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Skew metrics valued in Sugihara semigroups

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Abstract. We consider skew metrics (equivalently, transitive relations that are tournaments, linear orderings) valued in Sugihara semigroups on autodual chains. We prove that, for odd chains and chains without a unit, skew metrics classify certain tree-like structures that we call perfect augmented plane towers. When the chain is finite and has cardinality $2K + 1$, skew metrics on a set X give rise to perfect rooted plane trees of height K whose frontier is a linear preorder of X.
Any linear ordering on X gives rise to an ordering on the set of its skew metrics valued in an arbitrary involutive residuated lattice Q. If Q satisfies the mix rule, then this poset is most often a lattice. We study this lattice for $X = \{1, \ldots, n\}$ and Q the Sugihara monoid on the chain of cardinality $2K + 1$. We give a combinatorial model of this lattice by describing its covers as moves on a space of words coding perfect augmented plane trees. Using the combinatorial model, we develop enumerative considerations on this lattice.

1 Introduction

Linear orders and trees are fundamental structures in Computer Science and Mathematics. We might consider linear orders using some object of truth values different from the classical two-element Boolean algebra. The theory of linear orders in an intuitionistic setting intrinsically suffers from the lack of a well-behaved negation; a striking consequence of this is the existence of different types of intuitionistic ordinals [11,24]. However, when the object of truth values is an involutive residuated lattice or a Girard quantale, negation is again fully operative, generalized linear orders are easily axiomatized, and a rich theory can be developed, capable to generalize non trivial results on classical linear orders.

For such object of truth values, linear orders can be equivalently defined either as some kind of metric valued in the quantale where the symmetry property of the distance is replaced by skewness—that is, we require $\delta(y, x) = \delta(x, y)^*$ where $(-)^*$ is the negation—or as transitive relations on the quantale obeying this law, analogous to the requirement that a binary relation is a tournament.

As part of a general investigation of these objects, see [9,20], we investigate in this paper skew metrics valued in Sugihara monoids and, more generally, in Sugihara semigroups. Research on Sugihara monoids can be traced back to [5] and constitutes nowadays a quite active domain, see e.g. [7,6]. More importantly,

Sugihara semigroups arise as the unique idempotent involutive residuated lattice structure that can be given to an autodual chain. Linear orders on the Sugihara chain with three elements—that is, linear preorders or pseudo-permutations—have already been investigated [14,2], partly motivated from complexity issues related to the representation of temporal reasoning [25]. The importance of linear preorders in relation with the combinatorics and geometry of Coxeter groups and hyperplane arrangements was remarked already in [2] and has been once more emphasized in [4]. For us, it is the canonicity of Sugihara semigroups and the use of these structures in combinatorics that motivates further investigations of linear orders on arbitrary Sugihara semigroups.

We focus in this paper on autodual chains C for which either the positive cone C^+ has no least element, or satisfying $C^+ \cap C^- \neq \emptyset$, C^- being the negative cone. We claim that the cases left can be easily studied from the present research. We show that skew metrics on a set X valued in the Sugihara semigroup on C are in bijection with some tree-like objects that we call augmented perfect towers and can be neatly described as functors from the poset C^+ to the category of linearly ordered sets with few additional properties and structure. In particular all the maps involved in such a functor T are surjective and, moreover, a cone from X to T (in the category of sets) is given, reflecting the fact that leaves are labeled by subsets of X.

Once this correspondence is established, we further study the case where $X = \{ 1, \ldots, n \}$ and $C = \{ -K, \ldots, -1, 0, 1, \ldots, K \}$, in which case augmented towers are indeed rooted plane trees, that are perfect (meaning that each branch has equal length) of height $K + 1$ and leaves are labelled by subsets of X, so the frontier forms an ordered partition of X. When X is linearly ordered, skew metrics defined on X can be ordered and most often this ordering yields a lattice. For such choice of X and C, we describe the poset of skew metrics and, by representing trees as words, we determine covers of this poset as moves or rewrites on these words.

Relying on the combinatorial description of skew metrics as words, we give enumerative results on these combinatorial objects and these posets such as determining the size and the length.

Let us mention that the combinatorial model obtained is unavoidably close to the one of [14,2]. In these works the combinatorial model is given and the algebra of Sugihara monoids is mostly used for proving the lattice property of pseudo-permutations. Here the flow has opposite direction: the algebraic framework is given, and the problem, solved, is to instantiate the algebra into the combinatorics.

In this work converge our previous research on the lattice structures that arise from linear orders [21,22,9,20] and research on the algebraic structures of logic that are in use in combinatorics, see e.g. [18,19]. W.r.t. the first line of research, a main advance here is the recognition of the primary role of the notion of skew metric or linear order on an involutive residuated lattice, compared to closed/open constructions, and its framing in a relational setting. W.r.t. the second line of research, we identify via [8] a connection of Sugihara monoids to

the combinatorics of hyperplane arrangements, thus witnessing once more the value of algebraic structures of logic in the realm of combinatorics.

The paper is structured as follows. In Section 2 we recall the definition of involutive residuated lattices, of Sugihara semigroups, state their canonicity and few properties needed in the rest of the paper. In Section 3 we develop and exemplify the elementary theory of skew metrics. In Section 4 we characterize skew metrics as perfect augmented towers. In Section 5 we recall results on the ordering on the set of skew metrics valued in an arbitrary involutive residuated lattice. In Section 6 we give a combinatorial model of this poset in the case $X = \{1, \dots, n\}$ and Q is the Sugihara monoid on a chain of size $2K + 1$. We conclude in Section 7 with enumerative results concerning these posets.

2 Sugihara semigroups on autodual chains

In this section we recall elementary facts on Sugihara semigroups, which are involutive residuated lattices on autodual chains. We take the view that residuated lattices might not have units, as indeed it will be the case for many autodual chains that we consider.

Definition 1. *An* involutive residuated lattice *is a structure* $\langle L, \wedge, \vee, \otimes, (-)^* \rangle$ *such that* $\langle L, \wedge, \vee \rangle$ *is a lattice,* $\otimes$ *is a semigroup operation on* L *compatible with the order,* $(-)^*$ *is an antitone involution such that*

$$x \otimes y \leq z \quad \textit{iff} \quad y \otimes z^* \leq x^* \quad \textit{iff} \quad z^* \otimes x \leq y^* . \tag{1}$$

We call the relations in (1) the shift relations. Let us define

$$x \oplus y := (y^* \otimes x^*)^* , \qquad x \backslash y := x^* \oplus y , \qquad x / y := x \oplus y^* .$$

It is easy to see that the shift relations are equivalent to asking that the semigroup operation is residuated in both variables:

$$x \otimes y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z / y , \qquad \text{for each } x, y, z \in L.$$

By an *autodual chain* we mean a totally ordered set C coming with an antitone involution $(-)^* : C \to C$. For C such an autodual chain we define the absolute value as expected: $|x| := \max(x, x^*)$, for each $x \in C$. Notice then that we also have $|x|^* = \min(x, x^*)$. If we let

$$C^+ := \{ x \in C \mid x^* \leq x \} , \qquad C^- := \{ x \in C \mid x \leq x^* \} ,$$

then $C = C^+ \cup C^-$, and $C^+ \cap C^-$ is either empty, or it is the singleton containing the unique fixed point of $(-)^*$. On C we can define the following two operations:

$$x \otimes y := \begin{cases} x , & |y| < |x| , \\ y , & |x| < |y| , \\ \min(x, y) , & |x| = |y| , \end{cases} \qquad x \oplus y := \begin{cases} x , & |y| < |x| , \\ y , & |x| < |y| , \\ \max(x, y) , & |x| = |y| . \end{cases} \tag{2}$$

These operations are associative, commutative, idempotent, and dual. Moreover, they satisfy the mix rule, meaning that $x \otimes y \leq x \oplus y$, for each $x, y \in C$.

Example 2. On the chain $\{-1, 0, 1\}$ these two idempotent semigroup structures are as follows:

$\otimes$	-1	0	1
-1	-1	-1	-1
0	-1	0	1
1	-1	1	1

$\oplus$	-1	0	1
-1	-1	-1	1
0	-1	0	1
1	1	1	1

Proposition 3. *For any autodual chain C, the structure $\langle C, \min, \max, \otimes, (-)^* \rangle$ is an involutive residuated lattice.*

The next statement, possibly part of the folklore, witnesses the canonicity of this semigroup structure.

Proposition 4. *If C is an autodual chain, then there is exactly one idempotent semigroup operation $\otimes$ on C making $\langle C, \min, \max, \otimes, (-)^* \rangle$ into an involutive residuated lattice.*

This canonical semigroup structure is known as the *Sugihara monoid* on the autodual chain C, see e.g. [7]. Indeed, units are usually considered in involutive residuated lattices, so we characterise next when such a semigroup structure has a unit.

Lemma 5. *The semigroup structure $\otimes$ has a unit ι if and only if the set C^+ has a greatest lower bound $\bigwedge C^+$. In either case, we have $\iota = \bigwedge C^+$.*

As a consequence of the lemma, each auto-dual chain C has at most one unital idempotent involutive residuated lattice structure on it, and exactly one if C is a complete chain.

In the following, we let $\mathbf{3}$ be the chain $\{-1, 0, 1\}$, which we consider with its Sugihara semigroup structure. For C an autodual chain and $k \in C^+$, we define $\chi_k : C \to \mathbf{3}$ as follows:

$$\chi_k(x) := \begin{cases} 1\,, & k < x\,, \\ 0\,, & k^* \leq x \leq k\,, \\ -1\,, & x < k^*\,. \end{cases} \tag{3}$$

The map χ_k is monotone, thus a lattice homomorphism. Let us remark that χ_k is not a semigroup homomorphism since, for example, if x, y, k are such that $x^* < y < k^* \leq k < y^* < x$, then $\chi_k(x \otimes y) = \chi_k(x) = 1$, while $\chi_k(x) \otimes \chi_k(y) = 1 \otimes -1 = -1$. Yet, χ_k satisfies the two properties stated in the following proposition, relevant for the considerations to come.

Proposition 6. *For each $x, y \in C$, we have*

$$\chi_k(x^*) = \chi_k(x)^*\,, \qquad \chi_k(x) \otimes \chi_k(y) \leq \chi_k(x \otimes y)\,.$$

3 Skew metrics valued in an involutive residuated lattice

In this section, we let $Q = \langle Q, \wedge, \vee, \otimes, (-)^* \rangle$ be a fixed involutive residuated lattice. For a set X, we let $\Delta_X := \{ (x, x) \mid x \in X \}$.

Definition 7. *A* Q-relation *on* X *is a map* $f : X^2 \setminus \Delta_X \to Q$. *A* Q*-relation* f *is*

- transitive *if* $f(x, y) \otimes f(y, z) \leq f(x, z)$,
- cotransitive *if* $f(x, z) \leq f(x, y) \oplus f(y, z)$,
- skew *if* $f(x, y) = f(y, x)^*$,

for each pairwise distinct $x, y, z \in X$.

The reader might be surprised about our choice of the domain of a Q-relation. Indeed, we could have defined a Q-relation as a map $f : X^2 \to Q$ and, for example, said it is reflexive if $1 \leq f(x, x)$, so a reflexive and transitive Q-relation is nothing else than a category enriched over Q, see [13]. However, we shall insist on the last property, skewness.[1] If f is skew and also defined on Δ_X, then $f(x, x) = f(x, x)^*$, for each $x \in X$, that is, the duality coming from Q has at least one fixed point. Moreover, if we ask the relation $1 \leq f(x, x)$ to hold, then $1 \leq f(x, x) = f(x, x)^* \leq 1^*$. This leaves out many involutive residuated lattices that either do not have units or, for example, for which $1^* < 1$. Our choice is therefore dictated by the aim to consider the largest number of examples. On the other hand, if Q has a unit 1 such that $1^* = 1$, we can freely assume that f is defined on the entire X^2 with $f(x, x) = 1$, for each $x \in X$.

Lemma 8. *If a* Q*-relation is skew, then it is transitive if and only if it is cotransitive.*

Definition 9. *A* skew Q-metric *(or* skew metric*, if* Q *is understood) on* X *is a cotransitive skew* Q*-relation.*

By the previous lemma, a skew metric is transitive. We prefer the name skew metric (to cotransitive skew Q-relation), since the conditions

$$f(x, z) \leq f(x, y) \oplus f(y, z)\,, \qquad\qquad f(y, x) = f(x, y)^*\,,$$

satisfied by a skew metric suggest that f is a distance where symmetry of a distance is being replaced by skewness, see e.g. [15,12]. The next examples explain why skew metrics are generalized linear orders.

Example 10. Let $\mathbf{2}$ be the two element Boolean algebra. Skew $\mathbf{2}$-metrics on X bijectively correspond to (strict) linear orders on X. Indeed, consider a function $f : X^2 \setminus \Delta_X \to \mathbf{2}$ and define $R_f := \{ (x, y) \mid f(x, y) = 1 \}$. Then $f(x, y) \wedge f(y, z) \leq f(x, z)$ holds iff R_f is transitive, $f(x, y) \leq \neg f(y, x)$ holds iff R_f is antisymmetric (where $\neg$ stands for Boolean complement), and $f(x, y) \geq \neg f(y, x)$ holds iff R_f is total (or linear). Indeed, skew $\mathbf{2}$-relations correspond to tournaments.

[1] In [12] a property analogous to skewness is considered. In this work the star operation appearing in the relation $f(y, x) = f(x, y)^*$ is *monotone*.

Example 11. This example is the most relevant for the following. A linear preorder on a set X is a transitive relation R which is total: for each x, y, xRy or yRx. Let $\mathbf{3}$ be the Sugihara monoid on the chain $\{-1, 0, 1\}$. Skew $\mathbf{3}$-metrics on X bijectively correspond to linear preorders on X via the mapping sending $f : X^2 \setminus \Delta(X) \to \mathbf{3}$ to $R_f := \{(x, y) \mid f(x, y) \geq 0\}$. Again, $f(x, y) \otimes f(y, x) \leq f(x, z)$ yields transitivity of R_f, while $f(x, y) = f(y, x)^*$, that is, $f(x, y) + f(y, x) = 0$, yields totality of R_f. Let us remark that, in turn, linear preorders bijectively correspond to ordered partitions of the set X. We can directly define an ordered partition of X from a skew metric $f : X^2 \setminus \Delta_X \to \mathbf{3}$ as follows. Say that $x \sim_0 y$ if $x = y$ or $f(x, y) = 0$. Then $\sim_0$ is an equivalence relation, so the blocks of the partition are the equivalence classes of $\sim_0$. If $x \sim_0 x'$ and $y \sim_0 y'$, then $f(x', y') = f(x, y)$, as witnessed by the following computation: $f(x, y) = f(x', x) \otimes f(x, y) \otimes f(y, y') \leq f(x', y') \leq f(x', x) \oplus f(x, y) \oplus f(y, y') = f(x, y)$. That is, we can define f on the set of equivalence classes $X/\sim_0$ and then the map from the quotient $f : (X/\sim_0)^2 \setminus \Delta_{(X/\sim_0)} \to \mathbf{2}$ yields a total ordering on the blocks.

Example 12. It was shown in [20] that if X is finite, then skew metrics valued in the involutive residuated lattice of sup-preserving maps from the unit interval $[0, 1]$ bijectively correspond to images of continuous (in the topological sense) maps $[0, 1] \to [0, 1]^X$ that are isotone, and preserve the endpoints. Alternatively, they correspond to maximal chains in the cube $[0, 1]^X$.

The statements below, whose proofs are straightforward, illustrate the elementary algebra that can be developed around skew metrics.

Lemma 13. *Let $f : X^2 \setminus \Delta_X \to Q$ be a skew metric. If $g : Y \to X$ is injective, then $f \circ g : Y^2 \setminus \Delta_Y \to Q$ is also a skew metric. If Q is commutative, then $f^* : X^2 \setminus \Delta_X \to Q$, defined by $f^*(x, y) := f(y, x)$, is a skew metric.*

Definition 14. *A* monoidal map *from an involutive residuated lattice Q to an involutive residuated lattice Q' is a function $h : Q \to Q'$ such that $h(x^*) = h(x)^*$ and $h(x) \otimes h(y) \leq h(x \otimes y)$.*

Lemma 15. *If $f : X^2 \setminus \Delta_X \to Q$ is a skew metric and $h : Q \to Q'$ is a monoidal function, then $h \circ f : X^2 \setminus \Delta_X \to Q'$ is a skew metric.*

Remark 16. Let C be an autodual chain and $k \in C^+$. As we have seen in Proposition 6, the map $\chi_k : C \to \mathbf{3}$ defined in equation (3) is monoidal. According to Lemma 15, $\chi_k \circ f : X^2 \setminus \Delta_X \to \mathbf{3}$ is a skew metric, for each skew metric $f : X^2 \setminus \Delta_X \to C$.

4 Augmented plane towers

We characterise in this section the combinatorial objects arising from skew metrics valued in Sugihara semigroups that are either unitless or odd. Let us make these notions precise.

Definition 17. *An autodual chain C is* even *if $C^+ \cap C^- = \emptyset$ and, otherwise, it is odd. We say that C is* interesting *if either C is even and $\bigwedge C^+$ does not exist (so the idempotent semigroup structure on C is unitless) or C is odd.*

The reader can easily verify that a *finite* autodual chain is odd if and only if it has odd cardinality and, otherwise, it is even. In the following we fix an interesting autodual chain C. Observe that if C is finite, then it is odd. Let us use LinOrd to denote the category of linearly ordered sets and order preserving maps, and $U : \mathsf{LinOrd} \to \mathsf{Set}$ to denote the forgetful functor from this category to the category of sets and functions. For the next definition, recall that a poset can be regarded as a category whose objects are the elements of the poset and for which there is exactly one arrow between two elements x, y when $x \leq y$.

Definition 18. *A* plane tower *is a functor $T : C^+ \to \mathsf{LinOrd}$. For X any set, an* augmented plane tower *on X is a pair (τ, T) with T a plane tower and $\tau : X \to U \circ T$ a cone.*

We spell out what the definition means. A plane tower T is a pair of collections $\{ T_k \mid k \in C^+ \}$ and $\{ T_{j,k} \mid j, k \in C^+, j \leq k \}$. For each $k \in C^+$, T_k is a linearly ordered set; for $j, k \in C^+$ and $j \leq k$, $T_{j,k} : T_j \to T_k$ is an order preserving map. These data satisfy the following constraints: $T_{k,k}$ is the identity and, for $j \leq k \leq u$, $T_{j,u} = T_{k,u} \circ T_{j,k}$. A cone $\tau : X \to U \circ T$ is a collection $\{ \tau_k \mid k \in C^+ \}$ of functions such that $\tau_k : X \to T_k$ and, for $j \leq k$, $\tau_k = T_{j,k} \circ \tau_j$. Let us insist on the fact that X is just a set, it is not linearly ordered, while all the sets T_k are linearly ordered.

Remark 19. For T a plane tower, let $El(T)$ be the poset whose elements are pairs (k, x) with $x \in T_k$ and for which $(k_1, x_1) \leq (k_2, x_2)$ if $k_2 \geq k_1$ and $x_2 = T_{k_1,k_2}(x_1)$. It easily verified that the C^+ is dually well-founded (e.g. Noetherian) if and only if $El(T)$ is a tree in the sense of set theory—that is, each downset $\downarrow(k, x) = \{ (k', x') \mid k \leq k', x' = T_{k,k'}(x) \}$ is well-ordered. Thus, if C^+ is dually well-founded and in particular if C^+ is finite, then we call T a *plane tree* instead of a plane tower.

Definition 20. *An augmented plane tower (τ, T) is* perfect *if each map τ_i is surjective. An augmented plane tower is* complete *if, for each $x, y \in X$, the set $Eg(x, y) := \{ k \in C^+ \mid \tau_k(x) = \tau_k(y) \}$ has a least element.*[2]

It is an elementary exercise to verify that if an augmented plane tower (τ, T) is *perfect*, then also the maps $T_{j,k} : T_j \to T_k$, $j \leq k$, are surjective. The following lemma exemplifies some consequences of these conditions.

Lemma 21. *If (τ, T) is a perfect complete augmented tower from X, then, whenever $k = \bigwedge J$, the canonical map $T_k \to \lim_{j \in J} T_j$ is injective.*

[2] For finite binary trees, the adjectives perfect, full, and complete have precise yet distinct meanings. We adopt the wording perfect for a (non necessarily binary) tree (or a tower) all of whose branches have equal length. The wording complete refers here to a completeness property of the poset C^+.

Next, for a skew metric $f : X^2 \setminus \Delta_X \to C$, we give the following definitions:

$$x \sim_k^f y \;\text{ if }\; x = y \;\text{ or }\; |f(x,y)| \leq k\,, \qquad [x]_k^f := \{\, y \in X \mid y \sim_k^f x \,\}\,.$$
$$T_k^f := \{\, [x]_k^f \mid x \in X \,\}\,, \qquad [x]_k^f <_k^f [y]_k^f \;\text{ if }\; k < f(x,y)\,.$$

Notice that, for $j \leq k$, we have $[x]_j^f \subseteq [x]_k^f$ and so, for such j, k, we can define

$$T_{j,k}^f([x]_j^f) := [x]_k^f\,, \qquad \tau_k^f(x) := [x]_k^f\,.$$

Proposition 22. *$t^f := (\tau^f, T^f)$ is a perfect and complete augmented plane tower from X.*

Proof. Quite obviously T^f is a functor and τ^f is a cone from X to T_k^f.

Every set T_k^f is linearly ordered by $<_k^f$, since this ordering is induced by the skew metric $\chi_k \circ f : X \to \mathbf{3}$, see Example 11 and Remark 16. Moreover, for $j, k \in C^+$ and $j \leq k$, $j < f(x,y)$ and $|f(x,y)| \not\leq k$ imply $k < f(x,y)$. That is, for such j, k, $[x]_j^f <_j^f [y]_j^f$ implies $[x]_k^f \leq_k^f [y]_k^f$, so we can take as codomain of the functor T^f the category of linearly ordered sets and isotone functions. The maps τ_k^f are surjective, so t^f is perfect. For completeness, observe that $\bigwedge\{\, k \in C^+ \mid [x]_k^f = [y]_k^f \,\} = |f(x,y)|$, since by definition $[x]_k^f = [y]_k^f$ if and only if $|f(x,y)| \leq k$. □

For $t = (\tau, T)$ a complete augmented tower, we set

$$\delta_t(x,y) := \bigwedge\{\, k \in C^+ \mid \tau_k(x) = \tau_k(y) \,\}\,.$$

Notice that δ_t is an ultrametric on X valued in C^+, meaning that, for each $x, y, z \in X$, $\delta_t(x,y) = \delta_t(y,x)$ and $\delta_t(x,z) \leq \max(\delta_t(x,y), \delta_t(y,z))$. If C^+ has a least element 0, then the condition $\delta_t(x,y) = 0$ implies $x = y$ holds if and only if the map $\tau_0 : X \to T_0$ is injective.

Let now $t = (\tau, T)$ be a perfect and complete augmented plane tower. Observe that if $k < \delta_t(x,y)$, then we have either $\tau_k(x) < \tau_k(y)$, or $\tau_k(y) < \tau_k(x)$, but not both. Moreover, if $k, k' < \delta_t(x,y)$ and $\tau_k(x) < \tau_k(y)$, then $\tau_{k'}(x) < \tau_{k'}(y)$ as well. Indeed, if $\tau_k(x) < \tau_k(y)$ and $\tau_{k'}(y) < \tau_{k'}(x)$, then, for $K = \max(k, k')$, $\tau_K(x) \leq \tau_K(y)$ and $\tau_K(y) \leq \tau_K(x)$, thus $\tau_K(x) = \tau_K(y)$ with $K < \delta_t(x,y)$, a contradiction. Therefore, we define

$$\varsigma_t(x,y) := \begin{cases} -1\,, & \text{if, for some } k < \delta_t(x,y),\ \tau_k(y) < \tau_k(x)\,, \\ 1\,, & \text{otherwise}\,. \end{cases}$$

We define then $f_t : X^2 \setminus \Delta_X \to C$ by

$$f_t(x,y) := \varsigma_t(x,y) \cdot \delta_t(x,y)\,,$$

where the action of $\{-1, 1\}$ on C is as expected: $1 \cdot k = k$, and $-1 \cdot k = k^*$. Accordingly, we use the notation $-k$ as equivalent to k^*.

Proposition 23. *f_t so defined is a skew-metric on C.*

Proof. Firsty we argue that, for $x, y, z \in X$ arbitrary pairwise distinct, $f_t(x,y) \otimes f_t(y,z) \leq f_t(x,z)$. Recalling that $f_t(x,y) \otimes f_t(y,z) \in \{ f_t(x,y), f_t(y,z) \}$, we suppose that $f_t(x,y) \otimes f_t(y,z) = f_t(x,y)$ (if $f_t(x,y) \otimes f_t(y,z) = f_t(y,z)$, then the argument is similar). Under this assumption, we have either (i) $\delta_t(x,y) > \delta_t(y,z)$, or (ii) $\delta_t(x,y) = \delta_t(y,z)$ and $\varsigma_t(x,y) = -1$. Suppose (i). Since $\delta_t(x,y) > \delta_t(y,z)$, then $\delta_t(x,z) = \delta_t(x,y)$: indeed, $\delta_t(x,z) \leq \max(\delta_t(x,y), \delta_t(y,z)) = \delta_t(x,y)$, and $\delta_t(x,y) \leq \max(\delta_t(x,z), \delta_t(y,z))$ implies $\delta_t(x,y) \leq \delta_t(x,z)$. Therefore, in order to show that $f_t(x,y) \leq f_t(x,z)$, we need to argue that $\varsigma_t(x,y) = 1$ implies $\varsigma_t(x,z) = 1$. Assume therefore that $\varsigma_t(x,y) = 1$ and let $k = \delta_t(y,z)$; we have then $\tau_k(x) < \tau_k(y) = \tau_k(z)$, so $\varsigma_t(x,z) = 1$. We suppose now (ii), that is, $\delta_t(x,y) = \delta_t(y,z)$ and $\varsigma_t(x,y) = -1$. If $\varsigma_t(x,z) = 1$, then we obviously have $f_t(x,y) \leq f_t(x,z)$. Thus, we can assume that $\varsigma_t(x,z) = -1$. Considering that $\delta_t(x,z) \leq \max(\delta_t(x,y), \delta_t(y,z)) = \delta_t(x,y)$, then we immediately have $f_t(x,y) = -1 \cdot \delta_t(x,y) \leq -1 \cdot \delta_t(x,z) = f_t(x,z)$.

Next, we argue that $f_t(y,x) = f_t(x,y)^*$. Clearly, we have $\delta_t(x,y) = \delta_t(y,x)$. If $\varsigma_t(x,y) = -1$, then, for some $k < \delta_t(x,y)$, $\tau_k(y) < \tau_k(x)$, thus $\varsigma_t(y,x) = 1$ and $f_t(y,x) = -f_t(x,y)$. Suppose, therefore, that $\varsigma_t(x,y) = 1$, so $\tau_k(x) < \tau_k(y)$ for all $k < \delta_t(x,y)$. If $\delta_t(x,y)$ is not the least element of C^+, then we deduce $\varsigma_t(y,x) = -1$, so $f_t(y,x) = -f_t(x,y)$. Otherwise, $\delta_t(x,y)$ is the least element of C^+ and therefore $\varsigma_t(x,y) = \varsigma_t(y,x) = 1$, but also $C^+ \cap C^- = \{ \delta_t(x,y) \}$, since we assume that C is interesting, thus odd if C^+ has a least element. Then $f_t(x,y) = \delta_t(x,y) = -\delta_t(x,y) - \delta_t(y,x) = -f_t(y,x)$. □

Proposition 24. *For $f : X^2 \setminus \Delta_X \to C$ a skew metric, we have $f_{t^f} = f$.*

Proof. As we already observed, $\delta_{t^f}(x,y) = |f(x,y)|$. Moreover, $\varsigma_{t^f}(x,y) = -1$ iff for some $k < |f(x,y)|$ we have $[y]^f_k <^f_k [x]^f_k$, where the last inequality is equivalent, by definition, to $k < f(y,x)$. Now saying that, for some $k \in C^+$, $k < f(y,x)$, that is, $f(x,y) < -k$, is equivalent to saying that $f(x,y)$ is strictly negative, i.e. $f(x,y) \in C^- \setminus C^+$. Then, if $f(x,y)$ is strictly negative, then $f(x,y) = -|f(x,y)| = \varsigma_{t^f}(x,y) \cdot \delta_{t^f}(x,y) = f_{t^f}(x,y)$. If $f(x,y)$ is positive, then $\varsigma_{t^f}(x,y) = 1$, $\delta_{t^f}(x,y) = f(x,y)$, and again $f(x,y) = \varsigma_{t^f}(x,y) \cdot \delta_{t^f}(x,y) = f_{t^f}(x,y)$. □

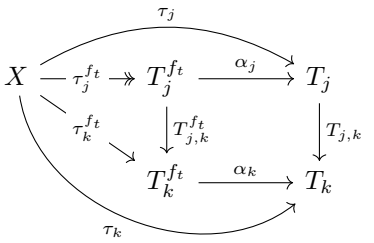
Let T, T' be two plane towers. Recall that a natural transformation $\alpha : T \to T'$ is a collection $\{ \alpha_k : T_k \to T'_k \mid k \in C^+ \}$ such that α_k is order preserving and such that, for $j \leq k$, $T'_{j,k} \circ \alpha_j = \alpha_k \circ T_{j,k}$. Such a natural transformation is a natural isomorphism if each α_k has an order preserving inverse. We say that $(\tau, T), (\tau', T')$ are isomorphic if there is such a natural isomorphism $\alpha : T \to T'$ such that $\tau' = \alpha \circ \tau$, that is, $\tau'_k = \alpha_k \circ \tau_k$ for each $k \in C^+$.

Proposition 25. *If $t = (\tau, T)$ is a perfect and complete augmented plane tower, then t^{f_t} is naturally isomorphic to t.*

Proof. Observe that $x \sim^{f_t}_k y$ iff $|f_t(x,y)| = \delta_{t^{f_t}}(x,y) \leq k$ iff $\tau_k(x) = \tau_k(y)$. That is, the equivalence relation $\sim^{f_t}_k$ is the kernel of τ_k. Since moreover τ_k is surjective, the function α_k sending $[x]^{f_t}_k$ to $\tau_k(x)$ is bijection from $T^{f_t}_k$ to T_k.

Something more can be said: $[x]^{f_t}_k <^{f_t}_k [y]^{f_t}_k$ iff $k < f_t(x,y)$, iff $\tau_k(x) < \tau_k(y)$. Threfore α_k is a bijective embedding of posets, whence an invertible map in the category of linear orders and isotone maps.

It is also obvious that $\tau = \alpha \circ \tau^{f_t}$, since this relation amounts to $\alpha_k(\tau^{f_t}_k(x)) = \alpha_k([x]^{f_t}_k) = \tau_k(x)$. From this relation and surjectivity of $\tau^{f_t}_k$ it also follows that $\alpha : T^{f_t} \to T$ is natural, which can be verified by inferring commutativity of the inner square from commutativity of the outer triangle in the diagram on the right.



□

5 The posets of skew metrics

In this section we consider again an arbitrary involutive residuated lattice Q and recall more advanced algebraic properties of skew metrics valued in Q. More precisely, we pinpoint that skew metrics can be ordered and that most often this ordering is a lattice.

Observe that it is not interesting to order skew metrics pointwise. For example, if $f, g : X^2 \setminus \Delta_X \to Q$ and $f(x,y) < g(x,y)$, then $g(y,x) = g(x,y)^* < f(x,y)^* = f(y,x)$. That is, a pointwise ordering is necessarily discrete (all the elements are incomparable). We can get a more interesting ordering if we assume that X is totally ordered. In this case, we let $\mathcal{I}_X := \{\, (x,y) \in X^2 \setminus \Delta_X \mid x < y \,\}$ and also introduce the following concept:

Definition 26. *A map $f : \mathcal{I}_X \to Q$ is* clopen *if, whenever $x < y < z \in X$,*

$$f(x,y) \otimes f(y,z) \leq f(x,z) \leq f(x,y) \oplus f(y,z)\,. \tag{4}$$

We let $\mathrm{Clop}_X(Q)$ be the set of clopen maps $f : \mathcal{I}_X \to Q$.

Roughly speaking (and up to a choice of a total ordering on X) clopen maps and skew metrics are the same kind of objects, as stated below:

Proposition 27. *Every clopen map $f : \mathcal{I}_X \to Q$ extends uniquely to a skew metric $f : X^2 \setminus \Delta_X \to Q$. Therefore, every total order on X determines a bijection from the set $\mathrm{Clop}_X(Q)$ to the set of skew metrics on X valued in Q.*

The set $\mathrm{Clop}_X(Q)$ can be ordered pointwise in a non trivial way. It was argued in [20, see Theorem 21] that if $X = \{\, 1, \ldots, n \,\}$ and Q is any involutive residuated lattice satisfying the mix rule, then $\mathrm{Clop}_X(Q)$ is a lattice. By inspecting the proof of this result, it is not difficult to generalize it as follows:

Theorem 28. *Let X be a totally ordered set. If Q is an involutive residuated lattice satisfying the mix rule, and every interval of X is finite or Q is complete as a lattice, then $\mathrm{Clop}_X(Q)$ is a lattice.*

As we have insisted on skew metrics, we can rephrase the previous statement in terms of skew metrics.

Theorem 29. *Let X be a totally ordered set, let Q be an involutive residuated lattice satisfying the mix rule. Let $\mathrm{SMet}_X(Q)$ be the set of skew metrics on X valued in Q. Order $\mathrm{SMet}_X(Q)$ as follows:*

$$f \leq g \quad \textit{iff} \quad g(x,y) \leq f(x,y)\,, \ \textit{for each } x,y \textit{ such that } x < y\,. \tag{5}$$

If every interval of X is finite or Q is complete as a lattice, then $\mathrm{SMet}_X(Q)$ is a lattice.

Let us remark that, for coherence with existing literature, we are considering in (5) the opposite ordering of the pointwise ordering on the restriction to $\mathcal{I}_X$.

Example 30. Suppose $X = \{1, \ldots, n\}$. If $Q = \mathbf{2}$, then $\mathrm{SMet}_X(Q)$ is the lattice all permutations of X, known as the Permutohedron or the weak Bruhat order on the symmetric group, see e.g. [10,3,22]. If $Q = \mathbf{3}$ is the Sugihara semigroup on the three element chain, then $\mathrm{SMet}_X(Q)$ is isomorphic to the lattice of pseudo-permutations of X, see [14,2,21]. For Q the lattice of sup-preserving maps from the chain $[0,1]$ to itself, the poset $\mathrm{Clop}_X(Q)$—and therefore $\mathrm{SMet}_X(Q)$—was studied in [9,20]. We study in the next section the lattice $\mathrm{SMet}_X(Q)$ for Q a Sugihara monoid on an odd finite chain.

Remark 31. For X a finite total order, the construction sending Q to $\mathrm{SMet}_X(Q)$ can be made into a limit preserving functor from the category of involutive residuated lattices satisfying the mix rule into the category of lattices, see [20]. As a consequence, given that a Sugihara monoid on the finite even chain $\mathbf{2k}$ can be embedded in the Sugihara monoid on the odd chain $\mathbf{2k+1}$, the lattice $\mathrm{SMet}_X(\mathbf{2k})$ can be described as a sublattice of $\mathrm{SMet}_X(\mathbf{2k+1})$. For this reason we have given priority to the investigation of the lattices of the form $\mathrm{SMet}_X(\mathbf{2k+1})$.

6 The poset of augmented plane trees

In this section we study the ordered set $\mathrm{SMet}_X(Q)$, with $X = \{1, \ldots, n\}$ and Q the Sugihara monoid on the finite chain of size $2K+1$, denoted henceforth by $\mathrm{SMet}_{n,K}$. The aim is to give a combinatorial model of this poset, by describing its covers as moves (i.e. elementary transformations or rewrite rules) on a set of combinatorial objects, in the spirit of [16]. Since skew metrics correspond—under the bijection described in Section 4—to $K+1$-level plane trees whose leaves are labelled by subsets of X, these subsets forming a partition of X,[3] we should describe the ordering directly on this kind of objects. However, mostly for compactness, we prefer to code trees as words and handle the latter.

[3] These objects are called $K+1$-level labeled linear rooted trees with n leaves on The On-Line Encyclopedia of Integer Sequences [1], cf. Figure 2.

Coding trees as words. We fix n and K and consider disjoint alphabets $\Sigma_0 := \{1, \ldots, n\}$ and $\Sigma_1 := \{|^1, \ldots, |^K\}$. We think of Σ_1 as an alphabet of walls of distinct heights. If $w \in (\Sigma_0 \cup \Sigma_1)^*$, then the walls from Σ_1 "split w into blocks". More precisely if w' is obtained from w by erasing letters from Σ_1, then w' can subdivided into blocks of contiguous letters from Σ_0. We define the set $Tw(n, K)$ (of tree-words) as the set of words w over the alphabet $\Sigma_0 \cup \Sigma_1$ satisfying the following conditions:

(i) The blocks of w are non-empty. That is, there are no contiguous walls, and walls do appear neither in first nor in last position.
(ii) The word obtained from w by erasing the walls is a permutation.
(iii) If two letters $x, y \in \Sigma_0$ are in the same block of w and $x < y$, then x appears on the left of y in w.

Example 32. The word $2|^2 13|^1 4$ belongs to $Tw(4, 2)$. The words $2|^2 134|^1$ and $2|^2|^3 13|^1 4$ violate the first constraint. The word $2|^2 23|^1 4$ violates the second constraint. The word $2|^2 31|^1 4$ violates the third constraint.

We take for granted that a word in $Tw(n, K)$ codes a perfect plane tree of height K augmented from $\{1, \ldots, n\}$, see Figure 1 for examples. Yet, a few remarks are due. By identifying a word $w \in Tw(n, K)$ with the augmented tree it codes, we have

$$\delta_w(x, y) = \max\{\, k \mid \text{the symbol } |^k \text{ separates } x \text{ from } y \text{ in } w \,\}\,,$$

and, for x, y such that $1 \leq x < y \leq n$,

$$\varsigma_w(x, y) = -1 \quad \text{iff} \quad y \text{ appears before } x \text{ in the permutation underlying } w\,,$$

where the latter relation is a consequence of the fact that letters belonging to the same block appear in increasing order.

Positive and negative walls, enabled walls. For $w \in Tw(n, K)$ and $k \in \{1, \ldots, K\}$, let $w{\uparrow_k}$ be the word in $Tw(n, K)$ obtained by first erasing all the walls $|^j \in \Sigma_1$ with $j < k$, and then by reorganising contiguous blocks so to satisfy the third constraint. For example, $(1|^1 3|^2 2){\uparrow_2} = 13|^2 2$. For $w \in Tw(n, K)$ and an occurrence of a letter $|^k \in \Sigma_1$ in w, the *left* (resp., right) *scope* of this occurrence is the block on its left (resp., right) in $w{\uparrow_k}$; we say that such an occurrence is *positive* (resp., *negative*) if all the letters in its left scope are smaller (resp., greater) than those in its right scope. If such an occurrence is either positive or negative, then we say that it is enabled. For example, by $1\underline{|^2}3|^1 2$ is positive, $2|^1 3\underline{|^2}2$ is negative, $2|^3 3\underline{|^2}2$ is neither negative nor positive, so it is not enabled.

Moves. An *erosion move* replaces a positive occurrence of a wall $|^k$ by $|^{k-1}$, if $k > 1$, or deletes it, if $k = 1$. For $k = 1$, we call such an instance of an erosion move a *join move*, for obvious reasons. Dually, a *build move* occurs when a negative occurrence of a wall $|^k$ (with $0 < k < K$) is replaced by $|^{k+1}$. We

can also consider the case when $k = 0$, which amounts to (i) inserting a wall $|^1$, thus splitting a block into two new blocks and then (ii) swapping the relative positions of these two new blocks. We call this a *split move*. Moves are illustrated in Figure 1.

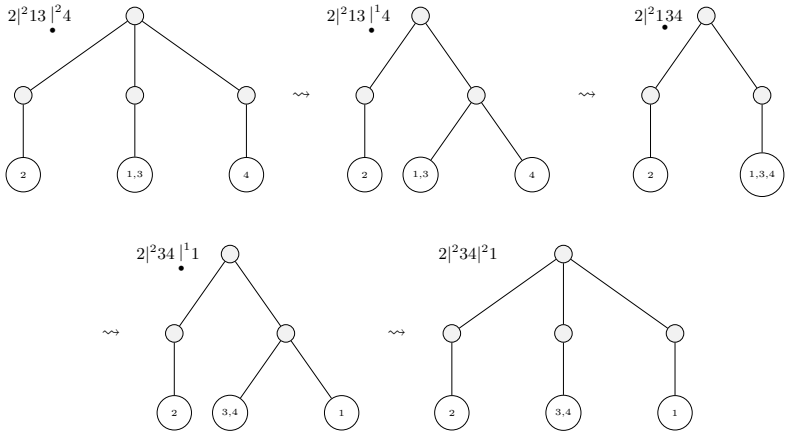


Fig. 1. Erosion, join, split, build moves, exemplified in the order

As from equation (5), we order $Tw(n, K)$ by saying that $w \leq u$ if, whenever $x < y$, $f_u(x, y) \leq f_w(x, y)$, where f_w, f_u are the skew metrics corresponding to w and u, respectively.

Theorem 33. *A word u is an upper cover of w in the poset $Tw(n, K)$ if and only if u is obtained from w by one of these moves.*

The theorem is an immediate consequence of the following two propositions, together with the straightforward observation that distinct moves from the same word yield incomparable words.

Proposition 34. *For each $w, u \in Tw(n, K)$, if u can be obtained from w by any of these moves, then $w < u$.*

Proof. Let $w, u \in Tw(n, K)$ be as stated, we need to show that, for each $(x, y) \in \mathcal{I}_X$, $f_u(x, y) \leq f_w(x, y)$, and $f_u(x, y) < f_w(x, y)$ for some $(x, y) \in \mathcal{I}_X$. Notice that if x, y are on the opposite scopes of the wall $|^k$ whose value k is being decreased or increased, then $\delta_u(x, y) \neq \delta_w(x, y)$. Therefore, we shall show that, for all $(x, y) \in \mathcal{I}_X$, $f_u(x, y) \leq f_w(x, y)$.

If the relative positions of x, y are not changed, that is, if $\varsigma_u(x, y) = \varsigma_w(x, y)$, and if $\delta_u(x, y) = \delta_w(x, y)$, then $f_u(x, y) = f_w(x, y)$. If $\varsigma_u(x, y) \neq \varsigma_w(x, y)$, then this happens with a split move and in this case we also have $\delta_w(x, y) \neq \delta_u(x, y)$. We suppose therefore that $\delta_w(x, y) \neq \delta_u(x, y)$.

If $\delta_u(x,y) < \delta_w(x,y)$, then u is obtained from w by erosion of a positive occurrence of a wall $|^k$ with $k := \delta_w(x,y)$. Then, x, y appear in the scopes of this wall, and since this occurrence is positive, x appear on the left and y appears on the right of the wall. Thus, we have $\varsigma_w(x,y) = \varsigma_u(x,y) = 1$ and $f_u(x,y) < f_w(x,y)$.

If $\delta_u(x,y) > \delta_w(x,y)$, then u is obtained from w by a build move of a negative occurrence of a wall $|^k$ with $k := \delta_w(x,y)$. Let us suppose first that $k > 0$. Thus, x, y appear in the scopes of this occurrence and, since this occurrence is negative, x appears in the right scope and y in the left scope. Thus we have $\varsigma_w(x,y) = \varsigma_w(x,y) = -1$, $\delta_u(x,y) = k+1$, and therefore $f_u(x,y) < f_w(x,y)$. Let us suppose finally that $k = 0$, that is, u is the result of a split move, so $\delta_w(x,y) = 0$ and $\delta_u(x,y) = 1$. Therefore, $f_w(x,y) = 0$, x, y belong to the same block of w, while x, y are separated by a wall $|^1$ in u, x being on its right scope and y being on its left scope. We have therefore $\varsigma_u(x,y) = -1$, $\delta_u(x,y) = 1$, and therefore $f_u(x,y) = -1 < 0 = f_w(x,y)$. □

Proposition 35. *If $w < u$, then there exists w', obtained from w by one of these moves, such that $w' \leq u$.*

Proof. For this proof, recall that $x \leq_0^w y$ if and only if $0 \leq f_w(x,y)$. Even if this is just a preorder, but we can still define standard notions, such as the closed interval $[x,y]_0^w := \{\, z \mid x \leq_0^w z \leq y \,\}$, using which, the block of x is $[x]_0^w := [x,x]_0^w$.

Since $w < u$, the set $A := \{\, (x,y) \in \mathcal{I}_X \mid f_u(x,y) < f_w(x,y) \,\}$ is non-empty. Consider a pair $(x,y) \in A$ minimizing the function δ_w on A. Moroever, among all such pairs, choose (x,y) such that the cardinality of $[x,y]_0^w \cup [y,x]_0^w$ is minimum. We suppose firstly that $\delta_w(x,y) = 0$, that is, x, y belong to the same block $[x]_0^w$. We split this block in two so to obtain w' with $f_u \leq f_{w'}$. To achieve this, consider that the restriction of the equivalence relation $\sim_0^u$ to $[x]_0^w$ splits it into blocks, say $b_1, \ldots, b_m$. If $x', y' \in [x]_0^w$, $x' < y'$, and $x' \in b_i, y' \in b_j$ with $i \neq j$, then $\varsigma_u(x', y') = -1$. This is a consequence of $\delta_u(x', y') > 0$ (since $[x']_0^u \neq [y']_0^u$) and $f_u(x', y') \leq f_w(x', y') = f_w(x, y) = 0$. Therefore, we can order the blocks so that $b_i < b_j$ if, for some $y' \in b_i$, $x' \in b_j$, $x' < y'$. Without loss of generality, we can assume that $b_1 < b_2 < \ldots < b_m$. Therefore, if we let $s := \mathrm{card}(b_m)$, then we can split $[x]_0^w$ at position s to obtain w'. We have $f_{w'}(x', y') = -1$ for each $(x', y') \in \mathcal{I}_X$ with $x' \in b_m$ and $y' \in [x]_0 \setminus b_m$ and, otherwise, $f_{w'}(x', y') = f_w(x', y')$. This shows that $f_u \leq f_{w'}$.

Suppose now that $\delta_w(x,y) > 0$ and let $k := \delta_w(x,y)$. By minimality of the cardinal of $[x,y]_0^w \cup [y,x]_0^w$, it follows that there is at most one wall $|^k$ separating x from y in $w\!\uparrow_k$. We claim that this occurrence is enabled. Indeed, take x', y' on the opposite scopes of the wall and observe that $\max(\delta_w(x,x'), \delta_w(y,y')) < \delta_w(x,y) = \delta_w(x',y')$. From this, it follows that

$$f_w(x', y') = f_w(x, x') \otimes f_w(x', y') \otimes f_w(y', y) \leq f_w(x, y)\,,$$

and, dually, $f_w(x,y) \leq f_w(x', y')$. Thus $f_w(x,y) = f_w(x', y')$ and, consequently, $\varsigma_w(x', y') = \varsigma_w(x,y)$. Therefore, if $\varsigma_w(x,y) = 1$, that is, if $x \leq_0^w y$, the occurrence

of $|^k$ is positive, and if $\varsigma_w(x,y) = -1$ (i.e. $y \leq_0^w x$), the occurrence of $|^k$ is negative. Suppose that the occurrence of $|^k$ is positive. We have $f_{w'}(x',y') = k-1$, for each $(x',y') \in \mathcal{I}_X$ with x' in the left scope of this wall and y' in its right scope and, otherwise, $f_{w'}(x',y') = f_w(x',y')$. This shows that $f_u \leq f_{w'}$. A similar argument shows that $f_u \leq f_{w'}$ if the occurrence of $|^k$ is negative. □

7 Enumerative considerations

Several enumerative questions concerning the lattices $\mathrm{SMet}_{n,K}$ may be answered via the combinatorial model. We can determine the length of the posets $\mathrm{SMet}_{n,K}$, that is, the length of a longest chain. It is easily seen that a chain cannot have length greater than $2K\frac{n(n-1)}{2}$ and we claim that this is the length of some chain. We construct such a chain in $\mathrm{SMet}_{n+1,K}$ by concatenating a longest chain $\mathrm{SMet}_{n,K}$ with n sequences of $2K$ moves switching contiguous letters, as suggested below:

$$1|^K 2 \ldots |^K n|^K n+1 \rightsquigarrow^* n|^K \ldots 2|^K 1|^K n+1$$
$$\rightsquigarrow^{2K} n|^K \ldots 2|^K n+1|^K 1 \rightsquigarrow^{(n-1)2K} n+1|^K n|^K n-1 \ldots |^K n+1\,.$$

Letting $\ell_{n,K}$ be the length of such a sequence, we have the recurrence $\ell_{1,K} = 0$ and $\ell_{n+1,K} = \ell_{n+1,K} + 2nK$, yielding $\ell_{n,K} = Kn(n-1)$. Notice that a minimal sequence of moves from the bottom to the top elements of this poset has length $2K(n-1)$, so in particular these posets are not ranked.

The cardinalities $f(n,K) := \mathrm{card}(\mathrm{SMet}_{n,K})$ can be computed by

$$f(n,K) \overset{(a)}{=} \sum_{i=1}^{n} i! \left\{ {n \atop i} \right\} K^{i-1} \overset{(b)}{=} \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle K^i (K+1)^{n-1-i}\,,$$

where in these equalities $\left\{ {n \atop i} \right\}$ is the Stirling number of the second kind, counting the number of partitions of an n-element set into i blocks, while $\left\langle {n \atop i} \right\rangle$ is the Eulerian number, counting the number of permutations of n-elements with i descent positions.[4] Both formulas for $f(n,K)$ immediately follows from the correspondence with words in $Tw(n,K)$ given in the previous section. Equality (a) can be understood as follows. The number $j!\left\{ {n \atop j} \right\}$ counts the number of ordered partitions of an n-element set into j blocks and K^{j-1} counts the ways we can assign heights to the separating walls. Equality (b) stems from a well-known relation between ordered partitions and permutations. It can be read out as follows: given a permutation with i descent positions, we construct a word in $Tw(n,K)$ by (i) inserting a wall at each descent position and chosing an height for it in K different ways, (ii) for the other $n-1-i$ positions, either we do not insert a wall

[4] A descent position in a permutation $\sigma_1\sigma_1\ldots\sigma_n$ is an index $i \in \{1,\ldots,n-1\}$ such that $\sigma_i > \sigma_{i+1}$. The numbers $\left\langle {n \atop i} \right\rangle$ can be easily computed via the alternating formula $\left\langle {n \atop i} \right\rangle = \sum_{j=0}^{i} (-1)^j \binom{n+1}{j} (k+1-j)^n$, see e.g. [17].

or we insert a wall and assign it an height, resulting in $K+1$ choices. Inspecting the values of the function $f(n,K)$ on The On-Line Encyclopedia of Integer Sequences [1], see Figure 2, we came across the reference [8]. This work, which

n/K	1	2	3	4	5	6	7	OEIS
2	3	5	7	9	11	13	15	
3	13	37	73	121	181	253	337	A003154
4	75	365	1015	2169	3971	6565	10095	A193252
5	541	4501	17641	48601	108901	212941	378001	
6	4683	66605	367927	1306809	3583811	8288293	16984815	
7	47293	1149877	8952553	40994521	137595781	376372333	890380177	
OEIS	A000670	A050351	A050352	A050353				

Fig. 2. Cardinalities of $\mathrm{SMet}_{n,K}$

also pinpoints the recursion $f(1,K)=1$, $f(n+1,K)=1+K\sum_{i=1}^{n} f(i,K)$, allows to establish a connection between skew metrics on Sugihara monoids and the geometry of hyperplane arrangements, see e.g. [23], a connection already known for the Sugihara monoid $\mathbf{3}$, see e.g. [2,4]. It is proved in [8] that $f(n,K)$ is the number of maximal elements of the intersection poset of the affine braid arrangement $\{\, H_{i,j,k} \mid 1 \le i < j \le n, -K \le k \le K \,\}$, with $H_{i,j,k}$ being the affine hyperplane of equation $x_j = x_i + k$. Sugihara monoids (and, more generally, involutive residuated lattices, as argued in [20]) therefore appear to have a pervasive role in this realm of geometry and in the related combinatorics. It is still a long way towards making this role fully explicit, but surely it is a research path that we want to pursue.

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