# Time Warps, from Algebra to Algorithms 

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#### Abstract

Graded modalities have been proposed in recent work on programming languages as a general framework for refining type systems with intensional properties. In particular, continuous endomaps of the discrete time scale, or time warps, can be used to quantify the growth of information in the course of program execution. Time warps form a complete residuated lattice, with the residuals playing an important role in potential programming applications. In this paper, we study the algebraic structure of time warps, and prove that their equational theory is decidable, a necessary condition for their use in real-world compilers. We also describe how our universal-algebraic proof technique lends itself to a constraint-based implementation, establishing a new link between universal algebra and verification technology.


Keywords: Residuated lattices • Universal algebra • Decision procedures • Graded modalities • Type systems • Programming languages.

## 1 Introduction

Program types are almost as old as programs themselves. Their initial role was to allow compilers to determine data sizes at compilation time, e.g., distinguishing machine integers from double precision numbers [1]. Type system research has developed tremendously since these humble beginnings, benefiting from close connections to logic [15]. For example, dependent types are expressive enough to serve as specification languages for program results [23,24].

Another line of research into type systems aims to classify not only what programs compute, but also how they do so. Such type systems describe the effect of a program - e.g., which parts of memory it modifies [18]-or the resources it requires-e.g., how long it takes to run [12]. Recently, graded modalities [7,8] have emerged as a unified setting for describing effect- and resource-annotated types. A graded modality $\square$ allows programmers to form a new type $\square_{f} A$ from a type $A$ and a grading $f$. The meaning of $\square_{f} A$ depends on the system at hand, but can generally be understood as a modification of $A$ that includes the behavior prescribed by $f$.

[^0]In many cases, gradings come equipped with an ordered algebraic structure that is relevant for programming applications. Most commonly, they form a monoid whose binary operation corresponds to a notion of composition such that $\square_{g f} A$ is related to $\square_{f} \square_{g} A$. It is also often the case that gradings can be ordered by some sort of precision ordering along which the graded modality acts contravariantly. That is, we have a generic program of type $\square_{g} A \rightarrow \square_{f} A$ if $f \leq g$, allowing us to freely move from more to less precise types. As a consequence, the structure of this ordering is reflected by the operations available on types; for example, when the infimum of $f$ and $g$ exists, it permits the conversion of two values of types $\square_{f} A$ and $\square_{g} B$ into a single value of type $\square_{f \wedge g}(A \times B)$.

The additional flexibility and descriptive power gained by adopting graded modalities in a programming language comes at a price, however. The language implementation must now be able to manipulate gradings in various ways; in particular, it should be able to decide the ordering between gradings in order to distinguish between well-typed and ill-typed programs. In this paper, we address this issue for a specific class of gradings known as time warps: sup-preserving functions on $\omega^{+}=\omega \cup\{\omega\}$, or, equivalently, monotonic functions $f: \omega^{+} \rightarrow \omega^{+}$ satisfying $f(0)=0$ and $f(\omega)=\bigvee\{f(n) \mid n \in \omega\}$ [13]. Informally, time warps describe the growth of data along program execution. In this setting, any type $A$ describes a family of sets $\left(A_{n}\right)_{n \in \omega}$, where $A_{n}$ is the set of values classified by $A$ at execution step $n$. The type $\square_{f} A$ classifies the set of values of $A_{f(n)}$ at step $n$. This typing discipline generalizes a long line of works on programming languages for embedded systems [5] and type theories with modal recursion operators [21,2].

Let us denote the set of time warps by $\mathscr{W}$. Then $\langle\mathscr{W}, o, i d\rangle$ is a monoid, where $f g:=f \circ g$ denotes the composition of $f, g \in \mathscr{W}$, and $i d$ is the identity function. Moreover, equipping $\mathscr{W}$ with the pointwise order, defined by

$$
f \leq g: \Longleftrightarrow f(p) \leq g(p) \text { for all } p \in \omega^{+},
$$

yields a complete distributive lattice $\langle\mathscr{W}, \wedge, \vee\rangle$ satisfying, for all $f, g_{1}, g_{2}, h \in \mathscr{W}$,

$$
f\left(g_{1} \vee g_{2}\right) h=f g_{1} h \vee f g_{2} h \text { and } f\left(g_{1} \wedge g_{2}\right) h=f g_{1} h \wedge f g_{2} h,
$$

with a least element $\perp$ that maps all $p \in \omega^{+}$to 0 , and a greatest element $\top$ that maps all $p \in \omega^{+} \backslash\{0\}$ to $\omega$. Note that the operation $\circ$ is a double quasi-operator on this lattice in the sense of $[10,11]$, and that the structure $\langle\mathscr{W}, \wedge, \vee, \mathrm{o}, i d\rangle$ belongs to the family of unital quantales of sup-preserving functions on a complete lattice studied in [22].

The monoidal structure of time warps plays the expected role in programming applications. In particular, $\square_{g f} A$ and $\square_{f} \square_{g} A$ are isomorphic, as are $\square_{i d} A$ and $A$. However, time warps also admit further additional algebraic structure of interest for programming. Since they are sup-preserving, there exist binary operations $\backslash, /$ on $\mathscr{W}$, called residuals, satisfying for all $f, g, h \in \mathscr{W}$,

$$
f \leq h / g \Longleftrightarrow f g \leq h \Longleftrightarrow g \leq f \backslash h .
$$

From a programming perspective, residuals play a role similar to that of weakest preconditions in deductive verification. The type $\square_{h / g} A$ can be seen as the most
general type $B$ such that $\square_{h} A$ can be sent generically to $\square_{g} B$. Similarly, $f \backslash h$ is the most general (largest) time warp $f^{\prime}$ such that $\square_{h} A$ can be sent generically to $\square_{f} \square_{f} A$. Such questions arise naturally when programming in a modular way [13], justifying the consideration of residuated structure in gradings.

The algebraic structure $\mathbf{W}=\langle\mathscr{W}, \wedge, \vee, \circ, \backslash, /, i d, \perp, \top\rangle$, referred to here as the time warp algebra, belongs to the family of (bounded) residuated lattices, widely studied as algebraic semantics for substructural logics $[3,9,19]$. The main goal of this paper is to prove the following theorem, a necessary condition for the use of time warps in real-world compilers:

Theorem 1. The equational theory of the time warp algebra $\mathbf{W}$ is decidable.
A time warp term is a member of the term algebra over a countably infinite set of variables of the algebraic language with binary operation symbols $\wedge, \vee, \circ, \backslash, /$, and constant symbols $i d, \perp, \top$, and a time warp equation consists of an ordered pair of terms $s, t$, denoted by $s \approx t$. Let $s \leq t$ denote the equation $s \wedge t \approx s$, noting that $\mathbf{W} \models s \approx t$ if, and only if, $\mathbf{W} \models s \leq t$ and $\mathbf{W} \models t \leq s$, and, by residuation, $\mathbf{W} \models s \leq t$ if, and only if, $\mathbf{W} \models i d \leq t / s$. Clearly, to prove Theorem 1, it will suffice to provide an algorithm that decides $\mathbf{W} \models i d \leq t$ for any time warp term $t$.

## Overview of the proof of Theorem 1

We prove Theorem 1 by describing an algorithm with the following behavior:
Input. A time warp term $t$ in the variables $x_{1}, \ldots, x_{k}$.
Output. If $\mathbf{W} \models i d \leq t$, the algorithm returns 'Valid'; if $\mathbf{W} \not \vDash i d \leq t$, the algorithm returns 'Invalid at $\left(\hat{f}_{1}, \ldots, \hat{f}_{k}, p\right)$ ' for some $p \in \omega^{+}$and finite descriptions $\hat{f}_{1}, \ldots, \hat{f}_{k}$ of time warps $f_{1}, \ldots, f_{k}$, such that $\llbracket t \rrbracket(p)<p$, where $\llbracket t \rrbracket$ is the time warp obtained from $t$ by mapping each $x_{i}$ to $f_{i}$.
We now give a high-level overview of the three main steps of the algorithm; the details and the proof of its correctness will occupy us for the rest of the paper.
I. Pre-processing into a disjunction of basic terms. In Section 2, we show how to effectively obtain for any time warp term $t$, a time warp term

$$
t^{\prime}:=\bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_{i}} t_{i, j}
$$

such that $\mathbf{W} \models t \approx t^{\prime}$, where each $t_{i, j}$ is a basic term, constructed using $\circ$, id, $\perp$, and the defined operations $s^{\ell}:=i d / s, s^{r}:=s \backslash i d$, and $s^{\circ}:=\top \backslash s$ (Theorem 9). Since $\mathbf{W} \models i d \leq t$ if, and only if, $\mathbf{W} \models i d \leq \bigvee_{j=1}^{n_{i}} t_{i, j}$ for each $i \in\{1, \ldots, m\}$, our task is reduced to giving an algorithm with the required behavior for terms of the form $t_{1} \vee \cdots \vee t_{n}$, where each $t_{i}$ is a basic term. Once we have an algorithm that solves this case, we can run it for each of the $m$ conjuncts of $t^{\prime}$ in turn, returning 'Invalid at $\left(\hat{f}_{1}, \ldots, \hat{f}_{k}, p\right)$ ' whenever this is the result of one of these runs, and otherwise 'Valid'.
II. Finitary characterization through diagrams. The crucial step in our algorithm is the finitary characterization of 'potential counterexamples' for an equation of the form $i d \leq t_{1} \vee \cdots \vee t_{n}$, where each $t_{i}$ is a basic term. Our main tool for providing these finitary characterizations is the notion of a diagram. ${ }^{3}$

Let us give an example to illustrate the basic idea. To falsify the equation $i d \leq x y x^{\ell} \vee y^{\ell}$ in $\mathbf{W}$, it suffices to find time warps $f_{x}$ and $f_{y}$, and an element $p \in \omega^{+}$, such that $\left(f_{x} \circ f_{y} \circ f_{x}^{\ell}\right)(p)<p$ and $f_{y}^{\ell}(p)<p$. Although time warps are, as functions on $\omega^{+}$, infinite objects, only finitely many of the values of $f_{x}$ and $f_{y}$ are relevant for falsifying the equation. Moreover, an upper bound for the number of values required for such a counterexample can be computed. The condition $\left(f_{x} \circ f_{y} \circ f_{x}^{\ell}\right)(p)<p$ is 'unravelled' by stating that there exist $\alpha_{1}, \alpha_{2}, \alpha_{3} \in \omega^{+}$such that $\alpha_{3}<p$, where $\alpha_{1}:=f_{x}^{\ell}(p), \alpha_{2}:=f_{y}\left(\alpha_{1}\right)$, and $\alpha_{3}:=f_{x}\left(\alpha_{2}\right)$. More formally, using a 'time variable' $\kappa$ to refer to the value $p$, we build a finite sample set $\Gamma_{1} \supseteq$ $\left\{\kappa, x^{\ell}[\kappa], y\left[x^{\ell}[\kappa]\right], x\left[y\left[x^{\ell}[\kappa]\right]\right]\right\}$, where $\Gamma_{1}$ is 'saturated' with extra conditions used to describe, e.g., the behavior of $f_{x}^{\ell}$ at relevant values. Similarly, we obtain a finite saturated sample set $\Gamma_{2} \supseteq\left\{\kappa, y^{\ell}[\kappa]\right\}$ for the condition $f_{y}^{\ell}(p)<p$. The problem of deciding if there exists a counterexample to $i d \leq x y x^{\ell} \vee y^{\ell}$ then becomes the problem of deciding if there exists a suitable function $\delta: \Gamma_{1} \cup \Gamma_{2} \rightarrow \omega^{+}$satisfying $\delta\left(x\left[y\left[x^{\ell}[\kappa]\right]\right]\right)<\delta(\kappa)$ and $\left.\left.\delta\left(y^{\ell}[\kappa]\right]\right]\right)<\delta(\kappa)$. In particular, $\delta$ should determine partial sup-preserving functions $\hat{f}_{x}$ and $\hat{f}_{y}$ on $\omega^{+}$satisfying $\hat{f}_{x}(\delta(\alpha))=\delta(x[\alpha])$ for all $x[\alpha] \in \Gamma_{1} \cup \Gamma_{2}$, and $\hat{f}_{y}(\delta(\alpha))=\delta(y[\alpha])$ for all $y[\alpha] \in \Gamma_{1} \cup \Gamma_{2}$.

Clearly, not every function $\delta$ from a saturated sample set to $\omega^{+}$extends to a valuation in $\mathbf{W}$; e.g., if $\delta(\kappa)=0$, then we must also have $\delta(x[\kappa])=0$. Moreover, although time warp equations in the residual-free language can be decided by considering an algebra of sup-preserving functions on a finite totally ordered set, this is not the case for the full language. ${ }^{4}$ Section 3 develops a general theory that precisely characterizes the functions-called diagrams - that extend to valuations and can be used to falsify a given equation. This allows us to prove that there exists a counterexample to $i d \leq t_{1} \vee \cdots \vee t_{n}$ if, and only if, there exists a diagram $\delta: \Gamma \rightarrow \omega^{+}$satisfying $\delta(\kappa)>\delta\left(t_{i}[\kappa]\right)$ for each $i \in\{1, \ldots, n\}$, where $\Gamma$ is the finite saturated sample set extending $\left\{t_{1}[\kappa], \ldots, t_{n}[\kappa]\right\}$ (Theorem 31).
III. Encoding as a satisfiability query. In the last step of the algorithm, described in Section 4, we use the decidability of the satisfiability problem in the first-order logic of natural numbers with the natural ordering and successor. More precisely, we show that the existence of a diagram in Theorem 31 can be encoded as an existential first-order sentence in that signature. Concretely, our algorithm constructs a quantifier-free formula which is satisfiable in the structure $(\mathbb{N}, \leq, S, 0)$ if, and only if, there exists a diagram as specified by Theorem 31.

[^1]Moreover, a satisfying assignment can be converted into a valuation into $\mathbf{W}$ that provides a counterexample to the equation $i d \leq t_{1} \vee \cdots \vee t_{n}$.

## 2 A normal form for time warps

The main aim of this section is to provide a normal form for time warp terms. Our first step is to provide a more precise description of the left and right residuals of time warps. Note that to prove that two time warps are equal, it suffices to show that they coincide on every non-zero natural number, since for any time warp $f$, it is always the case that $f(0)=0$ and $f(\omega)=\bigvee\{f(n) \mid n \in \omega\}$.

Lemma 2. For any time warps $f, g$ and $p \in \omega^{+}$,
(a) $(f \backslash g)(p)= \begin{cases}0 & \text { if } p=0 ; \\ \bigvee\left\{q \in \omega^{+} \mid f(q) \leq g(p)\right\} & \text { if } p \in \omega \backslash\{0\} ; \\ \bigvee\left\{q \in \omega^{+} \mid(\exists m \in \omega)(f(q) \leq g(m))\right\} & \text { if } p=\omega\end{cases}$
(b) $(g / f)(p)=\bigwedge g\left[\left\{q \in \omega^{+} \mid p \leq f(q)\right\}\right]$.

Proof. (a) Let $h$ denote the function defined by cases on the right of the equation. Clearly, $h$ is monotonic and satisfies $h(0)=0$ and $h(\omega)=\bigvee\{h(n) \mid n \in \omega\}$, so $h$ is a time warp. Moreover, since $f$ preserves arbitrary joins, $f h \leq g$, and hence $h \leq f \backslash g$. For the converse, just observe that for any $n \in \omega \backslash\{0\}$, since $f((f \backslash g)(n)) \leq g(n)$, also $(f \backslash g)(n) \leq h(n)$. So $h=f \backslash g$.
(b) Let $h$ be the function defined by $h(p):=\bigwedge g\left[\left\{q \in \omega^{+} \mid p \leq f(q)\right\}\right]$. Clearly, $h$ is monotonic and satisfies $h(0)=0$ and $h(\omega)=\bigvee\{h(n) \mid n \in \omega\}$, so $h$ is a time warp. Moreover, $h f \leq g$, and hence $h \leq g / f$. For the converse, let $n \in \omega \backslash\{0\}$. If $q \in \omega^{+}$satisfies $n \leq f(q)$, then $(g / f)(n) \leq(g / f)(f(q)) \leq g(q)$, and hence $(g / f)(n) \leq \bigwedge g\left[\left\{q \in \omega^{+} \mid n \leq f(q)\right\}\right]=h(n)$. So $h=g / f$.

Next, we show that residuals of time warps distribute over joins and meets.
Lemma 3. For any time warps $f, g, h$,
(a) $f \backslash(g \wedge h)=(f \backslash g) \wedge(f \backslash h)$
(e) $(g \wedge h) / f=(g / f) \wedge(h / f)$
(b) $(g \wedge h) \backslash f=(g \backslash f) \vee(h \backslash f)$
(f) $f /(g \wedge h)=(f / g) \vee(f / h)$
(c) $f \backslash(g \vee h)=(f \backslash g) \vee(f \backslash h)$
(g) $(g \vee h) / f=(g / f) \vee(h / f)$
(d) $(g \vee h) \backslash f=(g \backslash f) \wedge(h \backslash f)$
(h) $f /(g \vee h)=(f / g) \wedge(f / h)$.

Proof. Parts (a), (d), (e), and (h) hold in any residuated lattice (see, e.g., [3]). For (b), consider any $n \in \omega \backslash\{0\}$. Using Lemma 2(a),

$$
\begin{aligned}
((g \wedge h) \backslash f)(n) & =\bigvee\left\{q \in \omega^{+} \mid(g \wedge h)(q) \leq f(n)\right\} \\
& =\bigvee\left\{q \in \omega^{+} \mid g(q) \leq f(n) \text { or } h(q) \leq f(n)\right\} \\
& =\bigvee\left\{q \in \omega^{+} \mid g(q) \leq f(n)\right\} \vee \bigvee\left\{q \in \omega^{+} \mid h(q) \leq f(n)\right\} \\
& =((g \backslash f) \vee(h \backslash f))(n)
\end{aligned}
$$

For (f), consider any $n \in \omega \backslash\{0\}$. Using Lemma 2(b),

$$
\begin{aligned}
(f /(g \wedge h))(n) & =\bigwedge f\left[\left\{q \in \omega^{+} \mid n \leq(g \wedge h)(q)\right\}\right] \\
& =\bigwedge f\left[\left\{q \in \omega^{+} \mid n \leq g(q) \text { and } n \leq h(q)\right\}\right] \\
& =\bigwedge f\left[\left\{q \in \omega^{+} \mid n \leq g(q)\right\}\right] \vee \bigwedge f\left[\left\{q \in \omega^{+} \mid n \leq h(q)\right\}\right] \\
& =((g / f) \vee(h / f))(n)
\end{aligned}
$$

Parts (c) and (g) are proved similarly.
It follows from Lemma 3 that every time warp term is equivalent to a meet of joins of terms constructed using the operations $\circ, \backslash, /, i d, \perp$, and $T$. However, we can take this simplification process one step further by expressing the residuals of time warps in terms of their restrictions to certain unary operations.

Definition 4. For any time warp $f$, let

$$
f^{\ell}:=i d / f, \quad f^{r}:=f \backslash i d, \quad \text { and } \quad f^{\circ}:=\top \backslash f .
$$

Lemma 5. For any time warps $f, g$,
(a) $f \backslash g=f^{r} g \vee(\top f)^{r} \vee g^{\circ}$
(b) $g / f=g f^{\ell} \vee\left(f^{\ell}\right)^{\circ}$.

Proof. For (a), note first that clearly $f^{r} g \vee(\top f)^{r} \vee g^{\circ} \leq f \backslash g$. For the converse, consider any $n \in \omega \backslash\{0\}$. If $g(n)=0$, then, by Lemma 2(a),

$$
(f \backslash g)(n)=\bigvee\left\{q \in \omega^{+} \mid f(q) \leq 0\right\}=\bigvee\left\{q \in \omega^{+} \mid \top f(q) \leq i d(n)\right\}=(\top f)^{r}(n)
$$

If $g(n) \in \omega \backslash\{0\}$, then, by Lemma 2(a),
$(f \backslash g)(n)=\bigvee\left\{q \in \omega^{+} \mid f(q) \leq g(n)\right\}=\bigvee\left\{q \in \omega^{+} \mid f(q) \leq i d(g(n))\right\}=f^{r}(g(n))$.
Finally, if $g(n)=\omega$, then, by Lemma 2(a),

$$
\left.\left.(f \backslash g)(n)=\bigvee\left\{q \in \omega^{+} \mid f(q) \leq \omega\right)\right\}=\omega=\bigvee\left\{q \in \omega^{+} \mid \top(q) \leq \omega\right)\right\}=g^{\circ}(n)
$$

So $f \backslash g=f^{r} g \vee(\top f)^{r} \vee g^{\circ}$.
For (b), note first that clearly $g f^{\ell} \vee\left(f^{\ell}\right)^{\circ} \leq g / f$. For the converse, consider any $n \in \omega \backslash\{0\}$. If $\left\{q \in \omega^{+} \mid n \leq f(q)\right\}=\emptyset$, then, by Lemma 2(b),

$$
(g / f)(n)=\bigwedge g[\emptyset]=\omega=\left(\left(f^{\ell}\right)^{\circ}\right)(n)
$$

Otherwise, $\left\{q \in \omega^{+} \mid n \leq f(q)\right\} \neq \emptyset$ and, by Lemma 2(b),
$(g / f)(n)=\bigwedge g\left[\left\{q \in \omega^{+} \mid n \leq f(q)\right\}\right]=g\left(\bigwedge i d\left[\left\{q \in \omega^{+} \mid n \leq f(q)\right\}\right]\right)=\left(g f^{\ell}\right)(n)$.
So $g / f=g f^{\ell} \vee\left(f^{\ell}\right)^{\circ}$.

To gain a better understanding of these defined unary operations, we observe that Lemma 2 yields for any $n \in \omega \backslash\{0\}$,

$$
\begin{aligned}
f^{\circ}(n) & =\max \left\{m \in \omega^{+} \mid \omega \leq f(n)\right\} \\
f^{r}(n) & =\max \left\{m \in \omega^{+} \mid f(m) \leq n\right\} \\
f^{\ell}(n) & =\bigwedge\left\{m \in \omega^{+} \mid n \leq f(m)\right\}
\end{aligned}
$$

The following lemmas collect some simple consequences of these observations.
Lemma 6. For any time warp $f$ and $n \in \omega \backslash\{0\}$,

$$
\begin{aligned}
f^{\circ}(n)=0 & \Longleftrightarrow f(n)<\omega \\
f^{\circ}(n)=\omega & \Longleftrightarrow f(n)=\omega \\
f^{\circ}(\omega)=0 & \Longleftrightarrow f(k)<\omega \text { for all } k \in \omega \\
f^{\circ}(\omega)=\omega & \Longleftrightarrow f(k)=\omega \text { for some } k \in \omega .
\end{aligned}
$$

Lemma 7. For any time warp $f, n \in \omega \backslash\{0\}$, and $m \in \omega$,

$$
\begin{aligned}
f^{r}(n)=m & \Longleftrightarrow f(m) \leq n<f(m+1) \\
f^{r}(n)=\omega & \Longleftrightarrow f(\omega) \leq n \\
f^{r}(\omega)=m & \Longleftrightarrow f(m+1)=\omega \text { and } f^{r}(k)=m \text { for some } k \in \omega \\
f^{r}(\omega)=\omega & \Longleftrightarrow f(\omega)<\omega \text { or }(f(\omega)=\omega \text { and } \forall k \in \omega: f(k)<\omega) .
\end{aligned}
$$

Lemma 8. For any time warp $f, n \in \omega \backslash\{0\}$, and $m \in \omega$,

$$
\begin{aligned}
f^{\ell}(n)=m & \Longleftrightarrow f(m-1)<n \leq f(m) \\
f^{\ell}(n)=\omega & \Longleftrightarrow f(\omega)<n \\
f^{\ell}(\omega)=m & \Longleftrightarrow f(m)=\omega \text { and } f^{\ell}(k)=m \text { for some } k \in \omega \\
f^{\ell}(\omega)=\omega & \Longleftrightarrow f(\omega)<\omega \text { or }(f(\omega)=\omega \text { and } \forall k \in \omega: f(k)<\omega) .
\end{aligned}
$$

Note also that $\top=\perp^{\ell}$. We call a time warp term basic if it is constructed using only $\circ, i d, \perp$, and the defined operations $t^{\ell}:=i d / t, t^{r}:=t \backslash i d$, and $t^{\circ}:=\top \backslash t$. Our normal form theorem now follows, using Lemma 5 to remove residuals from a time warp term, then Lemma 3 and other distributivity properties of $\mathbf{W}$ to push out meets and joins, preserving equivalence in $\mathbf{W}$ at every step.

Theorem 9. There is an effective procedure that given any time warp term $t$, produces positive integers $m, n_{1}, \ldots, n_{m}$ and a set of basic time warp terms $\left\{t_{i, j} \mid 1 \leq i \leq m ; 1 \leq j \leq n_{i}\right\}$ satisfying $\mathbf{W} \models t \approx \bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_{i}} t_{i, j}$.

Corollary 10. The equational theory of $\mathbf{W}$ is decidable if, and only if, there exists an effective procedure that decides for any finite non-empty set of basic time warp terms $\left\{t_{1}, \ldots, t_{n}\right\}$ if $\mathbf{W} \models i d \leq t_{1} \vee \cdots \vee t_{n}$.

We conclude this section by introducing a further notion that will be useful for providing finitary characterizations of time warps.

Definition 11. For any time warp $f$, let

$$
\operatorname{last}(f):=\bigwedge\left\{p \in \omega^{+} \mid f(p)=f(\omega)\right\}
$$

Observe that last $(f)<\omega$ if, and only if, $f$ is eventually constant, i.e., increases a finite number of times, and that last $(f)$ can be defined equivalently in the language of time warps as $\left(f^{\ell} f\right)(\omega)$. For future reference, we record the following easy consequences of this definition.

Lemma 12. For any time warps $f, g$,
(a) $\operatorname{last}(f g)=\omega \Longleftrightarrow(\operatorname{last}(f)=\omega$ and $\operatorname{last}(g)=\omega)$
(b) $\operatorname{last}(f)=\omega \Longleftrightarrow \operatorname{last}\left(f^{r}\right)=\omega \Longleftrightarrow \operatorname{last}\left(f^{\ell}\right)=\omega$.

## 3 Diagrams

In this section, we define diagrams as finitary characterizations of 'potential counterexamples' for equations of the form $i d \leq t_{1} \vee \cdots \vee t_{n}$, where each $t_{i}$ is a basic time warp term. This definition is obtained by considering relevant properties of time warps assigned to variables in a refuting valuation, and it therefore follows easily that if $\mathbf{W} \neq i d \leq t_{1} \vee \cdots \vee t_{n}$, then there exists a suitable refuting diagram. The more challenging direction is to show that every refuting diagram extends to a refuting valuation witnessing $\mathbf{W} \not \vDash i d \leq t_{1} \vee \cdots \vee t_{n}$.

Note first that, using Theorem 9, we may without loss of generality express validity in $\mathbf{W}$ using a simplified language where the restricted residuals are taken as fundamental operations. Let $\mathscr{T}_{V}$ be a countably infinite set of term variables, with elements denoted by $x, y, z$, etc.
Definition 13. A basic term belongs to the grammar

$$
\mathscr{T} \ni t, u::=x|t u| t^{\circ}\left|t^{\ell}\right| t^{r}|i d| \perp .
$$

We also define valuations and interpretations explicitly for basic terms.
Definition 14. A valuation $\theta$ is a map $\mathscr{T}_{V} \rightarrow \mathscr{W}$. The interpretation of a basic term $t$ under $\theta$, denoted by $\llbracket t \rrbracket_{\theta}$, is the time warp defined inductively by

$$
\llbracket x \rrbracket_{\theta}:=\theta(x), \quad \llbracket t u \rrbracket_{\theta}:=\llbracket t \rrbracket_{\theta} \llbracket u \rrbracket_{\theta}, \quad \llbracket t^{\star} \rrbracket_{\theta}:=\llbracket t \rrbracket_{\theta}^{\star} \text { for } \star \in\{\mathrm{o}, \ell, \mathrm{r}\}
$$

Corollary 10 tells us that the equational theory of $\mathbf{W}$ is decidable if, and only if, there exists an effective procedure that decides, for any finite set of basic terms $T$, if there exists a valuation $\theta$ and $p \in \omega^{+}$such that $\llbracket t \rrbracket_{\theta}(p)<p$ for all $t \in T$. To refer to this element $p$, we let $\mathscr{I}_{V}$ be a countably infinite set of time variables containing elements denoted by $\kappa, \kappa^{\prime}$, etc, noting that in fact only one time variable will be required for the proofs in this paper. We now define a new language of 'samples' that will be used to refer to values considered in a diagram.
Definition 15. A sample belongs to the grammar (where $t$ is any basic term)

$$
\mathscr{I} \ni \alpha::=\kappa|t[\alpha]| \mathbf{s}(\alpha)|\mathrm{p}(\alpha)| \operatorname{last}(t)
$$

Although samples are purely syntactic, the notation is indicative of their intended meaning. Given an initial sample set $\left\{t_{1}[\kappa], \ldots, t_{n}[\kappa]\right\}$, obtained from the equation $i d \leq t_{1} \vee \cdots \vee t_{n}$, the idea is to 'saturate' this set by adding further samples required to describe the existence of a counterexample.

Definition 16. A sample set $\Delta$ is called saturated if whenever $\alpha \in \Delta$ and $\alpha \rightsquigarrow \beta$, also $\beta \in \Delta$, where $\rightsquigarrow$ is the relation between samples defined by

$$
\begin{aligned}
t[\alpha] & \rightsquigarrow \alpha & & t^{\circ}[\alpha]
\end{aligned}>t[\alpha], \text { s }(\alpha) \rightsquigarrow \alpha, ~ t^{r}[\alpha] \rightsquigarrow t\left[t^{r}[\alpha]\right], t\left[\mathbf{s}\left(t^{r}[\alpha]\right)\right] .
$$

The saturation of a sample set $\Delta$ is

$$
\Delta^{\rightsquigarrow}:=\left\{\beta \mid \exists \alpha \in \Delta, \alpha \rightsquigarrow^{*} \beta\right\},
$$

where $\rightsquigarrow *$ denotes the reflexive transitive closure of $\rightsquigarrow$.
A proof of the following result can be found in Appendix A.1.
Lemma 17. The saturation of a finite sample set is finite.
Let us fix, until after Definition 25 , a saturated sample set $\Delta$.
Definition 18. A $\Delta$-prediagram is a map $\delta: \Delta \rightarrow \omega^{+}$.
We now give a list of conditions for a $\Delta$-prediagram to be a $\Delta$-diagram.
Definition 19. For $p \in \omega^{+}$, let

$$
p \ominus 1:=\left\{\begin{array}{ll}
p-1 & \text { if } p \in \omega \backslash\{0\} \\
p & \text { if } p \in\{0, \omega\}
\end{array}, \quad p \oplus 1:=\left\{\begin{array}{ll}
p+1 & \text { if } p \in \omega \\
p & \text { if } p=\omega
\end{array} .\right.\right.
$$

Definition 20. A $\Delta$-prediagram $\delta$ is called structurally-sound if

$$
\left.\left.\begin{array}{l}
\forall t[\alpha], t[\beta] \in \Delta, \delta(\alpha) \leq \delta(\beta) \Rightarrow \delta(t[\alpha]) \leq \delta(t[\beta]) \\
\forall t[\alpha] \in \Delta, \delta(\alpha)=0 \Rightarrow \delta(t[\alpha])=0 \\
\forall \mathrm{p}(\alpha) \in \Delta, \delta(\mathrm{p}(\alpha))=\delta(\alpha) \ominus 1 \\
\forall \mathrm{~s}(\alpha) \in \Delta, \delta(\mathrm{s}(\alpha))=\delta(\alpha) \oplus 1 \\
\forall t[\alpha]
\end{array}\right] \Delta, \delta(\operatorname{last}(t)) \leq \delta(\alpha) \Leftrightarrow \delta(t[\alpha])=\delta(t[\operatorname{last}(t)])\right] .
$$

Definition 21. A $\Delta$-prediagram $\delta$ is called logically-sound if

$$
\begin{align*}
& \forall i d[\alpha] \in \Delta, \delta(i d[\alpha])=\delta(\alpha)  \tag{7}\\
& \forall \perp[\alpha] \in \Delta, \delta(\operatorname{last}(\perp))=0  \tag{8}\\
& \forall t u[\alpha] \in \Delta, \delta(t u[\alpha])=\delta(t[u[\alpha]])  \tag{9}\\
& \forall t u[\operatorname{last}(t u)] \in \Delta, \delta(\operatorname{last}(t u))=\omega \Rightarrow \delta(\operatorname{last}(t))=\delta(\operatorname{last}(u))=\omega \tag{10}
\end{align*}
$$

Definition 22. A $\Delta$-prediagram $\delta$ is called o-sound if

$$
\begin{align*}
\forall t^{\circ}[\alpha] & \in \Delta, \delta\left(t^{\circ}[\alpha]\right)=0 \text { or } \delta\left(t^{\circ}[\alpha]\right)=\omega  \tag{11}\\
\forall t^{\circ}[\alpha] & \in \Delta, \delta(\alpha)<\omega \Rightarrow\left(\delta\left(t^{\circ}[\alpha]\right)=\omega \Leftrightarrow \delta(t[\alpha])=\omega\right)  \tag{12}\\
\forall \operatorname{last}\left(t^{\circ}\right) & \in \Delta, \delta\left(\operatorname{last}\left(t^{\circ}\right)\right)<\omega  \tag{13}\\
\forall t[\alpha], t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right] & \in \Delta,\left(\delta\left(t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right]\right)<\omega \text { and } \delta(\alpha)<\omega\right) \Rightarrow \delta(t[\alpha])<\omega . \tag{14}
\end{align*}
$$

Definition 23. A $\Delta$-prediagram $\delta$ is called $r$-sound if

$$
\begin{align*}
& \forall t\left[t^{r}[\alpha]\right] \in \Delta, \delta\left(t\left[t^{r}[\alpha]\right]\right) \leq \delta(\alpha)  \tag{15}\\
& \forall t^{r}[\alpha] \in \Delta, \quad\left(0<\delta(\alpha)<\omega \text { and } \delta\left(t^{r}[\alpha]\right)<\omega\right) \Rightarrow \delta(\alpha)<\delta\left(t\left[\mathbf{s}\left(t^{r}[\alpha]\right)\right]\right)  \tag{16}\\
& \forall t^{r}\left[\operatorname{last}\left(t^{r}\right)\right] \in \Delta, \delta\left(\operatorname{last}\left(t^{r}\right)\right)=\omega \Rightarrow \delta(\operatorname{last}(t))=\omega  \tag{17}\\
& \forall t^{r}\left[\operatorname{last}\left(t^{r}\right)\right] \in \Delta, \delta\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)<\omega \Rightarrow \delta\left(t\left[\mathbf{s}\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)\right]\right)=\omega . \tag{18}
\end{align*}
$$

Definition 24. A $\Delta$-prediagram $\delta$ is called $\ell$-sound if

$$
\begin{align*}
& \forall t\left[t^{\ell}[\alpha]\right] \in \Delta, \delta\left(t^{\ell}[\alpha]\right)<\omega \Rightarrow \delta(\alpha) \leq \delta\left(t\left[t^{\ell}[\alpha]\right]\right)  \tag{19}\\
& \forall t^{\ell}[\alpha] \in \Delta, \quad\left(0<\delta(\alpha)<\omega \text { and } \delta\left(t^{\ell}[\alpha]\right)<\omega\right) \Rightarrow \delta\left(t\left[\mathrm{p}\left(t^{\ell}[\alpha]\right)\right]\right)<\delta(\alpha)  \tag{20}\\
& \forall t\left[t^{\ell}[\alpha]\right] \in \Delta, \quad\left(\delta(\alpha)<\omega \text { and } \delta\left(t^{\ell}[\alpha]\right)=\omega\right) \Rightarrow \delta\left(t\left[t^{\ell}[\alpha]\right]\right)<\delta(\alpha)  \tag{21}\\
& \forall t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right] \in \Delta, \delta\left(\operatorname{last}\left(t^{\ell}\right)\right)=\omega \Rightarrow \delta(\operatorname{last}(t))=\omega  \tag{22}\\
& \forall t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right] \in \Delta, \delta\left(t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right)<\omega \Rightarrow \delta\left(t\left[t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right]\right)=\omega \text {. } \tag{23}
\end{align*}
$$

Definition 25. A $\Delta$-prediagram $\delta$ is called a $\Delta$-diagram if it is structurally sound, logically sound, $o$-sound, $\ell$-sound, and $r$-sound.

It follows from the next proposition that any counterexample to the validity of an equation in $\mathbf{W}$ restricts to a finite diagram witnessing this failure. More precisely, if $\mathbf{W} \not \vDash i d \leq t_{1} \vee \cdots \vee t_{n}$, where each $t_{i}$ is a basic term, and $\Delta$ is the saturation of the sample set $\left\{t_{1}[\kappa], \ldots, t_{n}[\kappa]\right\}$, then there exists a $\Delta$-diagram $\delta$ satisfying $\delta(\kappa)>\delta\left(t_{i}[\kappa]\right)$ for each $i \in\{1, \ldots, n\}$.
Proposition 26. Let $T$ be a set of basic terms, $\kappa$ a time variable, and $\Delta$ the saturation of the sample set $\{t[\kappa] \mid t \in T\}$. Then for any valuation $\theta$ and $p \in \omega^{+}$, there exists a $\Delta$-diagram $\delta$ such that $\delta(\kappa)=p$ and $\delta(t[\kappa])=\llbracket t \rrbracket_{\theta}(p)$ for all $t \in T$.

Proof. We define the map $\delta: \Delta \rightarrow \omega^{+}$recursively by

$$
\begin{aligned}
\delta(\kappa) & :=p \\
\forall t[\alpha] \in \Delta, \quad \delta(t[\alpha]) & :=\llbracket t \rrbracket_{\theta}(\delta(\alpha)) \\
\forall \operatorname{last}(t) \in \Delta, \delta(\operatorname{last}(t)) & :=\operatorname{last}\left(\llbracket t \rrbracket_{\theta}\right) \\
\forall \mathrm{p}(\alpha) \in \Delta, \quad \delta(\mathrm{p}(\alpha)) & :=\delta(\alpha) \ominus 1 \\
\forall \mathrm{~s}(\alpha) \in \Delta, \quad \delta(\mathrm{s}(\alpha)) & :=\delta(\alpha) \oplus 1 .
\end{aligned}
$$

The map $\delta$ is well-defined since $\alpha \in \Delta$ if, and only if, there exist samples $\alpha_{1}, \ldots, \alpha_{n}$ such that $\alpha_{1}=t[\kappa]$ for some $t \in T, \alpha_{n}=\alpha$, and $\alpha_{j} \rightsquigarrow \alpha_{j+1}$ for
each $j \in\{1, \ldots, n-1\}$. So $\delta$ is a $\Delta$-prediagram. A proof that $\delta$ is a $\Delta$-diagrami.e., that $\delta$ satisfies conditions (1) to (23) - may be found in Appendix A.2.

We now turn our attention to proving that every $\Delta$-diagram $\delta$ extends to a valuation $\theta$ satisfying $\llbracket t \rrbracket_{\theta}(\delta(\alpha))=\delta(t[\alpha])$ for all $t[\alpha] \in \Delta$. First, we use $\delta$ to define a partial sup-preserving function $\lfloor t\rfloor_{\delta}$ for each basic term $t$.

Definition 27. For any $\Delta$-diagram $\delta$ and basic term $t$, let

$$
\lfloor t\rfloor_{\delta}:=\{(\delta(\alpha), \delta(t[\alpha])) \mid t[\alpha] \in \Delta\}
$$

A time warp $f$ extends $\lfloor t\rfloor_{\delta}$ if $f(i)=j$ for all $(i, j) \in\lfloor t\rfloor_{\delta}$, and strongly extends $\lfloor t\rfloor_{\delta}$ if also

$$
\text { either }\lfloor t\rfloor_{\delta}=\emptyset \text { or }\left(\lfloor t\rfloor_{\delta} \neq \emptyset \text { and } \delta(\operatorname{last}(t))=\omega \Longrightarrow \operatorname{last}(f)=\omega\right)
$$

Lemma 28. There exists an effective procedure that produces for any finite $\Delta$ diagram $\delta$ and term variable $x$, an algorithmic description of a time warp $f$ that strongly extends $\lfloor x\rfloor_{\delta}$.

Proof. If $\lfloor x\rfloor_{\delta}=\emptyset$, then any time warp strongly extends it, so assume $\lfloor x\rfloor_{\delta} \neq \emptyset$. By (1), $\lfloor x\rfloor_{\delta}$ can be considered as a partial map from $\omega^{+}$to $\omega^{+}$. Moreover, since $\Delta$ is saturated, and, by (5), $\delta(x[\operatorname{last}(x)]) \geq j$ for all $(i, j) \in\lfloor x\rfloor_{\delta}$, we have $(\delta(\operatorname{last}(x)), \delta(x[\operatorname{last}(x)])) \in\lfloor x\rfloor_{\delta}$.

Let $X:=\lfloor x\rfloor_{\delta} \cup\{(0,0),(\omega, \delta(x[\operatorname{last}(x)]))\}$. This is still a partial map by (2) and (5). For each $i \in \omega$, there exists a unique pair $\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right) \in X$ such that $i_{1} \leq i<i_{2}$ and there is no $\left(i_{3}, j_{3}\right) \in X$ with $i_{1}<i_{3}<i_{2}$, and we define

$$
f(i):=\min \left(j_{2}, j_{1} \oplus\left(i-i_{1}\right)\right),
$$

where $n \oplus m:=\min \{\omega, n+m\}$. Let also $f(\omega):=\delta(x[\operatorname{last}(x)]))$.
Clearly $f$ is monotonic. It extends $\lfloor x\rfloor_{\delta}$, since $i=i_{1}<\omega$ implies $f\left(i_{1}\right)=$ $\min \left(j_{2}, j_{1}\right)=j_{1}$. In particular, $f(0)=0$. To confirm that $f$ is a time warp, it remains to show that $f(\omega)=\bigvee\{f(i) \mid i \in \omega\}$. If $\delta(x[\operatorname{last}(x)])=f(\omega)<\omega$, then, by (6), $\delta(\operatorname{last}(x))<\omega$ and, by monotonicity, $f(i)=f(\omega)$ for each $i \geq \delta(\operatorname{last}(x))$ and $f(\omega)=f(\delta(\operatorname{last}(x)))=\bigvee\{f(i) \mid i \in \omega\}$. If $f(\omega)=\omega$, then for each $j \in \omega$, there exists an $i \in \omega$ such that $f(i)>j$, and hence $\bigvee_{i<\omega} f(i)=\omega=f(\omega)$.

Finally, suppose that $\delta(\operatorname{last}(x))=\omega$. Then (6) yields $(\omega, \omega) \in\lfloor x\rfloor_{\delta}$ and for any $(i, j) \in\lfloor x\rfloor_{\delta}$, if $i \in \omega$, then also $j \in \omega$. Hence, $\operatorname{last}(f)=\omega$, by the definition of $f$. So $f$ strongly extends $\lfloor x\rfloor_{\delta}$.

Lemma 29. For every basic term $t$, valuation $\theta$, and $\Delta$-diagram $\delta$, if $\theta(x)$ strongly extends $\lfloor x\rfloor_{\delta}$ for every term variable $x$, then $\llbracket t \rrbracket_{\theta}$ strongly extends $\lfloor t\rfloor_{\delta}$.

Proof. By induction on $t$. The case $t=x$ is immediate and the other cases follow by a series of lemmas proved in Appendix A.3, and the induction hypothesis.

The next proposition is then a direct consequence of Lemmas 28 and 29.

Proposition 30. There is an effective procedure that produces for any finite $\Delta$-diagram $\delta$, an algorithmic description of a valuation $\theta$ satisfying $\llbracket t \rrbracket_{\theta}(\delta(\alpha))=$ $\delta(t[\alpha])$ for all $t[\alpha] \in \Delta$.

We are now ready to establish the main theorem of this section.
Theorem 31. Let $t_{1}, \ldots, t_{n}$ be basic terms, $\kappa$ a time variable, and $\Delta$ the saturation of the sample set $\left\{t_{1}[\kappa], \ldots, t_{n}[\kappa]\right\}$. Then $\mathbf{W} \not \vDash i d \leq t_{1} \vee \cdots \vee t_{n}$ if, and only if, there exists a $\Delta$-diagram $\delta$ such that $\delta(\kappa)>\delta\left(t_{i}[\kappa]\right)$ for all $i \in\{1, \ldots, n\}$.

Proof. Suppose first that $\mathbf{W} \not \vDash i d \leq t_{1} \vee \cdots \vee t_{n}$. Then there exist a valuation $\theta$ and $p \in \omega^{+}$such that $p=i d(p)>\llbracket t_{i} \rrbracket_{\theta}(p)$ for all $i \in\{1, \ldots, n\}$. Hence, by Proposition 26, there exists a $\Delta$-diagram $\delta$ such that $\delta(\kappa)=p>\llbracket t_{i} \rrbracket_{\theta}(p)=$ $\delta\left(t_{i}[\kappa]\right)$ for all $i \in\{1, \ldots, n\}$.

Now suppose that there exists a $\Delta$-diagram $\delta$ such that $\delta(\kappa)>\delta\left(t_{i}[\kappa]\right)$ for all $i \in\{1, \ldots, n\}$. Then, by Proposition 30, there exists a valuation $\theta$ such that $\llbracket t_{i} \rrbracket_{\theta}(\delta(\kappa))=\delta\left(t_{i}[\kappa]\right)$ for all $i \in\{1, \ldots, n\}$. So $i d(\delta(\kappa))=\delta(\kappa)>\llbracket t_{i} \rrbracket_{\theta}(\delta(\kappa))$ for all $i \in\{1, \ldots, n\}$. Hence $\mathbf{W} \not \vDash i d \leq t_{1} \vee \cdots \vee t_{n}$.

## 4 Decidability via Logic

Let $t_{1}, \ldots, t_{n}$ be basic terms, $\kappa$ a time variable, and $\Delta$ the saturation of the sample set $\left\{t_{1}[\kappa], \ldots, t_{n}[\kappa]\right\}$. Our aim in this section is to express the existence of a $\Delta$-diagram witnessing $\mathbf{W} \not \vDash i d \leq t_{1} \vee \ldots \vee t_{n}$, as stated in Theorem 31, via an existential sentence over the natural numbers with the ordering and successor relations. Since the first-order theory of this structure is decidable, it follows that the equational theory of $\mathbf{W}$ is decidable, concluding the proof of Theorem 1.

Note that in the logic encoding, we will no longer allow $\omega$ as a value for the variables. The theoretical reason why this is possible is that the ordinal $\omega^{+}$admits a first-order (even quantifier-free) interpretation in $\omega$. However, we will avoid relying upon such model-theoretic generalities here and just give the necessary concrete definitions.

Our construction of a first-order formula $\phi$ encoding the existence of a $\Delta$ diagram uses the samples in $\Delta$ as variables and proceeds in two steps:

1. We define a formula $\psi$ with variables in $\Delta$, intended to be interpreted in $\omega^{+}$, using the order relation symbol $\preceq$, the successor relation symbol $\mathcal{S}$, and two further unary relation symbols $\mathcal{O}$ and $\mathcal{I}$, where the intended interpretations of $\mathcal{O}(x)$ and $\mathcal{I}(x)$ are " $x=\omega$ " and " $x=0$ ", respectively.
2. We obtain $\phi$ by eliminating the symbols $\mathcal{O}$ and $\mathcal{I}$ from $\psi$ and re-interpreting $\preceq$ and $\mathcal{S}$ using an encoding of $\omega^{+}$in the structure $(\mathbb{N}, \leq, S, 0)$.

Let $\tau$ be the relational first-order signature with two binary relation symbols $\preceq$ and $\mathcal{S}$, and two unary relation symbols $\mathcal{O}$ and $\mathcal{I}$. We consider $\omega^{+}$as a $\tau$ structure by defining $\preceq^{\omega^{+}}$to be the natural ordering of $\omega^{+}, \mathcal{S}^{\omega^{+}}:=\{(n, n+1) \mid$ $n \in \omega\} \cup\{(\omega, \omega)\}, \mathcal{I}^{\omega^{+}}:=\{0\}$, and $\mathcal{O}^{\omega^{+}}:=\{\omega\}$. Note that a $\Delta$-prediagram is a valuation of the variables in $\Delta$ in this structure.

We define $\psi$ by translating the defining properties of being a $\Delta$-diagram into quantifier-free formulas of first-order logic in the signature $\tau$ with variables from $\Delta$. In the following definition, the symbols $\lambda$ and $\curlyvee$ denote the logical connectives 'and' and 'or', respectively, and the notation $a \prec b$ is shorthand for $a \preceq b \curlywedge \neg(b \preceq a)$. Note also that $\psi$ is well-defined, since $\Delta$ is finite by Lemma 17 .

Definition 32. Let $\psi$ be the first-order quantifier-free $\tau$-formula

$$
\curlywedge(\text { struct } \cup \log \cup \text { bounds } \cup \text { right } \cup \text { left } \cup \text { fail }),
$$

where the first five sets, corresponding to Definitions 20-24 in the definition of a diagram, and fail, expressing the failure of $i d \leq t_{1} \vee \ldots \vee t_{n}$ in $\mathbf{W}$ at the time variable $\kappa$, are defined as follows:

$$
\begin{aligned}
& \text { struct }:=\{\alpha \preceq \beta \Rightarrow t[\alpha] \preceq t[\beta] \mid t[\alpha], t[\beta] \in \Delta\} \cup \\
& \{\mathcal{I}(\alpha) \Rightarrow \mathcal{I}(t[\alpha]) \mid t[\alpha] \in \Delta\} \cup \\
& \{\mathcal{S}(\mathrm{p}(\alpha), \alpha) \curlyvee(\mathcal{I}(\mathrm{p}(\alpha)) \curlywedge \mathcal{I}(\alpha)) \mid \mathrm{p}(\alpha) \in \Delta\} \cup \\
& \{\mathcal{S}(\alpha, \mathbf{s}(\alpha)) \mid \mathbf{s}(\alpha) \in \Delta\} \cup \\
& \{\operatorname{last}(t) \preceq \alpha \Leftrightarrow t[\alpha]=t[\operatorname{last}[t]] \mid t[\alpha] \in \Delta\} \cup \\
& \{\mathcal{O}(\operatorname{last}(t)) \Rightarrow \mathcal{O}(t[\operatorname{last}(t)]) \mid t[\operatorname{last}(t)] \in \Delta\} \\
& \log :=\{i d[\alpha]=\alpha \mid i d[\alpha] \in \Delta\} \cup \\
& \{\mathcal{I}(\operatorname{last}(\perp)) \mid \perp[\alpha] \in \Delta\} \cup \\
& \{t u[\alpha]=t[u[\alpha]] \mid t u[\alpha] \in \Delta\} \cup \\
& \{\mathcal{O}(\operatorname{last}(t u)) \Rightarrow(\mathcal{O}(\operatorname{last}(t)) \curlywedge \mathcal{O}(\operatorname{last}(u))) \mid t u[\operatorname{last}(t u)] \in \Delta\} \\
& \text { bounds }:=\left\{\mathcal{I}\left(t^{\circ}[\alpha]\right) \curlyvee \mathcal{O}\left(t^{\circ}[\alpha]\right) \mid t^{\circ}[\alpha] \in \Delta\right\} \cup \\
& \left\{\neg \mathcal{O}(\alpha) \Rightarrow\left(\mathcal{O}\left(t^{\circ}[\alpha]\right) \Leftrightarrow \mathcal{O}(t[\alpha])\right) \mid t^{\circ}[\alpha] \in \Delta\right\} \cup \\
& \left\{\neg \mathcal{O}\left(\operatorname{last}\left(t^{\circ}\right)\right) \mid \operatorname{last}\left(t^{\circ}\right) \in \Delta\right\} \cup \\
& \left\{\left(\neg \mathcal{O}\left(t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right]\right) \curlywedge \neg \mathcal{O}(\alpha)\right) \Rightarrow \neg \mathcal{O}(t[\alpha]) \mid t[\alpha], t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right] \in \Delta\right\} \\
& \text { right }:=\left\{t\left[t^{r}[\alpha]\right] \preceq \alpha \mid t\left[t^{r}[\alpha]\right] \in \Delta\right\} \cup \\
& \left\{\left(\neg \mathcal{I}(\alpha) \curlywedge \neg \mathcal{O}(\alpha) \curlywedge \neg \mathcal{O}\left(t^{r}[\alpha]\right) \Rightarrow \alpha \prec t\left[\mathbf{s}\left(t^{r}[\alpha]\right)\right] \mid t\left[\mathbf{s}\left(t^{r}[\alpha]\right)\right] \in \Delta\right\} \cup\right. \\
& \left\{\mathcal{O}\left(\operatorname{last}\left(t^{r}\right)\right) \Rightarrow \mathcal{O}(\operatorname{last}(t)) \mid t^{r}\left[\operatorname{last}\left(t^{r}\right)\right] \in \Delta\right\} \cup \\
& \left\{\neg \mathcal{O}\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right) \Rightarrow \mathcal{O}\left(t\left[\mathbf{s}\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)\right]\right) \mid t\left[\mathbf{s}\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)\right] \in \Delta\right\} \\
& \text { left }:=\left\{\neg \mathcal{O}\left(t^{\ell}[\alpha]\right) \Rightarrow \alpha \preceq t\left[t^{\ell}[\alpha]\right] \mid t\left[t^{\ell}[\alpha]\right] \in \Delta\right\} \cup \\
& \left\{\left(\neg \mathcal{I}(\alpha) \curlywedge \neg \mathcal{O}(\alpha) \curlywedge \neg \mathcal{O}\left(t^{\ell}[\alpha]\right)\right) \Rightarrow t\left[\mathrm{p}\left(t^{\ell}[\alpha]\right)\right] \prec \alpha \mid t\left[\mathrm{p}\left(t^{\ell}[\alpha]\right)\right] \in \Delta\right\} \cup \\
& \left\{\left(\neg \mathcal{O}(\alpha) \curlywedge \mathcal{O}\left(t^{\ell}[\alpha]\right)\right) \Rightarrow t\left[t^{\ell}[\alpha]\right] \prec \alpha \mid t\left[t^{\ell}[\alpha]\right] \in \Delta\right\} \cup \\
& \left\{\mathcal{O}\left(\operatorname{last}\left(t^{\ell}\right)\right) \Rightarrow \mathcal{O}(\operatorname{last}(t)) \mid t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right] \in \Delta\right\} \cup \\
& \left\{\neg \mathcal{O}\left(\left\{t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right\} \Rightarrow \mathcal{O}\left(t\left[t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right]\right) \mid t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right] \in \Delta\right\}\right. \\
& \text { fail }:=\left\{t_{i}[\kappa] \prec \kappa \mid 1 \leq i \leq n\right\} \text {. }
\end{aligned}
$$

The next lemma then follows directly from the definition of a $\Delta$-diagram.
Lemma 33. Let $\delta: \Delta \rightarrow \omega^{+}$be a $\Delta$-prediagram. Then $\omega^{+}, \delta \models \psi$ if, and only if, $\delta$ is a $\Delta$-diagram such that $\delta\left(t_{i}[\kappa]\right)<\delta(\kappa)$ for each $i \in\{1, \ldots, n\}$.

Theorem 31 and Lemma 33 together show that $\mathbf{W} \not \vDash i d \leq t_{1} \vee \ldots \vee t_{n}$ if, and only, if $\psi$ is satisfiable in $\omega^{+}$. We could therefore conclude the proof of Theorem 1 at this point by appealing to classical decidability results on the first-order theory of ordinals [16]. Instead, however, we show explicitly how to interpret the $\tau$-structure $\omega^{+}$inside the standard model ( $\mathbb{N}, \leq, S, 0$ ), which is more commonly available in satisfiability solvers.

Consider the first-order signature $\sigma$ with two binary relation symbols $\leq$ and $S$, and one constant symbol 0 , and let $\mathbb{N}$ denote the $\sigma$-structure based on the natural numbers, where $\leq^{\mathbb{N}}$ is the usual order, $S^{\mathbb{N}}:=\{(n, n+1) \mid n \in \mathbb{N}\}$, and $0^{\mathbb{N}}:=0$. The following definition and lemma contain the crucial observation needed for encoding $\tau$-formulas over $\omega^{+}$into $\sigma$-formulas over $\mathbb{N} .^{5}$

Definition 34. Define the bijection $\iota: \mathbb{N} \rightarrow \omega^{+}$by $\iota(0):=\omega$, and $\iota(n):=n-1$ for each $n \in \omega \backslash\{0\}$.

For any valuation $w: \Delta \rightarrow \mathbb{N}$, let $\hat{w}: \Delta \rightarrow \omega^{+}$denote the function defined by $\hat{w}(x):=\iota(w(x))$. Note that the map $w \mapsto \hat{w}$ is a bijection between $\mathbb{N}^{\Delta}$ and $\left(\omega^{+}\right)^{\Delta}$, since $\iota$ is a bijection.

Lemma 35. Let $\chi$ be a quantifier-free $\tau$-formula. Define $\chi^{\prime}$ to be the quantifierfree $\sigma$-formula obtained from $\chi$ by making the following symbolic substitutions for every occurrence of an atomic formula in $\chi$ :
(i) $\mathcal{O}(x)$ is replaced by $x=0$
(ii) $\mathcal{I}(x)$ is replaced by $S(0, x)$
(iii) $\mathcal{S}(x, y)$ is replaced by $(x=0 \curlywedge y=0) \curlyvee(\neg(x=0) \curlywedge S(x, y))$
(iv) $x \preceq y$ is replaced by $y=0 \curlyvee(\neg(x=0) \curlywedge x \leq y)$.

Then, for any valuation $w: \Delta \rightarrow \mathbb{N}, \mathbb{N}, w \models \chi^{\prime}$ if, and only if, $\omega^{+}, \hat{w} \models \chi$.
Proof. By induction on the complexity of $\chi$. The induction step is immediate, and the atomic cases essentially follow from the definitions; we just show the proof for $x \preceq y$ as an example. For any valuation $w$, we have $\omega^{+}, \hat{w} \models x \preceq y$ if, and only if, $\hat{w}(y)=\omega$ or $(\hat{w}(x) \neq \omega$ and $\hat{w}(x) \leq \hat{w}(y))$ in $\omega^{+}$. Using the definition of $\hat{w}$, this is equivalent to $w(y)=0$ or $(w(x) \neq 0$ and $w(x) \leq w(y))$ in $\mathbb{N}$, that is, $\mathbb{N}, w \models y=0 \curlyvee(\neg(x=0) \curlywedge x \leq y)$.

Finally, we define our quantifier-free $\sigma$-formula $\phi$ encoding the non-validity of $i d \leq t_{1} \vee \cdots \vee t_{n}$ in $\mathbf{W}$.

Definition 36. Let $\phi:=\psi^{\prime}$, the $\sigma$-formula obtained from the $\tau$-formula $\psi$ (Definition 32) by performing the replacements in Lemma 35.

[^2]We are now ready to put everything together.
Theorem 37. The time warp equation id $\leq t_{1} \vee \cdots \vee t_{n}$ is valid in $\mathbf{W}$ if, and only if, the quantifier-free $\sigma$-formula $\phi$ is unsatisfiable in $\mathbb{N}$. Moreover, any valuation $w: \Delta \rightarrow \mathbb{N}$ such that $\mathbb{N}, w \models \phi$ effectively yields a valuation $\theta$ of the time warp variables occurring in $t_{1} \vee \cdots \vee t_{n}$ such that $\mathbf{W}, \theta \models i d \not \leq t_{1} \vee \cdots \vee t_{n}$.
Proof. By Theorem 31, the equation $i d \leq t_{1}, \vee \cdots \vee t_{n}$ is not valid in $\mathbf{W}$ if, and only if, there exists a $\Delta$-diagram $\delta$ such that $\delta\left(t_{i}[\kappa]\right)<\delta(\kappa)$ for all $i \in\{1, \ldots, n\}$. By Lemma 33 , the latter is equivalent to the existence of a valuation $v: \Delta \rightarrow \omega^{+}$ such that $\omega^{+}, v \models \psi$. By Lemma 35 , the latter is in turn equivalent to the existence of a valuation $w: \Delta \rightarrow \mathbb{N}$ such that $\mathbb{N}, w \models \phi$.

For the second claim, we retrace our steps. If $w: \Delta \rightarrow \mathbb{N}$ is a valuation such that $\mathbb{N}, w \models \phi$, define the function $\delta: \Delta \rightarrow \omega^{+}$by $\delta(\alpha):=\iota(w(\alpha))$ for $\alpha \in \Delta$. By Lemma $33, \delta$ is a $\Delta$-diagram such that $\delta\left(t_{i}[\kappa]\right)<\delta(\kappa)$ for each $i \in\{1, \ldots, n\}$. By Proposition $30, \delta$ effectively yields a valuation $\theta$ that falsfies $i d \leq t_{1} \vee \cdots \vee t_{n}$.

Theorem 1 follows now directly from Theorem 37 and the decidability of the first-order theory of $\mathbb{N}$ (see, e.g., [16]).

Concluding remark. The proof of Theorem 37, together with the normal form results of Section 2, provides a decision procedure for the equational theory of the time warp algebra, as explained in Section 1. We are currently in the process of implementing this decision procedure in a software tool. This tool is written in the OCaml functional programming language [17] and uses the Z3 theorem prover [20] to decide the satisfiability of the final logic formula. Our experiments with a preliminary implementation for basic time warp terms have been encouraging so far, and we hope to integrate a full version in a compiler for graded modalities. From a complexity perspective, the most challenging issue here is to deal with the potentially very large saturated sample sets and corresponding logic formulas produced by time warp equations. We therefore intend to consider encodings of the decision problem for time warps using alternative, possibly more efficient, data structures such as-following a helpful suggestion of one of the referees of this paper-arrays (see [4]) that are also supported by the Z 3 theorem prover.

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## A Appendix

## A. 1 Proof of Lemma 17

Definition A.1. The sample $\gamma_{\alpha}^{\beta}$ is defined inductively for samples $\alpha, \beta, \gamma$ by

$$
\begin{aligned}
\kappa_{\alpha}^{\beta}: & :\left\{\begin{array}{ll}
\beta & \text { if } \kappa=\alpha \\
\kappa & \text { otherwise; }
\end{array} \quad \operatorname{last}(t)_{\alpha}^{\beta}:= \begin{cases}\beta & \text { if last }(t)=\alpha \\
\operatorname{last}(t) & \text { otherwise } ;\end{cases} \right. \\
t[\gamma]_{\alpha}^{\beta}: & :\left\{\begin{array}{ll}
\beta & \text { if } t[\gamma]=\alpha \\
t\left[\gamma_{\alpha}^{\beta}\right] & \text { otherwise; }
\end{array} \quad \mathbf{q}(\gamma)_{\alpha}^{\beta}:=\left\{\begin{array}{ll}
\beta & \text { if } \mathbf{q}(\gamma)=\alpha \\
\mathbf{q}\left(\gamma_{\alpha}^{\beta}\right) & \text { otherwise }
\end{array} \quad \text { for } \mathbf{q} \in\{\mathbf{s}, \mathbf{p}\} .\right.\right.
\end{aligned}
$$

Note that $\left(\gamma_{\alpha}^{\beta}\right)_{\beta}^{\alpha}=\gamma$.
Definition A.2. For samples $\alpha, \beta_{1}, \ldots, \beta_{k}$, let $\mu\left(\beta_{1}, \ldots, \beta_{k}\right):=\left|\left\{\beta_{1}, \ldots, \beta_{k}\right\}^{\sim}\right|$ and $\mu_{\alpha}\left(\beta_{1}, \ldots, \beta_{k}\right):=\left|M_{\alpha}\left(\beta_{1}, \ldots, \beta_{k}\right)\right|$, where $M_{\alpha}\left(\beta_{1}, \ldots, \beta_{k}\right)$ denotes the set of samples $\beta \in\left\{\beta_{1}, \ldots, \beta_{k}\right\}^{\rightsquigarrow}$ such that whenever $\alpha_{1} \rightsquigarrow \cdots \rightsquigarrow \alpha_{n}$ with $\alpha_{1}=\beta_{j}$ and $\alpha_{n}=\beta$, there exists an $i \in\{1, \ldots, n\}$ such that $\alpha_{i}=\alpha$.

Note that clearly $\mu\left(\alpha_{1}, \ldots, \alpha_{k}\right) \leq \mu\left(\alpha_{1}\right)+\ldots+\mu\left(\alpha_{k}\right)$.
Lemma A.3. For any basic term $t, \mathrm{q} \in\{\mathrm{s}, \mathrm{p}\}$, samples $\alpha, \gamma_{1}, \gamma_{2}$, and time variable $\kappa$,

$$
\begin{aligned}
\mu\left(t[\alpha], t[\mathbf{q}(\alpha)], \gamma_{1}, \gamma_{2}\right) & \leq \mu\left(t[\kappa], t[\mathbf{q}(\kappa)], \gamma_{1}, \gamma_{2}\right)+\mu_{\alpha}\left(t[\alpha], t[\mathbf{q}(\alpha)], \gamma_{1}, \gamma_{2}\right) \\
\mu(t[\alpha]) & \leq \mu(t[\kappa])+\mu(\alpha) .
\end{aligned}
$$

In particular, $\mu(t[\operatorname{last}(u)]) \leq \mu(t[\kappa])$ for any basic term $u$.
Proof. If $\alpha=\operatorname{last}(u)$ for some basic term $u$, then clearly even the inequality $\mu\left(t[\alpha], t[\mathbf{q}(\alpha)], \gamma_{1}, \gamma_{2}\right) \leq \mu\left(t[\kappa], t[\mathbf{q}(\kappa)], \gamma_{1}, \gamma_{2}\right)$ holds. Suppose that $\alpha \neq \operatorname{last}(u)$. Let $A:=\{t[\kappa], t[\boldsymbol{q}(\kappa)]\} \cup\left\{\gamma_{1}, \gamma_{2}\right\}^{\aleph} \cup M_{\alpha}\left(t[\kappa], t[\mathbf{q}(\kappa)], \gamma_{1}, \gamma_{2}\right)$, where we assume for convenience of notation that these unions are disjoint. Define the function $K$ from $A$ to the set of all samples by

$$
K(\beta):= \begin{cases}\beta & \text { if } \beta \in M_{\alpha}\left(t[\kappa], t[\mathbf{q}(\kappa)], \gamma_{1}, \gamma_{2}\right) \cup\left\{\gamma_{1}, \gamma_{2}\right\}^{\leadsto} \\ \beta_{\kappa}^{\alpha} & \text { if } \beta \in\{t[\kappa], t[\mathbf{q}(\kappa)]\}^{\leadsto} .\end{cases}
$$

It suffices to show that $\left\{t[\alpha], t[q(\alpha)], \gamma_{1}, \gamma_{2}\right\}^{\leadsto}$ is contained in the image of $K$. Let $\beta \in\left\{t[\alpha], t[\mathbf{q}(\alpha)], \gamma_{1}, \gamma_{2}\right\}^{\leadsto}$. If $\beta \in\left\{\gamma_{1}, \gamma_{2}\right\}^{\leadsto}$, then clearly $\beta$ is in the image of $K$. So we may assume that $\beta \in\{t[\alpha], t[q(\alpha)]\}^{\leadsto \downarrow} \backslash\left\{\gamma_{1}, \gamma_{2}\right\}^{\leadsto}$. Then either there exist $\alpha_{1}, \ldots, \alpha_{n}$ with $\alpha_{1} \in\{t[\alpha], t[q(\alpha)]\}, \alpha_{n}=\beta$, and $\alpha_{1} \rightsquigarrow \ldots \rightsquigarrow \alpha_{n}$ such that $\alpha_{i} \neq \alpha$ for all $i \in\{1, \ldots, n\}$, or not. If not, then $\beta \in M_{\alpha}\left(t[\kappa], t[\mathbf{q}(\kappa)], \gamma_{1}, \gamma_{2}\right)$, i.e., $\beta=K(\beta)$ is in the image of $K$. Otherwise we want to show that $\alpha_{1}^{\kappa} \rightsquigarrow$ $\ldots \rightsquigarrow \alpha_{n}^{\kappa}$. Then, since $\alpha_{1}^{\kappa}=\{t[\kappa], t[\mathbf{q}(\kappa)]\}$, we have $\beta_{\alpha}^{\kappa} \in\{t[\kappa], t[\mathbf{q}(\kappa)]\}^{\rightsquigarrow}$ and $\beta=K\left(\beta_{\alpha}^{\kappa}\right)$ is in the image of $K$. We prove the claim by induction on $n$. If $n=1$, then there is nothing to prove. Suppose that the claim is proved for $n$ and we have $\alpha_{1} \in\{t[\alpha], t[\mathbf{q}(\alpha)]\}, \alpha_{n+1}=\beta$, and $\alpha_{1} \rightsquigarrow \ldots \rightsquigarrow \alpha_{n} \rightsquigarrow \alpha_{n+1}$. By the induction hypothesis we get $\alpha_{1}^{\kappa} \rightsquigarrow \ldots \rightsquigarrow \alpha_{n}^{\kappa}$. Since $\alpha_{n} \neq \alpha, \alpha_{n+1} \neq \alpha$ and $\alpha \neq \operatorname{last}(u)$ for any basic term $u$, it is clear from the saturation conditions that also $\alpha_{n}^{\kappa} \rightsquigarrow \alpha_{n+1}^{\kappa}$. For the second inequality the proof is analogous.

Proof of Lemma 17. It suffices to prove that the saturation of $\{\alpha\}$ is finite for any sample $\alpha$, i.e., that $\mu(\alpha)$ is finite. Clearly, $\mu(\mathrm{s}(\alpha)) \leq 1+\mu(\alpha)$ and $\mu(\mathrm{p}(\alpha)) \leq 1+\mu(\alpha)$. So, by Lemma A.3, it suffices to prove that $\mu(t[\kappa])$ is finite for every term $t$ and time variable $\kappa$, proceeding by induction on $t$. If $t \in \mathscr{T}_{V} \cup\{i d, \perp\}$, then $\{t[\kappa]\}^{\rightsquigarrow}=\{t[\kappa], \kappa, t[\operatorname{last}(t)]$, last $(t)\}$, so $\mu(t[\kappa])=4$.

If $t=a_{1} \cdots a_{n}$, where $a_{1}, \ldots, a_{n}$ are terms that are not products, then by the saturation conditions,

$$
\left.\{t[\kappa]\}^{\leadsto}=\{t[\kappa], t[\operatorname{last}(t)])\right\} \cup \bigcup_{i=1}^{n} \bigcup_{\alpha \in\{\kappa, \operatorname{last}(t)\}}\left\{a_{1} \cdots a_{i}\left[a_{i+1} \cdots a_{n}[\alpha]\right]\right\}^{\leadsto} .
$$

So, by Lemma A.3,

$$
\mu(t[\kappa]) \leq 2+2\left(\sum_{i=1}^{n} \mu\left(a_{1} \cdots a_{i}[\kappa]\right)+\mu\left(a_{i+1} \cdots a_{n}[\kappa]\right)\right)
$$

and, by the induction hypothesis, the right-hand-side is finite.
If $t=u^{\circ}$, then, by the saturation conditions,

$$
\{t[\kappa]\}^{\rightsquigarrow}=\{t[\kappa], t[\operatorname{last}(t)]\} \cup\{u[\kappa]\}^{\rightsquigarrow} \cup\{u[\operatorname{last}(t)]\}^{\rightsquigarrow} .
$$

So we get $\mu(t[\kappa]) \leq 2+2 \mu(u[\kappa])$ and, by the induction hypothesis, the right-hand-side is finite.

If $t=u^{r}$, then clearly

$$
\mu(t[\kappa])=\mu(u[\mathbf{s}(t[\kappa])], u[t[\kappa]], u[\mathbf{s}(t[\operatorname{last}(t)])], u[t[\operatorname{last}(t)]])
$$

and, by applying Lemma A. 3 for $\alpha=t[\kappa], \gamma_{1}=u[\mathbf{s}(t[\operatorname{last}(t)])]$, and $\gamma_{2}=$ $u[t[\operatorname{last}(t)]]$,

$$
\mu(t[\kappa]) \leq \mu\left(u[\mathbf{s}(\kappa)], u[\kappa], \gamma_{1}, \gamma_{2}\right)+\mu_{\alpha}\left(u[\mathbf{s}(\alpha)], u[\alpha], \gamma_{1}, \gamma_{2}\right)
$$

But applying Lemma A. 3 again for $\alpha^{\prime}=t[\operatorname{last}(t)]$ with $\gamma_{1}^{\prime}=u[\mathbf{s}(\kappa)], \gamma_{2}^{\prime}=u[\kappa]$, and a new time variable $\kappa^{\prime}$,

$$
\begin{aligned}
\mu\left(u[\mathbf{s}(\kappa)], u[\kappa], \gamma_{1}, \gamma_{2}\right) & \leq \mu\left(u\left[\mathbf{s}\left(\kappa^{\prime}\right)\right], u\left[\kappa^{\prime}\right], \gamma_{1}^{\prime}, \gamma_{2}^{\prime}\right)+\mu_{\alpha^{\prime}}\left(u\left[\mathbf{s}\left(\alpha^{\prime}\right)\right], u\left[\alpha^{\prime}\right], \gamma_{1}^{\prime}, \gamma_{2}^{\prime}\right) \\
& \leq 2 \mu(u[\mathbf{s}(\kappa)])+2 \mu(u[\kappa])+\mu_{\alpha^{\prime}}\left(u\left[\mathbf{s}\left(\alpha^{\prime}\right)\right], u\left[\alpha^{\prime}\right], \gamma_{1}^{\prime}, \gamma_{2}^{\prime}\right)
\end{aligned}
$$

In summary,

$$
\mu(t[\kappa]) \leq 2 \mu(u[\mathbf{s}(\kappa)])+2 \mu(u[\kappa])+\mu_{\alpha^{\prime}}\left(u\left[\mathbf{s}\left(\alpha^{\prime}\right)\right], u\left[\alpha^{\prime}\right], \gamma_{1}^{\prime}, \gamma_{2}^{\prime}\right)+\mu_{\alpha}\left(u[\mathbf{s}(\alpha)], u[\alpha], \gamma_{1}, \gamma_{2}\right)
$$

By the induction hypothesis, the sum $2 \mu(u[\mathrm{~s}(\kappa)])+2 \mu(u[\kappa])$ is finite. But also

$$
\begin{aligned}
M_{t[\kappa]}(u[\mathbf{s}(t[\kappa])], u[t[\kappa]], u[\mathbf{s}(t[\operatorname{last}(t)])], u[t[\operatorname{last}(t)]]) & =\{t[\kappa], \kappa\} \\
M_{t[\operatorname{last}(t)]}(u[\mathbf{s}(t[\operatorname{last}(t)])], u[t[\operatorname{last}(t)]], u[\mathbf{s}(\kappa)], u[\kappa]) & =\{t[\operatorname{last}(t)], \operatorname{last}(t)\}
\end{aligned}
$$

So $\mu(t[\kappa])$ is finite.
The case where $t=u^{\ell}$ is analogous to the case where $t=u^{r}$.
Note that this proof yields a rough upper-bound $\mu(t[\kappa]) \leq(6 \cdot c(t))^{c(t)}$, where $c(t)$ is the complexity of the term $t$.

## A. 2 Proof of Proposition 26

To conclude the proof of Proposition 26, it remains to prove that $\delta$ is a diagram, i.e., that $\delta$ satisfies conditions (1)-(23). For convenience, we assume without further mention that all samples used are in $\Delta$, and write $\llbracket t \rrbracket$ for $\llbracket t \rrbracket_{\theta}$.
(1) If $\delta(\alpha) \leq \delta(\beta)$, then, by the definition of $\delta$ and the fact that time warps are monotonic, $\delta(t[\alpha])=\llbracket t \rrbracket(\delta(\alpha)) \leq \llbracket t \rrbracket(\delta(\beta))=\delta(t[\beta])$.
(2) If $\delta(\alpha)=0$, then $\delta(t[\alpha])=\llbracket t \rrbracket(\delta(\alpha))=0$.
(3) By the definition of $\delta$.
(4) By the definition of $\delta$.
(5) By the definition of $\delta$,

$$
\delta(\operatorname{last}(t))=\operatorname{last}(\llbracket t \rrbracket)=\min \left\{n \in \omega^{+} \mid \llbracket t \rrbracket(n)=\llbracket t \rrbracket(\omega)\right\} .
$$

So clearly, for each $k \in \omega^{+}$,

$$
\operatorname{last}(\llbracket t \rrbracket) \leq k \Longleftrightarrow \llbracket t \rrbracket(\operatorname{last}(\llbracket t \rrbracket))=\llbracket t \rrbracket(k)
$$

Hence, for all $t[\alpha] \in \Delta$,

$$
\delta(\operatorname{last}(t)) \leq \delta(\alpha) \Longleftrightarrow \delta(t[\operatorname{last}(t)])=\delta(t[\alpha])
$$

(6) If last $(\llbracket t \rrbracket)=\delta(\operatorname{last}(t))=\omega$, then $\llbracket t \rrbracket(n)<\llbracket t \rrbracket(\omega)$ for all $n<\omega$, and $\delta(t[\operatorname{last}(t)])=\llbracket t \rrbracket(\omega)=\bigvee_{n<\omega} \llbracket t \rrbracket(n)=\omega$.
(7) $\delta(i d[\alpha])=\llbracket i d \rrbracket(\delta(\alpha))=\delta(\alpha)$.
(8) $\delta(\operatorname{last}(\perp))=\operatorname{last}(\llbracket \perp \rrbracket)=0$.
(9) $\delta(t u[\alpha])=\llbracket t u \rrbracket(\delta(\alpha))=\llbracket t \rrbracket(\llbracket u \rrbracket(\delta(\alpha)))=\delta(t[u[\alpha] \rrbracket)$.
(10) If $\delta(\operatorname{last}(t u))=\omega$, then last $(\llbracket t \rrbracket \llbracket u \rrbracket)=\omega$ and, by Lemma 12, $\delta(\operatorname{last}(t))=$ $\operatorname{last}(\llbracket t \rrbracket)=\omega$ and $\delta(\operatorname{last}(u))=\operatorname{last}(\llbracket u \rrbracket)=\omega$.
(11) By the definition of $\delta$, we have $\delta\left(t^{\circ}[\alpha]\right)=\llbracket t \rrbracket^{\circ}(\delta(\alpha))$. Moreover, by Lemma 6 , we have $\delta\left(t^{\circ}[\alpha]\right)=\llbracket t \rrbracket^{\circ}(\delta(\alpha))=0$ or $\delta\left(t^{\circ}[\alpha]\right)=\llbracket t \rrbracket^{\circ}(\delta(\alpha))=\omega$.
(12) If $\delta(\alpha)<\omega$, then, by Lemma 6 ,

$$
\delta\left(t^{\circ}[\alpha]\right)=\llbracket t \rrbracket^{\circ}(\delta(\alpha))=\omega \Longleftrightarrow \delta(t[\alpha])=\llbracket t \rrbracket(\delta(\alpha))=\omega
$$

(13) $\delta\left(\operatorname{last}\left(t^{\circ}\right)\right)=\operatorname{last}\left(\llbracket t \rrbracket^{\circ}\right)<\omega$, by Lemma 6.
(14) Suppose that $\llbracket t \rrbracket^{\circ}\left(\operatorname{last}\left(\llbracket t \rrbracket^{0}\right)\right)=\delta\left(t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right]\right)<\omega$. Then, since $\llbracket t \rrbracket^{\circ}\left(\operatorname{last}\left(\llbracket t \rrbracket^{0}\right)\right)=$ $\llbracket t \rrbracket^{\circ}(\omega)$, by Lemma 6 , we get $\llbracket t \rrbracket(k)<\omega$ for all $k<\omega$. So in particular for all $\delta(\alpha)<\omega$, we have $\delta(t[\alpha])=\llbracket t \rrbracket(\delta(\alpha))<\omega$.
(15) $\delta\left(t\left[t^{r}[\alpha]\right]\right)=\llbracket t \rrbracket\left(\llbracket t \rrbracket^{r}(\delta(\alpha))\right) \leq \delta(\alpha)$, by Lemma 7 .
(16) If $0<\delta(\alpha)<\omega$ and $\llbracket t \rrbracket^{r}(\delta(\alpha))=\delta\left(t^{r}[\alpha\rfloor\right)<\omega$, then $\delta(\alpha)<\llbracket t \rrbracket\left(\llbracket t \rrbracket^{r}(\delta(\alpha))+\right.$ $1)=\delta\left(t\left[\mathbf{s}\left(t^{r}[\alpha]\right)\right]\right)$, by Lemma 7.
(17) If last $\left(\llbracket t \rrbracket^{r}\right)=\delta\left(\operatorname{last}\left(t^{r}\right)\right)=\omega$, then $\delta(\operatorname{last}(t))=\operatorname{last}(\llbracket t \rrbracket)=\omega$, by Lemma 12 .
(18) If $\llbracket t \rrbracket^{r}\left(\operatorname{last}\left(\llbracket t \rrbracket^{r}\right)\right)=\delta\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)<\omega$, then $\llbracket t \rrbracket^{r}(\omega)=\llbracket t \rrbracket^{r}\left(\operatorname{last}\left(\llbracket t \rrbracket^{r}\right)\right)<\omega$ and $\delta\left(t\left[\mathbf{s}\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)\right]\right)=\llbracket t \rrbracket\left(\llbracket t \rrbracket^{r}(\omega)+1\right)=\omega$, by Lemma 7 .
(19) If $\llbracket t \rrbracket^{\ell}(\delta(\alpha))=\delta\left(t^{\ell}[\alpha]\right)<\omega$, then either $\delta(\alpha)=0$ and $\delta\left(t\left[t^{\ell}[\alpha]\right]\right)=\llbracket t \rrbracket\left(\left[t \rrbracket^{\ell}(0)\right)=\right.$ 0 , or $0<\delta(\alpha)<\omega$ and $\delta(\alpha) \leq \llbracket t \rrbracket\left(\llbracket t \rrbracket^{\ell}(\delta(\alpha))\right)=\delta\left(t\left[t^{\ell}[\alpha]\right]\right)$, by Lemma 8 .
(20) If $0<\delta(\alpha)<\omega$ and $\llbracket t \rrbracket^{\ell}(\delta(\alpha))=\delta\left(t^{\ell}[\alpha]\right)<\omega$, then $\delta\left(t\left[p\left(t^{\ell}[\alpha]\right)\right]\right)=$ $\llbracket t \rrbracket\left(\llbracket t \rrbracket^{\ell}(\delta(\alpha))-1\right)<\delta(\alpha)$, by Lemma 8 .
(21) If $\delta(\alpha)<\omega$ and $\llbracket t \rrbracket^{\ell}(\delta(\alpha))=\delta\left(t^{\ell}[\alpha]\right)=\omega$, then $\delta(\alpha)>0$ and $\delta\left(t\left[t^{\ell}[\alpha]\right]=\right.$ $\llbracket t \rrbracket(\omega)<\delta(\alpha)$, by Lemma 8.
(22) If last $\left(\llbracket t \rrbracket^{\ell}\right)=\delta\left(\operatorname{last}\left(t^{\ell}\right)\right)=\omega$, then $\delta(\operatorname{last}(t))=\operatorname{last}(\llbracket t \rrbracket)=\omega$, by Lemma 12 .
(23) If $\llbracket t \rrbracket^{\ell}\left(\operatorname{last}\left(\llbracket t \rrbracket^{\ell}\right)\right)=\delta\left(t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right)<\omega$, then $\llbracket t \rrbracket^{\ell}(\omega)=\llbracket t \rrbracket^{\ell}\left(\operatorname{last}\left(\llbracket t \rrbracket^{\ell}\right)\right)<\omega$ and $\delta\left(t\left[t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right]\right)=\llbracket t \rrbracket\left(\llbracket t \rrbracket^{\ell}(\omega)\right)=\omega$, by Lemma 8 .

## A. 3 Proof of Lemma 29

Recall that the proof of Lemma 29 proceeds by induction on $t$ and that the case $t=x$ follows by assumption. The other cases are direct consequences of the following lemmas and the induction hypothesis.

Lemma A.4. If $f_{1}$ strongly extends $\left\lfloor t_{1}\right\rfloor_{\delta}$ and $f_{2}$ strongly extends $\left\lfloor t_{2}\right\rfloor_{\delta}$, then $f_{1} f_{2}$ strongly extends $\left\lfloor t_{1} t_{2}\right\rfloor_{\delta}$.

Proof. Suppose that $f_{1}$ strongly extends $\left\lfloor t_{1}\right\rfloor_{\delta}$ and $f_{2}$ strongly extends $\left\lfloor t_{2}\right\rfloor_{\delta}$. Then for all $t_{1} t_{2}[\alpha] \in \Delta$,

$$
\begin{array}{rlr}
f_{1} f_{2}(\delta(\alpha)) & =f_{1}\left(f_{2}(\delta(\alpha))\right) & \\
& =f_{1}\left(\delta\left(t_{2}[\alpha]\right)\right) & \\
& =\delta\left(t_{1}\left[t_{2}[\alpha]\right]\right) & \\
& =\delta\left(t_{1} t_{2}[\alpha]\right) & \\
\text { (since definition) } \left.f_{2} \text { extends }\left\lfloor t_{2}\right\rfloor_{\delta}\right) \\
\text { (since } \left.f_{1} \text { extends }\left\lfloor t_{1}\right\rfloor_{\delta}\right) \\
\text { (by (9)). }
\end{array}
$$

So $f_{1} f_{2}$ extends $\left\lfloor t_{1} t_{2}\right\rfloor_{\delta}$, and it remains to show that the extension is strong. We can assume that $\left\lfloor t_{1} t_{2}\right\rfloor_{\delta}$ is non-empty, since otherwise there is nothing to prove. Suppose that $\delta\left(\operatorname{last}\left(t_{1} t_{2}\right)\right)=\omega$. Then $\delta\left(\operatorname{last}\left(t_{1}\right)\right)=\delta\left(\operatorname{last}\left(t_{2}\right)\right)=\omega$, by (10), and, since $f_{1}$ and $f_{2}$ strongly extend $\left\lfloor t_{1}\right\rfloor_{\delta}$ and $\left\lfloor t_{2}\right\rfloor_{\delta}$, respectively, also last $\left(f_{1}\right)=$ $\operatorname{last}\left(f_{2}\right)=\omega$. Hence last $\left(f_{1} f_{2}\right)=\omega$, by Lemma 12 .

Lemma A.5. If $f$ strongly extends $\lfloor t\rfloor_{\delta}$, then $f^{\circ}$ strongly extends $\left\lfloor t^{\circ}\right\rfloor_{\delta}$.
Proof. Suppose that $f$ strongly extends $\lfloor t\rfloor_{\delta}$ and consider any $t^{\circ}[\alpha] \in \Delta$. We prove that $f^{\circ}(\delta(\alpha))=\delta\left(t^{\circ}[\alpha]\right)$. Suppose first that $\delta(\alpha)<\omega$. We reason by cases for $\delta\left(t^{\circ}[\alpha]\right)$.
(i) $\delta\left(t^{\circ}[\alpha]\right)=\omega$. Then $\delta(t[\alpha])=\omega$, by (12), and, since $f$ extends $\lfloor t\rfloor_{\delta}$, also $f(\delta(\alpha))=$ $\omega$. Hence $f^{\circ}(\delta(\alpha))=\omega$, by Lemma 6 .
(ii) $\delta\left(t^{\circ}[\alpha]\right)<\omega$. Then $\delta\left(t^{\circ}[\alpha]\right)=0$, by (11), and hence $\delta(t[\alpha])<\omega$, by (12). Since $f$ extends $\lfloor t\rfloor_{\delta}$, also $f(\delta(\alpha))<\omega$. Hence $f^{\circ}(\delta(\alpha))=0$, by Lemma 6 .

Now suppose that $\delta(\alpha)=\omega$. Then $\delta\left(t^{\circ}[\alpha]\right)=\delta\left(t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right]\right)$, by (5), and $\delta\left(\operatorname{last}\left(t^{\circ}\right)\right)<$ $\omega$, by (13). As in the previous cases, $f^{\circ}\left(\delta\left(\operatorname{last}\left(t^{\circ}\right)\right)\right)=\delta\left(t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right]\right)$, recalling that by $(11)$, either $\delta\left(t^{\circ}[\alpha]\right)=\omega$ or $\delta\left(t^{\circ}[\alpha]\right)=0$.

1. $\delta\left(t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right]\right)=\delta\left(t^{\circ}[\alpha]\right)=\omega$. Then $f^{\circ}\left(\delta\left(\operatorname{last}\left(t^{\circ}\right)\right)\right)=\omega$, and $f^{\circ}(\omega)=\omega$.
2. $\delta\left(t^{\circ}\left[\operatorname{last}\left(t^{\circ}\right)\right]\right)=\delta\left(t^{\circ}[\alpha]\right)=0$. Then there are two cases. If $\delta(t[\alpha])<\omega$, then, since $f$ extends $\lfloor t\rfloor_{\delta}$, we have $f(\omega)<\omega$ and $f^{\circ}(\omega)=0$, by Lemma 6 . Otherwise, $\delta(t[\alpha])=\omega$. In this case, $\delta(t[\beta])<\omega$ for all $\delta(\beta)<\omega$ with $t[\beta] \in \Delta$, by (14), so $\delta(\operatorname{last}(t))=\omega$. Hence, since $f$ strongly extends $\lfloor t\rfloor_{\delta}$, we have last $(f)=\omega$ and $f^{\circ}(\omega)=0$, by Lemma 6 .

That $f^{\circ}$ strongly extends $\left\lfloor t^{\circ}\right\rfloor_{\delta}$ is clear, since $\delta\left(\right.$ last $\left.\left(t^{\circ}\right)\right)<\omega$, by (13).
Lemma A.6. If $f$ strongly extends $\lfloor t\rfloor_{\delta}$, then $f^{r}$ strongly extends $\left\lfloor t^{r}\right\rfloor_{\delta}$.
Proof. Let $t^{r}[\alpha] \in \Delta$. Note first that, by (2), if $\delta(\alpha)=0$, then $\delta\left(t^{r}[\alpha]\right)=0=$ $f^{r}(0)$. Hence assume that $\delta(\alpha)>0$. Suppose first that $\delta(\alpha)<\omega$. We reason by cases for $\delta\left(t^{r}[\alpha]\right)$.

1. $\delta\left(t^{r}[\alpha]\right)<\omega$. Then $\delta\left(t\left[t^{r}[\alpha]\right]\right) \leq \delta(\alpha)<\delta\left(t\left[\mathbf{s}\left(t^{r}[\alpha]\right)\right]\right)$, by (15) and (16). Since $f$ extends $\lfloor t]_{\delta}$, we have $f\left(\delta\left(t^{r}[\alpha]\right)\right)=\delta\left(t\left[t^{r}[\alpha]\right]\right) \leq \delta(\alpha)<\delta\left(t\left[s\left(t^{r}[\alpha]\right)\right]\right)=$ $f\left(\delta\left(\mathrm{~s}\left(t^{r}[\alpha]\right)\right)\right)$, and, by (4), also $f\left(\delta\left(\mathrm{~s}\left(t^{r}[\alpha]\right)\right)\right)=f\left(\delta\left(t^{\mathrm{r}}[\alpha]\right)+1\right)$. So $\delta\left(t^{\mathrm{r}}[\alpha]\right)=$ $f^{r}(\delta(\alpha))$, by Lemma 7.
2. $\delta\left(t^{r}[\alpha]\right)=\omega$. Then, since $\delta\left(t\left[t^{r}[\alpha]\right]\right) \leq \delta(\alpha)$, by (15), and $f$ extends $\lfloor t\rfloor_{\delta}$, we have $f(\omega)=f\left(\delta\left(t^{r}[\alpha]\right)\right) \leq \delta(\alpha)<\omega$. Hence $f^{r}(\delta(\alpha))=\omega$, by Lemma 7 .

Now suppose that $\delta(\alpha)=\omega$ and hence $\delta\left(t^{r}[\alpha]\right)=\delta\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)$. We reason by cases for $\delta\left(t^{r}[\alpha]\right)$.

1. $\delta\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)=\delta\left(t^{r}[\alpha]\right)<\omega$. Then $\delta\left(\operatorname{last}\left(t^{r}\right)\right)<\omega$ and $\delta\left(t\left[s\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)\right]\right)=$ $\omega$, by (6) and (18). So, using the previous cases, (4), and the fact that $f$ extends $\lfloor t\rfloor_{\delta}$, we get $f^{r}\left(\delta\left(\operatorname{last}\left(t^{r}\right)\right)\right)=\delta\left(t^{r}[\alpha]\right)$ and $f\left(\delta\left(t^{r}[\alpha]\right)+1\right)=\omega$. Hence $f^{r}(\delta(\alpha))=\delta\left(t^{r}[\alpha]\right)$, by Lemma 7 .
2. $\delta\left(t^{r}\left[\operatorname{last}\left(t^{r}\right)\right]\right)=\delta\left(t^{r}[\alpha]\right)=\omega$. Then there are two cases. If $\delta\left(\operatorname{last}\left(t^{r}\right)\right)<\omega$, then, using the previous cases, $f^{r}\left(\delta\left(\operatorname{last}\left(t^{r}\right)\right)\right)=\omega$, and hence $f^{r}(\omega)=\omega$ by Lemma 7. Otherwise $\delta\left(\operatorname{last}\left(t^{r}\right)\right)=\omega$. Then $\delta(\operatorname{last}(t))=\omega$, by (17), and since $f$ strongly extends $\lfloor t\rfloor_{\delta}$, we have last $(f)=\omega$. Hence $f^{r}(\omega)=\omega$, by Lemma 7 .

It remains to show that the extension is strong. Again we can assume that $\left\lfloor t^{r}\right\rfloor_{\delta}$ is non-empty. Suppose that $\delta\left(\operatorname{last}\left(t^{r}\right)\right)=\omega$. Then $\delta(\operatorname{last}(t))=\omega$, by (17), and, since $f$ strongly extends $\lfloor t\rfloor_{\delta}$, also last $(f)=\omega$. Hence last $\left(f^{r}\right)=\omega$, by Lemma 12 .

Lemma A.7. If $f$ strongly extends $\lfloor t\rfloor_{\delta}$, then $f^{\ell}$ strongly extends $\left\lfloor t^{\ell}\right\rfloor_{\delta}$.
Proof. Let $t^{\ell}[\alpha] \in \Delta$. Note first that, by (2), if $\delta(\alpha)=0$, then $\delta\left(t^{\ell}[\alpha]\right)=0=$ $f^{\ell}(0)$. Hence assume that $\delta(\alpha)>0$. Suppose first that $\delta(\alpha)<\omega$. We reason by cases for $\delta\left(t^{\ell}[\alpha]\right)$.

1. $\delta\left(t^{\ell}[\alpha]\right)<\omega$. Then $\delta\left(t\left[\mathrm{p}\left(t^{\ell}[\alpha]\right)\right]\right)<\delta(\alpha) \leq \delta\left(t\left[t^{\ell}[\alpha]\right]\right)$, by (19) and (20). Since $f$ extends $\lfloor t\rfloor_{\delta}$, we have $f\left(\delta\left(\mathrm{p}\left(t^{\ell}[\alpha]\right)\right)\right)=\delta\left(t\left[\mathrm{p}\left(t^{\ell}[\alpha]\right)\right]\right)<\delta(\alpha) \leq \delta\left(t\left[t^{\ell}[\alpha]\right]\right)=$ $f\left(\delta\left(t^{\ell}[\alpha]\right)\right)$. But also $f\left(\delta\left(\mathrm{p}\left(t^{\ell}[\alpha]\right)\right)\right)=f\left(\delta\left(t^{\ell}[\alpha]\right)-1\right)$, by (3), noting that $0<\delta\left(t^{\ell}[\alpha]\right)$, since $f(0)=0<\delta(\alpha)$. Hence $\delta\left(t^{\ell}[\alpha]\right)=f^{\ell}(\delta(\alpha))$, by Lemma 8 .
2. $\delta\left(t^{\ell}[\alpha]\right)=\omega$. Then $\delta\left(t\left[t^{\ell}[\alpha]\right]\right)<\delta(\alpha)$, by (21), and, since $f$ extends $\lfloor t\rfloor_{\delta}$, also $f(\omega)=f\left(\delta\left(t^{\ell}[\alpha]\right)\right)<\delta(\alpha)$. Hence $f^{\ell}(\delta(\alpha))=\omega$, by Lemma 8 .

Suppose now that $\delta(\alpha)=\omega$ and hence $\delta\left(t^{\ell}[\alpha]\right)=\delta\left(t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right)$. We reason by cases on $\delta\left(t^{\ell}[\alpha]\right)$.

1. $\delta\left(t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right)=\delta\left(t^{\ell}[\alpha]\right)<\omega$. Then $\delta\left(\operatorname{last}\left(t^{\ell}\right)\right)<\omega$ and $\delta\left(t\left[t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right]\right)=\omega$, by (6) and (23). So, by the previous cases and the fact that $f$ extends $\lfloor t\rfloor_{\delta}$, we have $f^{\ell}\left(\delta\left(\operatorname{last}\left(t^{\ell}\right)\right)\right)=\delta\left(t^{\ell}[\alpha]\right)$ and $f\left(\delta\left(t^{\ell}[\alpha]\right)\right)=\omega$. Hence $f^{\ell}(\delta(\alpha))=$ $\delta\left(t^{\ell}[\alpha]\right)$, by Lemma 8 .
2. $\delta\left(t^{\ell}\left[\operatorname{last}\left(t^{\ell}\right)\right]\right)=\delta\left(t^{\ell}[\alpha]\right)=\omega$. Then there are two cases. If $\delta\left(\operatorname{last}\left(t^{\ell}\right)\right)<\omega$, then by the previous cases, $f^{\ell}\left(\delta\left(\operatorname{last}\left(t^{\ell}\right)\right)\right)=\omega$, and $f^{\ell}(\omega)=\omega$ by Lemma 8 . If $\delta\left(\operatorname{last}\left(t^{\ell}\right)\right)=\omega$, then $\delta(\operatorname{last}(t))=\omega$, by (22), and, since $f$ strongly extends $\lfloor t\rfloor_{\delta}$, also last $(f)=\omega$ and $f^{\ell}(\omega)=\omega$, by Lemma 8 .

It remains to show that the extension is strong. Again we can assume that $\left\lfloor t^{\ell}\right\rfloor_{\delta}$ is non-empty. Suppose that $\delta\left(\operatorname{last}\left(t^{\ell}\right)\right)=\omega$. Then, by (22), we get $\delta(\operatorname{last}(t))=\omega$. So, since $f$ strongly extends $\lfloor t\rfloor_{\delta}$, also last $(f)=\omega$. Hence $\operatorname{last}\left(f^{\ell}\right)=\omega$, by Lemma 12.

Lemma A.8. The time warp id strongly extends $\lfloor i d\rfloor_{\delta}$.
Proof. The extension property follows from (7); the fact that it is strong follows from the fact that last $(i d)=\omega$.

Lemma A.9. The time warp $\perp$ strongly extends $\lfloor\perp\rfloor_{\delta}$.
Proof. The extension property follows from (5) and (8); the fact that it is strong is immediate by (8).


[^0]:    * Supported by Swiss National Science Foundation grant 200021_165850.

[^1]:    ${ }^{3}$ The name 'diagram' recalls a similar concept used to prove the decidability of the equational theory of lattice-ordered groups in [14].
    ${ }^{4}$ Indeed, the equational theory of the time warp algebra without residuals coincides with the equational theory of distributive lattice-ordered monoids [6], but an elegant (finite) axiomatization of the equational theory in the full language is not known.

[^2]:    ${ }^{5}$ We thank Thomas Colcombet for suggesting this idea.

