

The Power of Signaling and its Intrinsic Connection to the Price of Anarchy*

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Abstract

Strategic behaviors often render the equilibrium outcome inefficient. Recent literature on information design, a.k.a. *signaling*, looks to improve equilibria by selectively revealing information to players in order to influence their actions. Most previous studies have focused on the *prescriptive* question of designing optimal signaling schemes. This work departs from previous research by considering a *descriptive* question, and looks to quantitatively characterize the *power of signaling* (PoS), i.e., how much a signaling designer can improve her objective at the equilibrium outcome.

We consider four signaling schemes with increasing power: full information, optimal public signaling, optimal private signaling, and optimal *ex-ante* private signaling. Our main result is a clean and tight characterization of the additional power each signaling scheme has over its predecessors above in the general classes of cost-minimization and payoff-maximization games where: (1) all players minimize non-negative cost functions or maximize non-negative payoff functions; (2) the signaling designer (naturally) optimizes the sum of players' utilities. We prove that the additional power of signaling — defined as the worst-case ratio between the equilibrium objectives of *any* two signaling schemes in the above list — is bounded precisely by the well-studied notion of the *price of anarchy* (PoA) of the corresponding games. Moreover, we show that all these bounds are *tight*.

1 Introduction

A basic lesson from game theory is that strategic behaviors often render the equilibrium outcome inefficient. That is, the objective function value of an equilibrium outcome may be far from that of an optimal outcome in the absence of strategic behaviors. To reduce such inefficiency, one can “tune” the game equilibrium towards a more desirable outcome, and there are two primary ways to achieve this goal: through providing *incentives* or providing *information*. The

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former approach has been widely studied in the celebrated field of mechanism design [27, 29, 7, 9]. This paper, however, focuses on the second approach, namely, improving equilibrium by providing carefully designed information to influence players’ decisions. This falls into the recent flourishing literature on information design, a.k.a., *signaling* or *persuasion* [11, 19]. Researches in this literature so far have mainly focused on computing optimal signaling schemes for either fundamental setups [13, 14, 33, 6] or models motivated by varied applications including auctions [15, 3, 24], public safety and security [34, 30], conservation [35], privacy protection [36], voting [5], congestion games [2, 10, 4], recommender systems [25], robot design [21], etc.

Departing from the theme of all these previous works, this paper considers a different style of question. We look to characterize the *power of signaling (PoS)* — *how much a signaling designer can improve her objective function of the equilibrium and can we quantitatively characterize this power?* To our knowledge, this *descriptive* question has not been formally examined before in the literature of signaling, except for a few studies which implicitly show certain PoS-type results in the special case of non-atomic routing games [11, 10, 26]. Nevertheless, the study of the PoS is extremely well-motivated. It not only deepens our understanding about signaling as an important “knob” to influence equilibrium, but also justifies the value of previous prescriptive studies of optimal signaling design — after all, the designed optimal signaling schemes are useful in practice only when the power of signaling is not negligible. Thus PoS is an important measure when determining the adoption of a signaling scheme in practice, especially when its tradeoff with other potential drawbacks such as communication costs [17] and fairness concerns [18] need to be balanced.

We focus on the general classes of cost-minimization games and payoff-maximization games, where each player minimizes a non-negative cost function or maximizes a non-negative payoff function. These classes of games are often studied in the literature of the price of anarchy (PoA) [31], and include many widely studied examples such as routing games, congestion games, most formats of auctions, valid utility games [32], etc. Like all standard models of signaling, players’ utilities depend on a common random state of nature θ , which is drawn from a publicly known prior distribution. A signaling designer, referred to as the *sender*, has an informational advantage and can access the realized state θ . The sender is equipped with the natural objective of optimizing the total welfare, i.e., sum of players’ utilities.

Power of Signaling (PoS). Our goal is to formally quantify the relative power of different types of signaling schemes as they become less constrained. In particular, we consider four types of signaling schemes with increasing power: full information (FI), optimal public signaling (Pub), optimal private signaling (Pri), and optimal *ex-ante* private signaling (exP). FI is a natural *benchmark* without any strategic use of information whereas Pub, Pri, exP are arguably the three most widely studied schemes in previous literature.¹ The power of signaling of scheme B

¹Public and private signaling has been extensively studied in previous works. Several recent works study *ex-ante* private signaling with motivations from recommender systems [6, 4, 33].

over A for *any* A preceding B in list $\{\text{FI}, \text{Pub}, \text{Pri}, \text{exP}\}$ — termed $\text{PoS}(B:A)$ — is defined as the ratio of the sender’s utilities from scheme A and B . This ratio is *at least* 1 for cost-minimization games and *at most* 1 for payoff-maximization games (the same as the range of the price of anarchy). Moreover, the further it is from 1, the more powerful scheme B is than scheme A .

Characterizations of PoS. Our main result is a clean and tight characterization about the power of signaling. Concretely, for any cost-minimization game with a random state, we prove that all the aforementioned PoS ratios are upper bounded by the maximum PoA of its corresponding realized games. Moreover, all these upper bounds are tight in the following sense: for any ratio $r \geq 1$ and any scheme A preceding B in the list $\{\text{FI}, \text{Pub}, \text{Pri}, \text{exP}\}$, there exists a Bayesian cost-minimization game where all of its realized games have $\text{PoA} = r$ and moreover $\text{PoS}(B:A) = r$ as well. We show that exactly the same results hold for payoff-maximization games — the PoSs are similarly bounded by PoA and all the bounds are tight.²

Our results reveal the intrinsic connections between the power of signaling and price of anarchy. Prior to this work, it was not clear that these two concepts are inherently related — PoA characterizes the worst-case equilibrium welfare whereas PoS characterizes how much information can be strategically used to improve welfare. To our knowledge, recent work [26] is the only one to observe this connection but only for the power of public signaling over full information in non-atomic routing with affine latency functions. Our results are more systematic and general. An interesting computational implication of our characterization is that the full information scheme always serves as an r approximation simultaneously for optimal public signaling, optimal private signaling and optimal ex-ante private signaling for cost-minimization or reward-maximization games, where r is the worst PoA among realized games.

2 Preliminaries

2.1 Cost-Minimization/Payoff-Maximization Games

A cost-minimization game G is a standard strategic game where each player i minimizes a non-negative cost function $c_i \geq 0$. Let n denotes the number of players in the game. Each player $i \in [n] = \{1, \dots, n\}$ has action space S_i . Let $S = S_1 \times S_2 \cdots S_n$ denote the space of action profiles and $s \in S$ is a generic action profile. A (randomized) mixed strategy for player i is a distribution \mathbf{x}_i over S_i where $x_i(s_i)$ is the probability of taking action s_i . By convention, \mathbf{x} denotes the profile of mixed strategies for all players, and \mathbf{x}_{-i} denotes all the mixed strategies excluding i ’s. With slight abuse of notation, let $c_i(\mathbf{x}) = \mathbf{E}_{s_i \sim \mathbf{x}_i, \forall i} c_i(s)$ denote the expected utility of player i under mixed strategy \mathbf{x} . There is also a *global objective* which is simply to minimize the sum of the total costs $C(\mathbf{x}) = \sum_i c_i(\mathbf{x})$.

We adopt the standard mixed-strategy *Nash equilibrium* (NE) as the solution concept. A

²Instead of FI, another natural benchmark scheme is to reveal *no* information. The PoSs compared to this benchmark turn out to be unbounded, which we show in Appendix A.

strategy profile \mathbf{x}^* is a NE if for each player i , $c_i(\mathbf{x}^*) \leq c_i(\mathbf{x}_i, \mathbf{x}_{-i}^*)$ for any $\mathbf{x}_i \in \Delta(S_i)$. Let X^* denote the set of all NEs. The well-studied concept of the price of anarchy (for mixed equilibria) for a cost-minimization game is defined as follows [22, 23, 31, 16]

$$\text{POA} = \frac{\max_{\mathbf{x}^* \in X^*} C(\mathbf{x}^*)}{\min_{s \in S} C(s)} \in [1, \infty) \quad (1)$$

In other words, the POA is the ratio between the worst Nash equilibrium and the optimal social outcome.

Remark 1. *The PoA can also be defined with respect to pure Nash equilibrium in which case X^* consists of all pure equilibria. Since not every game admits a pure Nash equilibrium, in striving for generality, we choose to analyze the version w.r.t. to mixed equilibria since they always exist in finite games as well as in many infinite games. However, all our results — both upper and lower bound proofs — hold for pure equilibria as well, so long as they exist.*

Payoff-maximization games are defined similarly; here each player i maximizes expected payoff $u_i(\mathbf{x}) \geq 0$. The global objective is to maximize $U(\mathbf{x}) = \sum_i u_i(\mathbf{x})$. The price of anarchy here is defined similarly as $\text{POA} = \frac{\min_{\mathbf{x}^* \in X^*} U(\mathbf{x}^*)}{\max_{s \in S} U(s)}$, which now lies in $[0, 1]$.

This paper concerns games with uncertainty. Specifically, players' payoffs depend also on a *random* state of nature θ drawn from support Θ with distribution λ . We use $c_i^\theta(s)/u_i^\theta(s)$ to denote the cost/payoff function at state θ . Such a *Bayesian game* is denoted by $\{G^\theta\}_{\theta \sim \lambda}$. As is standard in information design, the prior distribution λ is publicly known to every player. We assume Θ to be finite for ease of notation, and use $\lambda(\theta)$ to denote the probability of state θ . However, all our results hold for infinite state space.

2.2 Signaling Schemes and Equilibrium Concepts

This paper adopts the perspective of an informationally advantaged *sender* who has privileged access to the *realized* state θ and would like to strategically signal this information to players in order to influence their actions. The sender is equipped with the natural objective of optimizing the sum of the players' utilities, i.e., the global objective $C(\mathbf{x})$ or $U(\mathbf{x})$, at equilibrium. We consider three natural types of signaling schemes with increasing generality.

Public Signaling. At a high level, a *public signalling scheme* constructs a random variable σ from support Σ — called the *signal* — that is correlated with the state of nature θ . The scheme then sends the sampled signal σ publicly to all players, which carries information about the state θ due to their correlation. Such a public scheme φ can be fully described by variables $\{\varphi(\sigma; \theta)\}_{\sigma \in \Sigma, \theta \in \Theta}$ where $\varphi(\sigma; \theta)$ is the probability of sending signal σ *conditioned* on state of nature θ . Adopting the standard information design assumption [20, 19], the sender commits to the signaling scheme before state θ is realized. Therefore, φ is publicly known to all players. The probability of sending signal σ equals $\Pr(\sigma) = \sum_\theta \lambda(\theta) \varphi(\sigma; \theta)$. Upon receiving signal σ , all

players perform a standard Bayesian update and infer the following posterior probability about the state θ : $\mathbf{Pr}(\theta|\sigma) = \lambda(\theta)\varphi(\sigma; \theta)/P(\sigma)$.

Since all players receive the same information, the game will be played according to the expected cost $\bar{c}_i(s; \sigma) = \sum_{\theta} \mathbf{Pr}(\theta|\sigma)c_i^{\theta}(s)$ or $\bar{u}_i(s; \sigma) = \sum_{\theta} \mathbf{Pr}(\theta|\sigma)u_i^{\theta}(s)$ for all i and signal σ . We assume that players will reach a NE of this average game. Let $C(\sigma)$ denote the sender's expected cost at equilibrium under signal σ and $C(\varphi) = \sum_{\sigma} \mathbf{Pr}(\sigma)C(\sigma)$ denote the expected sender cost under signaling scheme φ . Like the PoA literature, when there are multiple Nash equilibria, we always adopt the *worst* one in our analysis. Notations for payoff maximization are defined similarly.

Private Signaling. Private signaling relaxes public signaling by allowing the sender to send different, and possibly correlated, signals to different players. Specifically, let Σ_i denote the set of possible signals to player i and $\Sigma = \Sigma_1 \times \dots \times \Sigma_n$ denote the set of all possible signal profiles. With slight abuse of notation, a private signaling scheme can be similarly captured by variables $\{\varphi(\sigma; \theta)\}_{\theta \in \Theta, \sigma \in \Sigma}$. When signal profile σ is restricted to have the same signal to all players, this degenerates to public signaling. Private signaling leads to a truly Bayesian game where each player holds different information about the state of nature. The standard solution concept in this case is the *Bayes correlated equilibrium* (BCE) introduced by Bergemann and Morris [1], which consists of all outcomes that can possibly arise at Bayes Nash equilibrium under all possible signaling schemes. Standard revelation-principle type argument shows that signals of private signaling schemes in a BCE can be interpreted as *obedient action recommendations* [20, 1, 13]. That is, Σ_i can W.L.O.G. be S_i and $\Sigma = S$. An action recommendation s_i to player i is *obedient* if following this recommended action is indeed a best response for i , or formally, for any $s_i, s'_i \in S_i$ we have

$$\sum_{\theta \in \Theta, s_{-i} \in S_{-i}} \varphi(s_i, s_{-i}; \theta) \lambda(\theta) c_i^{\theta}(s_i, s_{-i}) \geq \sum_{\theta \in \Theta, s_{-i} \in S_{-i}} \varphi(s_i, s_{-i}; \theta) \lambda(\theta) c_i^{\theta}(s'_i, s_{-i}) \quad (2)$$

Ex-Ante Private Signaling. Motivated by recommender system applications, recent works [33, 6, 4] relax the obedience constraints (2) of BCE to a *coarse correlated equilibrium* type of obedience constraints, described as follows:

$$\sum_{\theta \in \Theta, s \in S} \varphi(s_i, s_{-i}; \theta) \lambda(\theta) c_i^{\theta}(s_i, s_{-i}) \geq \sum_{\theta \in \Theta, s \in S} \varphi(s_i, s_{-i}; \theta) \lambda(\theta) c_i^{\theta}(s'_i, s_{-i}), \forall s'_i \in S_i. \quad (3)$$

That is, for any player i , following the recommendation is better than opting out of the signaling scheme and acting just according to his prior belief. A signaling scheme satisfying Constraint (3) is dubbed an *ex-ante* private scheme [6, 4].

3 The Power of Signaling (PoS)

We now formalize the *Power of Signaling* (PoS) in cost-minimization and payoff-maximization games. Intuitively, the PoS characterizes how much additional power a class of signaling schemes has over another. Formally, let Φ^a and Φ^b be two classes of signaling schemes (e.g., public and private schemes). We say Φ^b is *less restricted* than Φ^a , conveniently denoted as $\Phi^a \subseteq \Phi^b$, if $\varphi \in \Phi^b$ whenever $\varphi \in \Phi^a$.

Definition 1 (PoS of Φ^b over Φ^a). *For any two classes of signaling schemes Φ^a, Φ^b where Φ^b is less restricted than Φ^a (i.e., $\Phi^a \subseteq \Phi^b$), the power of signaling of Φ^b over Φ^a , or $PoS(\Phi^b : \Phi^a)$ for short, is defined as*

$$PoS(\Phi^b : \Phi^a) = \frac{\min_{\varphi \in \Phi^a} C(\varphi)}{\min_{\varphi \in \Phi^b} C(\varphi)} \left(\text{or } \frac{\max_{\varphi \in \Phi^a} U(\varphi)}{\max_{\varphi \in \Phi^b} U(\varphi)} \right),$$

for cost-minimization (or payoff-maximization) games.

In other words, PoS is the ratio between the objectives of the optimal scheme from signaling class Φ^a and that from a less restricted class Φ^b . Similar to the PoA ratio, $PoS(\Phi^b : \Phi^a) \geq 1$ for cost-minimization games, and the larger this ratio is, the more powerful Φ^b is over Φ^a . In contrast, $PoS(\Phi^b : \Phi^a) \leq 1$ for payoff-maximization games, and the smaller this ratio is, the more powerful Φ^b is over Φ^a . If both the numerator and denominator are 0, we say the PoS is 1; if only the denominator is 0, the PoS is $+\infty$.

Though PoS is well-defined for any two classes of signaling schemes, in this paper we primarily consider the following well-studied classes of signaling schemes:

- Φ^1 or **FI**: full information ;
- Φ^2 or **Pub**: public signaling schemes;
- Φ^3 or **Pri**: private signaling schemes;
- Φ^4 or **exP**: ex-ante private signaling schemes.

The full information class **FI** only contains a single signaling scheme, i.e., fully revealing the state θ . This serves as a benchmark scheme where information is not strategically signaled. Another natural benchmark scheme is to reveal *no* information. We show in Appendix A that the PoS compared to this benchmark turns out to be unbounded.

4 PoS in Cost-Minimization Games

The main result of this section is the following tight characterization about the PoS ratios in cost-minimization games.

Theorem 1. For any Bayesian cost-minimization game $\{G^\theta\}_{\theta \sim \lambda}$, let $\text{PoA}_{\max} = \max_{\theta} \text{PoA}(G^\theta)$ denote the worst PoA ratio among game G^θ s. We have

$$\text{PoS}(\Phi^j : \Phi^i) \leq \text{PoA}_{\max}, \quad \forall 1 \leq i < j \leq 4. \quad (4)$$

Moreover, these upper bounds are all tight in the following sense: for any $r \geq 1$ and $1 \leq i < j \leq 4$, there exists a Bayesian cost-minimization game $\{G^\theta\}_{\theta \sim \lambda}$ with $\text{PoA}(G^\theta) = r$ for any θ and $\text{PoS}(\Phi^j : \Phi^i) = r$ as well.

The remainder of this section is devoted to the proof of Theorem 1. The following simple observation follows from Definition 1 of the PoS, and will be useful for proving the tightness of the bounds in Inequality (4).

Fact 1. For any $1 \leq i < j < j' \leq 4$, we have

$$\text{PoS}(\Phi^j : \Phi^i) \leq \text{PoS}(\Phi^{j'} : \Phi^i).$$

As a consequence of Fact 1, if we prove the tightness of $\text{PoS}(\Phi^2 : \Phi^1)$, i.e., $\text{PoS}(\text{Pub:FI})$ by constructing an example with $\text{PoS}(\text{Pub:FI}) = \text{PoA}_{\max}$, the example must satisfy $\text{PoS}(\text{Pri:FI}) = \text{PoS}(\text{exP:FI}) = \text{PoA}_{\max}$ as well, implying their tightness. Therefore, to prove the tightness of Inequality (4), we only need to prove the tightness of $\text{PoS}(\text{Pub:FI})$, $\text{PoS}(\text{Pri:Pub})$ and $\text{PoS}(\text{exP:Pri})$.

4.1 A Simultaneous Proof of all the PoS Upper Bounds

We first prove all the upper bounds in Inequality (4) through a unified result, summarized in the following theorem.

Theorem 2. For any Bayesian cost-minimization game $\{G^\theta\}_{\theta \sim \lambda}$, we have $\text{PoS}(\Phi^b : \Phi^a) \leq \max_{\theta} \text{PoA}(G^\theta)$ for any two classes of signaling schemes Φ^a, Φ^b satisfying $\Phi^a \subseteq \Phi^b$ and that the full information scheme is contained in Φ^a .

Proof. Let $\text{PoA}_{\max} = \max_{\theta} \text{PoA}(G^\theta)$ be the worst (i.e., the maximum) price of anarchy ratio among game G^θ s. Denote by $C^*(G_\theta) = \min_{s \in S} C(s)$ the minimum total social cost among all outcomes (not necessarily an equilibrium) for game G_θ ; let $s^{\theta*}$ be an strategy profile that achieves $C^*(G_\theta)$. φ^0 denotes the full information revelation scheme.

Observe that for any signaling scheme φ we must have $C(\varphi) \geq \sum_{\theta} \lambda(\theta) C^*(G^\theta)$ because regardless of how players act in the scheme φ , its expected total cost can never be less than the

minimum possible total cost $\sum_{\theta} \lambda(\theta) C^*(G^{\theta})$. Now since Φ^a contains φ^0 , we thus have

$$\begin{aligned} \min_{\varphi' \in \Phi^a} C(\varphi') &\leq C(\varphi^0) \\ &= \sum_{\theta \in \Theta} \lambda(\theta) C(G_{\theta}) \\ &\leq \sum_{\theta \in \Theta} \lambda(\theta) \cdot r C^*(G_{\theta}) \\ &\leq r C(\varphi), \text{ for any scheme } \varphi \end{aligned}$$

where $C(G_{\theta})$ is the worst (i.e., maximum) equilibrium cost of game G^{θ} and the second inequality is by the definition of the price of anarchy. As a result, $\text{PoS}(\Phi^b : \Phi^a) = \frac{\min_{\varphi' \in \Phi^a} C(\varphi')}{\min_{\varphi \in \Phi^b} C(\varphi)} \leq r$, as desired. \square

4.2 Tightness of the Upper-Bound for PoS(Pub:FI)

Non-Atomic Routing. It turns out that all the PoS bounds in Theorem 1 are tight in a special and well-studied class of cost-minimization games, i.e., non-atomic routing. The game takes place on a directed graph $G = (V, E)$ with V as the vertex set and E as the edge set. There is a continuum of players, each controlling a negligible amount of flow characterized by a pair of nodes (s, t) where $s \in V$ is the starting node of the flow and $t \in V$ is its destination. In non-atomic routing with incomplete information, each edge $e \in E$ can be described by a congestion function $c_e^{\theta}(x)$ which depends on the total amount of flow x on edge e as well as a random state of nature $\theta \in \Theta$. Each player (s, t) optimizes her own utility by taking a minimum-cost directed path from s to t . The sender minimizes overall congestion cost. There is an essentially unique pure Nash equilibrium for non-atomic routing under a public scheme. Therefore, equilibrium selection is not an issue in non-atomic routing.

We now show the tightness of PoS(Pub:FI) for any ratio $r \geq 1$ via a non-atomic routing game example, which implies the tightness of PoS(Pri:FI) and PoS(exP:FI) by Fact 1. Consider a variant of Pigou's example [28], as depicted in Figure 1, where cost functions are described on each edge. The traffic demand from s_1 to t is set as $d(\alpha) = \left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}}$ whereas the demand from s_2 to t is $1 - d(\alpha)$. Each state of nature occurs with probability 0.5.



Figure 1: A Tight Example for PoS(Pub:FI)

First, we compute the price of anarchy (PoA) of each game, as a function of α . Clearly, at equilibrium no flow will pass through the edge with cost 2 since deviating to the edge with cost 0 is strictly better. It is easy to see that at equilibrium all flow will go through the edge with cost x^α , leading to total congestion 1 at equilibrium. The optimal flow, however, is that all flow at s_1 goes through edge (s_1, t) and all flow at s_2 goes through (s_2, t) , leading to minimum total congestion $\left(\frac{1}{\alpha+1}\right)^{\frac{(\alpha+1)}{\alpha}} + 1 - \left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}}$. Therefore, the PoA as a function of α in this instance is

$$\text{PoA} = \frac{1}{\left(\frac{1}{\alpha+1}\right)^{\frac{(\alpha+1)}{\alpha}} + 1 - \left(\frac{1}{\alpha+1}\right)^{\frac{1}{\alpha}}}, \quad (5)$$

which is a continuous function of $\alpha > 0$. Standard analysis shows that this function tends to ∞ as $\alpha \rightarrow \infty$ and tends to 1 as $\alpha \rightarrow 0^+$. For the special case of $\alpha = 0$, it can be directly verified that the PoA ratio is 1. Therefore, this PoA ratio can take any value $r \geq 1$ with a proper choice of α .

We now consider the cost of the full information scheme FI. In this case, all flow will always go through the edge with cost 1 at equilibrium, leading to total cost 1. The optimal public scheme in this example happens to be revealing no information. Without being able to distinguish the zero-cost edge from the edge of cost 2, all flow at s_2 will take the (s_2, t) path. This achieves the minimum total cost, rendering the PoS(Pub:FI) ratio equal the PoA for any n .

4.3 Tightness of the Upper-Bound for PoS(Pri:Pub)

We now show the tightness of PoS(Pri:Pub) for any ratio $r \geq 1$, which implies the tightness of PoS(exP:Pub) by Fact 1. We construct a Bayesian non-atomic routing game as depicted in Figure 2, which can be viewed as another variant of Pigou's example.³ There is a 1 unit of flow demand from s to t . Each state has equal probability 0.5.



Figure 2: A Tight Example for PoS(Pri:Pub)

Similarly to the calculation for the Example in Figure 1, the PoA for each game here also equals that as described in Equation (5). We now argue that the expected cost of any public signaling scheme will equal 1 in this example — i.e., all public schemes are equally bad and will not be able to reduce any congestion. Any public signal gives the same information about

³This example generalizes an earlier example observed by Cheng, Dughmi and Xu [11]. It also avoided their use of the (unrealistic) ∞ flow cost, and relies on a less trivial analysis due to the finite edge cost.

the game state to all players. Let $\lambda \in [0, 1]$ denote the posterior probability of θ_1 given any public signal. W.l.o.g., consider the case $\lambda \geq 0.5$ since the other case is symmetric. The top edge will have expected cost $2\lambda + x^\alpha(1 - \lambda) > 1$ for any $x > 0$, therefore this edge will never be taken since the bottom edge is a strictly better choice. Consequently, players will be choosing between the middle edge, with expected cost function $f(x) =: x^\alpha\lambda + 2(1 - \lambda)$, and the bottom edge with fixed cost 1. Note that $f(0) = 2(1 - \lambda) \leq 1$ and $f(1) = 1 - \lambda \geq 1$. Therefore, at the unique equilibrium, the amount of flow through the middle edge will be exactly the x^* such that $f(x^*) = 1$ whereas the remaining flow will be through the bottom edge. The expected total cost at this equilibrium is 1.

Finally, we show that the optimal private signaling will be able to induce the optimal flow, concluding our tightness proof. Consider the following private signaling scheme: revealing full information to a randomly selected $x^* = (1/(\alpha + 1))^{1/\alpha}$ fraction of the players, and revealing no information to the remaining players. The x^* fraction of players given the full information has a dominant action of taking the edge with cost x^α at the told state. For the remaining players with no information, their cost of taking either the top or the middle edge will be at least $2 \times (1/2) + (x^*)^\alpha \times (1/2) > 1$. Therefore, their optimal response will be taking the bottom edge. This leads to exactly the optimal flow for each state, as desired.

4.4 Tightness of the Upper-Bound for PoS(exP:Pri)

Finally, we prove the tightness of PoS(exP:Pri). Consider the non-atomic routing game depicted in Figure 3.⁴ There is one unit of flow from s to t and the two states θ_1, θ_2 occurs with equal probability 0.5.



Figure 3: A Tight Example for PoS(exP:Pri)

Similar to the analysis for previous examples, the PoA for each game equals also the function described in Equation (5), which takes value in $[1, \infty)$ as we vary the parameter α .

Next we argue that the optimal private signaling scheme is full-revelation, with cost 1. Recall from the preliminary section, any private scheme can be viewed as obedient action recommendations. Numbering the edges from the top to the bottom as edge 1, 2, 3, 4, we claim that any obedient action recommendation should never recommend edge 2 and 3. This is because if a non-zero amount of players are recommended, e.g., to edge 2, switching to edge 1 or 4 will be strictly

⁴This example generalizes an earlier example observed by Cheng [8] after removing (unrealistic) ∞ cost edges. Note that the analysis of the example becomes less obvious without the use of ∞ costs.

better. In particular, if the players are certain that they are at state θ_1 , they will prefer to switch to edge 4. Otherwise, there is non-zero probability that they are at state θ_2 . In this case, switching to edge 1 is strictly better. Consequently, the optimal private scheme can be captured by two variables: (1) x : the amount of flow recommended to edge 4 at state θ_1 (thus edge 1 consumes the remaining $1 - x$ amount); (2) y : the amount of flow recommended to edge 1 at θ_2 . It can be shown that the parameterized total expected cost $[x^{\alpha+1} + (\alpha+1)(1-x) + y^{\alpha+1} + (\alpha+1)(1-y)]/2$ is minimized at $x = 1, y = 1$, i.e., the full information scheme.

Finally, we show that the optimal ex-ante private scheme achieves the minimum possible total social cost, concluding the tightness proof of $\text{PoS}(\text{exP:Pri})$. In particular, consider the ex-ante private scheme that induces the optimal flow by recommending a randomly chosen $(1/(\alpha+1))^{1/\alpha}$ amount of the flow to the edge with cost x^α at any state and the remaining flow to the edge with cost 1. This is indeed obedient in the ex-ante sense because opting out from this scheme and taking any path will lead to cost at least $(\alpha+1 + \frac{1}{\alpha+1})/2$, which is at least 1 and thus is larger than its expected utility in the scheme (less than 1).

5 PoS in Payoff-Maximization Games

In this section, we show that a similar tight characterization as in Theorem 1 holds for payoff-maximization games as well.

Theorem 3. *For any Bayesian payoff-maximization game $\{G^\theta\}_{\theta \sim \lambda}$, let $\text{PoA}_{\min} = \min_{\theta} \text{PoA}(G^\theta)$ denote the worst-case PoA ratio among game G^θ s. We have*

$$\text{PoS}(\Phi^j : \Phi^i) \geq \text{PoA}_{\min}, \quad \forall 1 \leq i < j \leq 4. \quad (6)$$

Moreover, these lower bounds are all tight in the following sense: for any $r \in (0, 1]$ and any $1 \leq i < j \leq 4$, there exists a Bayesian payoff-maximization game $\{G^\theta\}_{\theta \sim \lambda}$ with $\text{PoA}(G^\theta) = r$ for any θ and $\text{PoS}(\Phi^j : \Phi^i) = r$ as well.

Remark 2. *In payoff-maximization games, the smaller PoS is, the more powerful signaling is. Therefore, Inequality (6) is a lower bound for the PoS ratio but an upper bound for the power of signaling. Similar discrepancies also arise in the definition of the PoA for payoff-maximization game [31].*

The remaining of this section is devoted to the proof of Theorem 3. A simultaneous proof of all the PoS lower bounds in Inequality (6) follow an analogous argument as that for Theorem 2, and thus is omitted here. We only prove their tightness. One might wonder whether the tightness proof here can be adapted from that for cost-minimization games by simply reversing minimizing cost functions to be maximizing their negations (plus a large constant to make it positive). The answer turns out to be *no*. We illustrate the detailed reasons in the Appendix B, but at a high level there are at least two reasons. First, the optimal flow for congestion

minimization may not be optimal any more in the negation of the game. Second, some PoA ratios cannot be achieved in the negation of routing games.

Our tightness proof here requires carefully constructed payoff-maximization games and analysis. Thanks to Fact 1, we will only need to prove the tightness of $\text{PoS}(\text{Pub:FI})$, $\text{PoS}(\text{Pri:Pub})$ and $\text{PoS}(\text{exP:Pri})$.

5.1 Tightness of the Lower-Bound for $\text{PoS}(\text{Pub:FI})$

Consider the following game played by two players P1, P2 where α, ϵ are parameters satisfying $\alpha - 1 > 2\epsilon > 0$. Each player has two actions, conveniently denoted as A, B . There are two states θ_1, θ_2 with equal probability 0.5. The only difference of the two states is the payoffs for action profile (B, A) , which is $(1, \alpha)$ at state θ_1 and $(\alpha, 1)$ at state θ_2 .

		P2	
		A	B
P1	A	$(1 + \epsilon, 1)$	$(1 - \epsilon, 1 - \epsilon)$
	B	$(1, \alpha)$	$(1, 1 + \epsilon)$
		θ_1	

		P2	
		A	B
P1	A	$(1 + \epsilon, 1)$	$(1 - \epsilon, 1 - \epsilon)$
	B	$(\alpha, 1)$	$(1, 1 + \epsilon)$
		θ_2	

We first consider full information (FI). At state θ_1 , action A is a strictly dominant action for Player 2. This implies that both players choosing action A is the only Nash equilibrium, resulting in sender utility $2 + \epsilon$. However, the optimal outcome at state θ_1 is that Player 1 chooses B and Player 2 chooses A , leading to total payoff $\alpha + 1$ ($> 2 + 2\epsilon$ since $\alpha - 1 > 2\epsilon$). State θ_2 is symmetric. The expected utility of full information is $2 + \epsilon$, and the PoA of each game is $\frac{2+\epsilon}{\alpha+1}$.

We now show that the optimal public signaling scheme can maximize total payoff. Consider the scheme which reveals no information at all. The only change is that the expected payoffs of action profile (B, A) becomes $\frac{\alpha+1}{2}$ for both players. We see then, that A remains a strictly dominant strategy for Player 2 and B becomes a strictly dominant action for Player 1. The only equilibrium is the action profile (B, A) , resulting in expected sender utility $\alpha + 1$. Therefore, $\text{PoS}(\text{Pub:FI}) = \frac{2+\epsilon}{\alpha+1}$, equaling the price of anarchy. If $\alpha \rightarrow 1$, the PoS ratio tends to 1, and $\alpha \rightarrow \infty$ with a fixed ϵ makes the PoS ratio continuously tend to 0. Setting $\alpha = 1 + \epsilon$ gives a trivial case where the PoS and PoA are both 1. Thus, the ratio achieves all possible values within $(0, 1]$.

5.2 Tightness of the Lower-Bound for $\text{PoS}(\text{Pri:Pub})$

The Robber's Game. As a means of illustrating our constructed game, consider two robbers robbing a bank. They have triggered the alarm, and are pressed for time. As they enter the vault, they have to make a decision. In the vault are two *safes*, and a huge pile of cash. There are three options: attempt to crack Safe 1 (action S_1), Safe 2 (action S_2), or simply take as

much cash as they can carry (action C), worth α . In order to protect the bank's valuable items, one of the safes is a decoy safe, and is *empty*. Inside the other safe are two objects: a gold bar worth $\alpha + \epsilon$, where $1 \geq \alpha > \epsilon > 0$, and an extremely delicate but valuable crystal worth 1. The two robbers (players) have very different skill sets. Player 1 (P1) is a lock picking expert, and Player 2 (P2) is a demolitions expert. The players' payoffs equal whatever they individually steal from the vault. Only P2 is capable of carrying the crystal without breaking it. If P1 cracks the safe using his lockpicking, P2 can take the crystal on the way out, leaving P1 with the gold bar. However, if P2 cracks the safe, the crystal will be destroyed due to his explosions, leaving him only the gold bar. In addition, if both players try to crack the same safe, they get in each others' way and are forced to leave with nothing.

		P2		
		S_1	C	S_2
P1	S_1	(0, 0)	(0, α)	(0, $\alpha + \epsilon$)
	C	(α , 0)	(α , α)	(α , $\alpha + \epsilon$)
	S_2	($\alpha + \epsilon$, 1)	($\alpha + \epsilon$, 1)	(0, 0)

θ_1

		P2		
		S_1	C	S_2
P1	S_1	(0, 0)	($\alpha + \epsilon$, 1)	($\alpha + \epsilon$, 1)
	C	(α , $\alpha + \epsilon$)	(α , α)	(α , 0)
	S_2	(0, $\alpha + \epsilon$)	(0, α)	(0, 0)

θ_2

Figure 5: Payoffs of the robber's game achieving tight PoS(Pri:Pub); each state occurs with probability 0.5.

Concretely, the payoff matrix for the aforementioned game is in the above tables. At θ_1 , safe 1 is empty, and at θ_2 , safe 2 is empty. Note that θ_2 simply exchanges the payoffs of strategies S_1 and S_2 symmetrically for both players. A-priori, both robbers do not know the state and share the common uniform random prior. The sender is a heist leader and knows which safe is which. The sender gets to pocket a cut of the total haul, so naturally she is interested in maximizing the total utility. In public signaling, both robbers use the same radio to communicate with the sender, but in private signaling each player has his own communication radio.

We first calculate the PoA for the game at each state. W.l.o.g., we consider equilibria for state θ_1 as θ_2 is symmetric. Since action C dominates action S_1 (i.e., cracking the empty safe) for both players, without loss of generality we will assume both players do not play action S_1 in our following analysis. It is easy to see that (S_2, C) and (C, S_2) are the only two pure Nash equilibria after excluding action S_1 . We now consider mixed equilibrium in this game. Note that since both players have only two actions, both players must randomize in any mixed equilibrium. Let $\lambda_1, \lambda_2 \in (0, 1)$ denote the probability that player 1 and 2 play C , respectively. We have $\lambda_2(\alpha + \epsilon) = \alpha$ since player 1's both actions must be equally good, and similarly $\lambda_1\alpha + (1 - \lambda_1)(\alpha + \epsilon) = \alpha$. This implies $\lambda_1 = 1/(1 + \epsilon)$ and $\lambda_2 = \alpha/(\alpha + \epsilon)$. The total expected payoff of this mixed strategy equilibrium is $\alpha + (\alpha + \epsilon)/(1 + \epsilon)$, which is the smallest equilibrium total payoff. The largest social payoff is however $\alpha + \epsilon + 1$. Therefore, the PoA of this game is $\frac{2\alpha + \epsilon + \alpha\epsilon}{(1 + \alpha + \epsilon)(1 + \epsilon)}$.

We now show that the optimal public signalling scheme is to reveal full information, assuming worst-case equilibrium selection. We prove this by arguing that for any public signal with

posterior probability $p \in [0, 1]$ of state θ_1 , the sender's utility is at most $U_0 = \alpha + (\alpha + \epsilon)/(1 + \epsilon)$ in the worst equilibrium, which is the sender utility of full information revelation. This follows a case analysis, depending on whether p is between $\frac{\epsilon}{\alpha + \epsilon}$ and $\frac{\alpha}{\alpha + \epsilon}$, greater than or equal to $\frac{\alpha}{\alpha + \epsilon}$, or less than or equal to $\frac{\epsilon}{\alpha + \epsilon}$. By cases:

1. When $p(\alpha + \epsilon) < \alpha$ and $(\alpha + \epsilon)(1 - p) < \alpha$, i.e., $\frac{\epsilon}{\alpha + \epsilon} < p < \frac{\alpha}{\alpha + \epsilon}$ (recall that our parameter choice satisfies $\alpha > \epsilon$). In this case, for P1, the utility α of the safer action C is strictly larger than the best possible expected utility $(\alpha + \epsilon)(1 - p)$ of taking action S_1 and larger than the best possible expected utility $(\alpha + \epsilon)p$ of taking action S_2 as well. Given that P1 will always take C , P2 strictly prefers C as well as his utility $(\alpha + \epsilon)(1 - p)$ for S_1 and utility $(\alpha + \epsilon)p$ for S_2 are both smaller. Therefore, both players taking action C is the unique equilibrium, leading to sender utility 2α which is less than U_0 since $U_0 > \alpha + (\alpha + \epsilon)/(1 + \epsilon) = 2\alpha$.
2. When $p(\alpha + \epsilon) \geq \alpha$. In this case, state θ_1 is very likely and action S_1 is strictly *dominated* by action C for P1 and thus will not be taken by P1. We show that there exists a mixed strategy that has sender utility worse than U_0 . In particular, consider the following mixed strategies: (1) P1 chooses action C with probability $p_C = \frac{p + \alpha(1 - p)}{\epsilon p + p}$ and action S_2 with remaining probability $1 - p_C$; (2) P2 chooses action C with probability $q_C = \frac{\alpha}{(\alpha + \epsilon)p}$ and action S_2 with remaining probability $1 - q_C$. We claim that this is a mixed strategy equilibrium. In particular, both action C and S_2 have expected utility α to P1 and both action C and S_2 have expected utility $p_C(\alpha + \epsilon)p$. Therefore, the sender's utility at the worst mixed equilibrium is at most

$$\begin{aligned} \alpha + \frac{p + \alpha(1 - p)}{\epsilon p + p}(\alpha + \epsilon)p &= \alpha + \frac{\alpha + \epsilon}{\epsilon + 1}[p + \alpha(1 - p)] \\ &\leq \alpha + \frac{\alpha + \epsilon}{\epsilon + 1} = U_0. \end{aligned}$$

3. When $(1 - p)(\alpha + \epsilon) \geq \alpha$, i.e. $p \leq \frac{\epsilon}{\alpha + \epsilon}$. This case is symmetric to Case 2, and thus has the same conclusion.

Finally, we argue that the optimal private scheme results in the optimal social outcome. Consider the private scheme that reveals no information to Player 2 but full information to Player 1. Given no information, Player 2 has a strictly dominant action C since $\epsilon < \alpha$. With full information, Player 1 will always open the non-empty safe. This leads to the optimal outcome and sender utility $1 + \alpha + \epsilon$. Therefore, the $\text{PoS}(\text{Pri:Pub})$ for this game equals the $\text{PoA} = \frac{2\alpha + \epsilon + \alpha\epsilon}{(1 + \alpha + \epsilon)(1 + \epsilon)}$. We see that when $\alpha = 1$ and $\epsilon \rightarrow 0$, the PoA tends to 1; when $\alpha \rightarrow 0$ and $\epsilon \rightarrow 0$, the PoA tends to 0. When $\alpha = 1$ and $\epsilon = 0$, we have a trivial case, with PoA and PoS 1. Thus its ratio takes any value within $(0, 1]$.

5.3 Tightness of the Lower-Bound for PoS(exP:Pri)

Finally, for any $r \in (0, 1]$, we show the tightness of PoS(exP:Pri). Consider the same two robbers, robbing the same bank. However, the bank has now upgraded its anti-theft countermeasures. Instead of a decoy safe, there is now an entire decoy vault. Naturally, any robber who goes into it will leave empty-handed. In the real vault, there is again a large pile of cash, but now (only) one safe. A robber can choose to take cash for a guaranteed payoff ($\alpha + \epsilon$ for Player 1, α for Player 2). The safe contains the same fragile, valuable crystal as in the previous section, but this time, no gold bar. If both players attempt to crack the safe, they get in each others' way and will fail. After cracking the safe, each player has enough time to take cash instead of the contents of the safe if they choose. We assume $1 \geq \alpha > \epsilon \geq 0$ and $\alpha + \epsilon < 1$. If Player 2 attempts to take the cash, and Player 1 cracks the safe, Player 2 can take the crystal on the way out, *instead of* the cash. However, if Player 2 does not go into the correct vault, he will leave with nothing. The detailed payoff is as in the following table. Here, θ_1 corresponds to the first vault being the decoy one, whereas θ_2 corresponds to the second vault being empty. Each state occurs with equal probability 0.5. The sender as the heist leader knows which vault is empty.

		P2			
		C_1	S_1	C_2	S_2
P1	C_1	$(0, 0)$	$(0, 0)$	$(0, \epsilon)$	$(0, \epsilon)$
	S_1	$(0, 0)$	$(0, 0)$	$(0, \alpha)$	$(0, \alpha)$
	C_2	$(\alpha + \epsilon, 0)$	$(\alpha + \epsilon, 0)$	$(\alpha + \epsilon, \alpha)$	$(\alpha + \epsilon, \alpha)$
	S_2	$(\alpha, 0)$	$(\alpha, 0)$	$(\alpha, 1)$	$(0, 0)$
		θ_1			
		P2			
		C_1	S_1	C_2	S_2
P1	C_1	$(\alpha + \epsilon, \alpha)$	$(\alpha + \epsilon, \alpha)$	$(\alpha + \epsilon, 0)$	$(\alpha + \epsilon, 0)$
	S_1	$(\alpha, 1)$	$(0, 0)$	$(\alpha, 0)$	$(\alpha, 0)$
	C_2	$(0, \alpha)$	$(0, \alpha)$	$(0, 0)$	$(0, 0)$
	S_2	$(0, \alpha)$	$(0, \alpha)$	$(0, 0)$	$(0, 0)$
		θ_2			

Figure 6: Payoffs of the robber's game variant with tight PoS(exP:Pri); each state occurs with probability 0.5.

We first calculate the PoA of each game. For the game at state θ_1 , it is easy to see that action C_2 strictly dominates all other actions for Player 1. This leads to (C_2, C_2) and (C_2, S_2) be the two unique NEs and the total payoff of any equilibrium is $(\alpha + \epsilon) + \alpha = 2\alpha + \epsilon$. The maximum total payoff however is $\alpha + 1$, under action profile (S_2, C_2) . Similar analysis holds for state θ_2 . The PoA for each game is thus $(2\alpha + \epsilon)/(1 + \alpha)$ (recall $\alpha + \epsilon < 1$ in our game).

We now argue that an optimal private scheme is to reveal full information. Crucially, Player 1's strictly dominant action is to take C_1 or C_2 , whichever is more likely to get the $\alpha + \epsilon$ cash

amount in the posterior distribution of his private signal. Given this, Player 2’s optimal action is to choose C_1/S_1 or C_2/S_2 , whichever is more likely to get the α payoff (from cash or from opening the safe) in the posterior distribution of his private signal. Consequently, any partial information will lead to Player 1 utility at most $\alpha + \epsilon$ and Player 2 utility at most α . This renders full information revelation optimal, leading to total player utility $2\alpha + \epsilon$.

Finally, we show that an ex-ante signaling scheme induces the optimal outcome. The scheme simply recommends (S_2, C_2) at state θ_1 and (S_1, C_1) at state θ_2 . This satisfies the ex-ante obedience constraint (3) because: (1) if Player 1 opts out and acts according to his prior belief, he gets expected utility at most $\frac{1}{2}(\alpha + \epsilon)$, which is strictly less than his utility α in the scheme; (2) Player 2 gets utility 1 in the scheme and certainly does not want to opt out. Therefore, the $\text{PoS}(\text{exP: Pri})$ ratio in this game equals precisely the PoA of each game $\frac{2\alpha + \epsilon}{1 + \alpha}$. This ratio tends to 1 as $\alpha + \epsilon \rightarrow 1$ and tends to 0 as $\alpha, \epsilon \rightarrow 0^+$. If $\alpha + \epsilon = 1$, the PoA and $\text{PoS}(\text{exP: Pri})$ are trivially equal to 1. Therefore, the ratio takes any value $r \in (0, 1]$.

6 Discussions and Future Work

In this paper, we initiate and formalize the concept of the power of signaling (PoS). In the general classes of cost-minimization and payoff-maximization games, we show that the PoS is inherently related to, in fact precisely characterized by, the *price of anarchy* (PoA).

There are many possibilities for future research. In our analysis, we use the full information scheme (FI) as the benchmark, since the no information scheme (NI) will lead to infinite bound. However, another natural and stronger benchmark is the *better* of these two schemes FI , NI . With such a stronger benchmark, the power of signaling will not increase. One interesting question is whether we will have strictly less power of signaling when compared to this strong benchmark. We observe that positive answers to this characterization question will have interesting algorithmic implications. For example, even restricting to non-atomic routing games with linear latency functions, if one can show that $\text{PoS}(\text{Pub: max}\{\text{FI}, \text{NI}\}) = r$ for some $r < 4/3$, this would imply that the better between FI and NI — which certainly can be computed efficiently — will serve as an r -approximation for optimal public signaling. However, it is proved in [2] that it is NP-hard to approximate optimal public signaling for this setting to be within a ratio better than $4/3$ in this setting. This shows that for non-atomic routing with linear latency functions, even $\text{PoS}(\text{Pub: max}\{\text{FI}, \text{NI}\})$ is strictly smaller than the PoA ratio $4/3$, it will be NP-hard to prove this conclusion since any proof implies an efficient approximation algorithm with ratio strictly better than $4/3$.

The above discussion considers stronger benchmark schemes. Another direction is to study the power of signaling for more restricted classes of signaling schemes, such as schemes with limited communication power [12] or schemes with costly communication [17]? For these classes of schemes, the power of signaling will also decrease. It is interesting to understand how much these restrictions limit the power of signaling. On the other hand, the sender’s objective con-

sidered in this paper is the total social welfare. In many applications of signaling, the sender’s objective may be different from the welfare, e.g., revenue as in auctions. In this case, tools beyond the price of anarchy may be needed since PoA mainly concerns welfare. It is an intriguing open direction to characterize the power of signaling in these settings.

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A PoS w.r.t the No Information (NI) Benchmark

In the main body of our paper, we choose full information (FI) as our benchmark scheme. One might wonder what happens if the no information scheme is used instead. It turns out that no information (NI) may lead to very bad social welfare, and examples are fairly easy to construct. Massicot and Langbort gave a Bayesian cost-minimization game with $\text{PoA} = 4/3$ for each game — more concretely, a non-atomic routing game example with affine latency function — such that $\text{PoS}(\text{Pub:NI}) \rightarrow \infty$ [26].

We now exhibit a simple reward-maximization game with $\text{PoA} = 1$ but $\text{PoS}(\text{Pub:NI}) \rightarrow 0$. Consider a (trivial) game with n actions $A_1, A_2, A_3, \dots, A_n$ and a single player. There are n equally likely states of nature, with each state of nature θ_i gives utility 1 to action A_i and 0 utility to all other actions. The price of anarchy of this game is trivially 1, since there is only one reward-maximizing player, and full information as the optimal public scheme achieves optimal welfare 1. However, in the case of no information, the player can only get expected utility $\frac{1}{n}$. As $n \rightarrow \infty$, $\text{PoS}(\text{Pub:NI}) \rightarrow 0$.

B Non-tightness of PoS in “Reverse” Routing

When proving the tightness for *payoff-maximization* games, a very natural first attempt is, perhaps, to convert the previously constructed cost-minimization routing games into payoff-maximization games by flipping the sign of cost functions and adding a large constant to make it positive. One example by reversing our game constructed for the tightness of $\text{PoS}(\text{Pri:Pub})$ is depicted in Figure 7. That is, any edge with cost function $c_e(x)$ in our original construction can be changed to instead having payoff $N - c_e(x)$ for large positive constant N . Clearly, this is a valid payoff-maximization game, which we term “reverse” routing game for convenience.

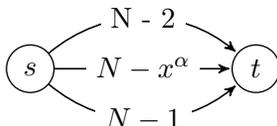


Figure 7: A reverse routing example

We argue this natural adaptation of our previous routing game constructions in this way does not produce an example with, e.g., tight $\text{PoS}(\text{Pri:Pub})$ ratio. This is why we must turn to new constructions of payoff-maximization games. There are two reasons. First, this adaption cannot lead to any price of anarchy ratio within $(0, 1)$. In particular, it can be verified that the PoA of the game in Figure 7 is at least $1/2$ since $N \geq 2$. The second major reason is that optimal routes in the standard cost-minimization routing game may not be optimal any more in its natural adaption to the reward-maximization situation. For example, in cost minimization, one never wants to route through a cycle but in its reward maximization variant, we would like to route through a cycle as much as possible to collect rewards (such examples are fairly easy to construct).