B_0-VPG Representation of AT-free Outerplanar Graphs

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Abstract

A *k-bend path* is a non-self-intersecting polyline in the plane made of at most $k+1$ axis-parallel line segments. *B_k-VPG* is the class of graphs which can be represented as intersection graphs of k-bend paths in the same plane. In this paper, we show that all AT-free outerplanar graphs are B_0-VPG, i.e., intersection graphs of horizontal and vertical line segments in the plane. Our proofs are constructive and give a polynomial time B_0-VPG drawing algorithm for the class.

Following a long line of improvements, Gonçalves, Isenmann, and Pennarun [SODA 2018] showed that all planar graphs are B_1-VPG. Since there are planar graphs which are not B_0-VPG, characterizing B_0-VPG graphs among planar graphs becomes interesting. Chaplick et al. [WG 2012] had shown that it is NP-complete to recognize B_k-VPG graphs within B_{k+1}-VPG. Hence recognizing B_0-VPG graphs within B_1-VPG is NP-complete in general, but the question is open when restricted to planar graphs. There are outerplanar graphs and AT-free planar graphs which are not B_0-VPG. This piqued our interest in AT-free outerplanar graphs. [1]

Keywords: Outerplanar graphs . AT-free . B_0-VPG . Connectivity augmentation . Outerpath . Linear outerplanar graph . Graph drawing.

1 Introduction

A *k-bend path* is a simple path in a two-dimensional grid with at most k bends. Geometrically, they are non-self-intersecting polylines in the plane made of at most $k+1$ axis-parallel (horizontal or vertical) line segments. *Vertex intersection graphs of Paths on a Grid (VPG)* (resp., *B_k-VPG*) is the class of graphs which can be represented as intersection graphs of (resp., k-bend) paths in a two-dimensional grid. The *bend number* of a graph G in VPG is the minimum k for which G is in B_k-VPG. One motivation to study B_k-VPG graphs comes from VLSI circuit design where the paths correspond to wires in the circuit. A natural concern in VLSI design is to reduce the number of bends in each path (wire) in the representation. A second motivation is that certain algorithmic tasks become easier when restricted to B_1-VPG or B_0-VPG graphs (cf. [26]).

Planar graphs have received the maximum attention from the perspective of B_k-VPG representations, some of which we describe in Section 1.3. Following up on a series of results and conjectures by various authors, Gonçalves, Isenmann, and Pennarun in 2018 showed that all planar graphs are B_1-VPG [20]. This is tight since many simple planar graphs like 4-wheel, 3-sun, triangular prism, to

*A part of this work was done while at Indian Institute of Technology Palakkad.

[1]A preliminary version of this work was presented at the 8^{th} International Conference on Algorithms and Discrete Applied Mathematics (CALDAM) 2022 [22]. In this extended version we show that a proper superclass of AT-free outerplanar graphs, which we name as *linear outerplanar graphs*, are B_0-VPG. We believe that his new class is interesting in its own right.

name a few, are not B_0-VPG. This makes the question of characterizing B_0-VPG planar graphs very appealing. Characterizing B_0-VPG outerplanar graphs will be a good step in this direction since some of the structures that forbid a planar graph from being B_0-VPG are also present among outerplanar graphs. Outerplanar graphs were known to be B_1-VPG [8] before the same was shown for planar graphs. Chaplick et al. [10] had shown that it is NP-complete to decide whether a given graph G is in B_k-VPG even when G is guaranteed to be in B_{k+1}-VPG. Hence recognizing B_0-VPG graphs with in B_1-VPG is NP-complete in general, but the question is open when restricted to planar graphs or outerplanar graphs.

This article is an outcome of our effort to characterize B_0-VPG outerplanar graphs. One can see from the geometry that the closed neighborhood of every vertex in a B_0-VPG graph is an interval graph [18]. We strengthen this necessary condition (Proposition 19) by identifying adjacent vertices which are forced to be represented by collinear segments in any B_0-VPG drawing. But this is still not sufficient to characterize B_0-VPG outerplanar graphs (Figure 6). However, we were able to show that, if the outerplanar graph itself is AT-free, then it is B_0-VPG (Theorem 18). We cannot extend this result to AT-free planar graphs since we have examples of AT-free planar graphs, like 4-wheel and triangular prism, which are not B_0-VPG.

While it is relatively easier to find a B_0-VPG drawing for biconnected AT-free outerplanar graphs, handling cutvertices turned out to be more challenging. Rather than trying to join B_0-VPG drawings of individual blocks, we found it easier to embed the given graph as an induced subgraph of a biconnected outerpath.

Definition 1 (Outerpath)**.** An *outerpath* is an outerplanar graph which admits a planar embedding whose weak dual is a path.

Note that outerpaths need not be biconnected. All the five graphs in Figure 3.(a) are outerpaths.

Our proof has essentially two parts with biconnected outerpaths forming the bridge between the two. The first part is a structural result which shows that any AT-free outerplanar graph can be realized as an induced subgraph of a biconnected outerpath. The second part is a B_0-VPG drawing procedure for biconnected outerpaths. Both the parts have a potential to be generalized. Since B_0-VPG is easily seen to be hereditary class, the result naturally extends to all induced subgraphs of biconnected outerpaths. This prompted us to name this class (Definition 2) and study it on its own merit.

Definition 2 (Linear Outerplanar Graph)**.** An outerplanar graph is *linear* if it is isomorphic to a subgraph of a biconnected outerpath.

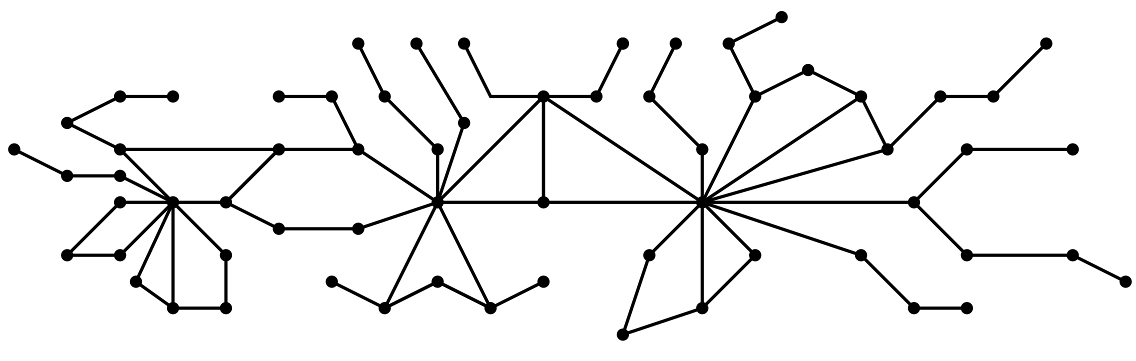


Figure 1: A Linear Outerplanar Graph

We give a complete characterization of this class in Theorem 7. Though the characterization seems technical, it is very easy to visualize and gives a poly-time recognition algorithm. As a pleasant surprise, we also discover that every graph in this class can be realized both as an induced subgraph as well as a spanning subgraph of (different) biconnected outerpaths. Figure 1 shows an example of a linear outerplanar graph.

The second part of our proof, the drawing procedure for biconnected outerpaths, can also be extended to a larger class of graphs than biconnected outerpaths, but this we set aside for a future work.

1.1 Organization

After recalling some standard graph theoretic terminology in Section 1.2 and a brief literature review in Section 1.3, we layout our proofs in three sections. Section 2 has the B_0-VPG drawing procedure for *biconnected outerpaths*. Section 3 contains the characterization of linear outerplanar graphs. In Section 4, we prove that all AT-free outerplanar graphs are *linear* thereby completing the proof of the titular result. We conclude with Section 5, where we describe some necessary conditions for the existence of a B_0-VPG representation. This may help in characterizing B_0-VPG outerplanar graphs.

1.2 Terminology

The *closed neighborhood* $N[v]$ of a vertex v in a graph G is the set containing v and and its neighbors in G. A set of three independent vertices is called an *asteroidal triple (AT)* when there exists a path among each pair of them containing no vertex from the closed neighborhood of the third vertex. An *AT-free* graph is a graph which does not have an AT.

A *plane graph* is an embedding of a planar graph in the plane with no crossing edges. Let G be a plane graph. The *dual* of G is a graph that has a vertex for each face of G and an edge between two of its vertices when the corresponding faces of G share an edge. The *weak dual* of G is obtained from its dual by removing the vertex corresponding to the outer face. An edge of G incident to the outer face of G is called a *boundary edge* and its endpoints are called *boundary neighbors* of each other. The remaining edges of G are called *internal edges*. A *leaf face* is a face with at most one internal edge. A planar graph is *outerplanar* if it has a plane embedding in which all the vertices are incident on the outer face. Outerplanar graphs will always be drawn in such a way that the outer face contains all the vertices, and the terminology of faces, duals, weak duals, boundary edges and internal edges will be used assuming such a plane drawing. Let G be an outerplanar graph. The weak dual of G is a forest [16] and we denote it by $\mathcal{T}_G$. Further, we call $\mathcal{T}_G$ a *linear forest* if each component in $\mathcal{T}_G$ is a path.

Let G be a graph. A k^+*-vertex* in G is a vertex having at least k neighbors in G. A *leaf edge* is an edge having one endpoint of degree one. A subgraph H of G is *spanning* if $V(H) = V(G)$, and *induced* if $E(H) = \{xy \mid x, y \in V(H),\ xy \in E(G)\}$. A graph induced by a subset S of the vertices of G is denoted by $G[S]$. A subset of vertices in a graph is called a *separator* if its removal increases the number of components of the graph. A vertex x is a *cutvertex* if $\{x\}$ is a separator. A graph is *k-connected* if it does not have a separator of size smaller than k. A graph is said to be *connected* (resp. *biconnected*) if it is 1-connected (resp. 2-connected). A *block* of a graph is a maximal biconnected subgraph of the graph. A *trivial block* is a block containing at most two vertices.

A graph G is *H-free* if G does not contain an induced subgraph isomorphic to H. A graph G is said to be *H-minor-free* if it does not contain a minor isomorphic to H. We use C_k to denote the simple cycle on k vertices. A cycle on k vertices $x_0, \ldots, x_{k-1}$ where each x_i is adjacent to x_{i+1} (addition is modulo k) can also be denoted as $x_0, \ldots, x_{k-1}, x_0$. A C_4 together with an additional vertex v adjacent to all the vertices of C_4 is called a *4-wheel*. A *triangular prism* is the complement of C_6. An *interval graph* is an intersection graph of a set of intervals on $\mathbb{R}$.

1.3 Literature

The class B_k-VPG was introduced by Asinowski et al. in 2012 [3]. Nevertheless, these graphs were previously studied in various forms. One of them is *grid intersection graphs* (GIG) which are equivalent to bipartite B_0-VPG graphs. The recognition problem for string graphs and hence VPG graphs is NP-complete [24, 28]. The recognition problem for B_0-VPG graphs is NP-complete [25]. B_0-VPG characterizations are known for block graphs [2], split graphs, chordal bull-free graphs, chordal claw-free graphs [18] and cocomparability graphs [27].

Segment intersection graphs are intersection graphs of line segments in the plane. Chalopin and Gonçalves in 2009 showed that every planar graph is a segment intersection graph [9], confirming a conjecture of Scheinerman from 1984 [29]. One way to refine the class of segment intersection graphs is to restrict the number of directions permitted for the segments. If the number of directions is limited to two, we rediscover B_0-VPG. *k-DIR* graphs are intersection graphs of line segments that can lie in at most k directions in the plane. It is known that bipartite planar graphs are 2-DIR [21, 13, 14]

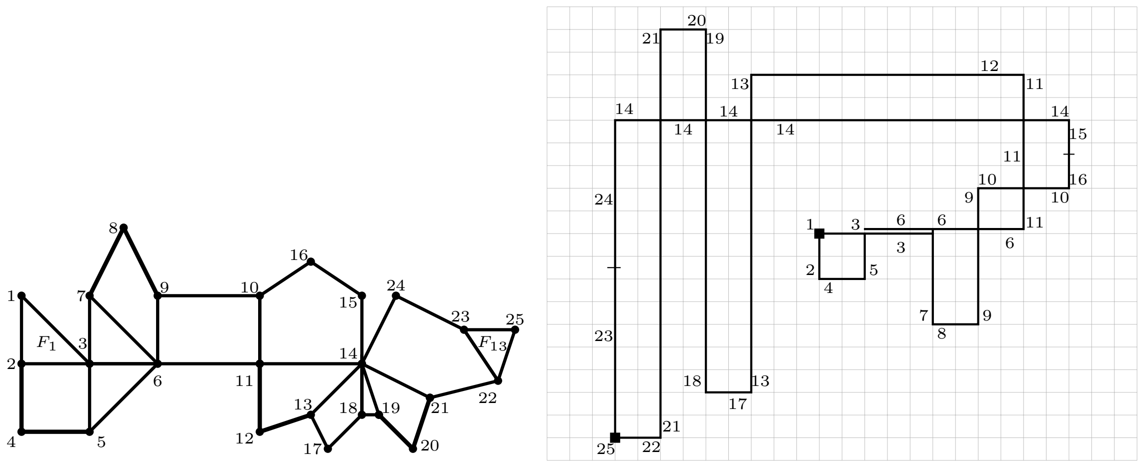


Figure 2: A biconnected outerpath G and a B_0-VPG representation of it. The collinear overlapping line segments are drawn a little apart for clarity. The point segments (for eg. vertices 1 and 25) are drawn as black squares.

and triangle-free planar graphs are 3-DIR [7]. West conjectured that any planar graph is 4-DIR [30] which was recently refuted by Gonçalves in 2020 [19]. Before the celebrated result by Gonçalves et al. that planar graphs are B_1-VPG [20], we had a chronology of results on B_k-VPG representation of planar graphs. Since 2-DIR graphs are equivalent to B_0-VPG, bipartite planar graphs B_0-VPG. In [3], Asinowski et al. showed that planar graphs are B_3-VPG and conjectured that it is tight. Disproving this conjecture, Chaplick and Ueckerdt proved that planar graphs have a B_2-VPG representation [11]. This adds to the appeal for characterizing B_0-VPG planar graphs.

Outerpaths have many geometric representations like *balanced circle-contact* representations [1], *geometric simultaneous embeddings* with a matching [6], and *partial geometric simultaneous embeddings* with another outerpath [15]. All these representations will extend to linear outerplanar graphs because of Theorem 7 and Remark 2. Babu et al. provides an algorithm to augment outerplanar graphs of pathwidth p to biconnected outerplanar supergraphs of pathwidth $\mathcal{O}(p)$ [4]. Connectivity augmentation of outerplanar graphs using minimum number of additional edges is studied in [17, 23]. Barát et al. have characterized the graphs with pathwidth at most two [5] and our class is a strict subclass of that.

2 B_0-VPG Representation of Biconnected Outerpaths

It's drawing time! In this section, we show that every biconnected outerpath is B_0-VPG (Theorem 3). The proof of Theorem 3 is constructive and it draws a B_0-VPG representation for any biconnected outerpath (cf. Figure 2 for example). Since B_0-VPG is easily seen to be a hereditary graph class (closed under induced subgraphs), and since every linear outerplanar graph can be represented as an induced subgraph of a biconnected outerpath (Theorem 7), it follows that all linear outerplanar graphs are B_0-VPG. Furthermore, since we show in Section 4 that all AT-free outerplanar graphs are linear (Lemma 17), the main result of this article follows.

Theorem 3. *Every biconnected outerpath is B_0-VPG.*

Proof. Let G be a biconnected outerpath with n faces labeled $F_1, \ldots, F_n$ such that the weak dual of G is the path $F_1, \ldots, F_n$. For each $i \in [n-1]$, the edge shared by F_i and F_{i+1} is denoted by e_i. For notational convenience, we set e_n to be any boundary edge of F_n. For each $i \in [n]$, let G_i denote the induced subgraph of G restricted to the faces $F_1, \ldots, F_i$.

In a B_0-VPG drawing D_i of G_i, we call a non-point horizontal (resp., vertical) line segment l in D_i *extendable* from a point $p \in l$ if at least one of the two infinite horizontal (resp., vertical) open rays starting at p (but not containing p) does not intersect any other line segment of D_i. A point segment l is said to be *extendable* from its location p if it is extendable from p both as a horizontal and a vertical line segment. An edge xy in G_i is said to be *extendable* in D_i if the line segments l_x and l_y representing the vertices x and y are extendable from a common point $p \in l_x \cap l_y$ either in the same direction or in orthogonal directions. Finally a B$_0$-VPG drawing D_i is said to be *extendable* if e_i is extendable and whenever F_i is a triangle, the vertex of F_i not incident to e_{i-1} is represented by a point segment.

If $F_1 \cong C_3$, then representing all the three vertices as point segments at the same point gives an extendable B$_0$-VPG drawing D_1 of G_1. If the length of F_1 is 4 or more, then we can represent F_1 as the intersection graph of line segments laid out on the boundary of an axis-parallel rectangle with the endpoints of e_1 being orthogonal (and hence sharing only a corner of the rectangle). This is an extendable B$_0$-VPG drawing D_1 of G_1. Let D_i, $i < n$, be an extendable B$_0$-VPG drawing of G_i. From D_i, we construct an extendable B$_0$-VPG drawing D_{i+1} of G_{i+1} as follows.

Case 1 (length of F_{i+1} is 4 or more). Let $F_{i+1} = v_0, \dots, v_k, v_0$, with $e_i = v_k v_0$ and $e_{i+1} = v_j v_{j+1}$, $j < k$. Since D_i is extendable, the edge $v_k v_0$ is extendable in D_i. Extend the line segments l_k and l_0 (representing v_k and v_0 respectively) in orthogonal directions to two points q_k and q_0 outside of the bounding box of D_i. Let q be the intersection point of the perpendiculars to l_k and l_0 at q_k and q_0 respectively. Represent the path $v_1, \dots v_{k-1}$ on the two line segments from q_0 to q and q to q_k such that v_1 is represented by a segment containing q_0, v_{k-1} by a segment containing q_k and v_j, v_{j+1} by orthogonal line segments sharing a point. The point shared by these two segments will be q_0 when $j = 0$, q_k when $j = k-1$ and q in all other cases. This gives the drawing D_{i+1}. It is clear that the new line segments added in this stage do not intersect any other line segments in D_i except l_0 and l_k. It is easy to verify that the edge $e_{i+1} = v_j v_{j+1}$ is extendable. Hence D_{i+1} is extendable.

Case 2 ($F_{i+1} \cong C_3$). Let $F_{i+1} = a, b, c, a$, with $e_i = ca$ and $e_{i+1} = ab$. Since D_i is extendable, the edge ca is extendable in D_i from a point p. If the line segments l_c and l_a are extendable in the same direction, then extend them to a point q outside the bounding box of D_i and represent b by a point segment l_b at q to obtain D_{i+1}. It is easy to check that the line segment l_a, the point segment l_b, and also the edge ab are extendable from q in D_{i+1}. Since ab is extendable and b is represented by a point segment, D_{i+1} is extendable. If l_c and l_a are extendable only in orthogonal directions, then neither of them is a point segment. Hence $F_i \ncong C_3$ and hence the vertices c and a have no common neighbor in G_i. So the point p is not contained in any line segment of D_i other than l_c and l_a. Represent b by a point segment l_b at p to get D_{i+1}. In both the subcases, it is clear that the new line segments added in this stage do not intersect any other line segments in D_i except l_c and l_a. It is easy to check that the line segment l_a, the point segment l_b, and also the edge ab are extendable from p in D_{i+1}. Since ab is extendable and b is represented by a point segment, D_{i+1} is extendable.

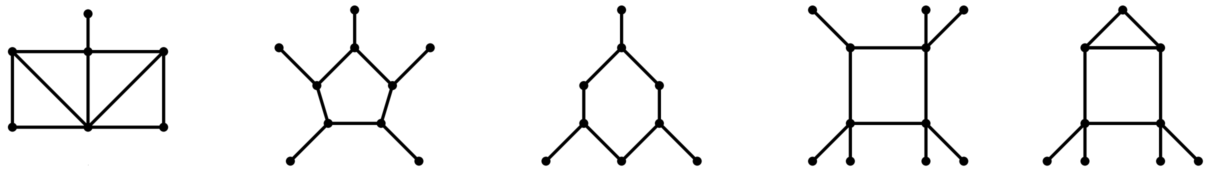
Repeating the above construction $n-1$ times gives a B$_0$-VPG drawing D_n of $G_n = G$. □

B$_0$-VPG is easily seen to be a hereditary graph class (that is, closed under induced subgraphs). Thus it follows from Theorem 3 that every induced subgraph of a biconnected outerpath is B$_0$-VPG. By the end of the next section, we extend this to every subgraph (not necessarily induced).

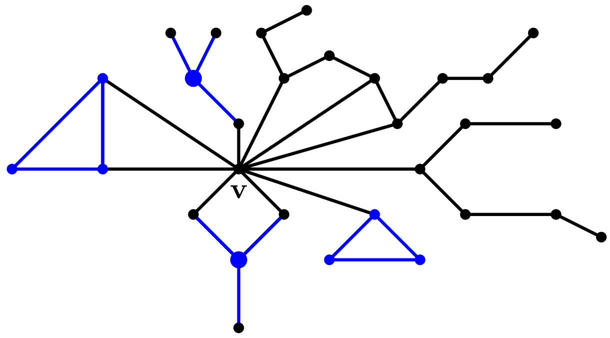
3 Characterization of Linear Outerplanar Graphs

In a preliminary version of the is article [22], we showed that every AT-free outerplanar graph can be identified as an induced subgraph of a biconnected outerpath. After that work, triggered by a question from Mathew C. Francis, we realized that the induced subgraphs of biconnected outerpaths can be more exotic than AT-free outerplanar graphs, and that this class deserves to be studied on its own. Our definition of *linear outerplanar graphs* in [22] was a technical choice made for the proof. That definition was more restrictive than the one here (Definition 2).

While the class of linear outerplanar graphs will inherit the rich collection of drawings and geometric representations available for biconnected outerpaths, the structure of a linear outerplanar graph is harder to describe than that of a biconnected outerpath. This section aims to do that. We first build the necessary terminology for stating and proving the characterization theorem (Theorem 7).



(a)



(b)

Figure 3: Examples of outerplanar graphs which are (a) not block-safe (b) not cut-safe. In (b), $\mathcal{C}_v$ has four big components.

Let v be a cutvertex in a graph G. For every component of C of $G \setminus v$, the subgraph of G induced on $V(C) \cup \{v\}$ is called a *component incident to* v. The set of components incident to v is denoted by $\mathcal{C}_v$. A component C incident to v is said to be *incident to a block* B if v is in B and C does not contain B.

Definition 4. Let v be a cutvertex in an outerplanar graph G and $C \in \mathcal{C}_v$. We call C *small for* v if $C \setminus v$ is a path and *big for* v otherwise. Further, when C is small for v, we call it a *tail at* v if C (including v) is a path.

Definition 5 (Cut-safety)**.** A cutvertex v in an outerplanar graph G is said to be *safe* if $\mathcal{C}_v$ contains at most two big components. The graph G is said to be *cut-safe* if every cutvertex in G is safe.

A set of at most two boundary edges of a block B in an outerplanar graph is called *antipodal* either when B is a single face or when the edges belong to different leaf faces of B.

Definition 6 (Block-safety)**.** A nontrivial block B in an outerplanar graph is called *safe* if there exist two antipodal edges a_0b_0 and a_1b_1 in B and the set of components incident to B can be partitioned into $\mathcal{C}_0$ and $\mathcal{C}_1$ such that

1. every component in $\mathcal{C}_i$ ($i \in \{0,1\}$) is incident to either a_i or b_i, and
2. at most one component in $\mathcal{C}_i$ ($i \in \{0,1\}$) is incident to a_i and it (if present) is a tail.

An outerplanar graph G is said to be *block-safe* if every nontrivial block in G is safe. The edges a_0b_0 and a_1b_1 are called *terminal edges* of B in G. A terminal edge is denoted as an ordered pair (x, y) where $x = a_i$ and $y = b_i$. The components in $\mathcal{C}_i$ are said to be *associated to* the terminal edge (a_i, b_i).

Theorem 7 (Characterization)**.** *An outerplanar graph G is linear if and only if G is cut-safe, block-safe and the weak dual of G is a linear forest. Moreover, if G is linear, then it can be realized both as an induced subgraph and as a spanning subgraph of (different) biconnected outerpaths.*

Remark 1. Note that the class of linear outerplanar graphs and outerpaths are incomparable. There are outerpaths which are not linear (cf. Figure 3.(a)) and linear outerplanar graphs which are not outerpaths (cf. Figure 4.(a)).

3.1 Proof of Theorem 7 (Necessity)

We first prove that if an outerplanar graph G is linear, then G is cut-safe, block-safe and $\mathcal{T}_G$ is a linear forest. Our strategy is to look at the edges of G which are forced to be internal edges in any biconnected outerplanar supergraph of G and then use the fact that the internal edges in a biconnected outerpath have a natural linear order. It is easy to see that every internal edge of G, at least one edge in each face of G (unless G itself is a cycle), and all but at most two edges incident to any vertex of G will all be internal edges in any biconnected outerplanar supergraph of G. We can say a bit more about edges incident to a cutvertex in a nontrivial block (Lemma 9) using the simple observation below. The extension of this simple observation to biconnected outerpaths (Observation 10) is a key to rest of this section.

Observation 8. *If uv is an internal edge in a biconnected outerplanar graph G, then $G \setminus \{u, v\}$ has exactly two components. Moreover, both these components contain exactly one boundary neighbor each of u and v.*

Lemma 9. *If an outerplanar graph G is an induced subgraph of a biconnected outerplanar graph G' then for any cutvertex v in a nontrivial block B of G, one of the two boundary edges incident to v in B is an internal edge in G'.*

Proof. Let uv and vw be the two boundary edges of B incident to v and let x be a neighbor of v outside B in G. If both uv and vw are boundary edges in G', then vx is an internal edge in G'. But u and w are in the same component of $G \setminus \{v, x\}$ and hence of its supergraph $G' \setminus \{v, x\}$. This contradicts Observation 8. ☐

For any three subsets X, Y, Z of the vertices of a graph, X *separates* Y and Z if every path between Y and Z contains a vertex from X.

Observation 10. *Let u_0v_0, u_1v_1, u_2v_2 be three distinct (but not necessarily disjoint) internal edges of a biconnected outerpath G. Then the endpoints of one of them, say $\{u_i, v_i\}$, separate $\{u_j, v_j\}$ from $\{u_k, v_k\}$ in G, where $\{i, j, k\} = \{0, 1, 2\}$.*

Lemma 11. *If an outerplanar graph G is linear, then $\mathcal{T}_G$ is a linear forest.*

Proof. Let G be a subgraph of a biconnected outerpath G'. If $\mathcal{T}_G$ is not a linear forest, then G has a face f with at least three internal edges. These remain internal edges in G' that violate the separation property in Observation 10. ☐

Lemma 12. *If an outerplanar graph G is linear, then G is cut-safe.*

Proof. Let G be a subgraph of a biconnected outerpath G'. Suppose G has a cutvertex v such that $\mathcal{C}_v$ contains three big components C_0, C_1, C_2. For each $i \in \{0, 1, 2\}$, since $C_i \setminus v$ is not a path, it either contains a face or a 3^+-vertex. In either case, $C_i \setminus v$ will contribute an internal edge e_i to G'. But e_0, e_1, e_2 violate Observation 10. ☐

Lemma 13. *If an outerplanar graph G is linear, then G is block-safe.*

Proof. Let G' be an edge-minimal biconnected outerpath which is a supergraph of G. If G itself is a biconnected outerpath, we are done. Otherwise, picture any nontrivial block B of G in G'. Let E' be the set of boundary edges of B which become internal edges in G'. Since $\mathcal{T}_{G'}$ is a path, it is easy to see that E' is antipodal. By Lemma 9, every cutvertex of B in G is an endpoint of an edge in E'. Let E' be $\{e_0\}$ if it is singleton, and $\{e_0, e_1\}$ otherwise. The proof will be complete if we can partition the set of components $\mathcal{C}_B$ incident to B into $\mathcal{C}_i$, $i \in \{0, 1\}$ and label the endpoints of e_i as a_i and b_i respecting the last condition in Definition 6.

By Observation 8, we get exactly two components in $G' \setminus V(e_i)$. Let G_i be the subgraph of G' induced by e_i and the component of $G' \setminus V(e_i)$ that does not contain any vertex of B. Let $\mathcal{C}_i$ denote the components in $\mathcal{C}_B$ that are captured by G_i in G'. Consider the leaf face f_i of G_i containing the edge e_i. Since f_i is not part of B, at least one edge e'_i of f_i is missing from G. By edge-minimality of G, this is a boundary edge of G' and hence G_i. Let P_i denote the (possibly trivial) path in $f_i \setminus e'_i$ between e_i and e'_i that does not contain an internal edge (if any) of G_i. We label the endpoint of e_i which meets P_i as a_i and the other as b_i. If P_i is trivial then there is no component in $\mathcal{C}_i$ incident to a_i. If P_i is not trivial, at most one component in $\mathcal{C}_i$, and that too a tail which is a subpath of P_i, is incident to a_i. It is easy to see that this labeling satisfies Definition 6. $\square$

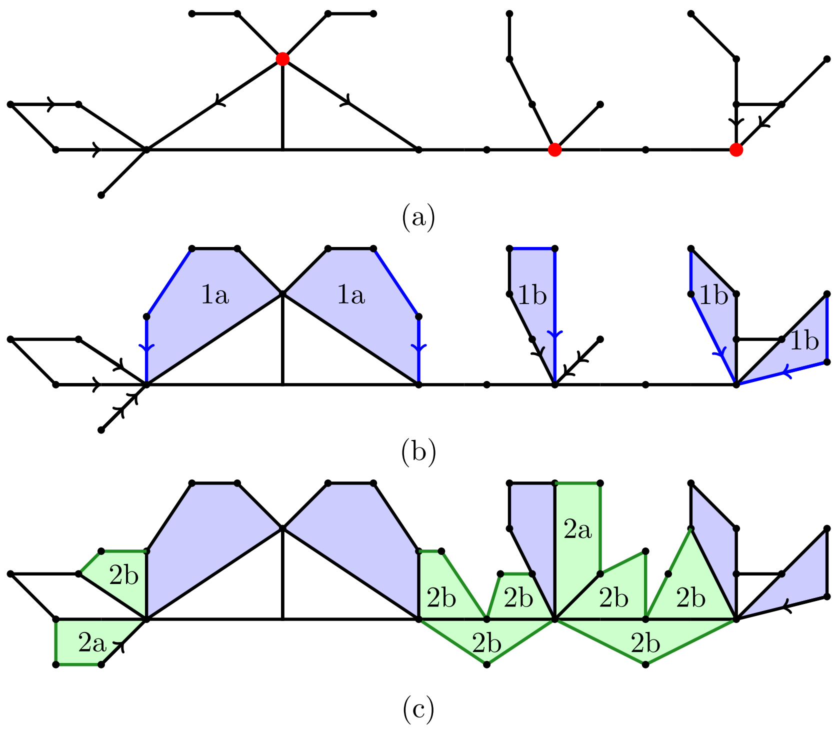


Figure 4: (a) A block-safe and cut-safe outerplanar graph G such that $\mathcal{T}_G$ is a linear forest. Terminal edges are shown oriented and cutvertices for Action 1 are highlighted. (b) Intermediate graph G_3. Faces created by the two cases of Action 1 are marked 1a and 1b respectively. (c) Final biconnected outerpath G'. Faces created by Action 2 are marked 2a if both the components merged are small and 2b otherwise. Since all the connectors are two-length paths, G is an induced subgraph of G'.

3.2 Proof of Theorem 7 (Sufficiency)

Now we prove the other direction of Theorem 7. Let G be a cut-safe and block-safe outerplanar graph such that $\mathcal{T}_G$ is a linear forest. We will construct a biconnected outerpath G' which contains G as a subgraph. A *connector* between two nonadjacent boundary vertices u and v of a plane graph H is either an edge uv or a two-length path (u, w, v) such that $w \notin V(H)$ drawn through the outer face. Hence the resultant graph is planar. In the following construction, if every connector used is an edge (resp. two-length path), then G is contained as a spanning (resp. induced) subgraph of G'. The construction of G' is done in two phases. Each phase consists of repeated applications of a single action.

Definition 14. For a cutvertex v, a small component $C \in \mathcal{C}_v$ is called a *maximal small component* if C is not a subgraph of a small component incident to another cutvertex.

Action 1 (Tuck the tails)**.** *Let v be a cutvertex in G and $C \in C_v$ be a maximal small component associated to a terminal edge (a_i, b_i) of a nontrivial block B. (a) If $v = a_i$ (in which case C is a tail at a_i), add a connector from the leaf of C to b_i. This merges the block B and the component C into a new block B'. Designate the remaining terminal edge of B and the last edge (v', b_i) of the connector as the two terminal edges of B'. (b) If $v = b_i$, add a connector between v and each endvertex of the path $C \setminus v$ which is not already adjacent to v to form a block B' containing C (but not B). If both the endvertices of $C \setminus v$ were adjacent to v, then $B' = C$. Designate (u, v) and (w, v) as the terminal* *edges of B', where uv and vw are the boundary edges of B' incident to v. If B' is a single edge uv, designate (u, v) as both the first and second terminal edge of B'.*

Let us call the resulting graph G_2. In Case (a), the weak dual $\mathcal{T}_{B'}$ of B' is a path formed by extending $\mathcal{T}_B$ with a leaf edge corresponding to the dual of a_ib_i. The new terminal edge in B' is in the new leaf face and hence the two terminal edges of B' are antipodal. Every component incident to B in G (except C) is incident to B' in G_2. Each such component associated to a terminal edge of B can be associated to the corresponding terminal edge of B'. In Case (b), the structure of B' is simple since every internal edge of B' is incident to v. It is easily verified that $\mathcal{T}_{B'}$ is a path and B' is safe with the given terminals. Next we argue that G_2 is cut-safe. In Case (a), the only change to $\mathcal{C}_v$ in G_2 is that C and the component C_B containing B merges into a single component $C_{B'}$. But if C_B is small for v in G, then $C_{B'}$ remains small for v in G_2 and hence v remains safe in G_2. In Case (b), since $B' \setminus v$ is a path, B' is a small component for v and v is safe in G_2. Let $u \neq v$ be a cutvertex in G_2 and let C_u be a small component incident to u. Since C is a maximal small component, C_u does not contain C and hence C_u remains unaffected (and hence small) in G_2. So u remains safe in G_2.

We perform Action 1 once for each maximal small component incident to a cutvertex in G to obtain a graph G_3. Note that for each cutvertex v in G_3, each small component in $\mathcal{C}_v$ is a block. Also, each component incident to a nontrivial block B in G_3 is incident at the second vertex of a terminal edge of B. For each trivial block $B = uv$ which is not a pendant edge of G_3, assign (u, v) to be the first and (v, u) to be the second terminal edge of B. Associate every component incident to B at u (resp. v) with terminal edge (v, u) (resp. (u, v)).

Action 2 (Bond with your sibling)**.** *Let v be a cutvertex in G_3 and let C_0 and C_1 be two components from $\mathcal{C}_v$, chosen prioritizing small components over big ones. For $i \in \{0, 1\}$, let B_i be the block in C_i which contains v, and (u_i, v) be a terminal edge of B_i. Add a connector from u_0 to u_1. This merges B_0 and B_1 into a new block B. The remaining terminal edges, one each from B_0 and B_1, are designated as the terminal edges of B.*

Let us call the resulting graph G_4. The weak dual $\mathcal{T}_B$ of B is a path obtained by connecting two leaf vertices of $\mathcal{T}_{B_0}$ and $\mathcal{T}_{B_1}$ through the dual vertex corresponding to the new bounded face. The new terminal edges of B are antipodal in B. Every cutvertex other than v in B_i ($i \in \{0, 1\}$) is contained in the second terminal edge of B_i which continues to be a terminal edge in B. If C_i is small for an $i \in \{0, 1\}$, then $B_i = C_i$ and the second terminal edge of B_i has v as its second vertex. If both C_0 and C_1 are big for v, since v is safe in G_3, there are no other components in $\mathcal{C}_v$. Hence v is no longer a cutvertex in G_4. Hence all the cutvertices of B are contained in its terminal edges and B is safe. Now we argue that G_4 is cut-safe. The difference from $\mathcal{C}_v$ in G_3 to $\mathcal{C}_v$ in G_4 is that C_0 and C_1 gets merged into a single component. If both C_0 and C_1 are small components, then one can easily check that the new component is B itself and it is small for v. Hence the number of big components in $\mathcal{C}_v$ does not increase and hence v remains safe in G_4 or ceases to be a cutvertex. Let $u \neq v$ be a cutvertex in G_4 and let C_u be a small component incident to u. Since C_u is a block, it does not contain the cutvertex v and hence remains unaffected (and thus small) in G_4. So u remains safe in G_4.

We repeat Action 2 as long as there is a cutvertex. Let G' denote the resulting biconnected graph. Since each action preserves cut-safety, block-safety and linearity of the weak dual, G' is a biconnected outerpath. This completes the proof of Theorem 7.

Remark 2. From the above construction, one can check that each connector added reduces the number of blocks at least by one. Hence the total number of new vertices added is less than the number of blocks in the original graph G.

Remark 3. Checking the cut-safety, block-safety and the linearity of weak dual of an outerplanar graph can be done in polynomial time. It can also be verified that the construction of G' from G can be done in polynomial time.

By Theorem 7, any linear outerplanar graph can be realized as an induced subgraph of a biconnected outerpath. This together with Theorem 3 gives the following corollary.

Corollary 15. *Every linear outerplanar graph is B_0-VPG.*

4 Linearity of AT-free outerplanar graphs

By Corollary 15, in order to prove that AT-free outerplanar graphs are B_0-VPG, it is enough to show that they are linear. Lemma 17 in this section asserts the same. Note that one may suspect whether all AT-free outerplanar graphs can be realized as induced subgraphs of biconnected AT-free outerplanar graphs. But this is incorrect. For example, let G be a C_5 together with a pendant vertex. While G is AT-free outerplanar, it is easy to see that any biconnected outerplanar graph G', containing G as an induced subgraph, is not AT-free.

The following observation is helpful in proving Lemma 17.

Observation 16. *Let B be a nontrivial block of an outerplanar graph G. Every leaf face f of B contains a vertex u of degree two in $G[B]$ which is not incident to any other bounded face of B.*

Justification: A face has at least three vertices and at most two vertices of a leaf face can be shared by another face. Hence all the remaining vertices are of degree two in $G[B]$ and are not incident to any other bounded face of B.

Lemma 17. *AT-free outerplanar graphs are linear.*

Proof. Let G be an AT-free outerplanar graph. If $\mathcal{T}_G$ is not a linear forest, then there exists three internal edges in one face f of G sharing with faces, say, f_1, f_2, f_3. One can verify that we can choose one vertex v_i $(1 \leq i \leq 3)$ each from $f_i \setminus f$ to form an AT.

If G is not cut-safe, then there exists a cutvertex v where $\mathcal{C}_v$ has more than two big components, say, C_1, C_2, C_3. That is, $C_i \setminus \{v\}$ $(1 \leq i \leq 3)$ is not a path and thus it has either a 3^+-vertex v_i or a face f_i. If all the neighbors of v_i are adjacent to v, then it is easy to see that C_i has $K_{2,3}$ as subgraph. Similarly if all the vertices of f_i are adjacent to v, then one can verify that C_i has K_4 as minor. Outerplanar graphs do not contain $K_{2,3}$ or K_4 as a minor [12]. Thus in both cases, there exists a vertex $v_i' \in C_i \setminus \{v\}$ nonadjacent to v in C_i. It is easy to see that $\{v_1', v_2', v_3'\}$ forms an AT.

It remains to check whether G is block-safe. Consider any nontrivial block B of G. Since $\mathcal{T}_G$ is a linear forest, B either has exactly two leaf faces f_1, f_2 or B itself is a face f_1. In the former case, Observation 16 guarantees that $G[B]$ has two vertices u_1 and u_2 of degree two in f_1 and f_2 respectively. For any cutvertex v in B, we denote an arbitrary neighbor of v outside B as v'. If there exists three cutvertices in B, say, v_1, v_2, v_3, then by using the path along the outer cycle (through the boundary) of B, one can verify that $\{v_1', v_2', v_3'\}$ is an AT. Thus there can be at most two cutvertices in B. If B itself is a face f_1, we can choose arbitrary boundary edges a_0b_0 and a_1b_1 of B such that b_0 and b_1 are the only cutvertices of B. These two edges are antipodal in B, and thus B satisfies the conditions of the terminal edges as per Definition 6, thereby showing that B is safe. Hence we assume in the rest of the proof that B contains more than one face. If one of the cutvertices, say v, is neither incident to f_1 nor f_2, then the vertices u_1 and u_2, together with v' form an AT. Hence the cutvertices incident to B, if any, must lie in the leaf faces of B. Moreover, we intend to associate each cutvertex v to a different leaf face of B containing v. If this is not possible, that is, both the cutvertices, say v_1, v_2 are not incident to one of the leaf faces, say f_2, then $\{v_1', v_2', u_2\}$ forms an AT. Thus we conclude that there exist at most two cutvertices incident to B, and they can be associated to different leaf faces of B containing them. We can choose arbitrary boundary edges a_0b_0 and a_1b_1 of B, one from each leaf face such that b_0 and b_1 are the only cutvertices of B. Clearly those edges are antipodal and their endpoints meet the conditions of the terminal edges as per Definition 6. Thus B is safe. $\square$

Lemma 17 together with Corollary 15 establish the main result.

Theorem 18. *Every AT-free outerplanar graph is B_0-VPG.*

5 Concluding Remarks

We have showed that all linear outerplanar graphs are B_0-VPG. However, it is easy to see that linearity is not necessary for B_0-VPG outerplanar graphs. For example, planar bipartite graphs, and hence outerplanar bipartite graphs are B_0-VPG [21]. But outerplanar bipartite graphs can be far from being linear, in the sense that their weak duals can be trees with arbitrarily large degrees for internal nodes.

One can see from a B_0-VPG drawing that the closed neighborhood of every vertex is an interval graph [18]. Next, we strengthen this necessary condition by identifying adjacent vertices which are forced to be represented by collinear segments in any B_0-VPG drawing. An induced C_4 has essentially a unique B_0-VPG representation [3]. A C_4 together with exactly one chord is called a *diamond*, and the chord is called the *diamond diagonal*. A diamond diagonal can only be drawn as the intersection of collinear line segments in every B_0-VPG representation [18]. In any B_0-VPG representation of an odd-cycle, an odd number of edges has to be represented as the intersection of collinear line segments. It is also easy to verify that in a given B_0-VPG representation D of G, the binary relation *collinearity* on the set of line segments of D is an equivalence relation. We combine these observations to obtain the following.

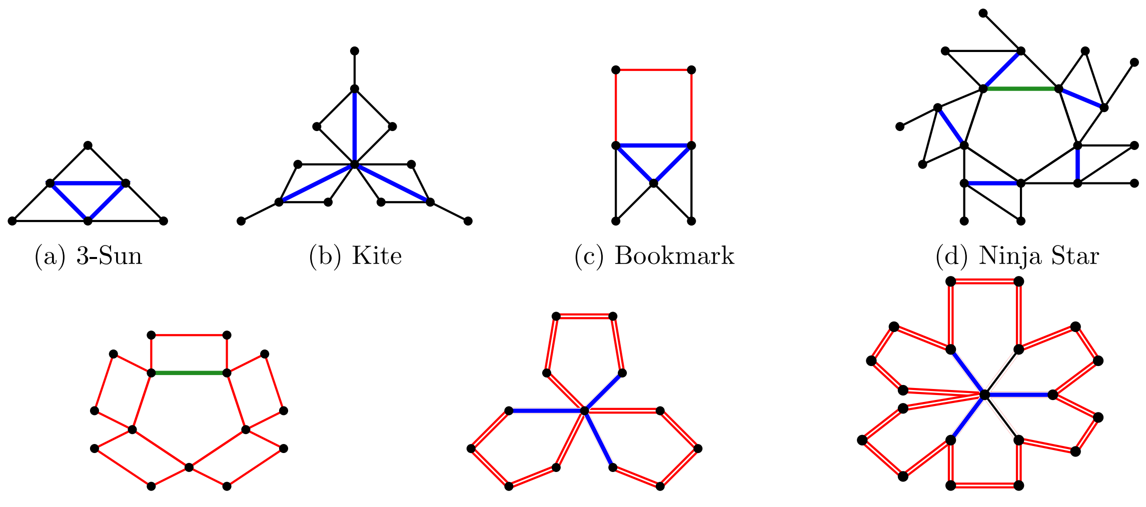
Proposition 19. *If a graph G is B_0-VPG, then there exists a subgraph H of G containing all diamond diagonals of G such that,*

(a) for every component C of H, the subgraph of G induced by the closed neighborhood $N[C]$ of C is an interval graph, and

(b) the minor of G obtained by contracting every component of H in G is a bipartite B_0-VPG graph.

Proof. Since G is B_0-VPG, there exists a B_0-VPG representation D of G. Let H be the spanning subgraph of G in which two vertices are adjacent if and only if the corresponding line segments in D are intersecting and collinear. Since any diamond diagonal of G is represented by the intersection of collinear line segments in D, their endpoints are adjacent in H too. Thus for proving (a), it remains to show that the closed neighborhood of any component of H is an interval graph. Let C be a component of H and let D_C be the drawing induced by the segments of D that represent the vertices in $N[C]$. In D_C, if we restrict the segments that represent the vertices in $N[C] \setminus C$ to point intervals at their intersection point with a vertex in C, we obtain an interval representation for the subgraph of G induced on $N[C]$. To obtain the minor G_H of G in (b), we contract every edge of H. Each component C of H, therefore becomes a branch set of G_H, and can be represented as a line segment obtained as the union of all the segments representing vertices in C. This gives a B_0-VPG representation of G_H. It is easily verified that it is a bipartite graph since there are no collinear intersections. $\square$

Remark 4. Note that since G_H is bipartite, H contains an odd number of edges from each odd cycle of G.



(a) 3-Sun (b) Kite (c) Bookmark (d) Ninja Star

(e) Odd cycle with C_4 petals (f) (g)

Figure 5: A few outerplanar graphs which are not B_0-VPG. The blue edges represent forced collinear edges. The green edges in (d) and (e) represent the edges arbitrarily chosen in an odd cycle to become collinear in a B_0-VPG drawing. The 4-cycles in (e) resemble petals and hence the name. The red double edges in (f) and (g) represent C_4 edges where the C_4 (petal) is not drawn to avoid cluttering.

Proposition 19 shows that the outerplanar graphs in Figure 5 are not B_0-VPG. Nonetheless, the above proposition is not sufficient to characterize B_0-VPG even among outerplanar graphs. Figure 6 is such an example.

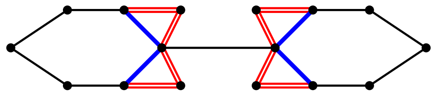


Figure 6: An outerplanar graph which is not B_0-VPG. This shows that the conditions in Proposition 19 was not sufficient. The red double edges represent C_4 edges where the C_4 is not drawn to avoid cluttering. The blue edges are forced to be collinear. Hence the bridge has to be realized as an orthogonal intersection and this prevents a non-crossing drawing of the two 6-cycles.

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References

[1] Alam, M., Eppstein, D., Goodrich, M.T., Kobourov, S.G., Pupyrev, S., et al.: Balanced circle packings for planar graphs. In: International Symposium on Graph Drawing. pp. 125–136. Springer (2014)

[2] Alcón, L., Bonomo, F., Mazzoleni, M.P.: Vertex intersection graphs of paths on a grid: characterization within block graphs. Graphs and Combinatorics **33**(4), 653–664 (2017)

[3] Asinowski, A., Cohen, E., Golumbic, M.C., Limouzy, V., Lipshteyn, M., Stern, M.: Vertex intersection graphs of paths on a grid. J. Graph Algorithms Appl. **16**(2), 129–150 (2012)

[4] Babu, J., Basavaraju, M., Chandran, L.S., Rajendraprasad, D.: 2-Connecting outerplanar graphs without blowing up the pathwidth. Theoretical Computer Science **554**, 119–134 (2014)

[5] Barát, J., Hajnal, P., Lin, Y., Yang, A.: On the structure of graphs with path-width at most two. Studia Scientiarum Mathematicarum Hungarica **49**(2), 211–222 (2012)

[6] Cabello, S., van Kreveld, M.J., Liotta, G., Meijer, H., Speckmann, B., Verbeek, K.: Geometric simultaneous embeddings of a graph and a matching. J. Graph Algorithms Appl. **15**(1), 79–96 (2011)

[7] de Castro, N., Cobos, F.J., Dana, J.C., Márquez, A., Noy, M.: Triangle-free Planar Graphs and Segment Intersection Graphs. J. Graph Algorithms Appl. **6**(1), 7–26 (2002)

[8] Catanzaro, D., Chaplick, S., Felsner, S., Halldórsson, B.V., Halldórsson, M.M., Hixon, T., Stacho, J.: Max point-tolerance graphs. Discrete Applied Mathematics **216**, 84–97 (2017)

[9] Chalopin, J., Gonçalves, D.: Every planar graph is the intersection graph of segments in the plane. In: Proceedings of the forty-first annual ACM symposium on Theory of computing. pp. 631–638 (2009)

[10] Chaplick, S., Jelínek, V., Kratochvíl, J., Vyskočil, T.: Bend-bounded path intersection graphs: Sausages, noodles, and waffles on a grill. In: International Workshop on Graph-Theoretic Concepts in Computer Science, 274–285. Springer (2012)

[11] Chaplick, S., Ueckerdt, T.: Planar graphs as VPG-graphs. In: International Symposium on Graph Drawing, 174–186. Springer (2012)

[12] Chartrand, G., Harary, F.: Planar permutation graphs. In: Annales de l'IHP Probabilités et statistiques. vol. 3, pp. 433–438 (1967)

[13] Czyzowicz, J., Kranakis, E., Urrutia, J.: A simple proof of the representation of bipartite planar graphs as the contact graphs of orthogonal straight line segments. Information Processing Letters **66**(3), 125–126 (1998)

[14] De Fraysseix, H., Ossona de Mendez, P., Pach, J.: Representation of planar graphs by segments. Intuitive Geometry **63**, 109–117 (1991)

[15] Evans, W., Kusters, V., Saumell, M., Speckmann, B.: Column planarity and partial simultaneous geometric embedding. In: International Symposium on Graph Drawing. pp. 259–271. Springer (2014)

[16] Fleischner, H.J., Geller, D.P., Harary, F.: Outerplanar graphs and weak duals. The Journal of the Indian Mathematical Society **38**(1-4), 215–219 (1974)

[17] García, A., Hurtado, F., Noy, M., Tejel, J.: Augmenting the Connectivity of Outerplanar Graphs. Algorithmica **56**(2), 160–179 (2010)

[18] Golumbic, M.C., Ries, B.: On the intersection graphs of orthogonal line segments in the plane: characterizations of some subclasses of chordal graphs. Graphs and Combinatorics **29**(3), 499–517 (2013)

[19] Gonçalves, D.: Not all planar graphs are in PURE-4-DIR. Journal of Graph Algorithms and Applications **24**(3), 293–301 (2020)

[20] Gonçalves, D., Isenmann, L., Pennarun, C.: Planar graphs as L-intersection or L-contact graphs. In: Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms. pp. 172–184. SIAM (2018)

[21] Hartman, I.B.A., Newman, I., Ziv, R.: On grid intersection graphs. Discrete Mathematics **87**(1), 41–52 (1991)

[22] Jain, S., Pallathumadam, S.K., Rajendraprasad, D.: B_0-VPG Representation of AT-free Outerplanar Graphs. In: Conference on Algorithms and Discrete Applied Mathematics. pp. 103–114. Springer (2022)

[23] Kant, G.: Augmenting Outerplanar Graphs. Journal of Algorithms **21**(1), 1–25 (1996)

[24] Kratochvíl, J.: String graphs. II. Recognizing string graphs is NP-hard. Journal of Combinatorial Theory, Series B **52**(1), 67–78 (1991)

[25] Kratochvíl, J., Matoušek, J.: Intersection graphs of segments. Journal of Combinatorial Theory, Series B **62**(2), 289–315 (1994)

[26] Mehrabi, S.: Approximation Algorithms for Independence and Domination on B_1-VPG and B_1-EPG Graphs. arXiv preprint arXiv:1702.05633 (2017)

[27] Pallathumadam, S.K., Rajendraprasad, D.: Characterization of B_0-VPG Cocomparability Graphs and a 2D Visualization of their Posets. Order pp. 1–20 (2022)

[28] Schaefer, M., Sedgwick, E., Štefankovič, D.: Recognizing string graphs in NP. Journal of Computer and System Sciences **67**(2), 365–380 (2003)

[29] Scheinerman, E.R.: Intersection classes and multiple intersection parameters of graphs. Princeton University (1984)

[30] West, D.: Open problems. SIAM J. Discrete Math. Newslett. **2**, 10–12 (1991)