at most a in G'. Note that $|A_{V \setminus S}| + |A_S| = n$. Moreover, since |S| = k and each vertex $v \in V$ has degree exactly r in G, $|A_E| \leq kr - 2$. Hence, A has capacity at most n + kr - 1 = a since $\omega(e') = 1$ for each $e' \in E'$. It remains to show that A is an (s, t)-cut in G'. Let $G^* := G' - A$. Note that $N_{G^*}(s) = S$ and $N_{G^*}(v) = \{s\}$ for each $v \in S \setminus e^*$.

Moreover, note that $N_{G^*}(v) = \{s, w_{e^*}\}$ for each $v \in e^*$ and $N_{G^*}(w_{e^*}) = e^*$. Hence, A is an (s, t)-cut in G' with capacity at most a. A contradiction.

Consequently, S is an independent set of size k in G and, therefore, I is a yes-instance of REGULAR-INDEPENDENT SET. \Box

By applying Corollary 1, we can extend the hardness results to MCP. Note that if k is odd, Corollary 1 replaces an edge with costs k + 1 by a path of even length and thus the resulting instance of MCP is not bipartite. Hence, to obtain W[1]-hardness in case of odd k, we set c(e) = k + 2 for edges not containing s.

Corollary 2. MCP is NP-complete and W[1]-hard when parameterized by d, even on bipartite graphs.

Next, we provide a complexity dichotomy for the classical complexity with respect to the maximum degree of the graph.

Lemma 5. ZWMCP can be solved in polynomial time on graphs of maximum degree two.

Proof. Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of ZWMCP where G has degree at most two. Recall that we can assume without loss of generality that G is connected. Observe that since G has degree at most two, G is either a path or a cycle.

G is a path. Let *P* be the unique (s,t)-path in *G* and let $E_A := \{e_i \in E(P) \mid \omega(e_i) \leq a\}$ be the set of edges of capacity at most *a*. Since $\{e_i\}$ is an (s,t)-cut of capacity at most *a* in *G* for every $e_i \in E_A$, we conclude that E_A is a subset of every solution of *I*. Consequently, *I* is a yes-instance of ZWMCP if and only if $d \geq c(E_A)$, since every (s,t)-cut $M \subseteq E \setminus E_A$ has capacity larger than *a*.

G is a cycle. Let P_1 and P_2 be the unique (s,t)-paths in *G*. Moreover, let $E_A := \{\{e_i^1, e_j^2\} \mid e_i^1 \in E(P_1), e_j^2 \in E(P_2), \omega(e_i^1) + \omega(e_j^2) \leq a\}$ be the set of minimal (s,t)-cuts of capacity at most *a* in *G*. Note that every other (s,t)-cut of capacity at most *a* is a superset of any (s,t)-cut in E_A . Hence, *I* is a yes-instance of ZWMCP if and only if there is a set $S \subseteq E(P_1) \cup E(P_2)$ with $c(S) \leq d$ such that $S \cap e \neq \emptyset$ for all $e \in E_A$. This is equivalent to the question if the graph *G'* with bipartition $(E(P_1), E(P_2))$ and edges E_A has a vertex cover of capacity at most *d* with *c* as the capacity function. This can be done in polynomial time.

Consequently, ZWMCP can be solved in polynomial time on graphs of degree at most two.

Lemma 6. WMCP is NP-hard and W[1]-hard when parameterized by d even on subcubic graphs and even if c(e) = 1 and $\omega(e) \in \mathcal{O}(|G|)$ for all $e \in E$.

Proof. We reduce from MCP which is W[1]-hard when parameterized by d due to Corollary 2. Let I = (G = (V, E), s, t, d, a) be an instance of MCP. Next, we construct an equivalent instance $I' = (G' = (V', E'), s', t', c', \omega', d', a')$ of WMCP as follows.

For each vertex $v \in V$ we add a path P_u consisting of |N(u)| vertices to G'. We denote the vertices of P_u by $p_u^1, \ldots p_u^{|N(u)|}$. In the following, we assume an arbitrary but fixed ordering on N(u). Thus, the *i*-th-vertex of N(u) is associated with vertex $p_u^i \in P_u$. Furthermore, if v is the *i*-th neighbor of u we also write p_u^v instead of p_u^i to access neighbor v more conveniently. We set c'(e) = 1 and $\omega'(e) = a + 1$ for each edge $e \in E(P_u)$. Furthermore, for each edge $\{u, v\} \in E(G)$ we add the edge $\{p_u^v, p_v^u\}$ to G' with cost and capacity equal to one. Next, we set $s' := p_s^1$ and $t' := p_t^1$. Finally, we set a' := a and d' := d.

Since each vertex in P_u has exactly one neighbor which is not in P_u , the graph G' is subcubic. Next, we prove that I is a yes-instance of MCP if and only if I' is a yes-instance of WMCP.

Let $v \in V$ and let e be an edge of P_v . By the fact that $\omega'(e) = a + 1$, e is not contained in any (inclusionminimal) (s', t')-cut of capacity at most a. Thus, by merging the endpoints of e, we obtain an equivalent instance of WMCP due to Lemma 3.

Let $I^* = (G^* = (V^*, E^*), s^*, t^*, c^*, \omega^*, d', a')$ be the instance of WMCP we obtain after merging the endpoints of all edges contained in any path P_v . Note that by Definition 1 it follows that G^* is isomorphic to G and $\omega^*(e) = c^*(e) = 1$ for each $e \in E^*$. Thus, I is a yes-instance of MCP if and only if I^* is a yes-instance of WMCP.

By Lemma 5 and Lemma 6 we obtain the following.

Theorem 1. WMCP can be solved in polynomial time on graphs of maximum degree two. WMCP is NP-hard and W[1]-hard when parameterized by d even on subcubic graphs and even if c(e) = 1 and $\omega(e) \in O(|G|)$ for all $e \in E$.

Next, we strengthen the NP-hardness of WMCP on subcubic graphs to MCP.

Lemma 7. MCP is NP-complete even on subcubic graph.

Proof. We reduce from MCP. Let I = (G = (V, E), s, t, d, a) be an instance of MCP. We first prove the statement for WMCP where $\omega(e) = 1$ and $c(e) \in n^{\mathcal{O}(1)}$. We do this intermediate step to emphasize the main idea of the reduction. Second, we apply Corollary 1 to each edge in the instance of WMCP to obtain an equivalent instance of MCP. Note that since Corollary 1 replaces an edge by a path, the resulting instance of MCP is also subcubic. Hence, it remains to prove the statement for the restricted version of WMCP.

Next, we construct an equivalent instance $I' = (G' = (V', E'), s', t', c', \omega', d', a')$ of WMCP as follows. For each vertex $v \in V$ we add a path P_u consisting of N(u) vertices to G'. We denote the vertices of P_u by $p_u^1, \ldots p_u^{|N(u)|}$. In the following, we assume an arbitrary but fixed ordering on N(u). Thus, the *i*-th-vertex of N(u) is associated with vertex $p_u^i \in P_u$. Furthermore, if v is the *i*-th neighbor of u we also write p_u^v instead of p_u^i to access neighbor v more convenient. We set $c'(e) := \omega'(e) := 1$ for each edge $e \in E(P_u)$. Furthermore, for each edge $\{u, v\} \in E(G)$ we add the edge $\{p_u^v, p_v^u\}$ to G' and set its costs to n^2 and its capacity to one. Next, we set $s' := p_s^1$ and $t' := p_t^1$. Finally, we set a' := a and $d' := dn^2 + n(n-1)$.

Since each vertex in P_u has exactly one neighbor which is not in P_u , the graph G' is subcubic. Next, we prove that I is a yes-instance of MCP if and only if I' is a yes-instance of WMCP.

 (\Rightarrow) Let $D \subseteq E$ be a solution with cost at most d of I. In the following, we construct a solution $D' \subseteq E'$ with cost at most d' of I'.

For each edge $\{u, v\} \in D$ we add the corresponding edge $\{p_u^v, p_v^u\}$ in G' to D'. Since each of these edges has $\cos n^2$, and $|D| \leq s$, these edges contribute at most dn^2 to to cost of D'. Furthermore, we add each edge in P_u for each $u \in V$ to D'. Since P_u has at most n-1 edges and each edge in P_u has cost one, all these edges contribute at most n(n-1) to the total costs. Hence, $|D'| \leq d'$. Assume towards a contradiction that G' has an (s', t')-cut $A' \subseteq E' \setminus D'$ with $\omega'(A') \leq a'$. Since $E(P_u) \subseteq D'$ for each $u \in V$, the (s, t)-cut A' contains only edges of the form $\{p_u^v, p_v^u\}$ between two different paths. We define the set A as the set of corresponding edges of A' in G. Since $|A'| \leq a$ we obtain $|A| \leq a$. Since there is no (s, t)-cut of capacity at most a in $G \setminus D$ and $A \cap D = \emptyset$, we conclude that there exists an (s, t)-path $(s, w_1, \ldots, w_\ell, t)$ in $G \setminus A$. Observe that $(p_s^1, \ldots, p_s^{w_1}, p_{w_1}^{w_1}, \ldots, p_{w_1}^{w_2}, \ldots, p_t^{w_\ell}, p_t^1)$ is an (s', t')-path in G' - A', a contradiction to the assumption that A' is an (s', t')-cut in G'.

 (\Leftarrow) Let $D' \subseteq E'$ be a solution with cost at most d' of I'. In the following, we construct a solution $D \subseteq E$ with cost at most d of I.

Since $d' = dn^2 + n(n-1)$, $c'(e) = n^2$ for each edge $e \notin E(P_u)$ and each $u \in V$ in G', and c(e) = 1 for each edge $e \in E(P_u)$ for some $u \in V$ in G', we can assume without loss of generality that $E(P_u) \subseteq D'$ for each $u \in V$. We start with an empty set D. For an edge $\{p_u^v, p_v^u\} \in D'$ between two different paths, we add the edge $\{u, v\}$ to D. Assume towards a contradiction that G has an (s, t)-cut $A \subseteq E \setminus D$ with $\omega(A) \leq a$. We define the set A' as the set of corresponding edges of A in G'. Note that since $\omega(e) = 1$ for each edge $e \in E'$ we have $|A'| \leq a = a'$. Since there is no (s', t')-cut of capacity at most a in $G' \setminus D'$ and $A' \cap D' = \emptyset$, we conclude that there exists an (s', t')-path $(p_s^1, \ldots, p_{s^1}^n, p_{w_1}^s, \ldots, p_{w_1}^{w_2}, p_{w_2}^{w_1}, \ldots, p_t^{w_e}, p_t^1)$ in G'. Thus, $(s, w_1, \ldots, w_\ell, t)$ is an (s, t)-path in $G \setminus A$, a contradiction to the assumption that A is an (s, t)-cut in G.

Theorem 2. WMCP can be solved in $a^d \cdot n^{\mathcal{O}(1)}$ time.

Proof. Let $J := (G, s, t, c, \omega, d, a)$ be an instance of WMCP. We prove the theorem by describing a simple search tree algorithm, that we initially call with $D := \emptyset$, where D represents the choice of the defender:

If c(D) > d, then return *no*. Otherwise, use the algorithm behind Lemma 1 to compute an (s, t)-cut $A = \{e_1, \ldots, e_z\} \subseteq E \setminus D$ for some $z \leq |A|$ with $\omega(A) \leq a$. If no such (s, t)-cut exists, then return *yes*. Otherwise, we branch into the cases where $D := D \cup \{e_i\}$ for each $i \in [1, z]$.

The correctness of the algorithm follows from the fact that for every (s,t)-cut $A \subseteq E \setminus D$ with $\omega(A) \leq a$, at least one of the edges of A must be contained in any solution of J. It remains to consider the running time of the algorithm. We have $\omega(e) \geq 1$ for every edge e in any WMCP instance. Hence, $|A| \leq a$ and therefore, the search tree algorithm branches into at most a cases. Furthermore, after every branching step, c(D) increases by at least 1, since we add one additional edge to D and we have $c(e) \geq 1$ for every edge e. Thus, the depth of the search tree is at most d. Together with the running time from Lemma 1, we obtain a total running time of $a^d n^{\mathcal{O}(1)}$. **Lemma 8.** MCP can be solved in $((d/2+1) \cdot \Delta(G))^d \cdot n^{\mathcal{O}(1)}$ time, where $\Delta(G)$ denotes the maximum degree of the input graph.

Proof. Let J := (G, s, t, d, a) be an instance of MCP. We prove the theorem by showing that $a \le d \cdot \Delta$ in non-trivial instances of MCP. Together with Theorem 2, we then obtain fixed-parameter tractability for $d + \Delta$.

If G contains an (s,t)-path with at most d edges, then J is a trivial yes-instance. Thus, we may assume that for every $D \subseteq E$ with $|D| \leq d$, there is no (s,t)-path in G that contains only edges from D. We use this assumption to prove the following claim.

Claim 2. If $a \ge (d/2 + 1) \cdot \Delta$, then J is a no-instance.

Proof. Let $D \subseteq E$ with $|D| \leq d$. We prove that there exists an (s, t)-cut $A \subseteq E$ of size at most a. To this end, consider the graph $G_D := (V, D)$ consisting only of the edges in D. Since $|D| \leq d$ we know that there is no (s, t)-path in G_D . Thus, s and t are in distinct connected components $C_s \subseteq V$ and $C_t \subseteq V$ in G_D . Furthermore, observe that in at least one of the induced graphs $G_D[C_s]$ or $G_D[C_t]$, there are at most d/2 edges. Without loss of generality, assume that this is the case for $G_D[C_s]$. Then, $|C_D| \leq d/2 + 1$. We define $A := \bigcup_{v \in C_D} X(v)$, where $X(v) \subseteq E \setminus D$ is the set of all edges in $E \setminus D$ that are incident with v in G. Note that $A \subseteq E \setminus D$ and that $|A| \leq |C_D| \cdot \Delta = (d/2+1) \cdot \Delta \leq a$. Moreover, A is an (s, t)-cut in G since $t \notin C_s$.

By Claim 2, we conclude that for every non-trivial instance of MCP, we have $a \le d \cdot \Delta$. Together with Theorem 2, we obtain that MCP can be solved in $((d/2 + 1) \cdot \Delta)^d \cdot n^{\mathcal{O}(1)}$ time.

By Lemma 7 and Lemma 8 we obtain the following.

Theorem 3. MCP is NP-complete even on subcubic graphs. Furthermore, MCP can be solved in $((d/2+1)\cdot\Delta(G))^d \cdot n^{\mathcal{O}(1)}$ time.

4 Parameterization by the Attacker Budget

In this section, we show that WMCP admits an FPT-algorithm for the parameter a. To this end, we first provide an algorithm with a running time of $a^{f(tw(G))} \cdot n$ for some computable function f, where tw(G) denotes the treewidth of the graph. Afterwards, we show that for every instance of WMCP we can obtain an equivalent instance I' of WMCP in polynomial time, where every edge is contained in an inclusion-minimal (s, t)-cut of size at most a in I'. Due to previous results [16, 20], the graph of I' then has treewidth at most g(a) for some computable function g. In combination with the algorithm for a and tw(G), we thus obtain the stated FPT-algorithm for the parameter a.

The algorithm with a running time of $a^{f(tw(G))} \cdot n$ relies on dynamic programming over a tree decomposition. Essentially, what the attacker can achieve in the current subgraph is to disconnect specific parts of the bag and thus obtain a cheap partition. Roughly speaking, the algorithm computes the minimum cost for an edge set D such that each choice of the attacker to obtain any partition disjoint from D is expensive. Hence, before we describe the algorithm, we first introduce some notations for partitions.

Let X be a set. We denote with B(X) the collection of all partitions of X. Let $P \in B(X)$ be a partition of X and let $v \in X$. Then, we define with $P - v := \{R \setminus \{v\} \mid R \in P\} \setminus \{\emptyset\}$ the partition of $X \setminus \{v\}$ after removing v from P. Analogously, for every $w \notin X$ we define $P + w := \{P' \in B(X \cup \{w\}) \mid P' - w = P\}$. Note that $B(X \setminus \{v\}) = \{P - v \mid P \in B(X)\}$ and $B(X \cup \{w\}) = \{P + w \mid P \in B(X)\}$. Moreover, we denote with P(v) the unique set of P containing v for a partition P of X and an element $v \in X$.

Let $(\mathcal{T} := (\mathcal{V}, \mathcal{A}, r), \beta)$ be a tree decomposition of a graph G. Recall that for anode $x \in \mathcal{V}$, we define with V_x the union of all bags $\beta(y)$, where y is reachable from x in $\mathcal{T}, G_x := G[V_x]$, and $E_x := E_G(V_x)$.

Let P be a partition of $\beta(x)$, then we call an edge set $A \subseteq E_x$ a *partition-cut for* P in G_x if v and w are in different connected components in $G_x - A$ for every pair of distinct vertices $\{v, w\}$ of $\beta(x)$ with $P(v) \neq P(w)$. Note that all edges between distinct sets of P are contained in every partition-cut for P in G_x .

Theorem 4. Let tw(G) denote the treewidth of G. Then, ZWMCP can be solved in $a^{tw(G)\mathcal{O}(tw(G))} \cdot n + m$ time.

Proof. Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of ZWMCP. In the following, we assume that there is no edge $\{s,t\} \in E$ since if $c(\{s,t\}) \leq d$, then $\{\{s,t\}\}$ is a valid solution with cost at most d and, thus, I is a trivial yes-instance of ZWMCP. Otherwise, this edge is contained in every (s,t)-cut and, thus, we can simply remove the edge from the graph and reduce a by $\omega(\{s,t\})$.

We describe a dynamic programming algorithm on a tree decomposition. First, we compute a nice tree decomposition $(\mathcal{T} = (\mathcal{V}, \mathcal{A}, r), \beta')$ of $G - \{s, t\}$ with $|\mathcal{V}| \leq 4n$ such that the bag of the root and the bag of each leaf is the empty set in tw^{$\mathcal{O}(tw^3)$} · n + m time [19, 4]. Next, we set $\beta(x) := \beta'(x) \cup \{s, t\}$ for each $x \in \mathcal{V}$. Note that (\mathcal{T}, β) is a tree decomposition of width at most tw + 2 for G. Recall that for a node $x \in \mathcal{V}$, the vertex set V_x is the union of all bags $\beta(y)$, where y is reachable from x in $\mathcal{T}, G_x := G[V_x]$, and $E_x := E_G(V_x)$.

The dynamic programming table T has entries of type $T[x, f_x, D_x]$ with $x \in \mathcal{V}$, $f_x : B(\beta(x)) \to [0, a+1]$, and $D_x \subseteq E(\beta(x))$. Each entry stores the minimal cost of an edge set $D \subseteq E_x$ with $D_x := D \cap E(\beta(x))$ such that for every $P \in B(\beta(x))$ the capacity of every partition-cut $A \subseteq E_x \setminus D$ of P in G_x is at least $f_x(P)$.

For each entry of T, we will sketch the proof of the correctness of its recurrence. The formal correctness proof is then direct and thus omitted.

We start to fill the table T by setting for each leaf node ℓ of \mathcal{T} :

$$T[\ell, f_{\ell}, \emptyset] := \begin{cases} 0 & \text{if } f_{\ell}(\{\{s\}, \{t\}\}) = f_{\ell}(\{\{s, t\}\}) = 0, \\ \infty & \text{otherwise.} \end{cases}$$

Recall that $\beta(\ell) = \{s, t\}$ and that we assumed that there is no edge between s and t in G. Hence, G_{ℓ} contains no edges and, thus, the empty set is a partition-cut for both $\{\{s\}, \{t\}\}\$ and $\{\{s, t\}\}\$, and has capacity zero.

To compute the remaining entries $T[x, f_x, D_x]$, we distinguish between the three types of non-leaf nodes.

Forget node: Let x be a forget node with child node y and let v be the unique vertex in $\beta(y) \setminus \beta(x)$. Then we compute the table entries for x by:

$$T[x, f_x, D_x] := \min_{E_v \subseteq E(v, \beta(x))} T[y, f_y, D_x \cup E_v]$$

where $f_y(P) := f_x(P - v)$ for each $P \in B(\beta(y))$.

The idea behind the definition of $f_y(P)$ is that every partition cut for P in G_y must be as expensive as the partition cut of the unique partition of $\beta(x)$ that agrees with P on $\beta(x)$. By the fact that $G_x = G_y$, it follows that for each partition $P \in B(\beta(x))$, an edge set $A \subseteq E_x$ is a partition-cut for P in G_x if and only if A is also a partition-cut for some $P' \in P + v$ in G_x . Since we are looking for the minimal costs of an edge set $D \subseteq E_x$ such that every partition-cut disjoint from D for P - v in G_x has capacity at least $f_x(P - v)$, it is thus necessary and sufficient that every partition-cut for P in G_y has capacity at least $f_x(P - v)$.

Introduce node: Let x be an introduce node with child node y and let v be the unique vertex in $\beta(x) \setminus \beta(y)$. Then we compute the table entries for x by:

$$T[x, f_x, D_x] := T[y, f_y, D_x \cap E(\beta(y))] + c(D_x \setminus E(\beta(y)))$$

where $f_y(P) := \max(\{0\} \cup \{f_x(P') - \omega(A_{P'}) \mid P' \in (P+v), D_x \cap A_{P'} = \emptyset\})$ for each $P \in B(\beta(y))$ and $A_{P'} := E(v, \beta(y) \setminus P'(v))$.

The idea behind the definition of $f_y(P)$ is that, since every partition in P + v agrees with P in $\beta(y)$, every partition cut for P in G_y must be sufficiently large to ensure that every partition cut for any partition in P + v is as least as expensive as desired. Since we are looking for the minimum cost of an edge set $D \subseteq E_x$ which intersects with $E(\beta(x))$ in exactly the set D_x , the cost of D is exactly $c(D \cap E(\beta(y))) + c(D_x \setminus E(\beta(y)))$. Let $P' \in B(\beta(x))$. Note that $A_{P'}$ is a subset of every partition-cut for P' in G_x . Hence, if $D_x \cap A_{P'} = \emptyset$, then $f_y(P' - v)$ has to be at least $f_x(P') - \omega(A_{P'})$. Otherwise, if $D_x \cap A_{P'} \neq \emptyset$, then there is no partition-cut for P' in G_x disjoint from D.

Join node: Let x be a join node with child nodes y and z. Then we compute the table entries for x by:

$$T[x, f_x, D_x] := \min_{f_y: B(\beta(y)) \to [0, a+1]} T[y, f_y, D_x] + T[z, f_z, D_x] - c(D_x)$$

where the mapping f_z is given by

$$f_z(P) := \max\left(0, \min\left(a+1, f_x(P) - f_y(P) + \omega(E(\beta(x)) \setminus E(P))\right)\right)$$

with $E(P) := \bigcup_{R \in P} E(R)$ for each $P \in B(\beta(z))$.

The idea behind the definition of $f_z(P)$ is that the no partition cut for P in G_z is more expensive than the sum of any combination of partition cuts for P in G_y and G_z minus the capacity of the cut-edges in the current bag. Recall that we are looking for the minimum cost of an edge set $D \subseteq E_x$ such that for each partition $P \in B(\beta(x))$, every partition-cut for P in G_x disjoint from D has capacity at least $f_x(P)$. Since $E_y \cap E_z = E(\beta(x))$ it follows that the cost of D is $c(S_y)+c(S_z)-c(D_x)$, where $S_y := E_y \cap D$ and $S_z := E_z \cap D$. Moreover, note that for every partition $P \in B(\beta(x))$,

every partition-cut $A_{\alpha} \subseteq E_{\alpha}$ for P in G_{α} has to contain all edges of $E(\beta(x)) \setminus E(P)$, where $\alpha \in \{x, y, z\}$. Thus, we have to guarantee that $f_y(P) + f_z(P) - \omega(E(\beta(x)) \setminus E(P)) \ge f_x(P)$, $f_y(P) > f_x(P)$, or $f_z(P) > f_x(P)$.

Then, there is a solution D of cost at most d of I if and only if $T[r, f_r, \emptyset] \le d$, where r is the root of \mathcal{T} , $f_r(\{\{s, t\}\}) = 0$ and $f_r(\{\{s\}, \{t\}\}) = a + 1$. Moreover, the corresponding set D can be found via traceback. It remains to show the running time.

For every node x of \mathcal{T} , there are $(a+2)^{|B(\beta(x))|} \cdot 2^{|\beta(x)|^2}$ entries. Since (\mathcal{T}, β) has at most 4n bags, each bag contains at most k := tw + 3 vertices, and $|B(X)| \leq |X|^{|X|}$, the dynamic programming table contains at most $4n \cdot (a+2)^{k^k} \cdot 2^{k^2}$ entries. Now, we bound the running times of the four types of bags.

- An entry for a leaf node can be computed in $\mathcal{O}(1)$ time.
- For a forget node, we can compute the function f_y in $k^k \cdot k^{\mathcal{O}(1)}$ time and iterate over all possible choices for E_y in 2^k time. Thus, an entry in $k^k \cdot 2^k \cdot k^{\mathcal{O}(1)}$ time.
- For an introduce node, we can compute the function f_y in $k^k \cdot k^{\mathcal{O}(1)}$ time and thus the entry in the same running time.
- For a join node, we have (a + 2)^{k^k} possibilities for f_y and for each of them, we can compute f_z in k^k · k^{O(1)} time. Hence, for a join node, we can compute an entry in (a + 2)^{k^k} · k^k · k^{O(1)} time.

The join nodes have the worst running time for any entry. Thus, we can compute all entries of T in $(a + 2)^{2(tw+3)^{tw+3}} \cdot (tw+3)^{tw+3} \cdot 2^{(tw+3)^2} \cdot tw^{\mathcal{O}(1)} \cdot n$ time and obtain the stated running time.

Next, we show that we can use Theorem 4 to obtain an FPT-algorithm for WMCP when parameterized by *a*. To this end, we first obtain the following corollary which follows from a result of Gutin et al. [16, Lemma 12].

Corollary 3. Let G = (V, E) be a graph, let s and t be distinct vertices of G, and let a be an integer. If every edge $e \in E$ is contained in an inclusion-minimal (s, t)-cut of size at most a, then $tw(G) \le g(a)$ for some computable function g.

Hence, to obtain an FPT-algorithm for WMCP with the parameter a, we only have to find an equivalent instance in polynomial time where each edge is contained in some inclusion-minimal (s, t)-cut of size at most a. Since each edge in an instance of WMCP has capacity at least one, by applying Rule 1 exhaustively we obtain an equivalent instance of WMCP where each edge is contained in some inclusion-minimal (s, t)-cut of *size* at most a. Hence, we obtain the following by combining Lemma 3, Corollary 3, and Theorem 4.

Theorem 5. WMCP is FPT when parameterized by a.

Note that this is not possible for ZWMCP due to Theorem 6.

Theorem 6. ZWMCP is W[1]-hard when parameterized by d + a even if $\omega(e) \in \{0, 1\}$ for all $e \in E$.

Proof. We describe a parameterized reduction from BICLIQUE which is known to be W[1]-hard when parameterized by k [10].

BICLIQUE **Input:** A bipartite graph $G = (X \cup Y, E)$ with partite sets X and Y and an integer k. **Question:** Does G contain a (k, k)-biclique?

Let $I = (G = (X \cup Y, E), k)$ be an instance of BICLIQUE. Now, we describe how to construct an instance $I' = (G' = (V', E'), s, t, c, \omega, d, a)$ of ZWMCP in polynomial time such that I is a yes-instance of BICLIQUE if and only if I' is a yes-instance of ZWMCP.

The graph G' contains the graph G as a copy together with two new vertices s and t and edges $F := \{\{s, x\} \mid x \in X\} \cup \{\{y, t\} \mid y \in Y\}$. Furthermore, each edge $\{x, y\} \in E$ is subdivided by a new vertex w_{xy} in G'. Hence, the graph G' contains the edges $\{x, w_{xy}\}$ and $\{w_{xy}, y\}$ instead of the edge $\{x, y\}$. We define $E_X := \{\{x, w_{xy}\} \mid x \in X\}$ and $E_Y := \{\{w_{xy}, y\} \mid y \in Y\}$. We set $d := (2k + 1)k^2 + 2k = 2k^3 + k^2 + 2k$ and for each edge $\{x, y\} \in E$ we set $c(\{x, w_{xy}\}) := d + 1$, $\omega(\{x, w_{xy}\}) := 1$, $c(\{w_{xy}, y\}) := 2k + 1$, and $\omega(\{w_{xy}, y\}) := 0$. Furthermore, for each edge $e \in F$ we set c(e) := 1, and $\omega(e) := 0$. Finally, we set $a := k^2 - 1$ which completes the construction of I'.

 (\Rightarrow) Suppose that I is a yes-instance of BICLIQUE. Then there exist sets $S_X \subseteq X$ and $S_Y \subseteq Y$ of size k each, such that $\{x, y\} \in E$ for all $x \in S_X$ and $y \in S_Y$. We set $D := \{\{w_{xy}, y\} \mid x \in S_X, y \in S_Y\} \cup \{\{s, x\} \mid x \in S_X\} \cup \{\{y, t\} \mid y \in S_Y\}$. Observe that $|D| = (2k+1)k^2 + 2k = d$. Next, we show that D is a solution of I'.

Consider for each $x \in S_X$ and for each $y \in S_Y$ the (s,t)-path (s,x,w_{xy},y,t) . Since $\{s,x\} \in D$, $\{w_{xy},y\} \in D$, and $\{y,t\} \in D$, each (s,t)-cut A has to contain the edge $\{x, w_{xy}\}$. By the fact that $\omega(\{x, w_{xy}\}) = 1$ for each $x \in S_X$ and each $y \in S_Y$ we observe that $\omega(A) \ge |S_X \times S_Y| = k^2 = a + 1$. Hence, I' is a yes-instance of ZWMCP.

(\Leftarrow) Let D be a solution with cost at most d of I'. Observe that for each edge $e \in E_X$ we have d(e) = d + 1. Thus, $E_X \cap D = \emptyset$. Furthermore, observe that for each edge $e \in E_Y$ we have d(e) = 2k + 1. Hence, for the set $E_D := D \cap E_Y$ we conclude that $|E_D| \le k^2$. Next, we define the vertex sets $S_X := \{x \mid \{s, x\} \in D\}$, the set of endpoints of edges in D incident with s, and $S_Y := \{y \mid \{y, t\} \in D\}$, the set of endpoints of edges in D incident with s, and $S_Y := \{y \mid \{y, t\} \in D\}$, the set of endpoints of edges in D incident with s, and $S_Y := \{y \mid \{y, t\} \in D\}$, the set of endpoints of edges in D incident with s, and $S_Y := \{y \mid \{y, t\} \in D\}$, the set of endpoints of edges in D incident with s, and $S_Y := \{y \mid \{y, t\} \in D\}$. Next, we partition A into two sets A_0 and A_1 , where $A_0 := (E_Y \cup F) \setminus D$ and $A_1 := \{\{x, w_{xy}\} \mid \{w_{xy}, y\} \in E_D\}$. Next, we show that A is an (s, t)-cut for G':

Observe that every (s,t)-path contains at least one subpath (x, w_{xy}, y) for a vertex $x \in X$ and a vertex $y \in Y$ as an induced subgraph. If $\{w_{xy}, y\} \in E_D$ then $\{x, w_{xy}\} \in A_1$, and otherwise if $\{w_{xy}, y\} \notin D_S$ then $\{w_{xy}, y\} \in A_0$. Furthermore, observe that the (s, t)-cut A has capacity $\omega(A_0) + \omega(A_1) = \omega(A_1) = |E_D| \le k^2$. Since D is a solution of I', we conclude that A has capacity at least $a + 1 = k^2$ and, thus, $|E_D| = \omega(A) = k^2$. Thus, $|D \cap F| \le 2k$.

Next, assume towards a contradiction that the set E_D contains an edge $\{w_{xy}, y\}$ such that $x \notin S_X$ or $y \notin S_Y$. Without loss of generality assume that $y \notin S_Y$. We set $A^* := A \setminus \{\{x, w_{xy}\}\}$ and show that A^* is an (s, t)-cut in G'. Since $\{y, t\} \in A^*$ and for each $x' \in N_G(y) \setminus \{x\}$ either $\{x', w_{x'y}\} \in A^*$ or $\{w_{x'y}, y\} \in A^*$, we obtain that A^* is an (s, t)-cut of capacity $\omega(A) - \omega(\{x, w_{xy}\}) = k^2 - 1 = a$. A contradiction. Hence, for each edge $\{w_{xy}, y\} \in D$ we have $\{s, x\} \in D$ and $\{y, t\} \in D$. Since $|E_D| = k^2$, we conclude that $|S_X| = k = |S_Y|$. Consequently, (S_X, S_Y) is a (k, k)-biclique in G and, thus, I is a yes-instance of BICLIQUE.

Together with Corollary 1 we obtain the following.

Corollary 4. ZWMCP is W[1]-hard when parameterized by d + a even if c(e) = 1 and $\omega(e) \in \{0, 1\}$ for all $e \in E$.

5 Parameterization by Vertex Cover Number

We investigate the parameterization by the vertex cover number vc(G). Observing that for MCP the number of protected edges d is at most 2vc(G) in nontrivial instances, eventually leads to the following FPT result.

Theorem 7. MCP can be solved in $2^{\mathcal{O}(vc(G)^2)} \cdot n^{\mathcal{O}(1)}$ time.

Proof. Let J := (G, s, t, d, a) be an instance of MCP, and let vc be the size of a minimum vertex cover of G. The algorithm that we describe here is based on two observations which we formalize in two claims. The first claim states that the defender budget d is upper bounded by $2 \cdot vc$.

Claim 3. If $d \ge 2 \cdot vc$, then J is a yes-instance.

Proof. Let S be a minimum vertex cover in G and let P be a shortest (s, t)-path in G. Then, for every pair v, w of consecutive vertices on P at least one of v and w is contained in S. Consequently, there are at most $2 \cdot vc(G)$ edges on P. If $d \ge 2 \cdot vc(G)$, then the set of edges on P is a solution of J of size at most d. Thus, J is a yes-instance.

Let S be a minimum vertex cover in G, and let $I := V \setminus S$ be the remaining independent set. With the next claim we state that only a bounded number of vertices in I is needed to find a minimal solution of J. To this end we introduce some notation: Given a subset $X \subseteq S$, we let $I_X := \{u \in I \mid N_G(u) = X\} \subseteq I$ denote the *neighborhood class* of X. Moreover, we let E_X denote the set of edges between X and I_X .

Claim 4. There exists a minimum solution D of J such that $|D \cap E_X| \leq |X|$ for every $X \subseteq D$.

Proof. Let D be a solution of J. If $|D \cap E_X| \le |X|$ for every $X \subseteq S$, nothing more needs to be shown. Thus, consider some $X \subseteq S$ such that $|D \cap E_X| > |X|$, and let $u \in I_X$. We then define $D' := (D \setminus E_X) \cup E(u, X)$. It then holds that $|D' \cap E_X| = |X|$. Moreover, observe that |D'| < |D| and $D' \setminus E_X = D \setminus E_X$.

We next show that D' is a solution of J. Let $A \subseteq E \setminus D'$ be an (s, t)-cut in G that is minimum among all (s, t)-cuts that avoid D'. We first prove that $A \cap E_X = \emptyset$. Obviously, $E(u, X) \cap A = \emptyset$ since $E(u, X) \subseteq D'$. Consider $u' \in I_X \setminus \{u\}$. Then, since N(u') = X, for every (s, t)-path P' containing u, there are two consecutive edges $\{x_1, u'\}$ and $\{u', x_2\}$

with $x_1, x_2 \in X$ on P'. Since N(u') = N(u), replacing u' with u defines another (s, t)-path P in G'. Then, since $\{x_1, u\}$ and $\{u, x_2\}$ are not contained in A, there exists another edge on P that is an element of A. Consequently, on every (s, t)-path P' containing u', there exists an edge in A that is not an element of E(u', X). Then, the fact that A is a minimum (s, t)-cut among all (s, t)-cuts that avoid D' implies $E(u', X) \cap A = \emptyset$. Therefore, $A \cap E_X = \emptyset$.

Then, since $A \cap E_X = \emptyset$ and $D' \setminus E_X = D \setminus E_X$ we have $A \subseteq E \setminus D$. Consequently, |A| > a since D is a solution of J.

Since $D' \setminus E_X = D \setminus E_X$, the modification of D described above can be applied on all neighborhood classes I_X independently. Therefore, there exists a minimum solution of J that has the described property.

Let $X \subseteq S$ and $I_X := \{v_1, \ldots, v_{|I_X|}\}$. We define I'_X by $I'_X := I_X$ if $|I_X| \leq |X|$ and $I'_X := \{v_1, \ldots, v_{|X|}\}$, otherwise. Due to Claim 4, there exists a minimum solution such that at most |X| vertices in I_X are endpoints of edges in S. Without loss of generality we may assume that all of these endpoints are from I'_X . Thus, we can assume that there is a minimum solution $D \subseteq E(S) \cup \bigcup_{X \subseteq S} E(X, I'_X)$. We use this assumption for the algorithm that we describe as follows.

- 1. If $d \ge 2 \cdot vc(G)$, then return yes.
- 2. Otherwise, we compute a minimum vertex cover S. Iterate over every possible edge-set $D \subseteq E(S \cup \bigcup_{X \subseteq S} E(X, I'_X)$ with $|D| \leq d$, and check with the algorithm behind Lemma 1 that every (s, t)-cut $A \subseteq E \setminus S$ in G has size bigger than a. If this is the case, then return yes.
- 3. If for none of the choices of D the answer yes was returned in Step 2, then return no.

The correctness of the algorithm is implied by Claims 3 and 4. It remains to analyze the running time. Obviously, Step 1 and Step 3 can be performed in linear time. Consider Step 2. A minimum vertex cover can be computed in $\mathcal{O}(1.28^{vc} + n \cdot vc)$ time [7]. Next, observe that

$$|E(S \cup \bigcup_{X \subseteq S} I'_X)| \le \operatorname{vc}^2 + \sum_{i=0}^{\operatorname{vc}} {\operatorname{vc}}_i^{\operatorname{vc}} i^2$$
$$\le \operatorname{vc}^2(1+2^{\operatorname{vc}-1}).$$

Since $d < 2 \cdot vc$, there are less than $(vc^2(1 + 2^{vc-1}))^{2vc}$ possible subsets $D \subseteq E(S \cup \bigcup_{X \subseteq S} I'_X)$ with $|D| \leq d$. Together with the running time from Lemma 1, Step 2 can be performed in $(vc^2(1+2^{vc-1}))^{2vc} \cdot n^{\mathcal{O}(1)}$ time. Altogether, the algorithm runs within the claimed running time.

Theorem 4 implies that WMCP can be solved in pseudopolynomial time on graphs with a constant treewidth and therefore on graphs with a constant vertex cover number. With the next two theorems we show that significant improvements of this result are presumably impossible.

Theorem 8. WMCP is weakly NP-hard on graphs with a vertex cover of size two.

Proof. We describe a polynomial time reduction from KNAPSACK which is known to be weakly NP-hard [14].

KNAPSACK **Input:** A set U, a size function $f : U \to \mathbb{N}$, a value function $g : U \to \mathbb{N}$, and two budgets $B, C \in \mathbb{N}$. **Question:** Is there a set of items $S \subseteq U$ such that $f(S) := \sum_{u \in S} f(u) \leq B$ and $g(S) := \sum_{u \in S} g(u) \geq C$?

Let I := (U, f, g, B, C) be an instance of KNAPSACK. We describe how to construct an equivalent instance $I' := (G = (V, E), s, t, c, \omega, d, a)$ of WMCP where G has a vertex cover of size two in polynomial time.

We set $V := U \cup \{s,t\}$ and $E := \{\{s,u\}, \{u,t\} \mid u \in U\}$. Note that $\{s,t\}$ is a vertex cover of size two in G. Next, we set d := B and a := |U| + C - 1. Finally, for every $u \in U$ we set $c(\{s,u\}) := f(u), \omega(\{s,u\}) := 1$, $c(\{u,t\}) := d + 1$, and $\omega(\{u,t\}) := g(u) + 1$.

Next, we show that I is a yes-instance of KNAPSACK if and only if I' is a yes-instance of WMCP.

 (\Rightarrow) Suppose that I is a yes-instance of KNAPSACK. Then, there is a set $S_U \subseteq U$ such that $f(S_U) \leq B = d$ and $g(S_U) \geq C$. We set $D := \{\{s, u\} \mid u \in S_U\}$. By construction, we obtain that $c(D) = f(S_U) \leq d$. Let $A \subseteq E \setminus D$ be an (s, t)-cut. We show that A has capacity larger than a.

Since $\{s, u\} \in D$ for all $u \in S_U$ it holds that $T := \{\{u, t\} \mid u \in S_U\} \subseteq A$. Note that $\omega(T) = \sum_{u \in S_U} (g(u) + 1) = g(S_U) + |S_U|$. Moreover, because of the path (s, u, t) for every $u \in U \setminus S_U$ we obtain that $\{s, u\} \in A$ or $\{u, t\} \in A$. Since both of these edges have capacity at least one, we obtain $\omega(A) \ge \omega(T) + |U \setminus S_U| = g(S_U) + |U| = a + 1$. Consequently, I' is a yes-instance of WMCP.

(\Leftarrow) Suppose that I' is a yes-instance of WMCP. Then, there is a solution $D \subseteq E$ with $c(D) \leq d$. By the fact that $c(\{u,t\}) = d + 1$ for all $u \in U$, it follows that $D \subseteq \{\{s,u\} \mid u \in U\}$.

Let $S_U := \{u \in U \mid \{s, u\} \in D\}$. By construction, $f(S_U) = c(D) \leq d = B$. We show that $g(S_U) \geq C$. Let $A \subseteq E \setminus D$ be an (s, t)-cut of minimum capacity. Recall that $\omega(A) \geq a + 1 = |U| + C$. Since A is an (s, t)-cut in G and disjoint to D, we know that $\{u, t\} \in A$ for all $u \in S_U$. Moreover, since $\omega(\{s, u\}) = 1 \leq \omega(\{u, t\})$ for all $u \in U$, we can assume without loss of generality, that $\{s, u\} \in A$ for all $u \in U \setminus S_U$. Hence, $a + 1 = |U| + C \leq \omega(A) = |U \setminus S_U| + \sum_{u \in S_U} (g(u) + 1) = g(S_U) + |U|$. Thus, $C \leq g(S_U)$. Consequently, I is a yes-instance of KNAPSACK.

Theorem 9. WMCP is W[1]-hard when parameterized by the vertex cover number vc(G) even if $c(e) + \omega(e) \in n^{\mathcal{O}(1)}$ and the graph is a biclique.

Proof. We describe a parameterized reduction from BIN PACKING which is W[1]-hard when parameterized by k even if the size of each item is polynomial in the input size [18].

BIN PACKING **Input:** A set U of items, a size-function $f: U \to \mathbb{N}$, and integers B and k. **Question:** Is there a k-partition (U_1, \ldots, U_k) of U with $\sum_{u \in U_i} f(u) = B$ for all $i \in [1, k]$?

Let I := (U, f, B, k) be an instance of BIN PACKING where the size of each item is polynomial in the input size. We can assume without loss of generality that $\sum_{u \in U} f(u) = Bk$, as, otherwise, I is a trivial no-instance of BIN PACKING. We construct an equivalent instance $I' := (G = (V, E), s, t, c, \omega, d, a)$ of WMCP where G has a vertex cover of size k + 1. The graph G is a biclique with bipartition $(\{s\} \cup B, \{t\} \cup U)$ where $\mathcal{B} := \{b_1, \ldots, b_k\}$. We set d := |U|, and

$$c(e) := \begin{cases} 1 & \text{if } e \in \{\{u, b\} \mid u \in U, b \in \mathcal{B}\}, \text{ and} \\ d+1 & \text{otherwise.} \end{cases}$$

Let $\lambda := 2B \cdot |U|$, we set

$$\omega(e) := \begin{cases} \lambda \cdot f(u) & \text{if } e = \{s, u\} \text{ with } u \in U, \\ \lambda \cdot B & \text{if } e = \{t, b\} \text{ with } b \in \mathcal{B}, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Finally, we set $a := |U| \cdot k + \lambda(Bk - 1)$. This completes the construction of I'. Figure 1 shows an example of the construction. Note that $\{s\} \cup B$ is a vertex cover of G of size k + 1. It remains to show that I is a yes-instance of BIN PACKING if and only if I' is a yes-instance of WMCP.

 (\Rightarrow) Suppose that I is a yes-instance of BIN PACKING. Then, there is a k-partition (U_1, \ldots, U_k) of U, such that $\sum_{u \in U_i} f(u) = B$ for all $i \in [1, k]$. We set $D := \{\{u, b_i\} \mid i \in [1, k], u \in U_i\}$. Note that c(D) = d. We next show that D is a solution.

Let $A \subseteq E \setminus D$ be an (s,t)-cut in G and let $i \in [1,k]$. Since for each $u \in U_i$, D contains the edge $\{u, b_i\}$, the (s,t)-path $P_u := (s, u, b_i, t)$ can only be cut if $\{s, u\} \in A$ or $\{b_i, t\} \in A$. Consequently, $\{\{s, u\} \mid u \in U_i\} \subseteq A$ or $\{b_i, t\} \in A$. Recall that $\sum_{u \in U_i} f(u) = B$. Hence, $\sum_{u \in U_i} \omega(\{s, u\}) = \sum_{u \in U_i} \lambda f(u) = \lambda B = \omega(\{b_i, t\})$. Since P_u and P_w are edge-disjoint if u and w are in distinct parts of the k-partition, we obtain that $\omega(A) \ge k\lambda B > a$ and, thus, I' is a yes-instance of WMCP.

(\Leftarrow) Suppose that I' is a yes-instance of WMCP. Then, there is a solution $D \subseteq E$ with $c(D) \leq d$. By construction, $D \subseteq E(U, \mathcal{B})$, since all other edges have cost d + 1.

Note that for each $u \in U$, there is some $b \in \mathcal{B}$, such that $\{u, b\} \in D$, as, otherwise $A := \{\{s, t\}\} \cup \{\{s, u'\} \mid u' \in U \setminus \{u\}\} \cup \{\{u, b'\} \mid b' \in \mathcal{B}\}\}$ is an (s, t)-cut in G with capacity $\lambda(Bk - f(u)) + k + 1 < a$. Since $|D| \leq d$, we obtain that for each $u \in U$, there is exactly one $b \in \mathcal{B}$, such that $\{u, b\} \in D$.

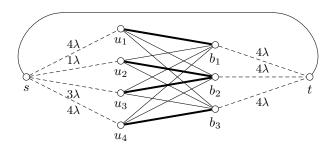


Figure 1: An example of the construction from the proof of Theorem 9 for a BIN PACKING instance with $f(u_1) = f(u_4) = 4$, $f(u_2) = 1$, $f(u_3) = 3$, B = 4, and k = 3. The thick edges represent a minimum solution D. The edge-labels represent all edge capacities that are bigger than one. Observe that every (s, t)-cut avoiding D contains dashed edges that have a capacity sum of at least 12λ .

We set $U_i := \{u \in U \mid \{u, b_i\} \in D\}$ for all $i \in [1, k]$. By the above, we obtain that (U_1, \ldots, U_k) is a k-partition of U. We show that $\sum_{u \in U_i} f(u) = B$ for all $i \in [1, k]$.

Assume towards a contradiction that $\sum_{u \in U_i} f(u) \neq B$ for some $i \in [1, k]$. This is the case if and only if there is some $j \in [1, k]$ with $\sum_{u \in U_j} f(u) < B$. We set $A := \{\{s, t\}\} \cup \{\{s, u\} \mid u \in U_j\} \cup \{\{b, t\} \mid b \in \mathcal{B} \setminus \{b_j\}\} \cup (E(U, \mathcal{B}) \setminus D)$. Note that $\omega(A) = 1 + \lambda(\sum_{u \in U_j} f(u)) + \lambda B(k-1) + |U| \cdot (k-1) \leq \lambda(B-1) + \lambda B(k-1) + |U| \cdot k = \lambda(Bk-1) + |U| \cdot k = a$, since $\sum_{u \in U_j} f(u) < B$. It remains to show that A is an (s, t)-cut in G. Observe that $N_{G-A}(t) = b_j$. Since $N_{G-A}(b_j) = \{t\} \cup U_j$ and $N_{G-A}(u) = \{b_j\}$ for each $u \in U_j$, we conclude that A is indeed an (s, t)-cut in G. This contradicts the fact that there is no (s, t)-cut disjoint to D in G of capacity at most a. As a consequence, $\sum_{u \in U_i} f(u) = B$ for all $i \in [1, k]$ and, thus, I is a yes-instance of BIN PACKING.

We use Theorem 9 to extend the W[1]-hardness to pw(G) + fvs(G) and thus vc(G) in the running time stated in Theorem 7 can presumably not be replaced by pw(G) + fvs(G). Hence, MCP is also W[1]-hard parameterized by tw(G).

Theorem 10. MCP is W[1]-hard when parameterized by pw(G) + fvs(G), where pw(G) denotes the pathwidth of the input graph and fvs(G) denotes the size of the smallest feedback vertex set of the input graph.

Proof. We reduce from WMCP which, due to Theorem 9, is W[1]-hard when parameterized by the vertex cover number vc(G) even if $c(e) + \omega(e) \in n^{\mathcal{O}(1)}$ and the graph is a biclique.

Let $I = (G = (V, E), s, t, c, \omega, d, a)$ be an instance of WMCP where $c(e) + \omega(e) \in n^{\mathcal{O}(1)}$ and the graph is a biclique. Moreover, let I' = (G' = (V', E'), s, t, d, a) be the equivalent instance of MCP we obtain obtain in polynomial time by applying the construction leading to Corollary 1. We show that both the size of the smallest feedback vertex set and the pathwidth of G' are upper-bounded by a function only depending on vc(G).

Let (X, Y) be the bipartition of G and let X be the smaller part. Thus, vc(G) = |X|. Moreover, let $x_1, \ldots, x_{|X|}$ be the elements of X and let $y_1, \ldots, y_{|Y|}$ be the elements of Y. Recall that we obtain I' by replacing every edge $e = \{u, v\} \in E$ by a subgraph $G_e = (V_e, E_e)$ which consists of vertex disjoint (u, v)-paths (besides u and v).

Note that G_e has a path decomposition \mathcal{B}_e of width at most three where each bag contains both endpoints of e. For every $y \in Y$ we set $\mathcal{B}_y := \mathcal{B}_{\{x_1,y\}} \cdot \ldots \cdot \mathcal{B}_{\{x_{|X|},y\}}$. Note that \mathcal{B}_y is a path decomposition of width at most three for $G_y = (\bigcup_{x \in X} V_{\{x,y\}}, \bigcup_{x \in X} E_{\{x,y\}})$.

Finally, let $\mathcal{B} := \mathcal{B}_{y_1} \cdot \cdots \cdot \mathcal{B}_{y_{|Y|}}$ and let \mathcal{B}' be the sequence of bags we obtain from \mathcal{B} by adding all vertices of X to each of the bags of \mathcal{B} . By construction, \mathcal{B}' is a path decomposition of width at most |X| + 3 for G'. Hence, $pw(G') \le |X| + 3 = vc(G) + 3$.

It remains to show that $fvs(G') \leq vc(G)$. Note that $G_{\{x,y\}} - \{x\}$ is acyclic for each $x \in X$ and $y \in Y$. Hence, G' - X is acyclic since Y is an independent set in G and for each pair of distinct edges $e_1, e_2 \in E$ it holds that $V_{e_1} \cap V_{e_2} = e_1 \cap e_2$. Consequently, $fvs(G') \leq vc(G)$ and, thus, MCP is W[1]-hard when parameterized by pw(G') + fvs(G').

6 On Problem Kernelization

6.1 A Polynomial Kernel for vc + a

On the positive side, we show that WMCP admits a polynomial kernel when parameterized by vc + a. The main tool for this kernelization is the merge of vertices according to Definition 1.

Let $J := (G = (V, E), s, t, c, \omega, d, a)$ be an instance of WMCP. We first provide two simple reduction rules that remove degree-one vertices.

Rule 2. If s has exactly one neighbor w and $\omega(\{s, w\}) \leq a$, then delete s, set s := w, and decrease d by $c(\{s, w\})$. Analogously, if t has exactly on neighbor v and $\omega(\{t, v\}) \leq a$, then delete t, set t := v, and decrease d by $c(\{t, v\})$.

The safeness of Rule 2 follows by the observation that, if s (or t, respectively) is incident with a unique edge e with $\omega(e) \leq a$, this edge must be part of every solution, since $M := \{e\}$ is an (s, t)-cut of capacity at most a.

Rule 3. If there exists a degree-one vertex $v \notin \{s, t\}$, then delete v.

It is easy to see that Rule 3 is safe. Since $v \neq s$ and $v \neq t$, the single edge incident with v is not contained in any inclusion-minimal (s, t)-cut and therefore not part of any minimal solution. The next reduction rule is the main idea behind the problem kernelization.

Rule 4. If there are vertices $u, v \in V$ such that a minimum (u, v)-cut has capacity at least a + 1, then merge u and v. Lemma 9. Rule 4 is safe.

Proof. Recall that due to Lemma 3 we can safely merge edges that are not contained in any inclusion-minimal (s, t)cut of capacity at most a. We show the safeness of Rule 4 by applying Lemma 3 on the vertex pair $\{u, v\}$. Note that uand v are not necessarily adjacent in G. Thus, we first transform J into an instance J' by adding an edge $\{u, v\}$ with cost d + 1 and capacity a + 1. Let c' and ω' be the cost functions and capacity functions of J'. We next show that Jis a yes-instance if and only if J' is a yes-instance.

 (\Rightarrow) Let S be a solution of J with $c(S) \leq d$. Since adding an edge might only increase the size of a cut, S is a solution of J'.

 (\Leftarrow) Let S' be a solution of J' with $c'(S) \leq d$. Then, $\{u, v\} \notin S'$ since $c'(\{u, v\}) = d + 1$. We show that S' is a solution of J. Let $M \subseteq E \setminus S'$ be an inclusion-minimal (s, t)-cut in G. We consider the corresponding partition (A, B) of V. If $u \in A$ and $v \in B$ or vice versa, then M is an (u, v)-cut in G and therefore $\omega(M) \geq a + 1$ by the condition of Rule 4. Otherwise, if u and v belong to the same partite set, then M is an (s, t)-cut in G'. Since S is a solution of J' we conclude $\omega(M) \geq a + 1$.

Thus, the instances J and J' are equivalent. Note that $\omega'(\{u, v\}) = a + 1$ implies that $\{u, v\}$ is not contained in any inclusion-minimal (s, t)-cut of capacity at most a. Then, Lemma 3 implies that u and v can safely be merged, which proves the safeness of Rule 4.

We now assume that J is reduced regarding Rules 2–4. Before we show that the number of edges in G is at most $2 \cdot vc(G) \cdot a$, we observe that there is no degree-one vertex in G: Since J is reduced regarding Rule 3, every vertex in $V \setminus \{s, t\}$ has degree at least two. Furthermore, since J is reduced regarding Rule 2 the vertices s and t are not incident with a unique edge of capacity at most a. Finally, since J is reduced regarding Rule 4, the vertices s and t are not incident with a unique edge $\{s, u\}$ (or $\{t, v\}$, respectively) of capacity at least a + 1 since the vertices u and v (or t and v) would have been merged by Rule 4.

To show that the number of edges in J is at most $2 \cdot vc(G) \cdot a$, we introduce *cut trees* which are special binary trees. Throughout this section, given an inner vertex x of a binary tree, we let x_{ℓ} denote its left child and x_r denote its right child.

Definition 2. Let G = (V, E) be a graph with a capacity function $\omega : E \to \mathbb{N}$, let $S \subseteq V$ be a vertex cover of G. Let $T = (\mathcal{V}, \mathcal{E})$ be a binary tree with root vertex $r \in \mathcal{V}$ and $\psi : \mathcal{V} \to 2^V$. Then, (T, ψ) is a cut tree of G with respect to S if

- 1. $\psi(r) = V$,
- 2. for every vertex $x \in \mathcal{V}$ with $|\psi(x) \cap S| \ge 2$, there exist vertices $u, v \in \psi(x) \cap S$ and a minimum (u, v)-cut M in $G[\psi(x)]$ with partitions (A, B) such that $\psi(x_{\ell}) = A$ and $\psi(x_r) = B$, and
- 3. every vertex $x \in \mathcal{V}$ with $|\psi(x) \cap S| = 1$ is a leaf.

Recall that we consider a reduced instance J with input graph G. In the following, let S be a minimum vertex cover of G. We consider a cut tree (T, ψ) of G with respect to S. Observe that there is no inner vertex of T that has exactly one child, and that $\{\psi(x) \mid x \text{ is a leaf of } T\}$ is a partition of V, where each set of the partition contains exactly one vertex from S. Thus, if S is a minimum vertex cover, then T consists of at most vc(G) inner vertices and vc(G) leaves. Furthermore, note that for each inner vertex x, the tuple $(\psi(x_\ell), \psi(x_r))$ is a partition of $\psi(x)$.

To give a bound on the number of edges of G, we associate an edge-set E_x with every $x \in \mathcal{V}$. If x is an inner vertex in T, then we define $E_x := E_G(\psi(x_\ell), \psi(x_r))$. Otherwise, if x is a leaf, then we define $E_x := E_G(\psi(x))$. Observe that for every inner vertex x the edge-set E_x is a minimum (u, v)-cut in G for a pair of vertices $u, v \in \psi(x)$. The size bound of the number of edges mainly relies on the following lemma.

Lemma 10. Let $(T = (\mathcal{V}, \mathcal{E}), \psi)$ be a cut tree of G = (V, E). Then, $E = \bigcup_{x \in \mathcal{V}} E_x$.

Proof. It clearly holds that $\bigcup_{x \in \mathcal{V}} E_x \subseteq E$ since each $E_x \subseteq E$. It remains to prove $E \subseteq \bigcup_{x \in \mathcal{V}} E_x$.

Let $e = \{u, v\} \in E$. If $e \in E_x$ for some leaf vertex x, nothing more needs to be shown. Otherwise, consider the leaf vertices x and y with $u \in \psi(x)$ and $v \in \psi(y)$ and let z be the first common ancestor of x and y. Then, $u \in \psi(z_\ell)$ and $v \in \psi(z_r)$ or vice versa. Consequently, $e \in E_G(\psi(z_\ell), \psi(z_r)) = E_z$.

We now prove the main result of this subsection.

Theorem 11. There is an algorithm that, given an instance of WMCP computes an equivalent instance in polynomial time, such that the graph consists of at most $2vc(G) \cdot a$ edges.

Proof. The algorithm is simply described as follows: Apply the Rules 2–4 exhaustively. Obviously, a single application of one rule can be done in polynomial time. Then, since after every application of one of the rules the number of vertices is decreased by one, Rules 2–4 can be applied exhaustively in polynomial time.

Let J be an instance of WMCP that is reduced regarding Rules 2–4. We next use Lemma 10 to prove that the input graph G consists of at most $2 \cdot vc \cdot a$ edges. Recall that for every pair (u, v) of vertices in G, there exists a (u, v)-cut of size at most a since J is reduced regarding Rule 4

Let $(T = (\mathcal{V}, \mathcal{E}), \psi)$ be a cut tree of G with respect to a minimum vertex cover S. Let $I := V \setminus S$ be the remaining independent set. Furthermore, let $\mathcal{L} \subseteq \mathcal{V}$ be the set of leaves of T and let $\mathcal{I} \subseteq \mathcal{V}$ be the set of inner vertices of T. Lemma 10 then implies

$$|E| \le |\bigcup_{x \in \mathcal{I}} E_x| + |\bigcup_{x \in \mathcal{L}} E_x|.$$

Since every (u, v)-cut in G has size at most a and $\omega(e) \ge 1$ for every edge e we conclude that $|E_x| \le a$ for every $x \in \mathcal{I}$. Thus, since T has at most vc(G) inner vertices, we have $|\bigcup_{x\in\mathcal{I}} E_x| \le vc(G) \cdot a$.

We next define an injective mapping $p : \bigcup_{x \in \mathcal{L}} E_x \to \bigcup_{x \in \mathcal{I}} E_x$. Observe that the existence of such a mapping implies $|\bigcup_{x \in \mathcal{L}} E_x| \le |\bigcup_{x \in \mathcal{I}} E_x|$ and thus $|E| \le 2 \cdot \operatorname{vc}(G) \cdot a$.

Let $\{u, v\} \in E_x$ for some leaf vertex x. Without loss of generality assume that $v \in S$ and $u \in I$. Since J is reduced regarding Rules 2–4, there are no degree-one vertices in G and thus, u has a neighbor $w \in S \setminus \{v\}$. We then define $p(\{u, v\}) := \{u, w\}$. Note that $p(\{u, v\}) \in E_G(S, I)$, and that both edges $\{u, v\}$ and $p(\{u, v\})$ are incident with the same vertex $u \in I$.

We first show that p is well-defined. That is, that $p(\{u,v\}) \in \bigcup_{x \in \mathcal{I}} E_x$ for every $\{u,v\} \in \bigcup_{x \in \mathcal{L}} E_x$. Since $|\psi(x) \cap S| = 1$, there exists another leaf vertex y with $w \in \psi(y)$. Let z be the first common ancestor of x and y. Then, $\{u,w\} \in E_z$. Since z is an inner vertex, we conclude that p is well-defined.

Next, we show that p is injective. Let $e := \{u, v\}$ and $e' := \{u', v'\}$ be edges in $\bigcup_{x \in \mathcal{L}} E_x$. Let p(e) = p(e'). We show that e = e'. Without loss of generality assume that $v, v' \in S$ and $u, u' \in I$. Then, all four edges e, e', p(e), and p(e') are incident with the same vertex of I and thus u = u'. Then, since $\{\psi(x) \mid x \in \mathcal{L}\}$ is a partition of V we conclude that e and e' are element of the same set E_x for some $x \in \mathcal{L}$. Then, $|\psi(x) \cap S| = 1$ implies v = v' and thus e = e'. Therefore, p is injective, which then implies $|E| \leq 2 \cdot vc(G) \cdot a$.

Technically, the instance from Theorem 11 is not a kernel since the encoding of d and the values of c(e) might not be bounded by some polynomial in a and vc. We use the following lemma to show that Theorem 11 implies a polynomial kernel for WMCP. **Lemma 11** ([12]). There is an algorithm that, given a vector $w \in \mathbb{Q}^r$ and some $W \in \mathbb{Q}$ computes in polynomial time a vector $\overline{w} = (w_1, \ldots, w_r) \in \mathbb{Z}^r$ where $\max_{i \in \{1, \ldots, r\}} |w_i| \in 2^{\mathcal{O}(r^3)}$ and an integer $\overline{W} \in \mathbb{Z}$ with total encoding length $\mathcal{O}(r^4)$ such that $w \cdot x \leq W$ if and only if $\overline{w} \cdot x \leq W$ for every $x \in \{0, 1\}^r$.

Corollary 5. WMCP admits a polynomial problem kernel when parameterized by vc + a.

Proof. Let $J := (G = (V, E), s, t, c, \omega, d, a)$ be the reduced instance from Theorem 11. Observe that both, the number of vertices n and the number of edges m of G are polynomially bounded in vc + a. We define r := m and w to be the r-dimensional vector where the entries are the values c(e) for each $e \in E$. Furthermore, let W := d. Applying the algorithm behind Lemma 11 computes a vector \overline{w} with the property stated in the lemma and an integer \overline{W} that has encoding length $\mathcal{O}(m^4)$.

Substituting all values c(e) with the corresponding entry in \overline{w} and substituting d by \overline{W} then converts J into an equivalent instance which has a size that is polynomially bounded in vc + a.

The algorithm behind Theorem 11 also implies a polynomial kernel for the unweighted problem MCP: We transform the unweighted instance into a weighted instance where all capacities and costs are one. Afterwards, we apply the algorithm from Theorem 11 to compute a reduced instance J'. In J' all costs are one, and the capacities are at most a + 1. We then use Corollary 1 to transform the reduced instance J' into an instance J of MCP. Due to the structure of J', the number of new vertices introduced in J is at most $m \cdot (a + 1)$, where m denotes the number of edges in J'. Since $m \leq 2vc(G) \cdot a$, we obtain the following corollary.

Corollary 6. MCP admits a polynomial problem kernel with $4vc(G) \cdot a^2$ edges.

6.2 Limits of Problem Kernelization

Let B_q be a full binary tree of height q. We denote the vertices on level ℓ as $b_{\ell,1}^q, \ldots, b_{\ell,2^\ell}^q$ for each $\ell \in [0,q]$. Hence, vertex $b_{\ell,i}^q$ for some $\ell \in [0,q-1]$ and some $i \in [1,2^\ell]$ has the neighbors $b_{\ell+1,2i-1}^q$ and $b_{\ell+1,2i}^q$ in the next level. The full binary tree R_q of height q with the vertices $r_{\ell,1}^q, \ldots, r_{\ell,2^\ell}^q$ on level $\ell \in [0,q]$ is defined analogously. A *mirror fully binary tree* M_q is the graph obtained after merging the vertices $b_{q,i}^q$ and $r_{q,i}^q$ for each $i \in [1, 2^q]$. By $\ln(G)$ we denote the length of a longest path in G.

Lemma 12. Let $q \ge 3$, then the longest path of a mirror fully binary tree M_q is $2q^2$.

Proof. By L_q we denote the length of each longest path with one endpoint being $b_{0,1}^q$ which does not contain vertex $r_{0,1}^q$. We prove the following statements inductively for $q \ge 3$.

- 1. $L_q = L_{q-1} + 2q 1$.
- 2. $|V(P_q) \cap \{b_{0,1}^q, r_{0,1}^q\}| = 1$ for each longest path P_q of M_q and $\ln(M_q) = 2L_q$.

Solving the recurrence implied by 1. and 2. leads to $lp(M_q) = 2q^2$. Hence, it remains to prove the two statements.

Base Case q = 3: By considering all possible longest paths in M_3 we show that the length of a longest path with one endpoint being $b_{0,1}^3$ and not containing $r_{0,1}^3$ is nine and that $lp(M_3) = 18$.

Inductive step $j - 1 \mapsto j$:

1. Let Z_j be a longest path starting at $b_{0,1}^j$ not containing vertex $r_{0,1}^j$. Without loss of generality, assume that $b_{1,1}^j \in Z_j$.

First, consider the case that $r_{1,1}^j \notin Z_j$. Then, the length of Z_j is at most the length of a longest path starting in vertex $b_{1,1}^j$ and not containing vertex $r_{1,1}^j$ in the mirror fully binary tree of height j-1 rooted in vertex $b_{1,1}^j$ plus one for the edge $\{b_{1,1}^j, b_{0,1}^j\}$. By inductive hypothesis we obtain that Z_j has length at most $L_{j-1} + 1$.

Second, consider the case that $r_{1,1}^j \in Z_j$. Since $b_{1,1}^j, r_{1,1}^j \in Z_j$ there exists a path connecting these two vertices. Without loss of generality, assume that vertices $b_{2,1}^j$ and $r_{2,1}^j$ are on this path of length exactly 2j-2. Note that $b_{1,1}^j$ has the neighbors $b_{0,1}^j$ and $b_{2,1}^j$ in the path and thus $b_{2,2}^j$ is not a neighbor of $b_{1,1}^j$. Thus, we can now use the inductive hypothesis. The length of each longest path starting at vertex $r_{1,1}^j$ and not containing

vertex $b_{1,1}^j$ in the mirror fully binary tree of height j-1 rooted in $r_{1,1}^j$ is at most L_{j-1} . Hence, the length of Z_j is at most $L_{j-1} + 2j - 1$. Thus, we obtain $L_j = L_{j-1} + 2j - 1$.

2. Consider the case that $|V(P_j) \cap \{b_{0,1}^j, r_{0,1}^j\}| = 1$. Then, by 1. we can construct a path Z_j of length $2L_j = 2L_{j-1} + 4j - 2$ by joining two paths where one endpoint is $b_{0,1}^j$ which both do not contain vertex $r_{0,1}^j$. Thus, $\ln(M_j) \ge 2L_j$. Next, assume towards a contradiction that $|V(P_j) \cap \{b_{0,1}^j, r_{0,1}^j\}| \ne 1$.

First, consider that case $|V(P_j) \cap \{b_{0,1}^j, r_{0,1}^j\}| = 0$. Then, the path P_j has length at most $\ln(M_{j-1}) = 2L_{j-1} < 2L_{j-1} + 4j - 2 = 2L_j$, a contradiction to the existence of the path Z_j .

Second, consider the case $|V(P_j) \cap \{b_{0,1}^j, r_{0,1}^j\}| = 2$. Let Q_j be the unique subpath of P_j with endpoints $b_{0,1}^j$ and $r_{0,1}^j$. Without loss of generality, $b_{1,1}^j \in V(Q_j)$ (and thus also $r_{1,1}^j \in V(Q_j)$). Since M_j is a mirror fully binary tree the subpath Q_j has length exactly 2j. Let M' be the mirror fully binary tree rooted at vertex $b_{1,1}^j$. Then $M' \cap V(P_j) = V(Q_j) \setminus \{b_{0,1}^j, r_{0,1}^j\}$. Furthermore, P_j can contain the edges $\{b_{0,1}^j, b_{1,2}^j\}$ and $\{r_{0,1}^j, r_{1,2}^j\}$, and a longest path in the mirror fully binary tree rooted in $b_{1,2}^j$ of height j - 1. Thus, the length of P_j is at most $lp(M_{j-1}) + 2j + 2 = 2L_{j-1} + 2j + 2 < 2L_{j-1} + 4j - 2 = 2L_j$, a contradiction to the existence of the path Z_j .

Hence, $lp(M_j) = 2L_j$.

On the negative side, we provide an OR-composition to exclude a polynomial kernel for the combination of almost all considered parameters with the exception of vc(G) and fvs(G).

Theorem 12. None of the problems MCP, WMCP, and ZWMCP admits a polynomial kernel when parameterized by $d + a + lp(G) + \Delta(G) + td(G)$, unless NP \subseteq coNP/poly, where td(G) denotes the treedepth of G.

Proof. Our strategy is as follows: First, we provide an OR-composition [5, 6] of 2^q instances of MCP to WMCP where $\omega(e) = 1$ and $c(e) \in (d+q)^{\mathcal{O}(1)}$ for each edge e. Second, we apply Lemma 1 exhaustively to transform the constructed instance of WMCP to an equivalent instance of MCP. Clearly, the budgets a and d do not change. In this transformation, each edge e with $c(e) \ge 2$ is replaced by a path with c(e) edges. Hence, the maximum degree does not increase. Furthermore, since $c(e) \in (d+q)^{\mathcal{O}(1)}$, the length of the longest path does only increase by a factor of $(d+q)^{\mathcal{O}(1)}$ and the tree-depth is only increased by $\mathcal{O}(\log(d+q))$. Thus, this transformation preserves all five parameters. It remains to show the statement for WMCP.

Now, we prove the no polynomial kernel result for WMCP by presenting an OR-composition from MCP. Let $I_1, I_2, \ldots, I_{2^q}$ be instances of MCP with the same budgets d and a, the same maximum degree $\Delta(G)$, the same length lp(G) of the longest path, and the same tree-depth td(G) for some integer $q \ge 3$. Moreover, let $I_j := (G_j = (V_j, E_j), s_j, t_j, d, a)$.

We describe how to construct an instance $I^* = (G^*, s^*, t^*, c^*, \omega^*, d^*, a^*)$ of WMCP in polynomial time, where $d^* + a^* + \ln(G^*) + \Delta(G^*) + \operatorname{td}(G^*) \in (d + a + \ln(G) + \Delta(G) + \operatorname{td}(G) + q)^{\mathcal{O}(1)}$ such that I^* is a yes-instance of WMCP if and only if I_j is a yes-instance of MCP for at least one $j \in [1, 2^q]$.

We add the following vertices and edges to the graph G^* :

- We add a copy of the graph G_j for each $j \in [1, 2^q]$ to G^* .
- Furthermore, we add 6q new vertices $Z_j := \{z_j^1, \ldots, z_j^{6q}\}$ and we add the the edges $\{s_j, z_j^i\}$ and $\{z_j^i, t_j\}$ for each $j \in [1, 2^q]$ and each $i \in [1, 6q]$ to G^* .
- Next, we add a full binary tree B with height q to G*. We denote the vertices on level l as b_l¹,..., b_l^{2^ℓ} for each l ∈ [0, q]. Hence, vertex b_lⁱ for some l ∈ [0, q-1] and some i ∈ [1, 2^ℓ] has the neighbors b_{l+1}²ⁱ⁻¹ and b_{l+1}²ⁱ in the next level. Now, we identify vertex s_j with the leaf b_q^j for each j ∈ [1, 2^q] and we identify vertex s^{*} with the root b₁⁰.
- Analogously, we add a full binary tree R of height q with the vertices $r_{\ell}^1, \ldots, r_{\ell}^{2^{\ell}}$ on level $\ell \in [0, q]$. Similarly, we identify vertex t^* with the root r_0^1 and identify vertex t_j with the leaf r_q^j for each $j \in [1, 2^q]$.

Next, we set $d^* := 2q(d+1) + d$ and $a^* := 7q + a$. Afterwards, we set $\omega^*(e) := 1$ for each edge $e \in E(G^*)$. We define the costs of each edge in $E(G^*)$ as follows:

- For each $e \in E_j$ for some $j \in [1, 2^q]$ we set c(e) := 1.
- We set $c(\{s_j, z_j^i\}) := d^* + 1$ and $c(\{z_j^i, t_j\}) := d^* + 1$ for each $j \in [1, 2^q]$ and each $i \in [1, 6q]$.
- For each edge e in one of the full binary trees we set c(e) := d + 1.

This completes the construction of I^* . Now, we prove that the parameters a^* , d^* , $\Delta(G^*)$, and $\ln(G^*)$ are bounded by $(a + d + \Delta(G) + \ln(G) + q)^{\mathcal{O}(1)}$.

- Since $d^* = 2q(d+1)$ and $a^* = 7q + a$, the statement is clear for a^* and d^* .
- Each vertex in B ∪ R except the leaves have degree at most three. Furthermore, each vertex z_jⁱ for some j ∈ [1, 2^q] and some i ∈ [1, 6q] as degree two and each vertex in V_j \ {s_j, t_j} has degree at most Δ(G). Note that vertex s_j and t_j for each j ∈ [1, 2^q] has degree at most 6q + 1 + Δ(G). Hence, the statement is true for Δ(G^{*}).
- Observe that the graph obtained from contracting all vertices in $W_j := V(G_j) \cup \{z_j^1, \ldots, z_j^{6q}\}$ in the graph G^* into one vertex for each $j \in [1, 2^q]$ is a mirror fully binary tree of height q. Observe that by construction, $\ln(G^*(W_j)) \in \mathcal{O}(\ln(G))$ for each $j \in [1, 2^q]$. Thus, by Lemma 12 we obtain that $\ln(G^*) \in \mathcal{O}(\ln(G) \cdot q^2)$. Hence, the statement is true for $\ln(G^*)$.
- Since td(G_j) = td(G) for each j ∈ [1, 2^q], the tree-depth of G'_j = G^{*}[V_j ∪ Z_j] is at most td(G) + 2. Hence, there is a directed tree T_j(V_j ∪ Z_j, A_j) of depth at most td(G) + 2 and root s_j, such that for each edge {u, w} ∈ E(G'_j) either u is an ancestor of w in T_j or vice versa. We define a directed tree T^{*} = (V(G^{*}), A^{*}) as follows. The tree T_j is a subtree of T^{*} for each j ∈ [1, 2^q]. The vertex b¹₀ is the root of T^{*} and for each ℓ ∈ [0, q-1] and each i ∈ [1, 2^ℓ], A^{*} contains the arcs (bⁱ_ℓ, rⁱ_ℓ), (rⁱ_ℓ, b²ⁱ_{ℓ+1}), and (rⁱ_ℓ, b²ⁱ⁺¹_{ℓ+1}). Recall that b^j_q = s_j. Since T_j is a subtree of T^{*} has depth at most 2q + td(G) we obtain the stated bound on the tree-depth of G^{*}.

Next, we prove the correctness. That is, we show that at least one instance I_j has a solution D_j with cost at most d for some $j \in [1, 2^q]$ if and only if I^* has a solution D with cost at most d^* .

 (\Rightarrow) Let D_j be a solution of I_j with cost at most d.

Let P_j^s be the unique (s^*, s_j) -path in B and let P_j^t be the unique (t_j, t^*) -path in R. We set $D := D_j \cup E(P_j^s) \cup E(P_j^t)$.

First, we show that $c^*(D) \leq d^*$. Since B and R are full binary trees of height q, both paths P_j^s and P_j^t consist of exactly q edges. Recall that each edge in both B and R has cost d + 1. Since $|D_j| \leq d$ and c(e) = 1 for each edge $e \in E_j$, we obtain $c^*(D) \leq 2q(d+1) + d$.

Second, we prove that there is no (s^*, t^*) -cut $A \subseteq E(G^*) \setminus D$ in G^* with $\omega^*(A) \leq a^*$. To this end, we present $a^* + 1$ many (s^*, t^*) paths in G^* whose edge sets may only intersect in D. Together with $\omega^*(e) = 1$ for each edge $e \in E(G^*)$ we then conclude that $\omega^*(M) \geq a^* + 1$. We use the following notation: For two paths $P_1 = (v_1, \ldots, v_k)$ and $P_2 = (w_1, \ldots, w_r)$ in G^* where $w_r = v_1$, we let $P_1 \multimap P_2 := (v_1, \ldots, v_k = w_1, \ldots, w_r)$ denote the *merge* of P_1 and P_2 .

- Let $P_j^{s,\ell}$ be the subpath of P_j^s until level $\ell \in [0, q-1]$. Since B is a full binary tree, let $b_{\ell+1}^i$ be the child of the endpoint of $P_j^{s,\ell}$ which is not contained in P_j^s . The subpath $P_j^{\ell,t}$ and the vertex $r_{\ell+1}^i$ are defined similarly. We consider the (s^*, t^*) -path $P_j^{s,\ell} \cdot (b_{\ell+1}^i, b_{\ell+2}^{2q-\ell}, \ldots, b_q^{2q-\ell}) = s_{2q-\ell,i}, z_{2q-\ell,i}^1, t_{2q-\ell,i} = r_q^{2q-\ell,i}, \ldots, r_{\ell+2}^{2i}, r_{\ell+1}^i) \cdot P_j^{\ell,t}$. Since $\ell \in [0, q-1]$, these are q paths in total.
- Observe that $P_j^s \multimap (s_j, z_j^i, t_j) \multimap P_j^t$ is an (s^*, t^*) -path for each $i \in [1, 6q]$. Hence, these are 6q paths in total.

Since D_j is a solution of I_j, there are a + 1 many (s_j, t_j)-paths P₁,..., P_{a+1} in G_j whose edge set may only intersect in D_j. Since D_j ⊆ D, P^s_j → P_i → P^t_j is an (s^{*}, t^{*})-path for each i ∈ [1, a + 1] such that P_i ⊆ E_j. Hence, these are a + 1 paths in total.

Thus, G^* contains at least a + 1 many (s^*, t^*) -paths whose edge set may only intersect in D and hence I^* is a yes-instance of WMCP.

(\Leftarrow) Conversely, let D be a solution with cost at most d^* of I^* . At the beginning, we prove the following statement.

Claim 5. For each solution D of I^* with cost at most d^* , there exists a $j \in [1, 2^q]$ such that $E(P_j^s) \subseteq D$ for the unique path P_j^s from s^* to s_j and $E(P_j^t) \subseteq D$ for the unique path P_j^t from t^* to t_j .

Proof. Assume towards a contradiction that this is not the case. We define $B_s := E(B) \cap D$ and $R_t := E(R) \cap D$ as the set of protected edges in the binary trees B and R. Note that since $c^*(e) = d + 1$ for each edge e in the binary trees B and R and $d^* = 2q(d+1) + d$, we have $|B_s| + |R_t| \le 2q$. By Z_s we denote the connected component of $G^*[B_s]$ containing vertex s^* . Since $|B_s| \le 2q$, we conclude that Z_s contains at most 2q + 1 vertices. Since B is a binary tree, each vertex in B has degree at most three. Recall that only vertices in level q of B have neighbors outside of B. We set $X := E_B(Z_s, N(Z_s))$. Note that $|X| \le 3 \cdot (2q+1) \le 6q+3$.

First, we consider the case that $G^*[B_s]$ does not contain the path P_j^s as an induced subgraph for any $j \in [1, 2^q]$. We set A := X and show that M is an (s^*, t^*) -cut in G^* . Note that $|M| \le 6q+3 \le a^*$. Observe that A avoids D since Z_s is a connected component in $G^*[D]$ and A contains only adjacent edges of Z_s . Thus, A is an (s^*, t^*) -cut in G^* that avoids D with $\omega^*(A) \le a^*$, a contradiction. Analogously, we can prove that $G^*[R_t]$ is a path $P_{i'}^t$ for some $j' \in [1, 2^q]$.

Second, we consider the case that $G^*[B_s]$ is a path P_j^s for some $j \in [1, 2^q]$ and that $G^*[R_t]$ is a path $P_{j'}^t$ for some $j' \in [1, 2^q]$ and $j \neq j'$. Recall that c(e) = d + 1 for each edge e in the binary trees B and R and that $d^* = 2q(d+1) + d$. Furthermore, note that $|B_s| = q = |R_t|$ and that Z_s contains q + 1 vertices. Thus, D contains no other edges of R than R_t . In particular, $e^* := \{r_q^j, r_{q-1}^{[j/2]}\} \notin D$. Recall that $t_j = r_q^j$. We define $A := \{e^*\} \cup X$. Clearly, A avoids D. Since $q \ge 2$, we conclude that $|A| \le 1 + 6q + 1 \le 7q$. It remains to show that A is an (s^*, t^*) -cut in G^* . Since $X \subseteq A$, every (s^*, t^*) -path P^* in $G^* - X$ starts with P_j^s followed by an (s_j, t_j) -path. Moreover, P^* has to contains the edge e^* . Since $e^* \in A$, A is indeed an (s^*, t^*) -cut in G^* with $|A| \le a^*$ that avoids D, a contradiction.

By Claim 5, let $j \in [1, 2^q]$ such that $E(P_j^s) \cup E(P_j^t) \subseteq D$. We define $D_j := D \cap E_j$. Observe that since c(e) = d + 1 for each edge in both binary trees B and R, each edge incident with vertex $z_{j'}^i$ for some $j' \in [1, 2^q]$ and some $i \in [1, 6q]$ has cost $d^* + 1$, and each edge in the copy of $G_{j'}$ for some $j' \in [1, 2^q]$ has costs one, we conclude that $|D_j| \leq d$. In the following, we show that D_j is a solution of I_j .

Assume towards a contradiction that there is an (s_j, t_j) -cut $A_j \subseteq E_j$ of size at most a in G_j that avoids D_j . We set $A := A_j \cup \{\{s_j, z_j^i\} \mid i \in [1, 6q]\} \cup X$, where $X := E_B(V(P_j^s), N(V(P_j^s)))$. Note that A avoids D. By the fact that B is a binary tree of depth q, it follows that $|M| \leq |M_j| + 6q + q \leq a + 7q = a^*$.

Since $X \subseteq A$, every (s^*, t^*) -path P^* in $G^* - X$ starts with P_j^s followed by an (s_j, t_j) -path. Moreover, since A_j is an (s_j, t_j) -cut in G_j and $\{\{s_j, z_j^i\} \mid i \in [1, 6q]\} \subseteq A$, A is an (s^*, t^*) -cut of capacity at most a^* in G^* , a contradiction. Hence, D_j is a solution with cost at most d of I_j and, thus, I_j is a yes-instance of WMCP.

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