On the *d*-Claw Vertex Deletion Problem*

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Abstract. Let *d*-claw (or *d*-star) stand for $K_{1,d}$, the complete bipartite graph with 1 and $d \ge 1$ vertices on each part. The *d*-claw vertex deletion problem, *d*-CLAW-VD, asks for a given graph *G* and an integer *k* if one can delete at most *k* vertices from *G* such that the resulting graph has no *d*-claw as an induced subgraph. Thus, 1-CLAW-VD and 2-CLAW-VD are just the famous VERTEX COVER problem and the CLUSTER VERTEX DELETION problem, respectively.

In this paper, we strengthen a hardness result in [M. Yannakakis, Node-Deletion Problems on Bipartite Graphs, SIAM J. Comput. (1981)], by showing that CLUS-TER VERTEX DELETION remains NP-complete when restricted to bipartite graphs of maximum degree 3. Moreover, for every $d \geq 3$, we show that d-CLAW-VD is NP-complete even when restricted to bipartite graphs of maximum degree d. These hardness results are optimal with respect to degree constraint. By extending the hardness result in F. Bonomo-Braberman et al., Linear-Time Algorithms for Eliminating Claws in Graphs, COCOON 2020, we show that, for every $d \geq 3$, d-CLAW-VD is NP-complete even when restricted to split graphs without (d + 1)-claws, and split graphs of diameter 2. On the positive side, we prove that *d*-CLAW-VD is polynomially solvable on what we call *d*-block graphs, a class properly contains all block graphs. This result extends the polynomial-time algorithm in [Y. Cao et al., Vertex deletion problems on chordal graphs, Theor. Comput. Sci. (2018)] for 2-CLAW-VD on block graphs to d-CLAW-VD for all $d \ge 2$ and improves the polynomial-time algorithm proposed by F. Bonomo-Brabeman et al. for (unweighted) 3-CLAW-VD on block graphs to 3-block graphs.

keyword:

Vertex Cover, Cluster Vertex Deletion, Claw Vertex Deletion, Graph Algorithm, NP-complete Problem

^{*} A preliminary version [12] of this article has appeared in Proceedings of the 27th International Computing and Combinatorics Conference (COCOON 2021); the second author discovered and corrected the incorrect proof of Theorem 3 in [12].

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1 Introduction

Graph modification problems are a very extensively studied topic in graph algorithm. One important class of graph modification problems is as follows. Let H be a fixed graph. The H VERTEX DELETION (H-VD for short) problem takes as input a graph G and an integer k. The question is whether it is possible to delete a vertex set S of at most kvertices from G such that the resulting graph is H-free, *i.e.*, G - S contains no induced subgraphs isomorphic to H. The optimization version asks for such a vertex set S of minimum size, and is denoted by MIN H VERTEX DELETION (MIN H-VD for short).

The case that H is a 2-vertex path, *i.e.*, an edge, is the famous VERTEX COVER problem, one of the basic NP-complete problems. The case that H is a 3-vertex path is well known under the name CLUSTER VERTEX DELETION (CLUSTER-VD for short). Very recently, the COCOON 2020 paper [2] addresses the case that H is the *claw* $K_{1,3}$, the complete bipartite graph with 1 and 3 vertices in each part, thus the CLAW-VD problem.

For any integer d > 0, let *d*-claw (or *d*-star) stand for $K_{1,d}$, the complete bipartite graph with 1 and *d* vertices on each part. In this paper, we go on with the CLAW-VD problem by considering the *d*-CLAW-VD problem for any given integer d > 0:

d-CLAW-VD

Instance: A graph G = (V, E) and an integer k < |V|.

Question: Is there a subset $S \subset V$ of size at most k such that G - S is d-claw free?

Thus, 1-CLAW-VD and 2-CLAW-VD are just the well-known NP-complete problems VER-TEX COVER and CLUSTER-VD, respectively, and 3-CLAW-VD is the CLAW-VD problem addressed in the recent paper [2] mentioned above.

While 1-CLAW-VD is polynomially solvable when restricted to perfect graphs (including chordal and bipartite graphs) [9], d-CLAW-VD is NP-complete for any $d \ge 2$ even when restricted to bipartite graphs [20]. When restricted to chordal graphs, it is shown in [2] that 3-CLAW-VD remains NP-complete even on split graphs. The computational complexity of 2-CLAW-VD on chordal graphs is still unknown [4,5]. Both 2-CLAW-VD and 3-CLAW-VD can be solved in polynomial time on block graphs [2,5], a proper subclass of chordal graphs containing all trees.

It is well known that the classical NP-complete problem VERTEX COVER remains hard when restricted to planar graphs of maximum degree 3 and arbitrary large girth. It is also known that, assuming ETH (Exponential Time Hypothesis), VERTEX COVER admits no subexponential-time algorithm in the vertex number [15] and, while MIN VERTEX COVER can be approximated within factor 2 by a simple 'textbook' greedy algorithm, no polynomial-time approximation with a factor better than 2 exists assuming UGC (Unique Games Conjecture) [13]. Very recently, it is shown in [1] that MIN CLUSTER-VD can be approximated within factor 2, and this is optimal assuming UGC.

As for MIN VERTEX COVER, MIN *d*-CLAW-VD can be approximated within a factor d + 1 but there is no polynomial-time approximation scheme [16]. From the results in [14] it is known that, for any $d \ge 2$, MIN *d*-CLAW-VD admits a *d*-approximation algorithm on bipartite graphs. This result was improved later by a result in [10], where the related problem *d*-CLAW-TRANSVERSAL was considered. Given a graph *G*, this problem asks

to find a smallest vertex set $S \subseteq V(G)$ such that G - S does not contain a *d*-claw as a (not necessarily induced) subgraph. In [10], it was shown that, in contrast to our MIN *d*-CLAW-VD problem, *d*-CLAW-TRANSVERSAL can be approximated within a factor of $O(\log(d+1))$. Since *d*-CLAW-VD and *d*-CLAW-TRANSVERSAL coincide when restricted to bipartite graphs, *d*-CLAW-VD admits an $O(\log(d+1))$ -approximation on bipartite graphs. The 2-CLAW-TRANSVERSAL is also known as P_3 VERTEX COVER (see, *e.g.*, [6,18]).

By a standard bounded search tree technique, d-CLAW-VD admits a parameterized algorithm running in $O^*((d+1)^k)$ time¹. The current fastest parameterized algorithm for VERTEX COVER and CLUSTER-VD has runtime $O^*(1.2738^k)$ [7] and $O^*(1.811^k)$ [19], respectively.

For the edge modification versions, there is a comprehensive survey [8].

In this paper, we first derive some hardness results by a simple reduction from VERTEX COVER to d-CLAW-VD, stating that d-CLAW-VD does not admit a subexponential-time algorithm in the vertex number unless the ETH fails, and that d-CLAW-VD remains NP-complete when restricted to planar graphs of maximum degree d + 1 and arbitrary large girth.

We then revisit the case of bipartite input graphs by showing that CLUSTER-VD remains NP-complete on bipartite graphs of maximum degree 3, and for $d \geq 3$, d-CLAW-VD remains NP-complete on bipartite graphs of maximum degree d and on bipartite graphs of diameter 3. These hardness results for d-CLAW-VD are optimal with respect to degree and diameter constraints, and improve the corresponding hardness results for d-CLAW-VD, $d \geq 2$, on bipartite graphs in [20].

Further, we extend the hardness results in [2] for CLAW-VD to d-CLAW-VD for every $d \geq 3$. We show that d-CLAW-VD is NP-complete even when restricted to split graphs without (d + 1)-claws and, assuming the UGC, it is hard to approximate MIN d-CLAW-VD to a factor better than d - 1. We obtain these hardness results by modifying the reduction from VERTEX COVER to CLAW-VD given in [2] to a reduction from HYPERGRAPH VERTEX COVER on to d-CLAW-VD.

We complement the negative results by showing that *d*-CLAW-VD is polynomial-time solvable on what we call *d*-block graphs, a class that contains all block graphs. As block graphs are 2-block graphs, and *d*-block graphs are (d + 1)-block graphs but not the converse, our positive result extends the polynomial-time algorithm for 2-CLAW-VD on block graphs in [5] to *d*-CLAW-VD for all $d \ge 2$, and for 3-CLAW-VD on block graphs in [2] to 3-block graphs.

The paper is organized as follows. In Section 2, we give some notation and terminologies used in this paper. Section 3 presents hardness results mentioned above. A polynomial result on *d*-block graphs is shown in Section 4. Section 5 concludes the paper with some remarks.

2 Preliminaries

Let G = (V, E) be a graph with vertex set V and edge set E. The neighborhood of a vertex v in G, denoted by $N_G(v)$, is the set of all vertices in G adjacent to v; if the

¹ The O^* notation hides polynomial factors.

context is clear, we simply write N(v). Let $\deg(v) = |N(v)|$ to denote the degree of the vertex v. The close neighborhood of v is denoted by N[v] that is $N(v) \cup \{v\}$.

We call a vertex *universal* if it is adjacent to all other vertices. Vertices of degree 1 are called *leaves*. The *distance* between two vertices in G is the length of a shortest path connecting the two vertices, the *diameter* is the maximal distance between any two vertices, and the *girth* is the length of a shortest cycle in G (if exists).

A center vertex of a d-claw H is a universal vertex of H; if $d \ge 2$, the center of a d-claw is unique. We say that a d-claw is centered at vertex v if v is the center vertex of that d-claw. Note that the 2-claw $K_{1,2}$ and the 3-vertex path P_3 coincide, and that the 3-claw $K_{1,3}$ is also called *claw*.

An independent set (respectively, a clique) in a graph G = (V, E) is a vertex subset of pairwise non-adjacent (respectively, adjacent) vertices in G. Graph G is a split graph if its vertex set V can be partitioned into an independent set and a clique. A graph is a cluster graph if it is a vertex disjoint union of cliques. Equivalently, cluster graphs are exactly those without induced 3-vertex path P_3 .

For a subset $S \subseteq V$, G[S] is the subgraph of G induced by S, and G - S stands for $G[V \setminus S]$. For simplicity, for a set S and an element v, we use S + v (respectively, S - v) to denote $S \cup \{v\}$ (respectively, $S \setminus \{v\}$).

Let H be a fixed graph. An H-deletion set is a vertex set $S \subseteq V(G)$ such that G - S is H-free. A $K_{1,1}$ -deletion set and a $K_{1,2}$ -deletion set are known as vertex cover and cluster deletion set, respectively. In other words, the cluster vertex deletion problem is the problem of finding a minimum cluster deletion set S on G such that the resulting graph G - S is a cluster graph.

A hypergraph G = (V, E) consists of a vertex set V and an edge set E where each edge $e \in E$ is a subset of V. Let $r \geq 2$ be an integer. A hypergraph is r-uniform if each of its edges is an r-element set. Thus, a 2-uniform hypergraph is a graph in usual sense. A vertex cover in a hypergraph G = (V, E) is a vertex set $S \subseteq V$ such that $S \cap e \neq \emptyset$ for any edge $e \in E$. The r-HYPERGRAPH VERTEX COVER (r-HVC for short) problem asks, for a given r-uniform hypergraph G = (V, E) and an integer k < |V|, whether G has a vertex cover S of size at most k. The optimization version asks for such a vertex set S of minimum size and is denoted by MIN r-HVC. Note that 2-HVC and MIN 2-HVC are the famous VERTEX COVER problem and MIN VERTEX COVER problem, respectively. It is known that r-HVC is NP-complete and MIN r-HVC is UGC-hard to approximate within a factor better than r [13].

3 Hardness results

Recall that d-CLAW-VD is NP-complete even on bipartite graphs [20]. We begin with two simple observations which lead to further hardness results on other restricted graph classes.

Observation 1 *d*-CLAW-VD remains NP-complete on graphs of diameter 2.

Proof. Given an instance (G, k) for d-CLAW-VD, let G' be obtained from G by adding a d-claw with center vertex v and joining v to all vertices in G. Then v is a universal

vertex in G' and hence G' has diameter 2. Moreover, $(G, k) \in d$ -CLAW-VD if and only if $(G', k + 1) \in d$ -CLAW-VD.

We remark that the graph G' in the proof above is a split graph whenever G is a split graph, and G' has only one vertex of unbounded degree whenever G has bounded maximum degree. The bipartite version of Observation 1 is:

Observation 2 For any $d \ge 2$, d-CLAW-VD remains NP-complete on bipartite graphs of diameter 3.

Proof. Let (G, k) be an instance for *d*-CLAW-VD, where G = (X, Y, E) is a bipartite graph. Let G' be the bipartite graph obtained from G by adding two *d*-claws with center vertices x and y, respectively, and joining x to all vertices in $Y \cup \{y\}$ and y to all vertices in $X \cup \{x\}$. Then G' has diameter 3. Moreover, $(G, k) \in d$ -CLAW-VD if and only if $(G', k+2) \in d$ -CLAW-VD.

We remark that the bipartite graph G' in the proof above has only two vertices of unbounded degree whenever G has bounded maximum degree.

We now describe a simple reduction from VERTEX COVER to d-CLAW-VD and some implications for the hardness of d-CLAW-VD. Let $d \ge 2$. Given a graph G = (V, E), construct a graph G' = (V', E') as follows.

- for each $v \in V$ let I(v) be an independent set of d-1 new vertices;

$$-V' = V \cup \bigcup_{v \in V} I(v);$$

 $-E' = E \cup \bigcup_{v \in V} \{vx \mid x \in I(v)\}.$

Thus, G' is obtained from G by attaching to each vertex v a set I(v) of d-1 leaves.

Fact 1 If S is a vertex cover in G, then S is a d-claw deletion set in G'.

Proof. This follows immediately from the construction of G'. Indeed, since G - S is edgeless, every connected component of G' - S is a single vertex (from I(v) for some $v \in S$) or a (d-1)-claw (induced by v and I(v) for some $v \notin S$). Thus, S is a d-claw deletion set in G'.

Fact 2 If S' is a d-claw deletion set in G', then G has a vertex cover S with $|S| \leq |S'|$.

Proof. Let S' be a *d*-claw deletion set in G'. We may assume that, for every $v \in V(G)$, S' contains no vertex in I(v). Otherwise, $(S' \setminus I(v)) \cup \{v\}$ is also a *d*-claw deletion set in G' of size at most |S'|. Thus $S' \subseteq V(G)$ and S = S' is a vertex cover in G. For otherwise, if uv were an edge in G - S then v and $\{u\} \cup I(v)$ would induce a *d*-claw in G' - S = G' - S' centered at v.

We now derive other hardness results for d-CLAW-VD from the previous reduction.

Theorem 1. Let $d \ge 2$ be a fixed integer. Assuming ETH, there is no $O^*(2^{o(n)})$ time algorithm for d-CLAW-VD on n-vertex graphs, even on graphs of diameter 2.

Proof. By Facts 1 and 2, and the known fact that, assuming ETH, there is no $O^*(2^{o(n)})$ time algorithm for VERTEX COVER on *n*-vertex graphs [15]. Since the graph G' in the construction has |V'| = |V| + (d-1)|V| = O(|V|) vertices, we obtain that there is no $O^*(2^{o(n)})$ time algorithm for *d*-CLAW-VD, too. By (the proof of) Observation 1, the statement also holds for graphs of diameter 2.

Theorem 2. Let $d \ge 2$ be a fixed integer.

- (i) d-CLAW-VD is NP-complete, even when restricted to planar graphs of maximum degree d+1 and arbitrary large girth.
- (ii) d-CLAW-VD is NP-complete, even when restricted to diameter-2 graphs with only one vertex of unbounded degree.

Proof. It is known (and it can be derived, *e.g.*, from [11,17]) that VERTEX COVER remains NP-complete on planar graphs G of maximum degree 3 and arbitrary large girth, and in which the neighbors of any vertex of degree 3 in G have degree 2.

Given such a graph G, let G' be obtained from G by attaching, for every vertex v of degree 2, d-1 leaves to v. Then G' is planar, has maximum degree d+1 and arbitrary large girth. Moreover, similarly to Facts 1 and 2, it can be seen that G has a vertex cover of size at most k if and only if G' has a d-claw deletion set of size at most k. This proves (i). Part (ii) follows from (i) and the reduction in the proof of Observation 1. \Box

We remark that the hardness result stated in Theorem 2 (ii) is optimal in the sense that graphs of bounded diameter and bounded vertex degree have bounded size. Hence d-CLAW-VD is trivial when restricted to such graphs.

Note that *d*-CLAW-VD is trivial on graph of maximum degree less than *d* (because such graphs contain no *d*-claws). Moreover, CLUSTER-VD is easily solvable on graphs of maximum degree 2. Thus, with Theorem 2 (i), the computational complexity of *d*-CLAW-VD, $d \geq 3$, on graphs of maximum degree *d* remains to discuss. We will show in the next subsections that the problem is still hard even on bipartite graphs of maximum degree *d*.

3.1 Bipartite graphs of bounded degree

Recall that VERTEX COVER is polynomially solvable on bipartite graphs, hence previous results reported above cannot be stated for bipartite graphs.

In this subsection, we first give a polynomial reduction from POSITIVE NAE 3-SAT to CLUSTER-VD showing that CLUSTER-VD is NP-complete even when restricted to bipartite graphs of degree 3. Then, we give another polynomial reduction from POSITIVE NAE 3-SAT to CLAW-VD showing that CLAW-VD is NP-complete even when restricted to bipartite graphs of maximum degree 3. From this, the hardness of *d*-CLAW-VD in bipartite graphs of maximum degree *d* will be easily derived for any d > 3. Thus, we obtain an interesting dichotomy for all $d \ge 3$: *d*-CLAW-VD is polynomial-time solvable on graphs of maximum degree less than *d* and NP-complete otherwise.

Recall that an instance for POSITIVE NAE 3-SAT is a 3-CNF formula $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ over *n* variables x_1, x_2, \ldots, x_n , in which each clause C_j consists of three distinct variables. The problem asks whether there is a truth assignment of the variables such that every clause in *F* has a true variable and a false variable. Such an assignment is called *nae assignment*. It is well known that POSITIVE NAE 3-SAT is NP-complete.

Cluster Vertex Deletion is hard in subcubic bipartite graphs Our reduction is inspired by a reduction from NAE 3-SAT to CLUSTER-VD on bipartite graphs in [20].

Let $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ over *n* variables x_1, x_2, \ldots, x_n , in which each clause C_j consists of three distinct variables. We will construct a subcubic bipartite graph *G* such that *G* has a cluster deletion set of size at most 2mn + 11m if and only if *F* admits a nae assignment. The graph *G* contains a gadget $G(v_i)$ for each variable v_i and a gadget $G(C_i)$ for each clause C_j .

Variable gadget. For each variable v_i we introduce m pairs of variable vertices v_{ij} and v'_{ij} one pair for each clause C_j , $1 \leq j \leq m$, as follows. First, take a cycle with 2mvertices v_{i1} , v'_{i1} , v_{i2} , v'_{i2} , ..., v_{im} , v'_{im} and edges $v_{i1}v'_{i1}$, $v'_{i1}v_{i2}$, $v_{i2}v'_{i2}$, ..., $v'_{i(m-1)}v_{im}$, $v_{im}v'_{im}$ and $v'_{im}v_{i1}$. Then subdivide every edge $v'_{ij}v_{i(j+1)}$ with 4 new vertices w_{ij} , x_{ij} , y_{ij} and z_{ij} to obtain a cycle on 6m vertices.

The obtained graph is denoted by $G(v_i)$. The case m = 3 is shown in Fig. 1.

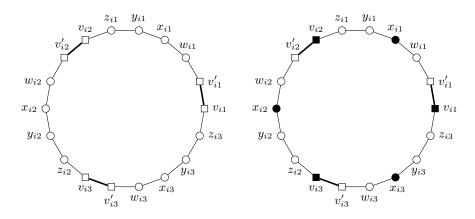


Fig. 1. The variable gadget $G(v_i)$ in case m = 3 (left) and an optimal cluster deletion set formed by the 2m black vertices (right).

The following properties of the variable gadget will be used:

Fact 3 $G(v_i)$ admits an optimal cluster deletion set of size 2m. Any optimal cluster deletion set S of $G(v_i)$ has the following properties:

- (a) S contains all or none of the variable vertices v_{ij} , $1 \le j \le m$. The same holds for the variable vertices v'_{ij} , $1 \le j \le m$;
- (b) For any $1 \le j \le m$, v_{ij} and v'_{ij} are not both in S. Moreover, each of v_{ij} and v'_{ij} has a neighbor outside S.

Proof. Observe that $G(v_i)$ can be partitioned into m induced $P_3 : v_{ij}v'_{ij}w_{ij}$ and m induced $P_3 : x_{ij}y_{ij}z_{ij}, 1 \leq j \leq m$. Therefore, any cluster deletion set in $G(v_i)$ must contain a vertex of each P_3 , hence has at least 2m vertices. Note also that $\{v_{ij}, x_{ij} \mid 1 \leq j \leq m\}$ and $\{w_{ij}, z_{ij} \mid 1 \leq j \leq m\}$ are cluster deletion sets of size 2m (see also Fig. 1 on the right hand).

In particular, an optimal cluster deletion set must contain exactly one vertex of each P_3 , showing (a) and (b).

Clause gadget. For each clause C_j consisting of variables c_{j1}, c_{j2} and c_{j3} , let $G(C_j)$ be the graph depicted on left hand side of Fig. 2; we call the six vertices labeled with c_{jk} and c'_{jk} , $1 \le k \le 3$, the *clause vertices*.

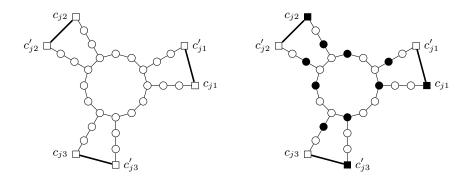


Fig. 2. The clause gadget $G(C_j)$ (left) and a cluster deletion set (black vertices) of size 11 (right).

Fact 4 $G(C_j)$ admits an optimal cluster deletion set of size 11. No optimal cluster deletion set of $G(C_j)$ contains all c_{j1} , c_{j2} and c_{j3} , or all c'_{j1} , c'_{j2} and c'_{j3} .

Proof. For $k \in \{1, 2, 3\}$, write x_{jk}, x'_{jk} for the two adjacent degree-3 vertices in $G(C_j)$ which together with c_{jk} and c'_{jk} belong to a 8-cycle. Let H be the 12-cycle containing all x_{jk} and x'_{jk} . Note that $G(C_j)$ minus H consists of three connected components each of which is a 6-vertex path with midpoints c_{jk} and c'_{jk} , $1 \le k \le 3$. Note also that any cluster deletion set of the 12-cycle H has at least 4 vertices, and any cluster deletion set of a 6-vertex path has at least 2 vertices.

Consider a cluster deletion set S in $G(C_j)$. If S contains at least 5 vertices from H then S has at least 11 vertices. Let us assume that S contains exactly 4 vertices from H. Then it can be verify that, for some $1 \le k \le 3$, both x_{jk}, x'_{jk} are outside S, and therefore, S contains at least 3 vertices from the 6-path with midpoints c_{jk}, c'_{jk} . Thus again, S contains at least 11 vertices. The first part follows now by noting that $G(C_j)$ admits a cluster deletion set of size 11 as depicted in Fig. 2.

Moreover, observe that $G(C_j)$ minus c_{j1}, c_{j2} and c_{j3} has a partition into 9 disjoint P_3 . Hence no optimal cluster deletion set in $G(C_j)$ can contain all c_{j1}, c_{j2} and c_{j3} , for otherwise it would contain at least 12 vertices. Similarly for c'_{j1}, c'_{j2} and c'_{j3} .

Finally, the graph G is obtained by connecting the variable and clause gadgets as follows: if variable v_i appears in clause C_j , i.e., $v_i = c_{jk}$ for some $k \in \{1, 2, 3\}$, then

- connect the variable vertex v_{ij} in $G(v_i)$ and the clause vertex c_{jk} in $G(C_j)$ by an edge; v_{ij} is the corresponding variable vertex of the clause vertex c_{jk} , and
- connect the variable vertex v'_{ij} in $G(v_i)$ and the clause vertex c'_{jk} in $G(C_j)$ by an edge; v'_{ij} is the corresponding variable vertex of the clause vertex c'_{jk} .

Fact 5 G has maximum degree 3 and is bipartite.

Proof. It follows from the construction that G has maximum degree 3. To see that G is bipartite, note that the bipartite graph forming by all variable gadgets $G(v_i)$ has a bipartition (A, B) such that all v_{ij} are in A and all v'_{ij} are in B, and the bipartite graph forming by all clause gadgets $G(C_j)$ has a bipartition (C, D) such that all c_{j1}, c_{j2}, c_{j3} are in C and all $c'_{j1}, c'_{j2}, c'_{j3}$ are in D. Hence, by construction, $(A \cup D, B \cup C)$ is a bipartition of G.

The following fact will be important for later discussion on cluster deletion sets in the clause gadget.

Fact 6 Let S be a cluster deletion set in G such that S contains exactly 2m vertices from each $G(v_i)$. Then, for any $1 \leq j \leq m$ and $1 \leq k \leq 3$, $c_{jk} \in S$ or $c'_{jk} \in S$. Moreover, if the clause vertex $c \in \{c_{jk}, c'_{jk}\}$ is not in S then the corresponding variable vertex of c is in S;

Proof. By Fact 3, the restriction of S on each $G(v_i)$ is an optimal cluster deletion set in $G(v_i)$. Hence, by Fact 3 (b), for every $1 \le j \le m$, some of the variable vertices v_{ij}, v'_{ij} is not in S. Thus, if both c_{jk} and c'_{jk} are not in S then with their corresponding variable vertices we have an induced P_3 outside S, a contradiction.

Moreover, if some $c \in \{c_{jk}, c'_{jk}\}$ is not in S then the corresponding variable vertex v is in S. For, otherwise c, v and a neighbor of v outside S (which exists by Fact 3 (b)) would induce a P_3 outside S.

Now suppose that G has a cluster deletion set S with at most 2nm + 11m vertices. Then by Facts 3 and 4, S has exactly 2nm + 11m vertices, and S contains exactly 2m vertices from each $G(v_i)$ and exactly 11 vertices from each $G(C_j)$.

Consider the truth assignment in which a variable v_i is true if all its associated variable vertices v_{ij} , $1 \leq j \leq m$, are in S. Note that by Fact 3 (a), this assignment is well-defined. For each $G(C_j)$, it follows from Fact 4 that some of c_{j1}, c_{j2}, c_{j3} is not in S and some of $c'_{j1}, c'_{j2}, c'_{j3}$ is not in S. Let $c_{j1} \notin S$, say. By Fact 6, $c'_{j1} \in S$. Hence $c'_{j2} \notin S$ or $c'_{j3} \notin S$, and again by Fact 6, $c_{j2} \in S$ or $c_{j3} \in S$. Let $c_{j2} \in S$; the case $c_{j3} \in S$ is similar. Let v_{rj} and v_{sj} be the corresponding variable vertices of c_{j1} and c_{j2} , respectively. Then by Fact 6 again, $v_{rj} \in S$ and $v'_{sj} \in S$. By Fact 3 (b), $v_{sj} \notin S$. That is, the clause C_j contains a true variable v_r and a false variable v_s . Thus, if G admits a cluster deletion set S with at most 2nm + 11m vertices then F has a nae assignment.

Conversely, suppose that there is a nae assignment for F. Then a cluster deletion set S of size 2nm + 11m for G is as follows. For each variable v_i , $1 \le i \le n$:

- If variable v_i is true, then put all 2m vertices $v_{ij}, x_{ij}, 1 \le j \le m$, into S.
- If variable v_i is false, then put all 2m vertices $v'_{ij}, y_{ij}, 1 \le j \le m$, into S.

For each clause C_j , $1 \le j \le m$, let v_{rj} , v_{sj} and v_{tj} be the corresponding variable vertices of c_{j1} , c_{j2} and c_{j3} , respectively. Extend S to 11 vertices of $G(C_j)$ as follows:

- If C_j has one true variable and two false variables, say v_r is true, v_s and v_t are false, then put the clause vertices c'_{j1} , c_{j2} and c_{j3} into S and extend S to another 8 vertices as indicated in Fig. 3 on the left hand.

- If C_j has two true variables and one false variable, say v_r and v_s are true, v_t is false, then put the clause vertices c'_{j1} , c'_{j2} and c_{j3} into S and extend S to another 8 vertices as indicated in Fig. 3 on the right hand.

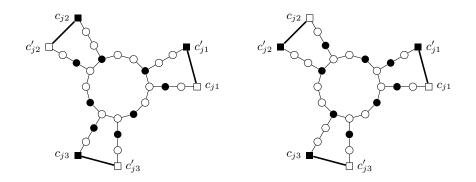


Fig. 3. An optimal cluster deletion set (black vertices) in $G(C_j)$; left hand: one true (c_{j1}) two false $(c_{j2} \text{ and } c_{j3})$ variables, right hand: two true $(c_{j1} \text{ and } c_{j2})$ one false (c_{j3}) variables.

By construction, S has exactly $n \times 2m + m \times 11$ vertices and, for every *i* and *j*, $G(v_i) - S$ and $G(C_j) - S$ are P_3 -free. Therefore, since a clause vertex is outside S if and only if the corresponding variable vertex is in S, G - S is P_3 -free. Thus, if F can be satisfied by a nae assignment then G has a cluster deletion set of size at most 2nm + 11m. In summary, we obtain:

Theorem 3. CLUSTER-VD is NP-complete even when restricted to bipartite graphs of maximum degree 3.

d-Claw Vertex Deletion is hard in bipartite graphs of maximum degree d We first give a polynomial-time reduction from POSITIVE NAE 3-SAT to 3-CLAW-VD. The case d > 3 will be easily derived from this case.

Let $F = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ over *n* variables x_1, x_2, \ldots, x_n , in which each clause C_j consists of three distinct variables. We will construct a subcubic bipartite graph *G* such that *G* has a claw deletion set of size at most 2mn + 16m if and only if *F* admits a nae assignment. The graph *G* contains a gadget $G(v_i)$ for each variable v_i and a gadget $G(C_j)$ for each clause C_j .

For any $1 \leq i \leq n$ and $1 \leq j \leq m$, we first consider an auxiliary 8-vertex graph H_{ij} depicted on the left hand side of Fig. 4 which will be useful in building our variable gadget.

We need the following property of H_{ij} which can be immediately verified:

Fact 7 Any optimal claw deletion set in H_{ij} contains exactly 2 vertices. Moreover, if S is an optimal claw deletion set containing v_{ij} then $S = \{v_{ij}, b_{ij}^2\}$.

We now are ready to describe the variable gadget.

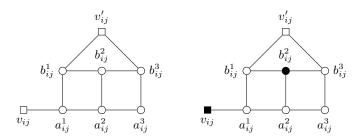


Fig. 4. Left: the auxiliary graph H_{ij} . Right: the unique optimal claw deletion set (back vertices) containing v_{ij} .

Variable gadget. For each variable v_i , take first m auxiliary graphs $H_{i1}, H_{i2}, \ldots, H_{im}$, one for each clause. Then, for all $1 \leq j < m$, connect the vertex a_{ij}^3 in H_{ij} with the vertex $v_{i(j+1)}$ in $H_{i(j+1)}$ by an edge. Last, connect the vertex a_{im}^3 in H_{im} with the vertex v_{i1} in H_{i1} by an edge. The obtained graph is denoted by $G(v_i)$. The case m = 3 is shown in Fig. 5. The 2m vertices v_{ij} and v'_{ij} , $1 \leq j \leq m$, in $G(v_i)$ are the variable vertices.

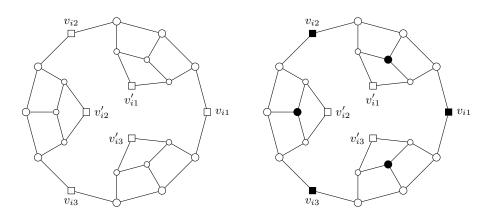


Fig. 5. The variable gadget $G(v_i)$ in case m = 3 (left) and an optimal claw deletion set formed by the 2m black vertices (right).

The following properties of the variable gadget will be used:

Fact 8 $G(v_i)$ admits an optimal cluster deletion set of size 2m. Moreover, if S is an optimal claw deletion set in $G(v_i)$ then the restriction of S on H_{ij} is an optimal claw deletion set in H_{ij} .

Proof. $G(v_i)$ consists of m disjoint H_{ij} , hence, by Fact 7, any claw deletion set in $G(v_i)$ must contain at least 2 vertices from each H_{ij} . Observe that the union of all $\{v_{ij}, b_{ij}^2\}$, $1 \leq j \leq m$, is a claw deletion set of $G(v_i)$ of size 2m. The second part follows from the first part and Fact 7. **Fact 9** Let S be an optimal claw deletion in $G(v_i)$. Then:

- (a) If some v_{ij} is contained in S then $S = \bigcup_{1 \le j \le m} \{v_{ij}, b_{ij}^2\}$; in particular, all v_{ij} are in S, and all v'_{ij} and their neighbors are outside S.
- (b) If some v_{ij} is not contained in S then all v_{ij} are outside S.

Proof. (a): By Fact 8, the restriction of S on H_{ij} is an optimal claw deletion set in H_{ij} , which is $\{v_{ij}, b_{ij}^2\}$ by Fact 7. In particular, $v'_{ij} \notin S$, and the P_3 : $a_{ij}^2 a_{ij}^3 b_{ij}^3$ is outside S as well, implying $v_{i(j+1)} \in S$. Similarly, by Fact 8 again, the restriction of S on $H_{i(j+1)}$ is an optimal claw deletion set in $H_{i(j+1)}$, which is $\{v_{i(j+1)}, b_{i(j+1)}^2\}$ by Fact 7. Apply Fact 7 for $H_{i(j+1)}$ we have $v'_{i(j+1)} \notin S$, $v_{i(j+2)} \in S$ and so on.

(b) follows from part (a).

Before presenting the clause gadget, we need other auxiliary graphs A_{jk} , A'_{jk} for any $1 \le j \le m$ and $1 \le k \le 3$ depicted in Fig. 6.

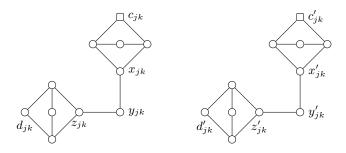


Fig. 6. The auxiliary graphs A_{jk} (left) and A'_{jk} (right).

Observing that any optimal claw deletion set in the complete bipartite graph $K_{2,3}$ consists of one degree-2 vertex and that A_{jk} contains two disjoint $K_{2,3}$, the following fact follows immediately:

Fact 10 Any claw deletion set in A_{jk} has at least 2 vertices; $\{x_{jk}, z_{jk}\}$ is the only optimal claw deletion set of size 2.

Note that Fact 10 holds accordingly for A'_{jk} . From the auxiliary graphs A_{jk} and A'_{jk} , we construct other auxiliary graphs H_j and H'_j , $1 \le j \le m$, as follows. H_j is obtained from A_{j1}, A_{j2} and A_{j3} by adding three additional edges $d_{j1}y_{j2}, d_{j2}y_{j3}$ and $d_{j3}y_{j1}$. H'_j is similarly defined; see also Fig. 7.

Fact 11 (a) Any claw deletion set in H_j has at least 8 vertices.

- (b) For any non-empty proper subset $T \subset \{c_{j1}, c_{j2}, c_{j3}\}$ there is an optimal claw deletion set of size 8 in H_j that contains T.
- (c) No optimal claw deletion set in H_j contains all c_{j1}, c_{j2} and c_{j3}, as well as a neighbor of any c_{jk}.

Proof. (a): Observe that, for each $1 \le k \le 3$, d_{jk} and its neighbors induce a claw which does not contain an x nor a z-vertex. Since a vertex of this claw must belong to any claw

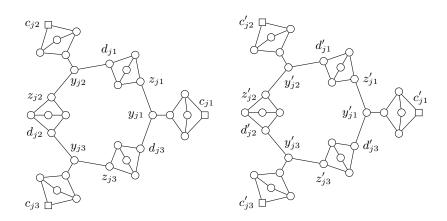


Fig. 7. The auxiliary graph H_j (left) and H'_j (right).

deletion set S, we have with Fact 10, that two of the restrictions S_{j1}, S_{j2} and S_{j3} of S on A_{j1}, A_{j2} and A_{j3} , respectively, have size at least 3. Hence $|S| \ge |S_{j1}| + |S_{j2}| + |S_{j3}| \ge 8$. (b): $\{c_{j1}, y_{j1}, d_{j1}, x_{j2}, z_{j2}, x_{j3}, y_{j3}, d_{j3}\}$ is an optimal claw deletion set containing c_{j1} ; similar for c_{j2} and for c_{j3} . $\{x_{j1}, z_{j1}, c_{j2}, y_{j2}, d_{j2}, c_{j3}, y_{j3}, d_{j3}\}$ is an optimal claw deletion set containing claw deletion set containing c_{j2} and c_{j3} ; similar for the pairs c_{j1}, c_{j2} and c_{j1}, c_{j3} .

(c): If all c_{j1}, c_{j2} and c_{j3} are in an optimal claw deletion set S then, by Fact 10, S has at least 3 vertices in each A_{jk} , and hence $|S| \ge 9$, contradicting (b). For the last part, let S_{j1}, S_{j2} and S_{j3} be the restrictions of S on A_{j1}, A_{j2} and A_{j3} , respectively. By (a) and (b), one of these sets is of size 2 and the other have size 3. Let $|S_{j1}| = 2$ and $|S_{j2}| = 3, |S_{j3}| = 3$, say. By Fact 10, $S_{j1} = \{x_{j1}, z_{j2}\}$. Then y_{j2} must belong to S_{j2} and it can be checked that S_{j2} cannot contain any neighbor of c_{j2} . Suppose S_{j3} contains a neighbor of c_{j3} . It follows that d_{j3} and its two neighbors in A_{j3} are not in S_{j3} , and thus d_{j3} is the center of a claw (containing y_{j1}) outside S, a contradiction.

Note that Fact 11 holds accordingly for H'_j . We are now ready to describe the clause gadget.

Clause gadget. For each clause C_j consisting of variables c_{j1}, c_{j2} and c_{j3} , let $G(C_j)$ be the graph consisting of two connected components, H_j and H'_j . We call the six vertices c_{jk} and c'_{jk} , $1 \le k \le 3$, the *clause vertices*.

From Fact 11, we immediately have:

Fact 12 $G(C_j)$ admits an optimal claw deletion set of 16 vertices. No optimal claw deletion set in $G(C_j)$ contains all c_{j1}, c_{j2} and c_{j3} , or all c'_{j1}, c'_{j2} and c'_{j3} , or a neighbor of any c'_{jk} .

Finally, the graph G is obtained by connecting the variable and clause gadgets as follows: if variable v_i appears in clause C_j , i.e., $v_i = c_{jk}$ for some $k \in \{1, 2, 3\}$, then

- connect the variable vertex v_{ij} in $G(v_i)$ and the clause vertex c_{jk} in $G(C_j)$ by an edge; v_{ij} is the corresponding variable vertex of the clause vertex c_{jk} , and

- connect the variable vertex v'_{ij} in $G(v_i)$ and the clause vertex c'_{jk} in $G(C_j)$ by an edge; v'_{ij} is the corresponding variable vertex of the clause vertex c'_{jk} .

Fact 13 G has maximum degree 3 and is bipartite.

Proof. It follows from the construction that G has maximum degree 3. To see that G is bipartite, note first that the bipartite graph forming by all variable gadgets $G(v_i)$ has a bipartition (A, B) such that all v_{ij} are in A and all v'_{ij} are in B, and the bipartite graph forming by all clause gadgets $G(C_j)$ has a bipartition (C, D) such that all c_{j1}, c_{j2}, c_{j3} are in C and all $c'_{j1}, c'_{j2}, c'_{j3}$ are in D. Hence, by construction, $(A \cup D, B \cup C)$ is a bipartition of G.

Fact 14 Let S be a claw deletion set of G such that S contains exactly 2m vertices from each $G(v_i)$. Suppose that, for some $1 \leq j \leq m$, S contains none of c_{j1}, c_{j2} and c_{j3} , or none of c'_{j1}, c'_{j2} and c'_{j3} . Then S contains at least 17 vertices from $G(C_j)$.

Proof. Consider the case that S contains none of c_{j1}, c_{j2} and c_{j3} ; the other case is similar. Suppose for the contrary that S contains at most 16 vertices from $G(C_j)$. Then by Facts 12 and 11, the restrictions of S on H_j and on H'_j are optimal claw deletion sets in H_j and H'_j , respectively.

Let v_{rj} , v_{sj} and v_{tj} be the corresponding variable vertices of c_{j1} , c_{j2} and c_{j3} , respectively. By Fact 11 (c), all neighbors of c_{j1} , c_{j2} and c_{j3} in H_j are outside S, implying all v_{rj} , v_{sj} and v_{tj} are in S. By Fact 9 (a), all v'_{rj} , v'_{sj} and v'_{tj} together with their neighbors in $G(v_r)$, $G(v_s)$ and $G(v_t)$ are outside S, implying all c'_{j1} , c'_{j2} and c'_{j3} are in S. This is a contradiction to Fact 11 (c).

Now suppose that G has a claw deletion set S with at most 2nm + 16m vertices. Then by Facts 8 and 11 (a), S has exactly 2nm + 16m vertices, and S contains exactly 2mvertices from each $G(v_i)$ and exactly 16 vertices from each $G(C_i)$.

Consider the truth assignment in which a variable v_i is true if all its associated variable vertices v_{ij} , $1 \le j \le m$, are in S. Note that by Fact 9 (a) and (b), this assignment is well-defined. For each $G(C_j)$, it follows from Facts 12 and 14 that some of c_{j1}, c_{j2}, c_{j3} is outside S and some of $c'_{j1}, c'_{j2}, c'_{j3}$ is outside S as well. That is, the clause C_j contains a true variables and a false variable. Thus, if G admits a claw deletion set S with at most 2nm + 16m then F has a nae assignment.

Conversely, suppose that there is a nae assignment for F. Then a claw deletion set S of size 2nm + 16m for G is as follows. For each variable v_i , $1 \le i \le n$:

- If variable v_i is true, then put all 2m vertices $v_{ij}, b_{ij}^2, 1 \le j \le m$, into S.
- If variable v_i is false, then put all 2m vertices $v'_{ij}, a^2_{ij}, 1 \le j \le m$, into S.

For each clause C_j , $1 \le j \le m$, let v_{rj} , v_{sj} and v_{tj} be the corresponding variable vertices of c_{j1} , c_{j2} and c_{j3} , respectively. Extend S to 16 vertices of $G(C_j)$ as follows; cf. Fact 11 (b).

- If C_j has one true variable and two false variables, say v_r is true, v_s and v_t are false, then put $x_{j1}, z_{j1}, c_{j2}, y_{j2}, d_{j2}, c_{j3}, y_{j3}, d_{j3}, c'_{j1}, y'_{j1}, d'_{j1}, x'_{j2}, z'_{j2}, x'_{j3}, y'_{j3}$ and d'_{j3} into S.
- If C_j has two true variables and one false variable, say v_r and v_s are true, v_t is false, then put $x_{j1}, z_{j1}, x_{j2}, y_{j2}, d_{j2}, c_{j3}, y_{j3}, d_{j3}, c'_{j1}, y'_{j1}, d'_{j1}, c'_{j2}, y'_{j2}, d'_{j2}, x'_{j3}$ and z'_{j3} into S.

By construction, S has exactly $n \times 2m + m \times 16$ vertices and, for every *i* and *j*, $G(v_i) - S$ and $G(C_j) - S$ are claw-free. Therefore, since a clause vertex is in S if and only if the corresponding variable vertex is not in S, G - S is claw-free. Thus, if F can be satisfied by a nae assignment then G has a claw deletion set of size at most 2nm + 16m.

In summary, we obtain:

Theorem 4. For any $d \ge 3$, d-CLAW-VD is NP-complete even when restricted to bipartite graphs of maximum degree d.

Proof. The case d = 3 follows from the previous arguments. For an fixed integer d > 3 and a bipartite graph G of maximum degree 3, let G' be obtained from G by adding, for each vertex v in G, d-3 new vertex v_1, \ldots, v_{d-3} all are adjacent to exactly v. Then G' is bipartite and has maximum degree d. It can be verified immediately that G has a claw deletion set of size at most k if and only if G' has a d-claw deletion set of size at most k.

Bipartite graphs of bounded diameter From Theorems 3 and 4, and (the proof of) Observation 2 we conclude:

Theorem 5. For any $d \ge 2$, d-CLAW-VD is NP-complete even when restricted to bipartite graphs of diameter 3 with only two unbounded vertices.

We remark that MIN *d*-CLAW-VD is polynomially solvable on bipartite graphs of diameter at most two. This can be seen as follows. Let G = (X, Y, E) be a bipartite of of diameter ≤ 2 ; such a bipartite graph is complete bipartite. Note first that X and Y are *d*-claw deletion sets for G. We will see that any optimal *d*-claw deletion set is X or Y or is of the form $(X \setminus X') \cup (Y \setminus Y')$ for some d-1-element sets $X' \subseteq X$ and $Y' \subseteq Y$. (In particular, all optimal *d*-claw deletion sets can be found in $O(n^{d-1})$ time.) Indeed, let S be an optimal *d*-claw deletion set. If $X \subseteq S$, then by the optimality of S, S = X. Similarly, if $Y \subseteq S$, then S = Y. So, let $X' = X \setminus S \neq \emptyset$ and $Y' = Y \setminus S \neq \emptyset$. Then $|X'| \leq d-1$ and $|Y'| \leq d-1$: if $|X'| \geq d$, say, then any vertex in Y' and d vertices in X' together would induce a d-claw in G - S. Thus, by the optimality of S, |X'| = |Y'| = d-1.

Unfortunately, we have to leave open the complexity of d-CLAW-VD on bipartite graphs of diameter 3 with only one vertex of unbounded degree.

3.2 Split graphs

In this subsection, we show that, for any $d \geq 3$, d-CLAW-VD is NP-complete even when restricted to split graphs. Note that split graphs have diameter 3. By Observation 1, however, we will see that d-CLAW-VD is hard even on split graphs of diameter 2. Recall that 1-CLAW-VD and 2-CLAW-VD are solvable in polynomial time on split graphs.

Let $d \geq 3$ be a fixed integer. We reduce (d-1)-HVC to d-CLAW-VD. Our reduction is inspired by the reduction from VERTEX COVER to 3-CLAW-VD in [2]. Let G = (V, E) be a (d-1)-uniform hypergraph with n = |V| vertices and m = |E| edges. We may assume that for any hyperedge $e \in E$ there is another hyperedge $f \in E$ such that $e \cap f = \emptyset$. For otherwise, G has a vertex cover of size $\leq |e| = d - 1$ and therefore (d-1)-HVC is polynomially solvable on such inputs G. We construct a split graph G' = (V', E') with $V' = Q \cup I$, where Q is a clique and I is an independent set, as follows:

- $\ I = \{ v' \mid v \in V \};$
- for each edge $e \in E$, let Q(e) be a clique of size n;
- all sets I and $Q(e), e \in E$, are pairwise disjoint;
- make $\bigcup_{e \in E} Q(e)$ to clique Q;
- for each $v' \in I$ and $e \in E$, connect v' to all vertices in Q(e) if and only if $v \in e$.

The description of the split graph G' is complete. Note that G' has nm + n vertices and $O(n^2m^2)$ edges, and can be constructed in $O(n^2m^2)$ time.

For each $e \in E$, write $e' = \{v' \in I \mid v \in e\}$. By construction, every vertex in Q(e) has exactly d-1 neighbors in I, namely the vertices in e'. Hence, every induced d-claw in G' is formed by a center vertex $x \in Q(e)$ for some $e \in E$ and $e' \cup \{y\}$, where y is any vertex in Q(f), $f \in E$, such that $f \cap e = \emptyset$. It follows that G' contains no induced (d+1)-claws.

Fact 15 If S is a vertex cover in the hypergraph G, then $S' = \{v' \mid v \in S\}$ is a d-claw deletion set in the split graph G'.

Proof. If C is a d-claw in G' with center vertex $x \in Q(e)$ for some $e \in E$ such that $C \cap S' = \emptyset$, then $e' \cap S' = \emptyset$. This means $e \cap S = \emptyset$, contradicting the fact that S is a vertex cover of the hypergraph G.

Fact 16 If S' is a d-claw deletion set in the split graph G' of size $\langle n, then S = \{v \mid v' \in S'\}$ is a vertex cover in the hypergraph G.

Proof. First, for each $e \in E$, $S' \cap e' \neq \emptyset$. For otherwise let $S' \cap e' = \emptyset$ for some $e \in E$. Since |S'| < n, there is a vertex $x \in Q(e) \setminus S'$ and a vertex $y \in Q(f) \setminus S'$ with $f \cap e = \emptyset$. Then x, y and e' induce a *d*-claw in G' - S', a contradiction. We have seen that, for each $e \in E$, $S' \cap e' \neq \emptyset$. Then, with $S = \{v \mid v' \in S'\}$, we have $S \cap e \neq \emptyset$ for all $e \in E$. That is, S is a vertex cover of the hypergraph G.

Fact 17 The size of a smallest vertex cover of G, $OPT_{VC}(G)$, and the size of a smallest d-claw deletion set in G', $OPT_{d-CLAW-VD}(G')$, are equal.

Proof. By Fact 15, $OPT_{d-CLAW-VD}(G') \leq OPT_{VC}(G)$. Let S' be a smallest d-claw deletion set in G'. Then |S'| < n because I minus an arbitrary vertex is a d-claw deletion set in G' with n-1 vertices. Hence, by Fact 16, $S = \{v \mid v' \in S'\}$ is a vertex cover in the hypergraph G with $|S| \leq |S'|$. Thus, $OPT_{VC}(G) \leq OPT_{d-CLAW-VD}(G')$.

We now derive hardness results for d-CLAW-VD and MIN d-CLAW-VD from the above reduction.

Theorem 6. For any fixed integer $d \ge 3$, d-CLAW-VD is NP-complete, even when restricted to

- (i) split graphs without induced (d+1)-claws, and
- (ii) split graphs of diameter 2.

Proof. Part (i) follows from Facts 15 and 16, and the fact that the split graph G' contains no induced (d+1)-claws. Part (ii) follows from (i) and (the proof of) Observation 1.

We remark that both hardness results in Theorem 6 are optimal in the sense that d-CLAW-VD is trivial for graphs without induced d-claws, in particular for graphs of diameter 1, *i.e.*, complete graphs. We also remark that Theorem 6 implies, in particular, that d-CLAW-VD is NP-complete on chordal graphs for any $d \ge 3$, while the complexity of 2-CLAW-VD on chordal graphs is still open (cf. [4,5]).

Since it is UGC-hard to approximate MIN (d-1)-HVC to a factor $(d-1) - \epsilon$ for any $\epsilon > 0$ [13], Fact 17 implies:

Theorem 7. Let $d \ge 3$ be a fixed integer. Assuming the UGC, there is no approximation algorithm for MIN d-CLAW-VD within a factor better than d - 1, even when restricted to split graphs without induced (d + 1)-claws.

We remark that for triangle-free graphs, in particular bipartite graphs, MIN *d*-CLAW-VD and *d*-CLAW-TRANSVERSAL coincide, hence a result in [10] implies that MIN *d*-CLAW-VD admits an $O(\log(d+1))$ -approximation when restricted to bipartite graphs.

4 A polynomially solvable case

In this section, we will show a polynomial-time algorithm solving MIN *d*-CLAW-VD for what we call *d*-block graphs. As *d*-block graphs generalize block graphs, this result extends the polynomial-time algorithm for 2-CLAW-VD on block graphs given in [5] to *d*-CLAW-VD for all $d \geq 2$ on block graphs, and improves the polynomial-time algorithm for MIN 3-CLAW-VD given in [2] on block graphs to 3-block graphs.

Recall that a *block* in a graph is a maximal biconnected subgraph. *Block graphs* are those in which every block is a clique. For each integer $d \ge 2$, the *d*-block graphs defined below generalize block graphs.

Definition 1. Let $d \ge 2$ be an integer. A graph G is d-block graph if, for every block B of G,

- -B is d-claw free,
- for every cut vertex v of G, $N(v) \cap B$ is a clique, and
- the cut vertices of G in B induce a clique.

Note that block graphs are exactly the 2-block graphs and d-block graphs are (d + 1)block graphs, but not the converse. Note also that, for $d \ge 3$, d-block graphs need not to be chordal since they may contain arbitrary long induced cycles. An example of a 3-block graph is shown in Fig. 8.

Let $d \ge 2$ and let G be a d-block graph. Recall that a block in G is an *endblock* if it contains at most one cut vertex. Vertices that are not cut vertices are called *endvertices*. Thus, if u is an endvertex then the block containing u (which may or may not be an endblock) is unique. We call a cut vertex u a *pseudo-endvertex* if u belongs to at most d-2 endblocks and exactly one non-endblock. Thus, for a pseudo-endvertex u, we say that B is the unique block containing u, meaning that B is the unique non-endblock that contains u.

In computing an optimal d-claw deletion set for G, we first observe that there is a solution that does not contain any endvertex. It is by the fact that each block is d-claw free and

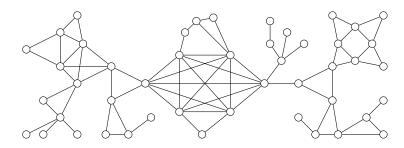


Fig. 8. A 3-block graph.

if an endvertex u is in a solution S then it must be a leaf of some claw centered at a cut vertex v. Then S - u + v is a solution that does not contain the endvertex u. The following lemmas are for pseudo-endvertices.

Lemma 1. Let u be a pseudo-endvertex. Then any d-claw C containing u, if any, is centered at a cut vertex $v \neq u$. Moreover,

- if B is the unique block containing u, then $C \cap B = \{u, v\}$;
- if B' is an endblock containing u, then $C \cap B' = \{u\}$.

Proof. This is because every block is d-claw-free and the neighbors of any cut vertex in any block induce a clique.

An optimal d-claw deletion set for G is also called *solution*.

Lemma 2. There is a solution that contains no pseudo-endvertices.

Proof. Let S be a solution for G and assume that S contains a pseudo-endvertex u. Let B be the unique block of G containing u. Since S - u is not a d-claw deletion set, there is some d-claw C of G outside $S \setminus \{u\}$. Then, of course,

$$C \cap S = \{u\}.\tag{1}$$

By Lemma 1, the center v of C is a cut vertex of G in B, and $C \cap B = \{u, v\}$. Thus, for every $w \in N(v) \cap B$, C - u + w is a d-claw, and by (1), $w \in S$. Hence

$$N(v) \cap B \subseteq S. \tag{2}$$

We now claim that S' = S - u + v is a *d*-claw deletion set (and thus S' is a solution). Indeed, let C' be an arbitrary *d*-claw. If $u \notin C'$ or $v \in C'$ then $C' \cap S' \neq \emptyset$. So let us consider the case in which $u \in C'$ and $v \notin C'$. Then, by Lemma 1, the center v' of C' is a cut vertex of G in B. Hence v' and v are adjacent, and by (2), $v' \in S'$.

We remark that Lemma 2 is the best possible in the sense that a cut vertex u belonging to exactly two non-endblocks may be contained in any solution; take the d-block graph that consists of two d-claws with exactly one common leaf u.

Lemma 3. Let v be a cut vertex and let B be a block containing v. If every vertex in $N(v) \cap B$ is a cut vertex, then $B = N[v] \cap B$. In particular, B is a clique.

Proof. Suppose the contrary that $B \setminus N[v] \neq \emptyset$. Then, as B - v is connected, there is an edge connecting a vertex $w \in N(v) \cap B$ and a vertex $w' \in B \setminus N[v]$. Now, as w is a cut vertex, $N(w) \cap B$ is a clique, implying that vw' is an edge. This is a contradiction, hence $B = N[v] \cap B$.

Lemma 4. If G has at most one non-endblock, then a solution for G can be computed in polynomial time.

Proof. If all blocks of G are endblocks, then G has at most one cut vertex. In this case, G contains a d-claw if and only if G has at least d blocks. If G has at least d blocks, then all d-claws in G are centered at the unique cut vertex v and $\{v\}$ is the solution.

So, let B be the block of G that is not an endblock. Write

- -U' for the set of endvertices in B,
- U for the set of pseudo-endvertices in B,
- -X for the set of vertices in B that belong to $\geq d$ endblocks,
- Y for the set of vertices v in B that belong to exactly d-1 endblocks and $N(v) \cap U \neq \emptyset$,
- $Z = B \setminus (U' \cup U \cup X \cup Y).$

Observe that $X \cup Y \cup Z$ is a *d*-claw deletion set for *G*, hence the size of an solution is at most |X| + |Y| + |Z|.

By Lemma 2, there is a solution not containing any pseudo-endvertex. Such a solution S must contain $X \cup Y$ because every vertex in $X \cup Y$ is the center of a *d*-claw in which all leaves are pseudo-endvertices. Thus, if $Z = \emptyset$, then $S = X \cup Y$.

So, let us assume that $Z \neq \emptyset$. Note that, as $d \ge 2$, every vertex $v \in Z$ is a cut vertex, and by definition of Z, $N(v) \cap B$ contains no pseudo-endvertices. Hence, by Lemma 3, B is a clique. Thus, by definition of Z again, $U = \emptyset$, and therefore $Y = \emptyset$. Now, observe that at most one vertex in Z may not be contained in $S: |Z \setminus S| \le 1$. Indeed, if z_1 and z_2 were two vertices in Z not belonging to S, then z_1, z_2 and d - 1 endvertices adjacent to z_1 would together induce a d-claw outside S. Thus, $|S| \ge |X| + |Z| - 1 = |B| - 1$. On the other hand, note that, for any $v \in B$, B - v is a d-claw deletion set for G. Thus, for any $v \in B$, S = B - v is a solution.

We are now ready to show that MIN *d*-CLAW-VD is polynomially solvable on *d*-block graphs. Our proof is inspired by the polynomial result for CLUSTER-VD on block graphs in [4, Theorem 10]. A *block-cut vertex tree* T of a (connected) graph G has a *node* for each block of G and for each cut vertex of G. There is an edge uv in T if and only if u corresponds to a block containing the cut vertex v of G. It is well known that the block-cut vertex tree of a graph can be constructed in linear time.

Theorem 8. MIN *d*-CLAW-VD *is polynomially solvable on d-block graphs.*

Proof. Let T be the block-cut vertex tree of G. Nodes in T corresponding to blocks in G are *block nodes*. For a block node u we use B(u) to denote the corresponding block in G. Nodes in T corresponding to cut vertices in G are *cut nodes* and are denoted by the same labels.

Choose a node r of T and root T at r. For a node $x \neq r$ of T, let p(x) denote the parent of x in T. Note that all leaves of T are block nodes, the parent of a block node is a cut node, and the parent of a cut node is a block node.

Let u be a leaf of T on the lowest level and let v = p(u) be the parent of u. Note that all children of v correspond to endblocks in G, and if r = p(v), then Lemma 4 is applicable. So, assume $r \neq p(u)$ and write u' = p(v), v' = p(u'). Note that by the choice of u, B' = B(u') is the unique non-endblock containing vertices in B' - v'.

If v has at most d - 2 children, then v is a pseudo-endvertex in G. By Lemma 2, we remove v and all children of v from T.

If v has at least d children, or v has exactly d-1 children and some vertex in $N_G(v) \cap B'$ is a pseudo-endvertex, then put v into the solution S and remove v and all children of v from T. Correctness follows again from Lemma 2.

It remains the case that v have exactly d-1 children and no vertex in $N_G(v) \cap B'$ is a pseudo-endvertex. In particular, every vertex in $N_G(v) \cap B'$ is a cut vertex. Hence B' is a clique by Lemma 3. Moreover, every vertex in $N_G(v) \cap (B' - v')$ belongs to at least d-1 endblocks. Now, note that all d-claws in G containing v contain a vertex in B' - v, and every solution for G not containing pseudo-endvertices must contain at least |B'|-1 vertices in B'. Thus, we put B' - v into solution S and remove the subtree rooted at v' from T, and for each other child $u_i \neq u$ of v', we solve the problem on the subgraph induced by $B(u_i)$ and its children. Note that, by the choice of u, all these subgraphs satisfy the condition of Lemma 4. Finally, it is not hard to check that all of these checks can be done in linear time by working with the block-cut vertex tree T of G.

5 Conclusion

This paper considers the *d*-claw vertex deletion problem, *d*-CLAW-VD, which generalizes the famous VERTEX COVER (that is 1-CLAW-VD) and the CLUSTER-VD (that is 2-CLAW-VD) problems and goes on with the recent study [2] on claw vertex deletion problem, 3-CLAW-VD. It is shown that CLUSTER-VD remains NP-complete on bipartite graphs of maximum degree 3 and, for each $d \geq 3$, *d*-CLAW-VD remains NP-complete on bipartite graphs of degree *d*, and thus a complexity dichotomy with respect to degree constraint. It is also shown that *d*-CLAW-VD remains NP-complete when restricted to split graphs of diameter 2 and to bipartite graphs of diameter 3 (with only two vertices of unbounded degree) and polynomially solvable on bipartite graphs of diameter 2, and thus another dichotomy with respect to diameter. We also define a new class of graphs called *d*-block graphs which generalize the class of block graphs and show that *d*-CLAW-VD is solvable in linear time on *d*-block graphs, extending the algorithm for CLUSTER-VD on block graphs in [5] to *d*-CLAW-VD, and improving the algorithm for (unweighted) 3-CLAW-VD on block graphs in [2] to 3-block graphs.

We note that VERTEX COVER and CLUSTER-VD have been considered by a large number of papers in the context of approximation, exact and parameterized algorithms. As a question for further research we may ask: which known results in case d = 1, 2 can be extended for all $d \ge 3$? We believe that the approaches in [3,19] for CLUSTER-VD could be extensible to obtain a similar parameterized algorithm for d-CLAW-VD for all $d \geq 3$. Finally, recall that VERTEX COVER is polynomially solvable on chordal graphs and CLUSTER-VD is polynomially solvable on split graphs, and that *d*-CLAW-VD is NPcomplete on chordal graphs for $d \geq 3$. Thus it would be interesting to clear the complexity of CLUSTER-VD on chordal graphs [4,5].

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