Free Modal Riesz Spaces are Archimedean: A Syntactic Proof

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Abstract. We prove, using syntactical proof–theoretic methods, that free modal Riesz spaces are Archimedean. Modal Riesz spaces are Riesz spaces (real vector lattices) endowed with a positive linear 1–decreasing operator, and have found application in the development of probabilistic temporal logics in the field of formal verification. All our results have been formalised using the Coq proof assistant.

1 Introduction

Riesz spaces, also known as real vector lattices, are real vector spaces equipped with a lattice order ($\leq$) such that the vector space operations of addition and scalar multiplication are compatible with the order in the following sense: (1) if $x \leq y$ then $x + z \leq y + z$ and (2) if $x \leq y$ then $rx \leq ry$, for all $r \in \mathbb{R}_{\geq 0}$.

The simplest example of Riesz space is the linearly ordered vector space of real numbers $(\mathbb{R}, \leq)$ itself. More generally, for a given set X, the space of all functions $\mathbb{R}^X$ with operations and order defined pointwise is a Riesz space. If X carries some additional structure, such as a topology or a σ–algebra, then the spaces of continuous and measurable functions both constitute Riesz subspaces of $\mathbb{R}^X$. For this reason, the study of Riesz spaces originated at the intersection of functional analysis, algebra and measure theory and was pioneered in the 1930's by F. Riesz, G. Birkhoff, L. Kantorovich and H. Freudenthal among others. Today, the study of Riesz spaces constitutes a well–established field of research. We refer to [LZ71, JR77] as standard references.

An important class of Riesz spaces is given by Archimedean Riesz spaces. A Riesz space $(A, \leq)$ is Archimedean if, for any given pair of elements $a, b \in A$,

$$\big(\forall n \in \mathbb{N}.\ na \leq b\big) \Longrightarrow a \leq 0.$$

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All the examples of Riesz spaces given above, given by collections of real valued functions, are Archimedean. For this reason the Archimedean property is of key importance in the theory of Riesz spaces.

It is well known that free Riesz spaces (i.e., free objects in the category of Riesz spaces and their homomorphisms) are Archimedean.

Modal Riesz spaces. In a series of recent works [MS17, MFM17, Mio18, FMM20] concerning the study and design of temporal logics for formal verification of probabilistic programs, the notion of *modal Riesz space* has been introduced as the algebraic semantics of the Riesz modal logic for Markov processes [FMM20].

A modal Riesz space (see Section 2.2) is a structure $(A, \leq, 1, \Diamond)$ where $(A, \leq)$ is a Riesz space, $1 \in A$ is a positive element $(1 \geq 0)$ and $\Diamond : A \to A$ is a unary operation which satisfies three axioms (see Figure 2): linearity $(\Diamond(r_1 x + r_2 y) = r_1\Diamond(x)+r_2\Diamond(y))$, positivity (if $x \geq 0$ then $\Diamond(x) \geq 0$) and 1–decreasing $(\Diamond(1) \leq 1)$.

Examples of modal Riesz spaces are given in Section 2.2 and more can be found in [FMM20]. The class of modal Riesz spaces, being defined by a set of equations, constitutes a variety and thus free objects exist. In [FMM20, §6.3] the authors left open the following problem regarding modal Riesz spaces: is the free modal Riesz space on the empty set of generators[3] Archimedean? The main contribution of this paper is to give a general answer, covering any possible sets of generators, to this question.

Theorem 1. *Free modal Riesz spaces are Archimedean.*

Our Syntactic Proof. An interesting aspect of our proof is that it is syntactic and based on the proof–theoretic machinery of the hypersequent calculus **HMR** for modal Riesz spaces developed in [LM19, LM20]. One of the novel results, obtained in [LM20, Thm 4.13] using the **HMR** machinery, is the decidability of the equational theory of modal Riesz spaces. This work further illustrates, by proving Theorem 1, the general usefulness of the proof theory. We first reformulate the Archimedean property in terms of derivability in **HMR** and then prove it using proof–theoretic techniques based on the results from [LM20] (like, e.g., a form of cut–elimination). Our main technical result (Theorem 2) establishes that derivability in **HMR** is continuous, in an appropriate sense.

After a preliminary Section 2 consisting of technical background on (modal) Riesz spaces, and Section 3 summarising the main notions and results regarding the hypersequent calculus **HMR** from [LM20], our proof of Theorem 1 is presented in Section 4. To better present the argument, we first prove, using the sequence of steps outlined above, the known fact that free (non–modal) Riesz spaces are Archimedean. To this end, rather than **HMR**, we use its subsystem **HR** (also introduced in [LM20, §3] and presented in Section 3.1), which is sound and complete for the theory of (non–modal) Riesz spaces. Once this is done, we prove Theorem 1 tackling in Section 4.2 the additional complexity of modal Riesz spaces using the system **HMR**.

[3] The focus in [FMM20] is on the free Riesz space on the empty set of generators because it is the initial object in the category of modal Riesz spaces.

Coq formalisation. All our definitions and proofs have been formalised using the Coq proof assistant [Luc21]. See Section 2.3 for a detailed discussion.

2 Technical background

In this section we present the basic definitions and results about Riesz spaces (Section 2.1), modal Riesz spaces (Section 2.2) and details about the Coq formalisation of the results of this work (Section 2.3).

2.1 Riesz Spaces

We refer to [LZ71, JR77] as standard references on the theory of Riesz spaces. The signature of Riesz spaces is given by $\Sigma_{\mathrm{RS}} = \big\{+, 0, \{r(\_)\}_{r\in\mathbb{R}}, \sqcup, \sqcap\big\}$ combining the signature of real vector spaces (addition, neutral element and scalar multiplication by reals) and of lattices (supremum and infimum). Given a set V, we denote with $\mathbf{T}_{\mathrm{RS}}(V)$ the set of Σ_{RS}–terms built from the set of atoms V. We use the letters ϕ and ψ to range over terms.

The class of Riesz spaces is the class of Σ_{RS}–algebras satisfying the axioms of Figure 1, each of which can be expressed as universally quantified equations.

1. Axioms of real vector spaces:
   - Abelian groups: $x + (y + z) = (x + y) + z$, $x + y = y + x$, $x + 0 = x$, $x - x = 0$,
   - Axioms of scalar multiplication: $r_1(r_2x) = (r_1 \cdot r_2)x$, $1x = x$, $r(x + y) = (rx) + (ry)$, $(r_1 + r_2)x = (r_1x) + (r_2x)$,
2. Lattice axioms: (associativity) $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$, $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$, (commutativity) $z \sqcup y = y \sqcup z$, $z \sqcap y = y \sqcap z$, (absorption) $z \sqcup (z \sqcap y) = z$, $z \sqcap (z \sqcup y) = z$.
3. Compatibility axioms:
   - if $x \leq y$ then $x + z \leq y + z$,
     expressed equationally as: $(x \sqcap y) + z \leq y + z$,
   - if $x \leq y$ then $rx \leq ry$, for all $r \geq 0$,
     expressed equationally as: $r(x \sqcap y) \leq ry$, for all $r \geq 0$.

   where $x \leq y$ can be expressed by the equality $x \sqcap y = x$.

Fig. 1. Axioms of Riesz spaces.

Example 1. The Riesz space $(\mathbb{R}, +, 0, \max, \min)$ is a main example. Furthermore, for any set V, the collection of functions $\mathbb{R}^V$ $(f : V \to \mathbb{R})$ is a Riesz space where operations on functions are defined pointwise: e.g., $(f + g)(v) = f(v) + g(v)$. Subalgebras of $\mathbb{R}^V$ are, therefore, also Riesz spaces. For instance, if V is a topological space, the collection of continuous functions on $\mathbb{R}$ is a Riesz space.

Given two terms $\phi, \psi \in \mathbf{T}_{\mathrm{RS}}(V)$, we write $\phi \equiv_{\mathrm{RS}} \psi$ (or just $\phi \equiv \psi$ if clear form the context) if ϕ and ψ can be proved equal, in the usual apparatus of equational logic, from the axioms of Riesz spaces in Figure 1.

Being definable purely by equations, the class of Riesz spaces is a variety in the sense of universal algebra. Therefore the category of Riesz spaces and their homomorphisms (functions preserving all Σ_{RS} operations) has free objects. Given a set V, we denote with $\mathbf{Free}_{\mathrm{RS}}(V)$ the free Riesz space on the set V. The following definition and proposition are standard.

Definition 1 (Term algebra). *Given a set V, the term algebra $\mathbf{T}_{\mathrm{RS}}(V)_{/\equiv}$ is the Riesz space whose elements are terms generated by V taken modulo the equivalence relation $\equiv_{\mathrm{RS}}$, and operations defined on equivalence classes as: $[\phi]_{\equiv}+[\psi]_{\equiv}=[\phi+\psi]_{\equiv}$, $r[\phi]_{\equiv}=[r\phi]_{\equiv}$, $[\phi]_{\equiv}\sqcup[\psi]_{\equiv}=[\phi\sqcup\psi]_{\equiv}$, $[\phi]_{\equiv}\sqcap[\psi]_{\equiv}=[\phi\sqcap\psi]_{\equiv}$.*

Proposition 1. *For any set V, the free Riesz space $\mathbf{Free}_{\mathrm{RS}}(V)$ and term Riesz space $\mathbf{T}_{\mathrm{RS}}(V)_{/\equiv_{\mathrm{RS}}}$ are isomorphic.*

We are now ready to define the Archimedean property of Riesz spaces (see, e.g., [LZ71, §22, Thm 22.2]).

Definition 2 (Archimedean Property). *A Riesz space A is* Archimedean *if, for any $a,b\in A$, it holds that: $\big(\forall n\in\mathbb{N}.\ na\leq b\big)\Longrightarrow a\leq 0$.*

The following result is well–known and follows from a theorem of Baker [Bak68, Thm 2.4] (see also [Ble73, Thm 2.3]) identifying the free Riesz space $\mathbf{Free}_{\mathrm{RS}}(V)$ with a Riesz subspace of $\mathbb{R}^V\to\mathbb{R}$, and the following simple facts (see, e.g., [H.74, §1.15]): (i) the Riesz space $\mathbb{R}^X$ is Archimedean for any set X (so in particular for $X=\mathbb{R}^V$) and (ii) any Riesz subspace of an Archimedean Riesz space is Archimedean.

Proposition 2. *For any set V, the Riesz space $\mathbf{Free}_{\mathrm{RS}}(V)$ is Archimedean.*

Syntactical conventions. We now introduce some convenient syntactical conventions. Rather than working with arbitrary scalar multiplications by $r\in\mathbb{R}$, it is often useful to introduce the derived *negation operator* $-\phi=(-1)\phi$ and restrict scalar multiplication only to strictly positive reals $r\in\mathbb{R}_{>0}$. Clearly this is not a restriction as one can, e.g., just rewrite $(-5)\phi$ to $-(5\phi)$ introducing the negation operator. Every Riesz term ϕ can be rewritten into a $\equiv_{\mathrm{RS}}$-equivalent term ψ in *negation normal form* (NNF), where negation is only applied to variables, using the following valid equalities: $-(\phi\sqcap\psi)=(-\phi)\sqcup(-\psi)$, $-(\phi\sqcup\psi)=(-\phi)\sqcap(-\psi)$, $-(-\phi)=\phi$, $-(\phi+\psi)=(-\phi)+(-\psi)$, $-0=0$, $0\phi=0$. We will use the capital letters A and B to range over Riesz terms in NNF, rather than ϕ and ψ. We write $\overline{A}$ for the NNF–term equivalent to the term $-A$. In particular, $\overline{x}=-x$. Note, therefore, that that terms in NNF can be seen as constructed, without negations, from the variables x and $\overline{x}$, with $x\in V$.

2.2 Modal Riesz spaces

In this section we introduce the notion of modal Riesz space, a concept which has emerged as relevant in recent works [MS17, MFM17, Mio18, FMM20] concerning the study and design of temporal logics for formal verification of probabilistic programs.

The signature of modal Riesz spaces is given by $\Sigma_{\text{MRS}} = \Sigma_{\text{RS}} \cup \{1, \Diamond\}$ where Σ_{RS} is the signature of Riesz spaces, 1 is a constant symbol and $\Diamond$ is a unary function symbol (we will often omit the parenthesis on $\Diamond$, since it is a unary operator). Given a set V, we denote with $\mathbf{T}_{\text{MRS}}(V)$ the set of Σ_{MRS}–terms build from the set of generators V. Note that $\mathbf{T}_{\text{RS}}(V) \subsetneq \mathbf{T}_{\text{MRS}}(V)$ since $\Sigma_{\text{RS}} \subsetneq \Sigma_{\text{MRS}}$. We use the letters ϕ, ψ also to range over $\mathbf{T}_{\text{MRS}}(V)$.

Definition 3 (Modal Riesz spaces). *The class of modal Riesz spaces is the equationally defined class of Σ_{MRS}–algebras generated by the universally quantified equational axioms of Figure 1 and the additional axioms of Figure 2.*

4. Positivity of $1 : 0 \leq 1$.
   expressed equationally: $0 \sqcap 1 = 0$.
5. Modal axioms:
   - Linearity: $\Diamond(r_1 x + r_2 y) = r_1 \Diamond(x) + r_2 \Diamond(y)$.
   - Positivity: if $x \geq 0$ then $\Diamond(x) \geq 0$
     expressed equationally: $\Diamond(0 \sqcup x) \sqcap 0 = 0$.
   - 1-decreasing: $\Diamond(1) \leq 1$.
     expressed equationally: $\Diamond(1) \sqcap 1 = \Diamond(1)$.

Fig. 2. Additional axioms of modal Riesz spaces.

Example 2. A typical example of modal Riesz space is $M = (\mathbb{R}^n, \leq, 1^M, \Diamond^M)$, the n-dimensional vector space $\mathbb{R}^n$ with vectors ordered pointwise where 1^M is the constant 1 vector, and $\Diamond^M$ is a linear map $\mathbb{R}^n \to \mathbb{R}^n$, hence representable as a square matrix (also denoted $\Diamond^M$ with some abuse of notation), such that all entries $r_{i,j}$ are non–strictly positive (due to the positivity axiom) and where all the rows sum up to a value ≤ 1, i.e., for all $1 \leq i \leq n$ it holds that $\sum_{j=1}^{k} r_{i,j} \leq 1$ (due to the 1–decreasing axiom):

$$1^M = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \Diamond^M = \begin{pmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n,1} & r_{n,2} & \cdots & r_{n,n} \end{pmatrix}$$

The modal Riesz space M can be seen as representing a discrete–time Markov chain, i.e., a probabilistic transition system having $\{1, \ldots, n\}$ states, where the probability of moving from state i to state j at the next step is $r_{i,j}$. The constraint $\sum_{j=1}^{k} r_{i,j} \leq 1$ represents the fact that there can be a nonzero probability of terminating the execution at the state i, thus not moving to any other state. These types of examples are the reason for the relevance of modal Riesz spaces to the axiomatisation of logics for expressing properties of probabilistic transition systems. In fact, the duality theory developed in [FMM20] shows that all Archimedean modal Riesz spaces with strong unit are topological generalisations of the example just presented and can be identified with discrete time Markov processes.

Remark 1. The choice of using the $\Diamond$ symbol for the unary operation of modal Riesz spaces might suggest the existence of a distinct De Morgan dual operator $\Box x = -\Diamond(-x)$. This is not the case since, due to linearity, $\Box x = \Diamond x$, i.e., $\Diamond$ is self dual. While using a different symbol such as $(\circ)$ might have been a better choice, we decided to stick to $\Diamond$ for backwards compatibility with previous works on modal Riesz spaces. Another source of potential ambiguity lies in the "modal" adjective itself. Of course other axioms for $\Diamond$ can be conceived (e.g., $\Diamond(x \sqcup y) = \Diamond(x) \sqcup \Diamond(y)$ instead of our $\Diamond(x+y) = \Diamond(x) + \Diamond(y)$, see, e.g., [DMS18]). Therefore different notions of modal Riesz spaces can be investigated, just like many types of classical modal logic exist (K, S4, S5, etc). Once again, our choice of terminology is motivated by backwards compatibility with previous works.

We denote with $\equiv_{MRS}$ (or just $\equiv$, if clear from the context) the equivalence relation on $\mathbf{T}_{\mathrm{MRS}}(V)$ which equates modal Riesz terms that are provably equal from the axioms of Definition 3. Being equationally defined, the class of modal Riesz spaces is a variety in the sense of universal algebra. Therefore the category of modal Riesz spaces and their homomorphisms (functions preserving all Σ_{MRS} operations) has free objects. Note that every modal Riesz space is a Riesz space (since it satisfies all axioms of Figure 1). Furthermore, any Riesz space R can be turned into a modal Riesz space by, e.g., defining $1^R = 0$ and $\Diamond^R = \mathrm{id}$, where $\mathrm{id} : R \to R$ is the identity map. Hence the notion of modal Riesz space is a conservative extension of that of Riesz space.

Given a set V, we denote with $\mathbf{Free}_{\mathrm{MRS}}(V)$ the free modal Riesz space on the set V and with $\mathbf{T}_{\mathrm{MRS}}(V)_{/\equiv_{\mathrm{MRS}}}$ the term algebra.

Proposition 3. *For any set V, the free modal Riesz space $\mathbf{Free}_{\mathrm{MRS}}(V)$ and the term modal Riesz space $\mathbf{T}_{\mathrm{MRS}}(V)_{/\equiv_{\mathrm{MRS}}}$ are isomorphic.*

The main result of this paper is Theorem 1, stating that $\mathbf{Free}_{\mathrm{MRS}}(V)$ is Archimedean. Our proof is presented in Section 4. This is a novel result and solves a problem left open [FMM20, §6.3]. We remark that free modal Riesz spaces can be rather complex objects. For instance, $\mathbf{Free}_{\mathrm{MRS}}(\emptyset)$ is not even finitely generated as a Riesz space [FMM20, §6]. For instance, the term $\Diamond^n 1$ can not equivalently be expressed by a Riesz combination of terms with $\Diamond$–depth (the maximum number of nested $\Diamond$ operators) lower than n.

Syntactical conventions. We extend the notion of negation normal form (NNF) from Riesz terms to modal Riesz space terms, taking in consideration the existence of the constant 1 in Σ_{RMS}. A modal Riesz term ϕ in $\mathbf{T}_{\mathrm{MRS}}(V)$ is in *negation normal form* (NNF) if the operator $(-)$ is only applied to atoms in V or the constant 1. Using the equality $(-\Diamond(\phi) = \Diamond(-\phi))$, every term ϕ in $\mathbf{T}_{\mathrm{MRS}}(V)$ is provably equal to a term in NNF. We will use the capital letters A and B to range over modal Riesz terms in NNF, rather than ϕ and ψ. We try to make it always clear if the term belong to $\mathbf{T}_{\mathrm{MRS}}(V)$ or just to $\mathbf{T}_{\mathrm{RS}}(V)$.

2.3 On the Coq formalisation

All the results of this paper have been formalised using the Coq proof assistant and are publicly available [Luc21]. Throughout the paper, we refer to specific points of the formalisation by highlighting with a grey background either some portions of Coq code (as in the definition of `Axiom IPP` below) or by specifying the name of the lemma and its path as: `Repository [Luc21]: (Lemma) in Path.`

Our formalisation is based on the following mathematical notions and results.

1. The real numbers $\mathbb{R}$, functions on them ($+$, $\times$, etc) and their basic properties.
2. The (strictly) positive real numbers $\mathbb{R}_{>0}$ with basic functions and properties.
3. The notion of polynomial expression, syntax and semantics.
4. Basic notions about limits of sequences of (tuples of) reals and (sequential) continuity of polynomial expressions.
5. The *infinitary pigeonhole principle*: for every sequence $u \in \mathbb{N}^{\mathbb{N}}$ bounded by some $m \in \mathbb{N}$ (i.e., $u_n < m$ for all n), there is a constant subsequence $(u_{\phi(n)})$ of u, i.e., there is $i \in [0..m[$ such that $u_{\phi(n)} = i$ for all n.
6. The *sequential compactness* of $\mathbb{R}$: if $u \in \mathbb{R}^{\mathbb{N}}$ is a sequence bounded by a lower bound $lb \in \mathbb{R}$ and a upper bound $ub \in \mathbb{R}$, then there is a subsequence $(u_{\phi(n)})$ of (u_n) and a real $l \in \mathbb{R}$ such that $\lim\limits_{n\in\mathbb{N}} u_{\phi(n)} = l$.

Regarding (1), we use the default Coq implementation of real numbers $\mathbb{R}$. For (2), strictly positive reals are implemented as dependent pairs where the first element is the real number and the second element is a proof that this real is strictly positive. Operations and basic properties on $\mathbb{R}_{>0}$ are easily derived from those of $\mathbb{R}$ (standard library). For (3), polynomial expressions over the variables $\alpha_1, \dots \alpha_n$ are simply defined by the grammar: $R, S := \alpha_i \mid r \in \mathbb{R} \mid R + S \mid RS$ (`Repository [Luc21]: (Poly : Type) in Utilities/polynomials.v.`) and interpreted as polynomial functions $P : \mathbb{R}^k \to \mathbb{R}$ as expected. Regarding (4), we use the `Coquelicot` library [BLM15] which provides definitions and results regarding uniform spaces (like $\mathbb{R}$), continuity, *etc.* In particular we are able to derive the following statement.

Proposition 4 (Sequential continuity of Polynomial expressions). *Let R be a polynomial expression. For all $j \in [1..k]$, let $(t_{i,j})_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $t_j \in \mathbb{R}$ such that $\lim\limits_{i\to+\infty} t_{i,j} = t_j$. Then $\lim\limits_{i\to+\infty} R(t_{i,1}, ..., t_{i,k}) = R(t_1, ..., t_k)$.*

Proof. `Repository [Luc21]: (Poly_lim) in Utilities/pol_continuous.v.`

Finally, the infinitary pigeonhole principle (5) and sequential compactness (6) are stated as follows and assumed as axioms:

```
Axiom IPP : forall (u : nat → nat) m,
  (forall n, u n < m) →
  {' (phi, i) & prod (subseq_support phi) (*∃ phi,i,(subseq phi)∧ *)
                ((i < m) * (* (i < m) ∧ *)
                (forall n, u (phi n) = i))}. (*(∀ n, u_{phi n} = i) *)
```

```
...
```

```
Axiom SequentialCompactness : forall (u : nat → R) lb ub,
(forall n, prod (lb <= u n) (u n <= ub)) →
  {' (phi , l) & prod (subseq_support phi) (*∃ phi,l,(subseq phi)∧ *)
          (is_lim_seq (fun n ⇒ u (phi n)) l)}. (* (lim u_{phi n} = l) *)
```

Repository [Luc21]: `Utilities/R_complements.v`

3 Hypersequent Calculi

In this section we introduce a structural proof system called **HMR** from [LM20] (see also [LM19]) for the theory of modal Riesz spaces. We also discuss a subsystem of **HMR**, called **HR**, also introduced in [LM20, §3] for the theory of Riesz spaces. A proof system is called structural if it manipulates terms (or formulas) having a certain specific structure. For instance, Gentzen's sequent calculus **LK** [Gen34] manipulates and allows for the derivation of *sequents* S of the form $A_1, \dots A_n \vdash B_1, \dots, B_m$ which are interpreted as the Boolean term $(\!|S|\!) = (A_1 \wedge \dots \wedge A_n) \Rightarrow (B_1 \vee \dots \vee B_m)$. We say that **LK** is sound and complete for the theory of Boolean algebras because a sequent S is derivable in **LK** if and only if $(\!|S|\!) = \top$ is a valid identity in the theory of Boolean algebras.

In a similar way, the proof system **HMR** is structural as it manipulates structured terms G, called *hypersequents*, which are interpreted as modal Riesz space terms $(\!|G|\!)$. The system **HMR** is sound and complete with respect to the theory of modal Riesz spaces in the sense that G is derivable in **HMR** if and only if $(\!|G|\!) \geq 0$ (or $(\!|G|\!) \sqcap 0 = 0$, written equationally) is a valid identity in the theory of modal Riesz spaces. Similarly, the subsystem **HR** of **HMR**, only manipulating (non–modal) Riesz space terms, is sound and complete with respect to the theory of (non–modal) Riesz spaces.

The key advantage of working with structural proof systems, compared to non–structural deductive systems such as equational logic, appears from results such as the cut–elimination system (called CAN elimination theorem in the context of **H(M)R**), which greatly simplify the analysis of proofs.

We now procede with the formal definitions. We first present the subsystem **HR** (Section 3.1) and then the full system **HMR** (Section 3.2).

All definitions and results regarding **HR** and **HMR** have been formalised: Repository [Luc21]: `folders: /hr and /hmr`.

3.1 Hypersequent calculus HR

In what follows, A and B range over Riesz terms in NNF (see end of Section 2.1) built from a set of variables V, ranged over by the letters x, y, z.

Definition 4 (Sequents and Hypersequents). *A* sequent *is a list of pairs (r, A) where $r \in \mathbb{R}_{>0}$ is a strictly positive real number and A is a formula in NNF. The sequent $\Gamma = ((r_1, A_1), ..., (r_n, A_n))$ is written as: $\vdash r_1.A_1, ..., r_n.A_n$. The empty sequent is denoted by $(\vdash)$. A* hypersequent *is a nonempty list of sequents. The hypersequent $G = [\Gamma_1, \Gamma_2, ..., \Gamma_n]$ is written as: $\vdash \Gamma_1 \mid \vdash \Gamma_2 \mid ... \mid \vdash \Gamma_n$.*

We use the letters Γ, Δ and the letters G, H to range over sequents and hypersequents, respectively. Note that $(\vdash \Gamma)$ can, ambiguosly, denote both the sequent $\vdash \Gamma$ and the hypersequent $[\vdash \Gamma]$ consisting of only one sequent. The context should always determine which of the two interpretations is intended.

The proof system **HR** allows for the derivation of hypersequents using the axioms and deductive rules of Figure 3. We write $\rhd_{\mathbf{HR}} G$ if the hypersequent G is derivable in the proof system **HR**. Before discussing the meaning of the rules and giving some examples, we define the interpretation of hypersequents and state the soundness and completeness of the proof system.

Definition 5 (Interpretation of Hypersequents). *We interpret sequents and hypersequents by Riesz terms as follows. A sequent* $\Gamma = (\vdash r_1.A_1, \dots, r_n.A_n)$ *is interpreted by the Riesz term* $(\!|\Gamma|\!) = r_1 A_1 + \cdots + r_n A_n$. *In particular, for the empty sequent,* $(\!|\vdash|\!) = 0$. *A hypersequent* $G = (\vdash \Gamma_1 \mid \cdots \mid \vdash \Gamma_n)$ *is interpreted by the Riesz term* $(\!|G|\!) = (\!|\vdash \Gamma_1|\!) \sqcup \cdots \sqcup (\!|\vdash \Gamma_n|\!)$.

Example 3. $(\!|\vdash 1.(x \sqcap y) \mid \vdash 1(\overline{x} \sqcup \overline{y}), 2.x|\!) = (1(x \sqcap y)) \sqcup (1(\overline{x} \sqcup \overline{y}) + 2x)$.

Lemma 1 (Soundness and Completeness [LM20, Thm 3.10 and 3.11]). *Let G be an hypersequent. Then* $\rhd_{\mathbf{HR}} G$ *if and only if* $(\!|G|\!) \geq 0$ *(or* $(\!|G|\!) \sqcap 0 = 0$*, written equationally) holds universally in all Riesz spaces.*

The meaning of most of the axioms and deductive rules of the hypersequent calculus **HR** is easy to grasp. For instance, the INIT rule allows to derive the

Axiom:

$$\frac{}{\vdash}\ \text{INIT}$$

ID and CAN rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.x, \vec{s}.\overline{x}}\ \text{ID}, \textstyle\sum r_i = \sum s_i \qquad \frac{G \mid \vdash \Gamma, \vec{s}.A, \vec{r}.\overline{A}}{G \mid \vdash \Gamma}\ \text{CAN}, \textstyle\sum r_i = \sum s_i$$

Structural rules:

$$\frac{G}{G \mid \vdash \Gamma}\ \text{W} \quad \frac{G \mid \vdash \Gamma \mid \vdash \Gamma}{G \mid \vdash \Gamma}\ \text{C} \quad \frac{G \mid \vdash \Gamma_1, \Gamma_2}{G \mid \vdash \Gamma_1 \mid \vdash \Gamma_2}\ \text{S} \quad \frac{G \mid \vdash \Gamma_1 \quad G \mid \vdash \Gamma_2}{G \mid \vdash \Gamma_1, \Gamma_2}\ \text{M} \quad \frac{G \mid \vdash r.\Gamma}{G \mid \vdash \Gamma}\ \text{T}$$

Logical rules:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \Gamma, \vec{r}.0}\ 0 \quad \frac{G \mid \vdash \Gamma, \vec{r}.A, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A+B)}\ + \quad \frac{G \mid \ \Gamma \vdash \Gamma, (s\vec{r}).A}{G \mid \ \Gamma \vdash \Gamma, \vec{r}.(sA)}\ \times$$

$$\frac{G \mid \vdash \Gamma, \vec{r}.A \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcup B)}\ \sqcup \quad \frac{G \mid \vdash \Gamma, \vec{r}.A \quad G \mid \vdash \Gamma, \vec{r}.B}{G \mid \vdash \Gamma, \vec{r}.(A \sqcap B)}\ \sqcap$$

Fig. 3. Inference rules of **HR** ([LM20]).

empty sequent $(\vdash)$ and indeed this is a sound rule since $(\!|\vdash|\!) = 0$ and $(\!|\vdash|\!) \geq 0$. The contraction rule (C) reflects the idempotency of the lattice operation $\sqcup$, which is used to interpret the (|) symbols of hypersequents. Similarly, the (+)–rule reflects the interpretation of commas in sequents as addition. In the T-rule, any sequent (in the hypersequent) can be multiplied by any positive scalar $r \in \mathbb{R}_{>0}$. This reflects the fact that if $rx \geq 0$ then $x \geq 0$, for every $r \in \mathbb{R}_{>0}$. Several rules adopt a vector notation ($\overrightarrow{r}$) to indicate that several terms are active in the rule. For example, the following is a valid instance of the rule ID:

$$\frac{G \mid \vdash \Gamma}{G \mid \vdash \frac{1}{2}.x, \frac{1}{3}.x, \frac{1}{6}.\overline{x}, \frac{2}{3}.\overline{x}} \text{ ID}, \quad \frac{1}{2} + \frac{1}{3} = \frac{1}{6} + \frac{2}{3}$$

because the proviso is satisfied. The ID rule expresses the fact that, since $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{6}(-x) + \frac{2}{3}(-x) = 0$, the terms can be cancelled out. Note that the CAN rule has the same interpretation, but in the reverse direction. Finally, the rules M and S which, in various forms have appeared in the proof–theory literature (not necessarily related to Riesz spaces, see, e.g., [MOG09, Avr96, FR94]) do not have an equally simple interpretation, but are sound [LM20, Thm 3.10].

The key results regarding the hypersequent calculus **HR** from [LM20], which are relevant for this work, are stated as the following lemmas.

Lemma 2 (CAN–elimination [LM20, Thm 3.14]). *If a hypersequent G is derivable in **HR** then G has a **HR** derivation that does not use the CAN rule.*

Lemma 3 ([LM20, Thm 3.12]). *The rules $\{0, +, \sqcup, \sqcap, \times\}$ are invertible: if the conclusion of an instance of one of these rules is derivable, then all its premises are also derivable.*

An hypersequent G is called *atomic* if all terms A appearing in G are either variables or covariables, i.e., $A = x$ or $A = \overline{x}$, for $x \in V$.

Lemma 4 (λ-property for HR [LM20, Lemma 3.43]). *For all atomic hypersequents G formed using the variables and covariables $x_1, \overline{x_1}, \ldots, x_k, \overline{x_k}$ of the form $\vdash \Gamma_1 \mid \ \ldots \mid \vdash \Gamma_m$, where for each $i \in [1 \ldots m]$,*

$$\Gamma_i = \overrightarrow{r}_{i,1}.x_1, \ldots, \overrightarrow{r}_{i,k}.x_k, \overrightarrow{s}_{i,1}.\overline{x_1}, \ldots, \overrightarrow{s}_{i,k}.\overline{x_{i,k}}$$

*then G is derivable in **HR** if and only if there exist numbers $t_1, \ldots, t_m \in [0, 1]$, one for each sequent in G, such that:*

1. *there exists $i \in [1 \ldots m]$ such that $t_i = 1$, and*
2. *for every (co)variable $(x_j, \overline{x_j})$, it holds that: $\sum_{i=1}^{m} t_i (\sum \overrightarrow{r}_{i,j} - \sum \overrightarrow{s}_{i,j}) = 0$.*

It is important to appreciate how Lemma 4 reduces the derivability problem of atomic hypersequents in **HR** to the existence of a solution in a linear arithmetic problem. The derivability problem of arbitrary hypersequents can also be reduced to linear arithmetic by invoking, in an iterative fashion, Lemma 3 which allows to simplify the term–complexity of the considered hypersequents.

3.2 Hypersequent calculus HMR

In this section we define the hypersequent calculus **HMR** from [LM20]. This is an extension of **HR** obtained by: (1) considering modal Riesz terms A, B (in NNF, see Section 2.2) rather than just (non–modal) Riesz terms, and (2) extending the set of rules of **HR** (Figure 3) with the two additional rules of Figure 3 dealing with the connectives $\{1, \Diamond\}$. The rule (1) is justified by the

Additional rules:

$$\frac{G \mid \vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}{G \mid \vdash \Gamma} \; 1, \sum s_i \leq \sum r_i \qquad \frac{\vdash \Gamma, \vec{r}.1, \vec{s}.\overline{1}}{\vdash \Diamond\Gamma, \vec{r}.1, \vec{s}.\overline{1}} \; \Diamond, \sum s_i \leq \sum r_i$$

Fig. 4. Additional inference rules of **HMR** ([LM20]).

axiom $0 \leq 1$ of modal Riesz spaces (Definition 3). The ($\Diamond$) rule is justified by the the positivity and linearity of the $\Diamond$ operator as well as the axiom $\Diamond 1 \leq 1$ (see [LM20, §4.3]).

We write $\triangleright_{\mathbf{HMR}} G$ if the hypersequent G (involving modal Riesz terms $A, B, \dots$) is derivable in the system **HMR**. By interpreting sequents and hypersequents as in Definition 5, the main results regarding **HR** extend to **HMR**: soundness, completeness, CAN–elimination and invertibility of the rules $\{0, +, \sqcup, \sqcap, \times\}$. Also a more sophisticated variant of the λ–property (Lemma 4) holds for **HMR** (see [LM20]), as we now state. A hypersequent G whose terms A are either atoms ($A = x$ or $A = \overline{x}$), or $A = 1$ or $A = \overline{1}$ or diamond–terms (i.e., $A = \Diamond B$, for some term B) is called a *basic hypersequent.*

Lemma 5 (λ-property of HMR [LM20, Lemma 4.44]). *For all basic hypersequents G formed using the variables and negated variables $x_1, \overline{x_1}, \dots, x_k, \overline{x_k}$ of the form*

$$\vdash \Gamma_1, \Diamond\Delta_1, \vec{r'}_1.1, \vec{s'}_1.\overline{1} \mid \;\dots \mid \vdash \Gamma_m, \Diamond\Delta_m, \vec{r'}_m.1, \vec{s'}_m.\overline{1}$$

*where $\Gamma_i = \vec{r}_{i,1}.x_1, \dots, \vec{r}_{i,k}.x_k, \vec{s}_{i,1}.\overline{x_1}, \dots, \vec{s}_{i,k}.\overline{x_{i,k}}$, for all $i \in [1 \dots m]$, then G is derivable in **HMR** if and only if there exist numbers $t_1, \dots, t_m \in [0, 1]$, one for each sequent in G, such that the following conditions hold:*

1. *there exists $i \in [1..m]$ such that $t_i = 1$,*
2. *for every (co)variable $(x_j, \overline{x_j})$ it holds that: $\sum_{i=1}^{m} t_i(\sum \vec{r}_{i,j} - \sum \vec{s}_{i,j}) = 0$*
3. *$0 \leq \sum_{i=1}^{m} t_i(\sum \vec{r'}_i - \sum \vec{s'}_i)$,*
4. *the following hypersequent (consisting of just one sequent) is derivable:* $\vdash t_1.\Delta_1, \dots, t_m.\Delta_m, (t_1\vec{r'}_1).1, \dots, (t_m\vec{r'}_m).1, (t_1\vec{s'}_1).\overline{1}, \dots, (t_m\vec{s'}_m).\overline{1}$.

3.3 Parametrised Hypersequents

The hypersequents of **HMR** (and its subsystem **HR**) are built out of expressions of the form $(r.A)$ where $r > 0$ is a concrete real number (see Definition

4). It is often useful, however, to state properties of parametrised families of hypersequents. For example, the hypersequent schema ($\vdash \alpha.x \mid\vdash \alpha.\overline{x}$), involving a variable α ranging over scalars, is derivable for all $\alpha > 0$.

Rather than just scalar variables (α, β), it is convenient to allow for even more general hypersequents schemas where in place of scalars we allow *polynomial expressions* over a certain number of variables α, β. We call such hypersequents *parametrised*. Given a parametrised hypersequent $G(\alpha_1, \ldots, \alpha_n)$, built using polynomial expressions R, S (see Section 2.3) involving scalar variables $\alpha_1, \ldots, \alpha_n$ we can obtain a concrete hypersequent (in the sense of Definition 4) $G(r_1, \ldots, r_n)$ by instantiating the scalar variables with concrete real numbers r_i and by evaluating the polynomial expressions as expected. Note, however, that since scalars r in expressions $(r.A)$ of concrete hypersequents are strictly positive real numbers, not all instantiations result in valid hypersequents. Therefore, when we write $G(r_1, \ldots, r_n)$, we implicitly mean that the substitution $[r_i/\alpha_i]$ results in a valid concrete hypersequent.

Example 4. $\vdash (\alpha_1 - 2\alpha_2).(x \sqcup y), (\alpha_1^2 - \frac{1}{2}).x$ is a parametrised hypersequent involving two scalar variables α_1 and α_2. The instance $[1/\alpha_1, -1/\alpha_2]$ results in the hypersequent $\vdash (3).(x \sqcup y), \frac{1}{2}.x$, and is therefore valid. The instance $[1/\alpha_1, 1/\alpha_2]$, instead, would result in $\vdash (-1).(x \sqcup y), \frac{1}{2}.x$ and is, therefore, not valid because (-1) is not a valid scalar.

Remark 2. Our main goal with the introduction of parametrised hypersequents is to express formally and schematically the last condition of the λ-property of **H(M)R** (Lemmas 4 and 5). To this end, even though scalars r in expressions $(r.A)$ of concrete hypersequents are strictly positive (Definition 4), it will be convenient to consider as valid also instances which results in the scalar 0. In this case, we use the convention $G \mid\vdash \Gamma, 0.A = G \mid\vdash \Gamma$, i.e., we remove every formula that has a weight equal to 0.

4 Main Result – Proof of Theorem 1

In this section we present our main result, a syntactic proof–theoretical proof of Theorem 1: free modal Riesz spaces are Archimedean.

```
Lemma FreeMRS_archimedean : forall A B,
    (forall n, (INRpos n) *S A ≤ B) → (* forall n, (n+1)*A ≤ B *)
    A ≤ MRS_zero. (* A ≤ 0 *)
```

hmr_archimedean/archimedean.v

The same type of proof technique can also be used to prove the known fact that free (non–modal) Riesz spaces are Archimedean (Proposition 2).

As a first step, we express the Archimedean property for (modal) Riesz spaces as a derivability problem in the hypersequent calculus proof system **H(M)R**.

Lemma 6. *For any set V, the free (modal) Riesz space $\mathbf{Free}_{(\mathrm{M})\mathrm{RS}}(V)$ has the Archimedean property if and only if, for any (modal) Riesz terms A and B it holds that:* $(\forall n, \triangleright_{\boldsymbol{H(M)R}} \vdash 1.\overline{A}, \frac{1}{n}.B) \Longrightarrow\ \triangleright_{\boldsymbol{H(M)R}} \vdash 1.\overline{A}$.

Proof. Recall that $\mathbf{Free}_{(\mathrm{M})\mathrm{RS}}(V)$ is isomorphic to $\mathbf{T}_{(\mathrm{M})\mathrm{RS}}(V)_{/\equiv}$ and therefore, by Definition 2, we have that $\mathbf{Free}_{(\mathrm{M})\mathrm{RS}}(V)$ is Archimedean if and only if the implication $(\forall n, n[A]_\equiv \leq [B]_\equiv) \Rightarrow [A]_\equiv \leq [0]_\equiv$ holds, for all (modal) Riesz terms A, B. Equivalently (using the identity $nx \leq y \Leftrightarrow -x + \frac{1}{n}y \geq 0$), $\mathbf{Free}_{(\mathrm{M})\mathrm{RS}}(V)$ is Archimedean if and only if $(\forall n, -[A]_\equiv + \frac{1}{n}[B]_\equiv \geq [0]_\equiv) \Rightarrow -[A]_\equiv \geq [0]_\equiv$holds. Finally, by the using the completeness and soundness of **H(M)R** (Theorem 1), this is equivalent to: $(\forall n, \rhd_{\mathbf{H(M)R}} \vdash 1.\overline{A}, \frac{1}{n}.B) \Rightarrow \rhd_{\mathbf{H(M)R}} \vdash 1.\overline{A}$. □

In order to establish the implication of Lemma 6 we prove a stronger result of independent interest about the hypersequent calculus **H(M)R**. This states that derivability in **H(M)R** is *continuous* in the sense that derivability preserves limits of scalars in hypersequents.

Theorem 2 (Continuity). *Let $G(\alpha_1, ..., \alpha_l)$ be a parametrized* ***H(M)R*** *hypersequent. Let $(s_{i,n}) \in \mathbb{R}^{\mathbb{N}}$ be a sequence of l–tuples of reals such that:*

1. *$G(s_{1,n}, ..., s_{l,n})$ is a valid instance of $G(\alpha_1, ..., \alpha_l)$, for all $n \in \mathbb{N}$, and*
2. *$\rhd_{\mathbf{H(M)R}} G(s_{1,n}, ..., s_{l,n})$ holds for all $n \in \mathbb{N}$,*
3. *For each $i \in \{1, \ldots, l\}$, the limit $\lim_{n \to +\infty} s_{i,n} = s_i$ exists.*

Then the limit instance $G(s_1, \ldots, s_l)$ is also valid and $\rhd_{\mathbf{H(M)R}} G(s_1, ..., s_l)$ holds.

Proof. Proofs for **HR** and **HMR** are presented in Sections 4.1 and 4.2. □

The Archimedean property of free (modal) Riesz spaces is a direct corollary of Lemma 6 and Theorem 2, considering the parametrised hypersequent $\vdash 1.\overline{A}, \alpha_1.B$ and the sequence $s_{1,n} = \frac{1}{n}$, so that $s_1 = \lim_{n \to +\infty} \frac{1}{n} = 0$.

Corollary 1. $\mathbf{Free}_{(\mathrm{M})\mathrm{RS}}(V)$ *has the Archimedean property.*

4.1 Proof of Continuity for HR

As a first step, we prove that Theorem 2 holds for all **HR** hypersequents $G(\alpha_1, ..., \alpha_l)$ that are atomic, i.e., such that all terms appearing in G are either variables or covariables, i.e., $A = x$ or $A = \overline{x}$. Intuitively, this fact follows from the last point of Lemma 4, which reduces derivability of atomic hypersequents to the existence of a solution of a system of polynomial inequalities, and polynomials expressions are continuous.

Lemma 7 (Atomic continuity of HR). *The statement of Theorem 2 for* ***HR*** *holds for all atomic hypersequents $G(\alpha_1, ..., \alpha_l)$.*

Proof. `Repository [Luc21]: (HR_atomic_lim) in archimedean.v.` Let $G = \vdash \Gamma_1 \mid \ \ldots \mid \vdash \Gamma_m$ with $\Gamma_i = \overrightarrow{R}_{i,1}.x_1, ..., \overrightarrow{R}_{i,k}.x_k, \overrightarrow{S}_{i,1}.\overline{x_1}, ..., \overrightarrow{S}_{i,k}.\overline{x_k}$, where all the indexed expressions R, $\overrightarrow{R}$, S and $\overrightarrow{R}$ are polynomials over $\alpha_1, \ldots, \alpha_l$.

By assumption, each $G(s_{1,n}, ..., s_{l,n})$ is a valid instance (the evaluation of all polynomial expressions results in a strictly positive scalar or 0 scalar, see Remark 2). Furthermore, $G(s_{1,n}, ..., s_{l,n})$ is assumed to be derivable in **HR** which means, by Lemma 4, that there exist real numbers $t_{1,n}, ..., t_{m,n} \in [0, 1]$, such that:

1. there exists $i \in [1..m]$ such that $t_i = 1$, and
2. for every variable and covariable pair $(x_j, \overline{x_j})$, it holds that $\sum_{i=1}^{m} t_{i,n}(\sum \overrightarrow{R}_{i,j}(s_{1,n}, ..., s_{l,n}) - \sum \overrightarrow{S}_{i,j}(s_{1,n}, ..., s_{l,n})) = 0$.

By the infinitary pigeon principle (see Section 2.3), there exists $i \in [1..m]$ such that $t_{i,n} = 1$ infinitely often and since $[0,1]^m$ is a compact space, by sequential compactness of $[0,1]$ (see Section 2.3), we can extract a subsequence $(t_{1,\sigma(j)}, ..., t_{m,\sigma(j)})_j$ converging to $(t_1, ..., t_m)$ with $t_{i,\sigma(j)} = 1$ for all j (and so $t_i = 1$). Finally, the identity $\sum_{i=1}^{m} t_i(\sum \overrightarrow{R}_{i,j}(s_1, ..., s_l) - \sum \overrightarrow{S}_{i,j}(s_1, ..., s_l)) = 0$ holds, because polynomial expressions are continuous and therefore preserve limits of converging sequences. Hence, we have that

1. there exists $i \in [1..m]$ such that $t_i = 1$, and
2. for every variable and covariable pair $(x_j, \overline{x_j})$, it holds that: $\sum_{i=1}^{m} t_i(\sum \overrightarrow{R}_{i,j}(s_1, ..., s_l) - \sum \overrightarrow{S}_{i,j}(s_1, ..., s_l)) = 0$,

and, according to Lemma 4, this implies that $G(s_1, ..., s_l)$ is derivable. $\square$

In order to conclude the proof of Theorem 2 for **HR**, we need to extend the result of Lemma 7 to arbitrary parametrised hypersequents $G(\alpha_1, \ldots, \alpha_l)$. This is done by showing that the continuity of parametrised hypersequents of a certain complexity can be reduced to the continuity of hypersequents of lower complexity, with the case of atomic hypersequent (Lemma 7) serving as base case. The main tool allowing this reduction is Lemma 3, which states that the logical rules of **HR** are invertible.

Definition 6 (Complexity). *The complexity of a sequent $\vdash \Gamma$, noted $|\vdash \Gamma|$, is the sum of all connectives $\{0, +, r(\_), \sqcup, \sqcap\}$ appearing in terms of Γ. The complexity of a (parametrized) hypersequent G, noted $|G|$, is a pair $(a,b) \in \mathbb{N}^2$ defined by: $a = \max_{\vdash \Gamma \in G} |\vdash \Gamma|$, the maximal complexity of sequents in G, and $b = |\{\vdash \Gamma \in G| \mid \vdash \Gamma| = a\}|$, the number of sequents in G with complexity a.*

Note that atomic hypersequents have complexity $|G| = (0,b)$. We are now ready to conclude the proof of Theorem 2.

Proof (general case). `Repository [Luc21]: (HR_lim) in archimedean.v.`
Let $G(\alpha_1, \ldots, \alpha_l)$ and $(s_{i,n}) \in \mathbb{R}^{\mathbb{N}}$ be as in the statement of Theorem 2. The proof goes by lexicographic induction on the complexity $|G|$.

If $|G| = (0,b)$, we can conclude with Lemma 7.

Otherwise $|G| = (a,b)$ for some $a, b > 1$. Hence G has the shape $G' \,|\!\vdash \Gamma, R.A$ where the complexity of $\vdash \Gamma, R.A$ is equal to a and A is a term with some outermost connective in $\{0, +, r(\_), \sqcup, \sqcap\}$. Here we only consider the case of $A = B \sqcup C$, the other cases being similar.

By assumption we know that, for all n, $\triangleright_{\mathbf{HR}}(G' \mid \vdash \Gamma, R.(B \sqcup C))(s_{1,n}, ..., s_{l,n})$, for all tuples $(s_{1,n}, ..., s_{l,n})$. The invertibility of the $\sqcup$ rule (Lemma 3) implies that the following hypersequents, for each tuple $(s_{1,n}, ..., s_{l,n})$, are also derivable:

$$\triangleright_{\mathbf{HR}}(G' \mid \vdash \Gamma, R.B \mid \vdash \Gamma, R.C)(s_{1,n}, ..., s_{l,n})$$

Note that the above hypersequents have complexity lower than $|G|$. Hence, by applying the induction hypothesis we obtain, by continuity, that

$$\triangleright_{\mathbf{HR}}(G' \mid \vdash \Gamma, R.B \mid \vdash \Gamma, R.C)(s_1, ..., s_l)$$

holds. We can then conclude the argument by deriving the desired hypersequent as follows, by one application of the $\sqcup$ rule.

$$\frac{(G' \mid \vdash \Gamma, R.B \mid \vdash \Gamma, R.C)(s_1, ..., s_l)}{(G' \mid \vdash \Gamma, R.(B \sqcup C))(s_1, ..., s_l)} \sqcup$$

$\square$

4.2 Proof of Continuity for HMR

The proof of Theorem 2 for **HMR** presented in this section has the same structure of the proof presented in Section 4.1 for **HR**. Namely, (1) we first prove a result similar to the atomic continuity Lemma 7 of Section 4.1 stating that Theorem 2 holds for certain "simple" hypersequents, and then (2) extend this result to arbitrary **HMR** hypersequents.

Regarding (1), the notion of "simple" is that of basic hypersequent associated with the λ–property of **HMR** (Lemma 5). Note, however, that unlike the corresponding λ–property of **HR** (Lemma 4), the statement of Lemma 5 reduces the derivability of basic **HMR** hypersequents not just to the existence of a solution in a system of polynomial inequalities, but also in terms of derivability of simpler (in terms of the number of $\Diamond$ operators in their terms) hypersequents. Hence, a slightly more sophisticated proof by induction on an appropriate notion of complexity of hypersequents is needed. Regarding (2), the key technical tool used (as in the proof of Section 4.1 for **HMR**) is the invertibility of the logical rules $\{0, +, r(\_), \sqcup, \sqcap\}$ of **HMR**, and is also based on an induction on the complexity of the hypersequent.

Definition 7 (Modal Depth and Outer Complexity). *The modal depth $\mathcal{D}(A)$ of a modal Riesz space term A is the maximum number of nested $\Diamond$'s in A, i.e., is defined inductively as: $\mathcal{D}(\Diamond B) = 1 + \mathcal{D}(B)$, $\mathcal{D}(A) = 0$ if $A \in \{x, \overline{x}, 1, \overline{1}, 0\}$, $\mathcal{D}(rB) = \mathcal{D}(B)$ and $\mathcal{D}(B \star C) = \max(\mathcal{D}(B), \mathcal{D}(C))$, for $\star \in \{+, \sqcup, \sqcap\}$.*

The outer complexity $\mathcal{O}(A)$ of a modal Riesz space term A is the total number of connectives $\{0, +, r(\_), \sqcup, \sqcap\}$ that do not appear under the scope of some $\Diamond$ in A, i.e., as: $\mathcal{O}(A) = 0$ if $A \in \{\Diamond B, x, \overline{x}, 1, \overline{1}\}$, $\mathcal{O}(0) = 1$, $\mathcal{O}(rB) = 1 + \mathcal{O}(B)$ and $\mathcal{O}(B \star C) = 1 + \mathcal{O}(B) + \mathcal{O}(C)$ for $\star \in \{+, \sqcup, \sqcap\}$.

Definition 8 (Hypersequent Complexity). *The modal depth and outer complexity of a sequent $\vdash \Gamma$ of the form $r_1.A_1, \dots r_n.A_n$ are defined as: $\mathcal{D}(\vdash \Gamma) = \max_{i=1}^{n} \mathcal{D}(A_i)$ and $\mathcal{O}(\vdash \Gamma) = \sum_{i=1}^{n} \mathcal{O}(A_i)$. The complexity of a (parametrized) hypersequent G, noted $|G|$, is a triplet $(a, b, c) \in \mathbb{N}^3$ defined by: $a = \max_{\vdash \Gamma \in G} \mathcal{D}(\vdash \Gamma)$ is the maximum modal depth of any sequent in G, $b = \max_{\vdash \Gamma \in G} \mathcal{O}(\vdash \Gamma)$ is the maximum outer complexity of any sequent in G, and $c = |\{\vdash \Gamma \in G | \mathcal{O}(\vdash \Gamma) = b\}|$ is the number of sequents in G having outer complexity b.*

Note that hypersequents G with $|G| = (a, 0, c)$ are basic hypersequents (see Section 3.2). Furthermore note that if $|G| = (0, 0, c)$, then the hypersequent only contains terms of the form $\{x, \overline{x}, 1, \overline{1}\}$, and the statement of Lemma 5 simplifies (point 4 becomes trivial) and it reduces the **HMR** derivability of G to the solution of a system of polynomial equations.

We are now ready to present the proof of Theorem 2 for **HMR**.

Proof. `Repository [Luc21]: (HMR_lim) in archimedean.v.` Let $G(\alpha_1, \ldots, \alpha_l)$ and $(s_{i,n}) \in \mathbb{R}^{\mathbb{N}}$ be as in the statement of Theorem 2. We prove the result by lexicographic induction on $|G|$.

If $|G| = (0, 0, c)$, then Lemma 7 of Section 4.1 can be easily adapted, using the λ–property (Lemma 5) of **HMR**, to prove that $\triangleright_{\mathbf{HMR}} G(s_1, ..., s_l)$.

If $|G| = (a, b, c)$ with $b > 0$, following the same technique presented in Section 4.1 based on the invertibility of the logical rules $\{0, +, r(\_), \sqcup, \sqcap\}$, we can reduce the complexity of $|G|$ to some (a, b', c') with $b' < b$, apply the induction hypothesis and finally deduce that $\triangleright_{\mathbf{HMR}} G(s_1, ..., s_l)$.

Lastly, assume that $|G| = (a, 0, c)$, i.e., that G is a basic hypersequent and has the form $\vdash \Gamma_1, \Diamond\Delta_1, \overrightarrow{R'}_1.1, \overrightarrow{S'}_1.\overline{1} \mid \ \ldots \mid \vdash \Gamma_m, \Diamond\Delta_m, \overrightarrow{R}_m.1, \overrightarrow{S'}_m.\overline{1}$ where $\Gamma_i = \overrightarrow{R}_{i,1}.x_1, ..., \overrightarrow{R}_{i,k}.x_k, \overrightarrow{S}_{i,1}.\overline{x_1}, ..., \overrightarrow{S}_{i,k}.\overline{x_{i,k}}$. By assumption $G(s_{1,n}, ..., s_{l,n})$ has a proof for all n. Thus, by the λ–property (Lemma 5), for all n there exist numbers $t_{1,n}, ..., t_{m,n} \in [0, 1]$ such that the following conditions hold:

1. there exists $i \in [1..m]$ such that $t_{i,n} = 1$,
2. for every variable and covariable pair $(x_j, \overline{x_j})$, it holds that: $\sum_{i=1}^{m} t_{i,n}(\sum \overrightarrow{R}_{i,j}(s_{1,n}, ..., s_{l,n}) - \sum \overrightarrow{S}_{i,j}(s_{1,n}, ..., s_{l,n})) = 0$,
3. $0 \leq \sum_{i=1}^{m} t_{i,n}(\sum \overrightarrow{R'}_i(s_{1,n}, ..., s_{l,n}) - \sum \overrightarrow{S'}_i(s_{1,n}, ..., s_{l,n}))$,
4. the following hypersequent, consisting of only one sequent, is derivable: $(\vdash t_{1,n}.(\Delta_1, \overrightarrow{R'}_1.1, \overrightarrow{S'}_1.\overline{1}), , ..., t_{m,n}.(\Delta_m, \overrightarrow{R'}_m.1, t_{m,n}\overrightarrow{S'}_m).\overline{1})(s_{1,n}, ..., s_{l,n})$.

From point (4), by considering m additional scalar variables $\alpha_{l+1}, \ldots, \alpha_{l+m}$ and by defining $s_{l+i,n} = t_{i,n}$, the parametrised hypersequent H defined as $\vdash \alpha_{l+1}.(\Delta_1, \overrightarrow{R'}_1.1, \overrightarrow{S'}_1.\overline{1}), , ..., \alpha_{l+m}.(\Delta_m, \overrightarrow{R'}_m.1, t_{m,n}\overrightarrow{S'}_m.\overline{1})$ has a proof for all n. Note that $|H| < |G|$, due to the lower modal–depth complexity.

By assumption $\lim\limits_{n \to +\infty} s_{i,n} = s_i$ exits for $i \in \{1, \ldots, l\}$ but the extended sequence $(s_1, \ldots, s_l, s_{l+1}, \ldots s_{l+m})$ might not have limits on the coordinates $l + 1, \ldots, l + m$. However, following the same argument of the proof of Lemma 7, by the infinitary pigeon principle and the sequential compactness of $[0, 1]^k$, we can extract a subsequence that converges on all coordinates and agrees with the existing limits s_i for $i \in \{1, \ldots, l\}$. Hence, by induction hypothesis, we can apply the continuity theorem to H and deduce that $H(s_1, ..., s_{l+m})$ has a proof.

Finally, in order to conclude the proof and prove that $\triangleright_{\mathbf{HMR}} G(s_1, ..., s_l)$, we apply once again the λ–property (Lemma 5) which states that $\triangleright_{\mathbf{HMR}} G(s_1, ..., s_l)$ is derivable if and only if the four points above hold, instantiated to $(s_1, ..., s_l)$. The fourth point has been established. The points (1–3) follow from the continuity of polynomial expressions as discussed in the proof of Lemma 7. $\square$

Bibliography

[Avr96] A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *From Foundations to Applications: European Logic Colloquium*, pages 1–32. Oxford University Press, 1996.

[Bak68] K. A. Baker. Free vector lattices. *Canadian Journal of Mathematics*, 20:58–66, 1968.

[Ble73] Roger D. Bleier. Free vector lattices. *Transactions of the American Mathematical Society.*, 1973.

[BLM15] Sylvie Boldo, Catherine Lelay, and Guillaume Melquiond. Coquelicot: A User-Friendly Library of Real Analysis for Coq. *Mathematics in Computer Science*, 9(1):41–62, March 2015.

[DMS18] Denisa Diaconescu, George Metcalfe, and Laura Schnüriger. A real-valued modal logic. *Log. Methods Comput. Sci.*, 14(1), 2018.

[FMM20] Robert Furber, Radu Mardare, and Matteo Mio. Probabilistic logics based on Riesz spaces. *Logical Methods in Computer Science*, Volume 16(1), 2020.

[FR94] Arnaud Fleury and Christian Retoré. The mix rule. *Mathematical Structures in Computer Science*, 4(2):273–285, 1994.

[Gen34] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:405–431, 1934.

[H.74] Fremlin D. H. *Topological Riesz Spaces and Measure Theory.* Cambridge University Press, 1974.

[JR77] E. De Jonge and A.C.M. Van Rooij. *Introduction to Riesz Spaces*, volume 78. Mathematical Centre Tracts, Amsterdam, 1977.

[LM19] Christophe Lucas and Matteo Mio. Towards a structural proof theory of probabilistic μ-calculi. In *FoSSaCS*, volume 11425 of *Lecture Notes in Computer Science*, pages 418–435. Springer, 2019.

[LM20] Christophe Lucas and Matteo Mio. Proof theory of Riesz spaces and modal Riesz spaces. *CoRR*, abs/2004.11185, 2020. (Submitted to peer review).

[Luc21] Christophe Lucas. A complete Coq formalisation is available at: https://doi.org/10.5281/zenodo.4897186, 2021.

[LZ71] W. A. J. Luxemburg and A. C. Zaanen. *Riesz Spaces*, volume 1. North-Holland Mathematical Library, 1971.

[MFM17] Matteo Mio, Robert Furber, and Radu Mardare. Riesz modal logic for Markov processes. In *32nd ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–12. IEEE, 2017.

[Mio18] Matteo Mio. Riesz modal logic with threshold operators. In *33rd Symposium on Logic in Computer Science (LICS)*, pages 710–719. ACM, 2018.

[MOG09] George Metcalfe, Nicola Olivetti, and Dov M. Gabbay. *Proof Theory for Fuzzy Logics*, volume 36 of *Applied Logic Series*. Springer Netherlands, 2009.

[MS17] Matteo Mio and Alex Simpson. Łukasiewicz μ-calculus. *Fundamenta Informaticae*, 150(3-4):317–346, 2017.