# Parameterized Complexity of Immunization in the Threshold Model 

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#### Abstract

We consider the problem of controlling the spread of harmful items in networks, such as the contagion proliferation of diseases or the diffusion of fake news. We assume the linear threshold model of diffusion where each node has a threshold that measures the node resistance to the contagion. We study the parameterized complexity of the problem: Given a network, a set of initially contaminated nodes, and two integers $k$ and $\ell$, is it possible to limit the diffusion to at most $k$ other nodes of the network by immunizing at most $\ell$ nodes? We consider several parameters associated to the input, including: the bounds $k$ and $\ell$, the maximum node degree $\Delta$, the treewidth, and the neighborhood diversity of the network. We first give $W[1]$ or $W[2]$-hardness results for each of the considered parameters. Then we give fixed-parameter algorithms for some parameter combinations.


Keywords: Parameterized Complexity, Contamination minimization, Threshold model

## 1 Introduction

The problem of controlling the spread of harmful items in networks, such as the contagion proliferation of diseases or the diffusion of fake news, has recently attracted much interest from the research community. The goal is to try to limit as much as possible the spreading process by adopting immunization measures. One such a measure consists in intervening on the network topology either blocking some links so that they cannot contribute to the diffusion process [28] or by immunizing some nodes [14]. In this paper we focus on the second strategy: Limit the spread to a small region of the network by immunizing a bounded number of nodes in the network. We study the problem in the linear threshold model where each node has a threshold, measuring the node resistance to the diffusion [27]. A node gets influenced/contaminated if it receives the item from a number of neighbors at least equal to its threshold. The diffusion proceeds in rounds: Initially only a subset of nodes has the item and is contaminated. At each round the set of contaminated nodes is augmented with each node that has a number of already contaminated neighbors at least equal to its threshold.

In the presence of an immunization campaign, the immunization operation on a node inhibits the contamination of the node itself. Thus, given a network and a subset of its nodes, called spreader set, that has
the malicious item to be diffused to the other nodes in the network, at each round the set of contaminated nodes is augmented only with the nodes for which the number of already contaminated neighbors is at least equal to the node threshold.

Under this diffusion model, we perform a broad parameterized complexity study of the following problem: Given a network, a spreader set, and two integers $k$ and $\ell$, is it possible to limit the diffusion to at most $k$ other nodes of the network by immunizing at most $\ell$ nodes?

### 1.1 Influence diffusion: Related Work

During the past decade the study of spreading processes in complex networks have experienced a particular surge of interest across many research areas from viral marketing, to social media, to population epidemics. Several studies have focused on the problem of finding a small set of individuals who, given the item to be diffused, allow its diffusion to a vast portion of the network, by using the links among individuals in the network to transmit the item itself to their contacts [32]. Threshold models are widely adopted by sociologists to describe collective behaviours [24] and their use to study of the propagation of innovations through a network was first considered in [27]. The linear threshold model has then been widely used in the literature to study the problem of influence maximization, which aims at identifying a small subset of nodes that can maximize the influence diffusion [4, 6, 7, 9, 13, 27].

Recently, some attention has been devoted to the important issue of developing strategies for reducing the spread of negative things through a network. In particular several studies considered the problem of what structural changes can be made to the network topology in order to block negative diffusion processes. Contamination minimization in linear threshold model by blocking some links has been studied in [16, 28]. Strategies for reducing the spread size by immunizing/removing nodes has been considered in several paper. As an example [2, 33] consider a greedy heuristic that immunize nodes in decreasing order of out-degree.

When all the node thresholds are 1 , the immunization can be obtained by a (multi)cut of the network. Some papers dealing with this problem are [5, 25, 26] in case of edge cuts and [19] in case of node cuts.

### 1.2 Parameterized Complexity

Parameterized complexity is a refinement to classical complexity theory in which one takes into account not only the input size, but also other aspects of the problem given by a parameter $p$. We recall that a problem with input size $n$ and parameter $p$ is called fixed parameter tractable $(F P T)$ if it can be solved in time $f(p) \cdot n^{c}$, where $f$ is a computable function only depending on $p$ and $c$ is a constant.

We study the parameterized complexity of the studied problem, formally defined in Section 2, We consider several parameters associated to the input: the bounds $k$ and $\ell$, the number $\zeta$ related to initially contaminated nodes, and some parameters of the underlying network: The maximum degree $\Delta$, the treewidth tw [35], and the neighborhood diversity nd [31]. The two last parameters, formally defined in Sections 3.4 and 3.5 respectively, are two incomparable parameters of a graph that can be viewed as representing sparse and dense graphs respectively [31]; they received much attention in the literature $[1,3,4,7,8,10,13,18,23,20,21,30]$.


Figure 1: A graph $G$ (node thresholds appear in red). (a) The diffusion process in $G$. (b) An example of $X$ whose $G[X]$ includes nodes not influeced. (c) An example of immunizing set $Y\left(X^{\prime}\right)=\left\{v_{3}\right\}$, which enables to confine the diffusion to $X^{\prime}=\left\{v_{1}, v_{5}\right\}$.

### 1.3 Road Map

In Section 2, we formally define the studied immunization problem and summarize our findings. In Section 3, we give hardness results for the considered parameters. In Section 4, we give fixed-parameter algorithms for some parameter combinations.

## 2 Problem statement

Denote by $G=(V, E, t)$ a undirected graph where $V$ is the nodes set, $E$ is the set of edges, and $t: V \rightarrow \mathbb{N}$ is a node threshold function. We use $n$ and $m$ to denote the number of nodes and edges in the graph, respectively. The degree of a node $v$ is denoted by $d_{G}(v)$. The neighborhood of $v$ is denoted by $\Gamma_{G}(v)=$ $\{u \in V \mid(u, v) \in E\}$. In general, the neighborhood of a set $V^{\prime} \subseteq V$ is denoted by $\Gamma_{G}\left(V^{\prime}\right)=\{u \in$ $\left.V \mid(u, v) \in E, v \in V^{\prime}, u \notin V^{\prime}\right\}$. The graph induced by a node set $V^{\prime}$ in $G$ is denoted $G\left[V^{\prime}\right]=\left(V^{\prime}, E^{\prime}, t^{\prime}\right)$ where $E^{\prime}=\left\{(u, v): u, v \in V^{\prime},(u, v) \in E\right\}$ and $t^{\prime}(v)=t(v)$ for each $v \in V^{\prime}$.

Given the network and a spreader set $S$, after one diffusion round, the influenced nodes are all those which are influenced by the nodes in $S$, that is, have a number of neighbors in $S$ at least equal to their threshold. Noticing that nodes in $S$ are already contaminated and cannot be immunized, we can then model the diffusion process as in a graph which represents the network except the spreader set. Namely, we consider the graph $G=(V, E, t)$ where: $V$ is the set of nodes of the network excluding those in the spreader set, $E \subseteq V \times V$ is the edge set, and $t$ is the threshold function $t: V \rightarrow \mathbb{N}$ with $t(v)$ equal to the original threshold of the node $v$ in the network decreased by the number of its neighbors in $S$.

Definition 1. The diffusion process in $G=(V, E, t)$ in the presence of a set $Y \subseteq V$ of immunized nodes is a sequence of node subsets $\mathrm{D}_{G, Y}[1] \subseteq \ldots \subseteq \mathrm{D}_{G, Y}[\tau] \subseteq \ldots \subseteq V$ with
$-\mathrm{D}_{G, Y}[1]=\{u \mid u \in V-Y, t(u)=0\}$, and
$-\mathrm{D}_{G, Y}[\tau]=\mathrm{D}_{G, Y}[\tau-1] \cup\left\{u\left|u \in V-Y,\left|\Gamma_{G}(u) \cap \mathrm{D}_{G}[\tau-1]\right| \geq t(u)\right\}\right.$.
The process ends at $\tau^{*}$ such that $\mathrm{D}_{G, Y}\left[\tau^{*}\right]=\mathrm{D}_{G}\left[\tau^{*}+1\right]$. We set $\mathrm{D}_{G, Y}=\mathrm{D}_{G, Y}\left[\tau^{*}\right]$.
We omit the subscript $Y$ when no node is immunized, that is, $\mathrm{D}_{G}=\mathrm{D}_{G, \emptyset}$. Moreover, we assume that for the input graph it holds $\mathrm{D}_{G}=V$; indeed, we could otherwise remove all the nodes that cannot be influenced, since they are irrelevant to the immunization problem. In particular, each remaining node $v \in V$
has $t(v) \leq d_{G}(v)$, otherwise it could not be influenced. An example is given in Fig. 1 (a). We are now ready to formally define our problem.

Influence-Immunization Bounding (IIB): Given a graph $G=(V, E, t)$ and bounds $k$ and $\ell$, is there a set $Y$ such that $|Y| \leq \ell$ and $\left|\mathrm{D}_{G, Y}\right| \leq k$ ?

For a given set $Y$ we are partitioning the nodes into three subsets: The set $\mathrm{D}_{G, Y}$ which contains the nodes that get influenced, the immunizing set $Y$, which has the property that, if all its nodes are immunized then the diffusion process is circumscribed to $\mathrm{D}_{G, Y}$, and the set $V-Y-\mathrm{D}_{G, Y}$ of the nodes that, by immunizing $Y$, are not influenced.
We will refer to the nodes in the above subsets as influenced, immunized and safe, respectively.
In some cases it will be easier to deal with a different formulation of IIB that starts from the set of nodes to which one wants to confine the diffusion. Given a set $X \subseteq V$, we define the immunizing set $Y(X)$ of $X$ as the set that contains all the nodes in $V-X$ that can be influenced in one round by those in $\mathrm{D}_{G[X]}$, that is, the nodes that get influenced in $X$ when $X$ is isolated from the rest of the graph, namely

$$
\begin{equation*}
Y(X)=\left\{u\left|u \in V-X,\left|\Gamma_{G}(u) \cap \mathrm{D}_{G[X]}\right| \geq t(u)\right\}\right. \tag{1}
\end{equation*}
$$

By the above definitions, we have

$$
\begin{equation*}
\mathrm{D}_{G[X]}=\mathrm{D}_{G, Y(X)}=\mathrm{D}_{G[V-Y(X)]} \subseteq X ; \tag{2}
\end{equation*}
$$

hence, the influenced, immunized and safe node sets are $\mathrm{D}_{G[X]}, Y(X), V-Y(X)-\mathrm{D}_{G[X]}$.
For some $X$, some nodes in $G[X]$ may be not influenced, even though they would in the whole graph $G$ (see Fig. 1 (b)). However, it is easy to see that for each $X$ the set $X^{\prime}=\mathrm{D}_{G[X]} \subseteq X$ is such that $\mathrm{D}_{G\left[X^{\prime}\right]}=X^{\prime}$ and $Y\left(X^{\prime}\right)=\left\{u\left|u \in V-X^{\prime},\left|\Gamma_{G}(u) \cap \mathrm{D}_{G\left[X^{\prime}\right]}\right| \geq t(u)\right\}=Y(X)\right.$. In the following, we will refer as minimal to a set $X$ such that $\mathrm{D}_{G[X]}=X$ (see Fig 1 (c)).

Fact 1. (IIB equivalent) $\langle G, k, \ell\rangle$ is a YES instance iff there is a minimal $X \subseteq V$ s.t.

$$
\begin{equation*}
|X|=\left|\mathrm{D}_{G[X]}\right| \leq k \text { and }|Y(X)| \leq \ell . \tag{3}
\end{equation*}
$$

### 2.1 Summary of results

In this paper we prove that Influence-Immunization Bounding is:
i) W[1]-hard with respect to any of the parameters $k$, tw or nd
ii) W[2]-hard with respect to the pairs $(\ell, \Delta)$, or $(\ell, \zeta)$;
iii) FPT with respect to any of the pairs $(k, \ell),(k, \zeta),(k, \mathrm{tw}),(\Delta, \mathrm{tw}),(k, \mathrm{nd}),(\ell, \mathrm{nd})$,
where tw and nd denote the tree width and the neighborhood diversity of the input graph and $\zeta=\mid\{v \mid v \in$ $V, t(v)=0\} \mid$ is the number of nodes with threshold 0 .

## 3 Hardness

In this section we give $W[1]$ or $W[2]$ hardness results for the considered parameters.

### 3.1 Parameter $k$

Theorem 1. IIB is $W$ [1]-hard with respect to $k$.
Proof. We give a reduction from the CUTting at most $k$ VERTICES with terminal (CVT- $k$ ) problem studied in [19]: Given a graph $H=(V(H), E(H))$, $s \in V(H)$, and two integers $k$ and $\ell$, is there a set $X_{H} \subseteq V(H)$ such that $s \in X_{H},\left|X_{H}\right| \leq k$, and $\left|\Gamma_{H}\left(X_{H}\right)\right| \leq \ell$ ?

To this aim, construct the instance $\langle G, k-1, \ell\rangle$ of IIB where $G=H[V(H)-\{s\}]$ and $t(v)=0$ for each node $v \in \Gamma_{H}(s)$ and $t(v)=1$ for each node $v \in V(H)-\{s\}-\Gamma_{H}(s)$.

Suppose $\langle G, k-1, \ell\rangle$ admits a solution. By $\langle 3\rangle$, there exists a minimal set $X$ such that $|X|=\left|\mathrm{D}_{G[X]}\right| \leq$ $k-1$ and $|Y(X)| \leq \ell$. Noticing that $\Gamma_{H}(s) \subseteq X \cup Y(X)$, one gets that for $X_{H}=X \cup\{s\}$ it holds $\Gamma_{H}(X \cup\{s\})=Y(X)$. Hence $X_{H}=X \cup\{s\}$ satisfies the inequalities $\left|X_{H}\right| \leq k$ and $\left|\Gamma_{H}\left(X_{H}\right)\right| \leq \ell$ and is a solution to CVT- $k$.

Suppose now $X_{H}=X \cup\{s\}$ is a minimum size solution to CVT- $k$. Then $H\left[X_{H}\right]$ is connected, otherwise the connected component containing $s$ would be a smaller solution. Recalling that in $G$ all thresholds are at most 1 , we have that all the nodes in the connected component of a node with threshold 0 get influenced. Hence,

$$
\begin{aligned}
Y(X) & =\left\{u\left|u \in V-X,\left|\Gamma_{G}(u) \cap \mathrm{D}_{G[X]}\right| \geq t(u)\right\}\right. \\
& =\{u \mid u \in V-X, t(u)=0\} \cup\left\{u\left|u \in V-X,\left|\Gamma_{G}(u) \cap X\right| \geq 1\right\}\right. \\
& =\Gamma_{H}(\{s\} \cup X) .
\end{aligned}
$$

As a consequence, $X$ is a solution to IIB. The theorem follows, since Theorem 3 in [19] proves that the latter problem is $W[1]$-hard whit respect to $k$.

The same reduction, recalling that Theorem 5 in [19] proves that CVT- $k$ is $W[1]$-hard with respect to $\ell$, also gives that IIB is $W$ [1]-hard with respect to $\ell$; however, a stronger result is given in the next section.

### 3.2 Parameters $\zeta$ and $\ell$

Theorem 2. IIB is $W$ [2]-hard with respect to the pair of parameters $\zeta$, the number of nodes with threshold 0 , and $\ell$.

Proof. We give a reduction from Hitting Set (HS), which is $W$ [2]-complete in the size of the hitting set: Given a collection $\left\{S_{1}, \ldots, S_{m}\right\}$ of subsets of a set $A=\left\{a_{1}, \ldots, a_{n}\right\}$ and an integer $h>0$, is there a set $H \subseteq A$ such that $H \cap S_{i} \neq \emptyset$, for each ${ }^{1} i \in[m]$ and $|H| \leq h$ ?

Given an instance $\left\langle\left\{S_{1}, \ldots, S_{m}\right\}, A=\left\{a_{1}, \ldots, a_{n}\right\}, h\right\rangle$ of HS, we construct an instance $\langle G, n+1, h\rangle$ of IIB. The graph $G=(V, E, t)$ has node set

$$
V=I \cup A \cup S,
$$

[^0]where $I=\left\{v_{0}, \ldots, v_{h}\right\}$ is a set of $h+1$ independent nodes, $A=\left\{a_{1}, \ldots, a_{n}\right\}$ is the ground set, and $S=\left\{s_{1}, \ldots, s_{m}\right\}$ (each $s_{j}$ represents the set $S_{j}$ ), edge set
$$
E=\left\{\left(v_{i}, a_{j}\right) \mid v_{i} \in I, a_{j} \in A\right\} \cup\left\{\left(a_{j}, s_{t}\right) \mid a_{j} \in A, s_{t} \in S, a_{j} \in S_{t}\right\},
$$
and threshold function defined by
\[

t(v)= $$
\begin{cases}0 & \text { if } v \in I \\ 1 & \text { if } v \in A \\ \left|S_{t}\right|=d_{G}\left(s_{t}\right) & \text { if } v=s_{t} \in S\end{cases}
$$
\]

Trivially, $\mathrm{D}_{G}[1]=I, \mathrm{D}_{G}[2]=I \cup A$, and $\mathrm{D}_{G}[3]=I \cup A \cup S=V$. We prove now that $\left\langle\left\{S_{1}, \ldots, S_{m}\right\}, A, h\right\rangle$ is a YES instance of HS iff $\langle G, n+1, h\rangle$ is a YES instance of IIB.

Suppose first there exists $H \subseteq A$ such that $|H| \leq h$ and $H \cap S_{t} \neq \emptyset$, for each $t \in[m]$. If we consider in $G$ the set of nodes $\tilde{Y} \subseteq A$ corresponding to the elements of $H$ then each node $s_{t} \in S$ is connected with a node in $\tilde{Y}$. Consequently, if all the nodes in $\tilde{Y}$ are immunized, then the number of influenced neighbors of $s_{t}$ cannot reach its threshold $t\left(s_{t}\right)=d_{G}\left(s_{t}\right)$. Hence, no node in $S$ can get influenced. Let then $Y$ be the set obtained by padding $\tilde{Y}$ with nodes in $A-\tilde{Y}$, so to have $|Y|=h$. Clearly, $\mathrm{D}_{G, Y}=I \cup(A-Y)$ with $\left|\mathrm{D}_{G, Y}\right|=n+1$.

Assume now there exists a solution $Y$ of IIB. We notice that:
a) $I \subseteq \mathrm{D}_{G, Y} \cup Y$ (having all the nodes in $I$ threshold 0 , they are immunized or influenced);
b) If there exists $v_{i} \in I \cap Y$, we can update $Y$ to $Y^{\prime}=Y \cup\{a\}-\left\{v_{i}\right\}$, for any $a \in A-Y$ (this implies that $\mathrm{D}_{G, Y^{\prime}} \subseteq \mathrm{D}_{G, Y} \cup\left\{v_{i}\right\}-\{a\}$ ).
c) If there exists $s_{t} \in S \cap Y$ we can update $Y$ to $Y^{\prime}=Y \cup\{a\}-\left\{s_{t}\right\}$, for any $a \in A \cap S_{t}$ (this implies that $\mathrm{D}_{G, Y^{\prime}} \subseteq \mathrm{D}_{G, Y}-\{a\}$ ).

Using a) and iterating b) and c), we can assume that $Y$ consists of at most $h$ nodes in $A$. As a consequence $I \cup(A-Y) \subseteq \mathrm{D}_{G, Y}$. If we assumed that $S \cap \mathrm{D}_{G, Y} \neq \emptyset$, then we would have $\left|\mathrm{D}_{G, Y}\right| \geq|I|+|A-Y|+\mid S \cap$ $\mathrm{D}_{G, Y} \mid>h+1+(n-|Y|) \geq n+1$. Being $S \cap \mathrm{D}_{G, Y}=\emptyset$ implies each node in $S$ has some neighbor in $Y$. Hence, the set $H$ of elements corresponding to the $h$ nodes in $Y$ satisfies $H \cap S_{t} \neq \emptyset$, for each $t \in[m]$.

### 3.3 Parameters $\Delta$ and $\ell$

Theorem 3. IIB is $W[2]$-hard with respect to the pair of parameters $\Delta$, the maximum node degree, and $\ell$.
Given an instance $\left\langle\left\{S_{1}, \ldots, S_{m}\right\}, A=\left\{a_{1}, \ldots, a_{n}\right\}, h\right\rangle$ of HS, we construct an instance $\langle G, k, \ell\rangle$ of IIB, where the maximum node degree is 3 . We start the construction of $G$ by inserting the nodes in $A \cup W \cup U \cup S$ where $A=\left\{a_{1}, \ldots, a_{n}\right\}$ is the ground set and $S=\left\{s_{1}, \ldots, s_{m}\right\}$ (each $s_{j}$ represents the set $S_{j}$ ), while $W$ and $U$ are two auxiliary sets, of at most $n m$ nodes each, that will be used to keep the degree bounded and, at the same time, simulating a complete bipartite connection between $A$ and $S$. We then add the following expansion, reduction and path gadgets.


Figure 2: (a) The expansion gadget. (b) The reduction gadget. (c) The graph $G$.

Expansion gadgets. For each $i \in[n]$, if the sets containing $a_{i}$ are exactly $S_{i_{1}}, S_{i_{2}}, \ldots, S_{i_{\delta_{i}}}$ then we encode this relationships with a gadget, which includes four new nodes for each $s_{i_{j}}$, for $j \in\left[\delta_{i}\right]$. Namely, we add $\delta_{i}$ nodes $\left\{w_{i, i_{1}}, w_{i, i_{2}}, \ldots w_{i, i_{\delta_{i}}}\right\}$ and the edges $\left(a_{i}, w_{i, i_{1}}\right)$ and $\left(w_{i, i_{j}}, w_{i, i_{j+1}}\right)$ for $j \in\left[\delta_{i}-1\right]$.

Reduction gadgets. For each $j \in[m]$, if $S_{j}=\left\{a_{j_{1}}, a_{j_{2}}, \ldots, a_{j_{\gamma_{j}}}\right\}$ then we encode this relationships with a gadget. Namely, we add $\gamma_{j}-1$ nodes $\left\{u_{j_{1}, j}, u_{j_{2}, j}, \ldots, u_{j_{\gamma_{j}-1}, j}\right\}$ and the edges $\left(w_{j_{r+1}, j}, u_{j_{r}, j}\right),\left(u_{j_{r}, j}, u_{j_{r+1}, j}\right)$, for $r \in\left[\gamma_{j}-2\right]$ and $\left(w_{j_{1}, j}, u_{j_{1}, j}\right),\left(w_{j_{\gamma_{j}}, j}, u_{j_{\gamma_{j}-1, j}}\right)$ and $\left(u_{\gamma_{\gamma_{j}-1}, j}, s_{j}\right)$. The reduction gadget is presented in Fig.2(b).

Path gadgets. A path $P_{j}$ of $p=n+2 n m$ nodes departs from each $s_{j} \in S$. See Fig 2 (c). Notice that, by construction the degree of nodes is upper bounded by 3 . We set now the thresholds of the nodes in $G$ as: $t(v)=0$ for each node $v \in A, t(v)=2$ for each node $v \in U$ and $t(v)=1$ for all the remaining nodes.

Lemma 1. $\left\langle\left\{S_{1}, \ldots, S_{m}\right\}, A, h\right\rangle$ is a YES instance of $H S$ iff $\langle G, p, h\rangle$ is a YES instance of IIB.
Proof. Suppose that there exists $H \subseteq A$ such that $|H| \leq h$ and $H \cap S_{j} \neq \emptyset$ for each $j \in[m]$. Consider in $G$ the set of nodes $Y$ corresponding to the elements of $H$. Since $H \cap S_{j} \neq \emptyset$, for each $j \in[m]$, we have that each node $s_{j} \in S$ is connected, through a reduction gadget, with a node in $w_{i, j}$ such that $a_{i} \in S_{j} \cap Y$. Consequently, if all the nodes in $Y$ are immunized, then at least one node in the reduction gadget associated to $s_{j}$ cannot reach the threshold and consequently $s_{j}$ will not be influenced. Hence, no node in $S$ as well as in the associated path gadgets can get influenced. We have $|Y| \leq h$ and $\left|\mathrm{D}_{G, Y}\right|<p$, where the last inequality follows noticing that $p=n+2 n m$ is greater than the number of nodes that remain in $G$ once we eliminate the nodes in $S$ and in the path gadgets.

Assume now there exists a solution $Y$ to IIB such that $|Y| \leq h$ and $\left|\mathrm{D}_{G, Y}\right| \leq p$. Without loss of generality, we can assume that $Y \subseteq A$. Indeed, if $Y$ contains either of the nodes $w_{i, i_{j}}, u_{i, i_{j}}, s_{i_{j}}$ or a node in the path $P_{i_{j}}$, for some $i \in[n]$, we could replace such a node by $a_{i} \in A$ without increasing neither the size of $Y$ nor $\mathrm{D}_{G, Y}$. Hence, we have that $Y$ consists of at most $h$ nodes in $A$. We argue that the set $H \subseteq A$ of the elements corresponding to the nodes in $Y$ satisfies $H \cap S_{j} \neq \emptyset$, for each $j \in[m]$. Indeed, assume by contradiction that there is a set $S_{j}$ such that $H \cap S_{j}=\emptyset$. This implies that in $G$ the node $s_{j}$ will be influenced. Indeed, $s_{j}$ is connected through gadgets, to all the nodes in $S_{j}$. Moreover each node in $S_{j}$ belongs to $A-Y$ and has threshold 0 . It follows that $s_{j}$ and, as a consequence, all the $p$ nodes on the associated path get influenced and we obtain the desired contradiction because this violate the bound on the size of $\mathrm{D}_{G, Y}$.

### 3.4 Graphs of bounded treewidth

Definition 2. A tree decomposition of a graph $G=(V, E)$ is a pair $\left(T,\left\{W_{u}\right\}_{u \in V(T)}\right)$, where $T$ is a tree in which each node $u$ is assigned a node subset $W_{u} \subseteq V$ such that:

1. $\bigcup_{u \in V(T)} W_{u}=V$.
2. For each edge $e=(v, w) \in E$, there exists $u$ in $T$ such that $W_{u}$ contains both $v$ and $w$.
3. For each $v \in V$, the set $T_{v}=\left\{u \in V(T): v \in W_{u}\right\}$, induces a connected subtree of $T$.

The width of a tree decomposition $\left(T,\left\{W_{u}\right\}_{u \in V(T)}\right)$ of a graph $G$, is $\max _{u \in V(T)}\left|W_{u}\right|-1$. The treewidth of $G$, denoted by $\operatorname{tw}(G)$, is the minimum width of a tree decomposition of $G$.

Theorem 4. IIB is $W[1]$-hard with respect to the treewidth of the input graph.
In order to prove Theorem 4, we present a reduction from Multi-Colored clique (MQ): Given a graph $G=(V, E)$ and a proper vertex-coloring $\mathbf{c}: V \rightarrow[q]$ for $G$, does $G$ contain a clique of size $q$ ?
Given an instance $\langle G, q\rangle$ of MQ, we construct an instance $\left\langle G^{\prime}=\left(V^{\prime}, E^{\prime}\right), k, \ell\right\rangle$ of IIB. We denote by $n^{\prime}=\left|V^{\prime}\right|$ the number of nodes in $G^{\prime}$. For a color $c \in[q]$, we denote by $V_{c}$ the class of nodes in $G$ of color $c$ and for a pair of distinct $c, d \in[q]$, we let $E_{c d}$ be the subset of edges in $G$ between a node in $V_{c}$ and one in $V_{d}$.

Our goal is to guarantee that any solution of IIB in $G^{\prime}$ encodes a clique in $G$ and vice-versa. Following some ideas in [4], we construct $G^{\prime}$ using the following gadgets:

Parallel-paths gadget: A parallel-paths gadget of size $h$, between nodes $x$ and $y$, consists of $h$ disjoint paths each made up by a connection node which is adjacent to both $x$ and $y$. In order to avoid cluttering, we draw such a gadget as an edge with label $h$ (cf. Fig. 3(a)).

Selection gadgets: The selection gadgets encode the selection of nodes (node-selection gadgets) and edges (edge-selection gadgets):

Node-selection gadget: For each $c \in[q]$, we construct a $c$-node-selection gadget which consists of a node $x_{v}$ for each $v \in V_{c}$; these nodes are referred as node-selection nodes. We then add a guard node $g_{c}$ that is connected to all the other nodes in the gadget; thus the gadget is a star centered at $g_{c}$.

Edge-selection gadget: For each $c, d \in[q]$ with $c \neq d$, we construct a $\{c, d\}$-edge-selection gadget which consists of a node $x_{u, v}$ for every edge $(u, v) \in E_{c d}$; these nodes are referred as edge-selection nodes. We then add a guard node $g_{c d}$ that is connected to all the other nodes in the gadget; thus the gadget is a star centered at $g_{c d}$.

Overall there are $n$ node-selection nodes with $q$ guard nodes and $m$ edge-selection nodes with $\binom{q}{2}$ guard nodes (cf. Fig. 3(b)).

Validation gadgets: We assign to every node $v \in V(G)$ two unique identifier numbers, $\operatorname{low}(v)$ and $\operatorname{high}(v)$, with $\operatorname{low}(v) \in[n]$ and $\operatorname{high}(v)=2 n-\operatorname{low}(v)$. For every pair of distinct $c, d \in[q]$, we construct two validation gadgets. One between the $c$-node-selection gadget and the $\{c, d\}$-edge-selection gadget and one between the $d$-node-selection gadget and the $\{c, d\}$-edge-selection gadget. We describe the validation gadget between the $c$-node-selection and $\{c, d\}$-edge-selection gadgets. It consists of two nodes. The first


Figure 3: (a) Parallel-paths gadget. (b) Representation of the graph $G^{\prime}$ for a trivial instance of the MQ problem $\left\langle G=\left(V_{1} \cup V_{2}, E_{1,2}\right), 2\right\rangle$.
one is connected to each node $x_{v}$, for $v \in V_{c}$, by parallel-paths gadgets of size $\operatorname{high}(v)$, and to each edgeselection node $x_{u, v}$, for $(u, v) \in E_{c d}$ and $v \in V_{c}$, by parallel-paths gadgets of size low $(v)$. The other node is connected to each node $x_{v}$, for $v \in V_{c}$, by parallel-paths gadgets of $\operatorname{size} \operatorname{low}(v)$, and to each edge-selection node $x_{u, v}$, for $(u, v) \in E_{c d}$ and $v \in V_{c}$, by parallel-paths gadgets of size $\operatorname{high}(v)$. Overall, there are $q(q-1)$ validation gadgets, each composed by two nodes.

Black-hole gadget: We add a set $B$ of $|B|=(n-q)(2 n q-2 n+1)+\left(m-\binom{q}{2}\right)(4 n+1)$ independent nodes and a complete bipartite graph between nodes in $B$ and the guard nodes.

To complete the construction, we specify the thresholds of the nodes in $G^{\prime}$

$$
t(x)= \begin{cases}0 & \text { if } x \text { is a selection node } \\ 1 & \text { if } x \text { is a connection node or } x \in B \\ d_{G^{\prime}}(x)-2 n+1 & \text { if } x \text { is a validation node } \\ \left|V_{c}\right| & \text { if } x=g_{c} \text { is a guard node for some } c \in[q] \\ \left|E_{c d}\right| & \text { if } x=g_{c d} \text { is a guard node for some } c, d \in[q]\end{cases}
$$

The complete construction of $G^{\prime}$ for an instance of the MQ problem appears in Fig. 3(b).
Lemma 2. $\langle G, q\rangle$ is a YES instance of $M Q$ if and only if $\left\langle G^{\prime}, k, \ell\right\rangle$, where $k=(n-q)(2 n q-2 n+1)+$ $\left(m-\binom{q}{2}\right)(4 n+1)$ and $\ell=q+\binom{q}{2}$ is a YES instance of IIB.

Proof. We first notice that a node $v$ can belong to the desired clique only if $\{v\} \cup \Gamma_{G}(v)$ contains at least one node from each color class. Hence, we can remove from $G$ all the nodes that do not satisfy such a property, since they are irrelevant to the problem.

Suppose that $K=(V(K), E(K))$ is a multi-colored clique in $G$ of size $q$. Let $C$ denote the set of connection nodes and $X_{K}=\left\{x_{v}: v \notin V(K)\right\} \cup\left\{x_{u, v}:(u, v) \notin E(K)\right\}$. We set

$$
X=X_{K} \cup\left\{c \in C: \Gamma_{G^{\prime}}(c) \cap X_{K} \neq \emptyset\right\} .
$$

We show that

$$
Y=\left\{x_{v}: v \in V(K)\right\} \cup\left\{x_{u, v}:(u, v) \in E(K)\right\}
$$

is the immunizing set of $X$, i.e., $Y=Y(X)$. Notice that $|Y|=q+\binom{q}{2}$.
We first observe that $\mathrm{D}_{G^{\prime}[X]}=X$. Indeed, nodes in $\left\{x_{v}: v \notin V(K)\right\} \cup\left\{x_{u, v}:(u, v) \notin E(K)\right\}$ have threshold 0 and their neighbors in $C$ have threshold 1 . Now we can easily evaluate the size of $X$. Indeed $X$ is composed by:

- $n-q$ nodes in the set of node-selection nodes and their $(n-q) 2 n(q-1)$ neighbors in $C$. Indeed, each node-selection node is connected with $q-1$ validation pair and, for each node $x_{u}$, we have $\operatorname{low}(u)+\operatorname{high}(u)=2 n$.
- $m-\binom{q}{2}$ nodes in the set of edge-selection nodes and their $\left(m-\binom{q}{2}\right) 4 n$ neighbors in $C$. Indeed, each edge-selection node is connected with two validation pair and for each node $x_{u, v}$ we have that $\operatorname{low}(u)+\operatorname{high}(u)=\operatorname{low}(v)+\operatorname{high}(v)=2 n$.

Overall the set $X$ has size

$$
\begin{equation*}
k=(n-q)(2 n q-2 n+1)+\left(m-\binom{q}{2}\right)(4 n+1) \tag{4}
\end{equation*}
$$

It remains to show that $Y=Y(X)$. First of all, we observe that $Y \subseteq Y(X)$ because all the nodes in $Y$ belongs to $V^{\prime}-X$ and have threshold 0 , hence, by $(1)$, each node in $Y$ belongs to $Y(X)$. We show now that for any $v \in V^{\prime}-X$ it holds $\left|\Gamma_{G^{\prime}}(v) \cap X\right|<t(v)$.

- Each guard node $g$ has a neighbor in $Y$ and its threshold is equal to the number of its neighbors belonging to its selection gadget. Hence, $\left|\Gamma_{G^{\prime}}(g) \cap \mathrm{D}_{G^{\prime}[X]}\right|<t(g)$.
- For each $b \in B$, it holds $\left|\Gamma_{G^{\prime}}(b) \cap X\right|=0<t(b)=1$.
- Consider now the validation nodes. Knowing that $K$ is a multi-colored clique, we have that for each validation pair there is exactly one node $u$ and one edge $(u, v)$ such that $x_{u}, x_{u, v} \in Y$. Hence, both nodes have exactly $\operatorname{low}(\cdot)+\operatorname{high}(\cdot)=2 n$ neighbors which do not belong to $X$. Since the threshold of each validation node $x$ is $t(x)=d_{G^{\prime}}(x)-2 n+1$, then $\left|\Gamma_{G^{\prime}}(x) \cap X\right|=d_{G^{\prime}}(x)-2 n<t(x)$.
- Finally, for each connection node $c \notin X$, we have $\left|\Gamma_{G^{\prime}}(c) \cap X\right|=0<t(c)=1$.

Assume now there exists a solution $Y$ to IIB such that $|Y| \leq \ell=q+\binom{q}{2}$ and

$$
\begin{equation*}
\left|\mathrm{D}_{G^{\prime}, Y}\right| \leq k=(n-q)(2 n q-2 n+1)+\left(m-\binom{q}{2}\right)(4 n+1) . \tag{5}
\end{equation*}
$$

Noticing that $k<|B|+1$ and all the nodes in $B$ get influences as soon as a guard node is, we have that the immunization of $Y$ saves all the guard nodes. Noticing that the number of guard nodes is exactly
$q+\binom{q}{2}$ and each guard node is connected to a separate set of selection nodes, we have that $|Y|=q+\binom{q}{2}$ and each node in $Y$ can save one guard node. Recalling that the thresholds of guard nodes is equal to the number of neighbors belonging to the corresponding selection gadget, we have that in order to save a guard node there are two options: Put the guard node in $Y$ or put in $Y$ one of its neighbors, belonging to the corresponding selection gadget. Without loss of generality, we can assume that $Y$ does not include any guard node. Indeed, if $Y$ contains a guard node we could replace such a node by one of its selection node neighbors without increasing neither the size of $Y$ nor of $\mathrm{D}_{G^{\prime}, Y}$.

We can then assume that $Y$ is composed by exactly $q$ node-selection nodes and $\binom{q}{2}$ edge-selection nodes. Let $V_{Y} \subseteq V$ be a set of $q$ nodes in $G$, defined by $V_{Y}=\left\{v \in V: x_{v} \in Y\right\}$. We argue that $G\left[V_{Y}\right]$ is a clique. By contradiction suppose that $G\left[V_{Y}\right]$ is not a clique. There are two nodes $u, v \in V_{Y}$ such that $(u, v) \notin E$. Let $c, d$ respectively the colors of $v$ and $u$. Let $x_{w, z}$ the node in $G^{\prime}$ which save the guard $g_{c d}$ associated to the pair $c, d$. Since $(u, v) \notin E$ we have that $w \neq u$ or $z \neq v$ or both. Without loss of generality, we can assume that $w \neq u$. Consider now the validation pair between the $c$-node- and $\{c, d\}$-edge-selection gadgets. Recalling that $Y$ contains exactly one node for each selection gadget, we have that both the nodes in the validation pair have all the neighbors influenced, except for the connections of the nodes $x_{u}$ and $x_{w, z}$. Since $w \neq u$, we have that one of the vertices in the validation pair will get influenced. This is because for any $w \neq u$ either $\operatorname{high}(w)+\operatorname{low}(u)<2 n$ or $\operatorname{low}(w)+\operatorname{high}(u)<2 n$. That is, there is a validation node $x$ having less than $2 n$ not influenced neighbors, while all the remaining neighbors get influenced. Recalling that the threshold of $x$ is $d_{G^{\prime}}(x)-2 n+1$, we have that $x$ get influenced.

Hence, $\left|\mathrm{D}_{G^{\prime}, Y}\right|=k+1$. Indeed $k$ are due to non immunized selection nodes and their connection neighbors (see (4)) plus at least one validation node. This contradicts (5).

Lemma 3. $G^{\prime}$ has treewidth $O\left(q^{2}\right)$.
Proof. We show now that $G^{\prime}$ admits a tree decomposition of width $O\left(q^{2}\right)$. The complete bipartite network defined by the guard nodes and the nodes in $B$ has treewidth $q+\binom{q}{2}$. Let $A$ be the set of the guard nodes of size $q+\binom{q}{2}$ and $b_{1}, b_{2}, \ldots, b_{\hat{n}}$ the nodes in $B$. The decomposition tree has $A$ as root and $A \cup b_{i}$ as children. Then we can add to this network the $q+\binom{q}{2}$ trees, rooted on the guard nodes and containing both selections and connection nodes, without increasing the treewidth. Finally we can add all $O\left(q^{2}\right)$ validation nodes, getting a tree decomposition of width $O\left(q^{2}\right)$ for $G^{\prime}$.

### 3.5 Graphs of bounded neighborhood diversity

Given a graph $G=(V, E)$, two nodes $u, v \in V$ are said to have the same type if $\Gamma_{G}(v) \backslash\{u\}=\Gamma_{G}(u) \backslash\{v\}$. The neighborhood diversity of a graph $G$, introduced by Lampis in [31] and denoted by $\mathrm{nd}(G)$, is the minimum number nd of sets in a partition $V_{1}, V_{2}, \ldots, V_{\text {nd }}$, of the node set $V$, such that all the nodes in $V_{i}$ have the same type, for $i \in[\mathrm{nd}]$. The family $\left\{V_{1}, V_{2}, \ldots, V_{\text {nd }}\right\}$ is called the type partition of $G$.
Notice that each $V_{i}$ induces either a clique or an independent set in $G$. Moreover, for each $V_{i}, V_{j}$ in the type partition, we get that either each node in $V_{i}$ is a neighbor of each node in $V_{j}$ or no node in $V_{i}$ has a neighbor in $V_{j}$. Hence, between each pair $V_{i}, V_{j}$, there is either a complete bipartite graph or no edges at all.

Theorem 5. IIB is W[1]-hard with respect to the neighborhood diversity of the input graph.
In order to prove Theorem 5, we use a reduction from Multi-Colored clique (MQ), defined in Section 3.4 As before, we refer to $V_{c}$ as a color class of $G$ and to $E_{c d}$ as the set of edges between nodes


Figure 4: An overview of the reduction. Each circle represents a bag. The number inside a bag is the number of nodes of the bag. The threshold of nodes in a bag is displayed in red.
in the color classes $V_{c}$ and $V_{d}$. Here we will use the fact that MQ remains W[1]-hard even if each color class has the same size and for each distinct colors $c, d \in[q]$, the set $E_{c d}$ has the same size [11]. We then denote by $r+1$ the size of each color class $V_{c}$ and by $s+1$ the size of each set $E_{c d}$, in particular we use the following notation

$$
\begin{equation*}
V_{c}=\left\{v_{0}^{c}, v_{1}^{c}, \ldots, v_{r}^{c}\right\}, \quad E_{c d}=\left\{e_{0}^{c d}, \ldots, e_{s}^{c d}\right\} \quad c, d \in[q], c \neq d \tag{6}
\end{equation*}
$$

and refer to $v_{i}^{c}$ and $e_{j}^{c d}$ as the $i$-th node in $V_{c}$ and the $j$-th edge in $E_{c d}$, respectively.
Let $\langle G, q\rangle$ be an instance of MQ. We describe a reduction from $\langle G, q\rangle$ to an instance $\left\langle G^{\prime}, k, \ell\right\rangle$ of IIB such that $\operatorname{nd}\left(G^{\prime}\right)$ is $O\left(q^{2}\right)$. The reduction runs in time poly $(|G|)$.

In order to present the reduction we introduce some gadgets that are used in the construction of $G^{\prime}$. They are inspired by those used in [13]. The rationale behind the construction is the following. First, we create two sets of gadgets (Selection and Multiple gadgets), which encode in $G^{\prime}$ the selection of nodes and edges as part of a potential multicolored clique in $G$. Then we create another set of gadgets (Incidence gadgets) that is used to check whether the selected sets of nodes and edges actually represent a multicolored clique in $G$. Our goal is to guarantee that any solution of IIB in $G^{\prime}$ encodes a clique in $G$ and vice-versa.

In the following we call bag an independent set of nodes of a graph sharing all neighbors. So, a connection between two bags points out a complete bipartite graph among the nodes in the bags. Fig. 4 shows the gadgets we are going to introduce and how they are connected.

Selection Gadget. For each $c \in[q]$, the selection gadget $L_{c}$ consists of three bags: $L_{c}$-neg and $L_{c}$-pos of $r$ nodes each, and $L_{c}$-guard of $\ell+1$ nodes (the value $\ell$, representing an upper bound on the number of nodes to be immunized, will be determined later). The bag $L_{c}$-guard is connected to both $L_{c}$-neg and $L_{c}$-pos. We set the threshold of each node $g$ in $L_{c}$-guard to $t(g)=r+1$ and the threshold of each node $v$ in $L_{c}$-neg $\cup L_{c}$-pos to $t(v)=0$. The selection gadget $L_{c}$ is connected to the rest of the graph $G^{\prime}$ using only nodes from $L_{c}$-neg $\cup L_{c}$-pos.

Multiple Gadget. For each $c, d \in[q]$ with $c \neq d$, we create a multiple gadget $M_{c d}$ consisting of six
 nodes each, and $M_{c d^{-}}$-guard of $\ell+1$ nodes. $M_{c d^{-}}$-guard is connected to the bags $M_{c d}$-pos and $M_{c d}$-neg. $M_{c d}$-pos is connected to $L_{c d}$-pos, and $M_{c d}$-neg is connected to $L_{c d}$-neg. Finally, the bag $L_{c d}$-guard is connected to both $L_{c d}$-pos and $L_{c d}$-neg. The rest of graph $G^{\prime}$ is connected only to the bags $L_{c d}$-pos and $L_{c d}$-neg. We set the threshold of each $g \in M_{c d}$-guard to $t(g)=s+1$. For each node $v \in L_{c d}$-pos $\cup L_{c d}$-neg, we set the threshold $t(v)=0$. Let $M_{c d}$-pos $=\left\{x_{0}, \ldots, x_{s}\right\}$ and $M_{c d}$-neg $=\left\{y_{0}, \ldots, y_{s}\right\}$; we set thresholds $t\left(x_{i}\right)=t\left(y_{i}\right)=2 r i+1$. Finally, for each $g \in L_{c d}$-guard, we set the threshold $t(g)=2 r s+1$.

Incidence Gadget. For each pair of distinct $c, d \in[q]$, we construct two incidence gadgets: $I_{c: c d}$ (connected with the gadgets $L_{c}$ and $M_{c d}$ ) and $I_{d: c d}$ (connected with the gadgets $L_{d}$ and $M_{c d}$ ). In the following we present the gadget $I_{c: c d}$ which has the same structure of the gadget $I_{d: c d}$. The incidence gadget $I_{c: c d}$ has three bags $I_{c: c d}$-pos and $I_{c: c d}$-neg of $s+1$ nodes each, and $I_{c: c d}$-guard of $\ell+1$ nodes. We connect $I_{c: c d^{-} \text {-guard }}$ to $I_{c: c d}$-pos and $I_{c: c d}$-neg. Furthermore, we connect $I_{c: c d}$-pos to $L_{c}$-pos and $L_{c d}$-pos. Similarly, we connect $I_{c: c d}$-neg to $L_{c}$-neg and $L_{c d}$-neg. We set the threshold of each $g \in I_{c: c d}$-guard to $t(g)=s+1$. Recalling that there are $s+1$ edges in the set $E_{c d}$, and that there are $s+1$ nodes in $I_{c: c d}$-pos and $I_{c: c d}$-neg, we create one-to-one correspondences between $E_{c d}$ and $I_{c: c d}$-pos and between $E_{c d}$ and $I_{c: c d}$-neg. Namely, for each $j=$ $0, \ldots s$, we associate the $j$-th edge $e_{j}^{c d}$ in $E_{c d}$ (cfr. (6)) to a node $u_{j} \in I_{c: c d}$-pos and to a node $w_{j} \in I_{c: c d}$-neg (with $u_{j} \neq u_{j^{\prime}}$ and $w_{j} \neq w_{j^{\prime}}$, for $j \neq j^{\prime}$ ). Moreover, if the endpoint of $e_{j}^{c d}$ of color $c$ is the $i$ th node $v_{i}^{c}$ of $V_{c}$ (cfr. 6) then we set $\quad t\left(u_{j}\right)=i+1+2 r j, \quad t\left(w_{j}\right)=r-i+1+2 r(s-j)$.
It is worth observing that the nodes in $I_{c: c d}$-pos (respectively, $I_{c: c d}$-neg) have different thresholds. Indeed, the numbers $i+1+2 r j$ (respectively, $r-i+1+2 r(s-j)$ ) are all different, for $0 \leq i \leq r$ and $0 \leq j \leq s$.
Black-hole Gadget. Finally we add a gadget, which will force the immunizing set $Y$ to contain a specific number of nodes for selection ( $r$ nodes) and multiple gadgets ( $2 r s$ nodes). We add a bag $B$ of $|B|=$ $q r+\binom{q}{2}(2 r+3) s$ nodes and connect it to the guard bags in all the selection, multiple and incidence gadgets. For each $v \in B$, we set $t(v)=1$.

Lemma 4. $\langle G, q\rangle$ is a YES instance of MQ iff $\left\langle G^{\prime}, k, \ell\right\rangle$ is a YES instance of IIB, where $k=q r+\binom{q}{2}(2 r+3) s$ and $\ell=q r+\binom{q}{2} 2 r s$.

The proof of Lemma 4 will follow by Claims 1, 2 proved below.
Claim 1. If $\langle G, q\rangle$ is a YES instance of MQ then $\left\langle G^{\prime}, k, \ell\right\rangle$ is a YES instance of IIB.
Proof. Let $K=(V(K), E(K))$ be a multicolored clique of $G$. We will show how to select nodes to be added to the immunizing set $Y$ according to the nodes in $K$. First of all notice that, all the nodes in the bags $L_{c}$-pos, $L_{c}$-neg, $L_{c d}$-pos, and $L_{c d}$-neg belong to $Y \cup \mathrm{D}_{G^{\prime}, Y}$, as they all have threshold zero.

For each $c \in[q]$, if the unique node of color $c$ in $K$ is $v_{i}^{c}$, the $i$-th node in $V_{c}$, then we add $i$ nodes of $L_{c}$-neg and $r-i$ nodes of $L_{c}$-pos to $Y$. For each pair of distinct $c, d \in[q]$, if the unique edge with endpoints of colors $c$ and $d$ in $K$ is $e_{j}^{c d}$, then we add $2 r j$ nodes of $L_{c d}$-neg and $2 r(s-j)$ nodes of $L_{c d}$-pos to $Y$. Overall, $|Y|=\ell=q r+\binom{q}{2} 2 r s$. We now prove that $\left|\mathrm{D}_{G^{\prime}, Y}\right|=k=q r+\binom{q}{2}(2 r+3) s$.

Consider the diffusion process in $V\left(G^{\prime}\right)-Y$. At the first round, all non immunized nodes with threshold zero are influenced; hence $\mathrm{D}_{G^{\prime}, Y}[1]$ contains: $i$ nodes of $L_{c}$-pos, for all $c \in[q]$ and $r-i$ nodes of $L_{c}$-neg, $2 r j$ nodes of $L_{c d}$-pos, $2 r(s-j)$ nodes of $L_{c d}$-neg, for all $c, d \in[q]$ with $c \neq d$.

We claim that, at the second round, the additional influenced nodes (in the neighborhood of $\mathrm{D}_{G^{\prime}, Y}[1]$ ) are exactly: $s$ nodes in $M_{c d}$-pos $\cup M_{c d}$-neg, $s$ nodes in $I_{c: c d}$-pos $\cup I_{c: c d}$-neg, and $s$ nodes in $I_{d: c d}$-pos $\cup I_{d: c d}$-neg, for each pair of distinct $c, d \in[q]$. Indeed, let $M_{c d}$-pos $=\left\{x_{0}, \ldots, x_{s}\right\}$ and $M_{c d}$-neg $=\left\{y_{0}, \ldots, y_{s}\right\}$. Since at the end of the first round the nodes in $M_{c d}$-pos have $2 r j$ influenced neighbors in $L_{c d}$-pos and the nodes in $M_{c d}$-neg have $2 r(s-j)$ influenced neighbors in $L_{c d}$-neg, recalling that $t\left(x_{j}\right)=t\left(y_{j}\right)=2 r j+1$, we have that nodes $x_{0}, \ldots, x_{j-1}$ in $M_{c d}$-pos and nodes $y_{0}, \ldots, y_{s-j-1}$ in $M_{c d}$-neg get influenced. Overall $s$ nodes in $M_{c d}$-pos $\cup M_{c d}$-neg are influenced at the second round.
Consider now the incidence gadgets. Since there are $2 r j+i$ influenced nodes in $L_{c}$-pos $\cup L_{c d}$-pos that are in neighborhood of the nodes in $I_{c: c d}$-pos, recalling that the thresholds of nodes in $I_{c: c d}$-pos are:

$$
\begin{aligned}
t\left(u_{j}\right) & =2 r j+i+1>2 r j+i \text { and } \\
t\left(u_{h}\right) & =2 r h+h^{\prime}+1 \text { for each } 0 \leq h \leq s, h \neq j, \text { and } 0 \leq h^{\prime} \leq r
\end{aligned}
$$

we have

$$
\begin{array}{lll}
t\left(u_{h}\right) \leq 2 r h+r+1 \leq 2 r(j-1)+r+1=2 r j-r+1 \leq 2 r j+i & \text { if } h<j \\
t\left(u_{h}\right) \geq 2 r h+1 \geq 2 r(j+1)+1>2 r j+2 r+1>2 r j+i & \text { if } h>j
\end{array}
$$

Hence, nodes $u_{0}, \ldots, u_{j-1}$ in $I_{c: c d}$-pos are influenced at the second round.
We now make a similar analysis for the nodes in $I_{c: c d}$-neg. Since there are $r-i+2 r(s-j)$ influenced nodes in $L_{c}$-neg $\cup L_{c d}$-neg that are in neighborhood of the nodes in $I_{c: c d}$-neg, recalling that the threshold of nodes in $I_{c: c d}$-pos are:

$$
\begin{aligned}
& t\left(w_{j}\right)=2 r(s-j)+r-i+1>2 r(s-j)+r-i \text { and } \\
& t\left(w_{h}\right)=2 r(s-h)+r-h^{\prime}+1 \text { for some } 0 \leq h^{\prime} \leq r
\end{aligned}
$$

we have

$$
\begin{aligned}
& t\left(w_{h}\right) \geq 2 r(s-h)+1 \geq 2 r(s-j)+2 r+1>2 r(s-j)+r-i \quad \text { for } h<j \\
& t\left(w_{h}\right) \leq 2 r(s-h)+n+1 \leq 2 r(s-j)-r+1 \leq 2 r(s-j)+r-i \quad \text { for } h>j
\end{aligned}
$$

Hence, nodes $w_{j+1}, \ldots, w_{s}$ in $I_{c: c d}$-neg are influenced at the second round. Overall, we have that $s$ nodes in $I_{c: c d}$-pos $\cup I_{c: c d}$-neg are influenced at the second round.
Using exactly the same argument we can show that $s$ nodes in $I_{d: c d}$-pos $\cup I_{d: c d}$-neg are influenced at the second round.

Finally, the nodes in $L_{c^{-}}$guard (resp. $L_{c d}$-guard) have $r$ (resp. $2 r s$ ) influenced neighbors at the end of the first round and since all of them have threshold $r+1$ (resp. $2 r s+1$ ), we have that none of them gets influenced at the second round.

We notice now that only the nodes in $M_{c d^{-}}$guard and $I_{c: c d^{-}}$-guard have neighbors in $\mathrm{D}_{G^{\prime}, Y}[2]$. However, they cannot be influenced (indeed, each of them has threshold $s+1$ but it has only $s$ influenced neighbors in $\mathrm{D}_{G^{\prime}, Y}[2]-$ in $M_{c d^{\prime}}$-pos $\cup M_{c d}$-neg or in $I_{c: c d}$-pos $\cup I_{c: c d}$-neg). We have that $\mathrm{D}_{G^{\prime}, Y}[3]=\mathrm{D}_{G^{\prime}, Y}[2]$ and the diffusion process stops.

Summarizing, $\mathrm{D}_{G^{\prime}, Y}$ contains: $r$ influenced nodes for each of the $q$ nodes in the clique $K$ (those that are influenced in the selection gadgets $L_{c}$ for $\left.c \in[q]\right), 2 r s+s$ influenced nodes for each of the $\binom{q}{2}$ edges
in $K$ (those in the multiple gadgets $M_{c d}$, for $c, d \in[q]$ ) and $2 s$ influenced nodes, for each of the $\binom{q}{2}$ edges in $K$ (those in the incidence gadgets $I_{c: c d}$ and $I_{d: c d}$, for distinct $c, d \in[q]$ ). Hence, the set $\mathrm{D}_{G^{\prime}, Y}$ contains $k=q r+\binom{q}{2}(2 r+3) s$ nodes.

Let $Y$ be an immunizing set such that $|Y| \leq \ell=q r+\binom{q}{2} 2 r s$ and $\left|\mathrm{D}_{G^{\prime}, Y}\right| \leq k=q r+\binom{q}{2}(2 r+3) s$. In the following we derive some useful constraints on the nodes contained in $Y$ and $\mathrm{D}_{G^{\prime}, Y}$.

Proposition 1. For distinct $c, d \in[q]$, no node in $L_{c^{-}}$-guard, $L_{c d^{-}}$-guard, $I_{c: c d}{ }^{-}$guard, $I_{d: c d^{-}}$guard, $M_{c d}$-guard can be in $\mathrm{D}_{G^{\prime}, Y}$.

Proof. Since the threshold of each $v \in B$ is $t(v)=1$, it is sufficient that at least one guard node $g \in$ $L_{c}$-guard $\cup L_{c d}$-guard $\cup I_{c: c d}$-guard $\cup I_{d: c d}$-guard $\cup M_{c d}$-guard is influenced to influence the whole $B$. However this cannot be since $|B|+1=k+1>\left|\mathrm{D}_{G^{\prime}, Y}\right|$.

Proposition 2. For distinct $c, d \in[q]$, both $Y$ and $\mathrm{D}_{G^{\prime}, Y}$ contain
(1) exactly $r$ nodes of $\left(L_{c}\right.$-pos $\cup L_{c}$-neg $)$,
(2) exactly 2 rs nodes of ( $L_{c d}$-pos $\cup L_{c d}$-neg),
(3) a multiple of $2 r$ nodes of $L_{c d}$-pos and $L_{c d}$-neg.

Proof. First of all consider that all the nodes in $L_{c}$-pos, $L_{c}$-neg, $L_{c d}$-pos and $L_{c d}$-neg have threshold zero, and so all of them are in $Y \cup \mathrm{D}_{G^{\prime}, Y}$. We claim that at most $r$ of the nodes of ( $L_{c}$-pos $\cup L_{c}$-neg) can be in $\mathrm{D}_{G^{\prime}, Y}$. Indeed, if $\mathrm{D}_{G^{\prime}, Y}$ contains at least $r+1$ nodes in ( $L_{c}$-pos $\cup L_{c}$-neg) then each node $g \in L_{c}$-guard (recall $t(g)=r+1$ ) either is influenced (i.e., $g \in \mathrm{D}_{G^{\prime}, Y}$ ) or is immunized (i.e., $g \in Y$ ). By Proposition 1 , no node in $L_{c}$-guard can be influenced. On the other hand, it cannot occur that all the nodes in $L_{c}$-guard are immunized, since $\mid L_{c}$-guard $|=\ell+1>|Y|$.
Using the same argument we can prove that at most $2 r s$ of the nodes of ( $L_{c d}$-pos $\cup L_{c d}$-neg) can be in $\mathrm{D}_{G^{\prime}, Y}$. Assume on the contrary that $\mid \mathrm{D}_{G^{\prime}, Y} \cap\left(L_{c d}\right.$-pos $\cup L_{c d}$-neg $) \mid \geq 2 r s+1$. Having each node in $L_{c d}$-guard threshold $2 r s+1$, we have that either the node is influenced or it must be immunized. However, by Proposition 1 we know that the nodes in $L_{c d}$-guard are not influenced; moreover they cannot all be immunized since $\mid L_{c d}$-guard $|=\ell+1>|Y|$.

This allows to say that $Y$ contains at least $r$ nodes of ( $L_{c}$-pos $\cup L_{c}$-neg) and at least $2 r s$ nodes of ( $L_{c d}$-pos $\cup L_{c d}$-neg). However, if there exists a $c \in[q]$ or a pair of distinct $c, d \in[q]$ such that $Y$ contains strictly more than $r$ nodes of ( $L_{c}$-pos $\cup L_{c}$-neg) or $2 r s$ nodes of ( $L_{c d}$-pos $\cup L_{c d}$-neg), then $|Y|>q r+\binom{q}{2} 2 r s$ and this is not possible. Hence, (1) and (2) follow.

To prove (3) we proceed by contradiction. Suppose that $\mathrm{D}_{G^{\prime}, Y}$ contains $2 r a+z$ nodes of $L_{c d}$-pos, where $a<s$ and $0<z<2 r$. By (2) we have that $\mathrm{D}_{G^{\prime}, Y}$ contains $2 r(s-a)-z$ nodes of $L_{c d}$-neg. Write $M_{c d}$-pos $=\left\{x_{0}, \ldots, x_{s}\right\}$ and $M_{c d}$-neg $=\left\{y_{0}, \ldots, y_{s}\right\}$. Recalling that the nodes in $M_{c d}$-pos are neighbors of those in $L_{c d}$-pos, the nodes in $M_{c d}$-neg are neighbors of those in $L_{c d}$-neg and $t\left(x_{i}\right)=t\left(y_{i}\right)=2 r i+1$, we have that nodes $x_{0}, \ldots, x_{a}$ of $M_{c d}$-pos and nodes $y_{0}, \ldots, y_{s-a-1}$ of $M_{c d}$-neg get influenced. Since these $s+1$ influenced nodes are neighbors of each node $g \in M_{c d^{-}}$-guard, whose threshold is $t(g)=s+1$, we have that either $g$ is influenced or it is immunized. By Proposition 1, no node in $M_{c d}$-guard can be influenced. On the other hand, it cannot occur that all the nodes in $M_{c d}$-guard are immunized, since $\mid M_{c d}$-guard $\mid=\ell+1>$ $|Y|$.
Claim 2. If $\left\langle G^{\prime}, k, \ell\right\rangle$ is a YES instance of IIB then $\langle G, q\rangle$ is a YES instance of MQ.

Proof. Being $\left\langle G^{\prime}, k, \ell\right\rangle$ a YES instance of IIB, there exists an immunizing set $Y$ of size at most $\ell=q r+$ $\binom{q}{2} 2 r s$ such that $\left|\mathrm{D}_{G^{\prime}, Y}\right| \leq k=q r+\binom{q}{2}(2 r+3) s$.

We proceed by identifying the clique $K$ of $G$ according to the number of nodes that are in $L_{c}$-neg $\cap Y$ for each $c \in[q]$ and in $L_{c d}$-neg $\cap Y$, for each distinct $c, d \in[q]$. Namely, we select:

- the node $v_{i}^{c} \in V_{c}$, such that $\mid L_{c}$-neg $\cap Y \mid=i$, for some $0 \leq i \leq r$, and
- the edge $e_{j}^{c d} \in E_{c d}$ such that $\mid L_{c d}$-neg $\cap Y \mid=2 r j$, for some $0 \leq j \leq s$.

The above selection is correct since, by Proposition 2, we know that $\mid Y \cap\left(L_{c}\right.$-pos $\cup L_{c}$-neg $) \mid=r$ and $\mid Y \cap\left(L_{c d}\right.$-pos $\cup L_{c d}$-neg) $\mid=2 r s$ (in particular, $Y$ contains a multiple of $2 r$ nodes of both $L_{c d}$-pos and $L_{c d}$-neg).

Let $V(K)$ be the set of the $q$ selected nodes and $E(K)$ be the set of the $\binom{q}{2}$ selected edges. We argue that $K=(V(K), E(K))$ is a clique. By contradiction assume there are two distinct colors $c, d \in[q]$ such that $v_{i}^{c} \in V(K)$ and $e_{j}^{c d} \in E(K)$ but $v_{i}^{c}$ is not an endpoint of $e_{j}^{c d}$. Consider the incidence gadget $I_{c: c d}$. Let $I_{c: c d}$-pos $=\left\{u_{0}, \ldots, u_{s}\right\}$ and $I_{c: c d}$-neg $=\left\{w_{0}, \ldots, w_{s}\right\}$. Assume that $v_{h}^{c}$ is the endpoint of color $c$ of $e_{j}^{c d}$. Recall that nodes $u_{j}$ and $w_{j}$ represent the edge $e_{j}^{c d}$ and that, by the construction of $G^{\prime}$, it holds $t\left(u_{j}\right)=2 r j+h+1$ and $t\left(w_{j}\right)=2 r(s-j)+r-h+1$. Since the nodes of $I_{c: c d}$-pos have $2 r j+i$ influenced neighbors (those in $\mathrm{D}_{G^{\prime}, Y} \cap\left(L_{c}\right.$-pos $\cup L_{c d}$-pos)) and the nodes of $I_{c: c d}$-neg have $2 r(s-j)+r-i$ influenced neighbors, (those in $\mathrm{D}_{G^{\prime}, Y} \cap\left(L_{c}\right.$-neg $\cup L_{c d}$-neg) ) by an analysis similar to that in the proof of Lemma 1 , we have that nodes $u_{0}, \ldots, u_{j-1}$ in $I_{c: c d}$-pos and nodes $w_{j+1}, \ldots, w_{s}$ in $I_{c: c d}$-neg all get influenced. It remains to analyze the nodes $u_{j}$ and $w_{j}$. We will prove that at least one of them gets influenced: If $h<i$ then $t\left(u_{j}\right)=2 r j+h+1 \leq 2 r j+i$ and $t\left(w_{j}\right)=2 r(s-j)+r-h+1>2 r(s-j)+r-i$ and $u_{j}$ is influenced; if $h>i$ then $t\left(u_{j}\right)=2 r j+h+1>2 r j+i$ and $t\left(w_{j}\right)=2 n(s-j)+n-h+1 \leq 2 r(s-j)+r-i$ and $w_{j}$ is influenced. This allows to say that if $v_{h}^{c} \in e_{j}^{c d}$ then $s+1$ nodes among those in $I_{c: c d}$-pos and $I_{c: c d}$-neg are influenced. As a consequence, each node $g \in I_{c: c d}$-guard, whose threshold is $t(g)=s+1$, must either be influenced or immunized. By Proposition 1, no node in $I_{c: c d}$-guard can be influenced. On the other hand, it cannot occur that all the nodes in $I_{c: c d}{ }^{-}$guard are immunized, since $\mid I_{c: c}$-guard $|=\ell+1>|Y|$.

Lemma 5. $G^{\prime}$ has neighborhood diversity $O\left(q^{2}\right)$.
Proof. Since each bag in $G^{\prime}$ is a type set in the type partition of $G^{\prime}$ and, since for each $c \in[q]$, there are three bags in $L_{c}$ and, for each $c, d \in[q]$ with $c \neq d$ there are six bags in $M_{c d}$, and three bags in both $I_{c: c d}$ and $I_{d: c d}$, we have that the neighborhood diversity of $G^{\prime}$ is $3 q+12\binom{q}{2}$.

## 4 FPT Algorithms

In this section, we present FPT algorithm for several pairs of parameters.

### 4.1 Parameters $k$ and $\ell$

Theorem 6. IIB can be solved in time $2^{k+\ell}(k+\ell)^{O(\log (k+\ell))} \cdot n^{O(1)}$.
Proof. The fixed parameter tractability of IIB with respect to $k+\ell$ can be proved by the arguments used in Theorem 1 in [19] for the problem cutting at most $k$ Vertices with terminal. For sake of completeness, the complete proof is given in the following.

Let $\langle G, k, \ell\rangle$ be the input instance of IIB. Consider a random labelling of the nodes of $G$, where each node is independently assigned either 0 or 1 with equal probability. Let now $H=G\left[V_{1}\right]$ be the graph induced by the set $V_{1}$ of nodes having label 1. Consider the set $\mathrm{D}_{H}$ of influenced nodes when we run the diffusion process on $H$. If $\left|\mathrm{D}_{H}\right| \leq k$ and $\left|Y\left(\mathrm{D}_{H}\right)\right| \leq \ell$ then $(3)$ holds for $X=\mathrm{D}_{H}$ and we can answer Yes.

We estimate now the number of needed iterations of random labelling. Suppose $G$ contains a set $X$ satisfying (3). For such a set, it holds $|X|=\left|\mathrm{D}_{G[X]}\right| \leq k$ and $|Y(X)| \leq \ell$, then a random labelling identifies a solution of IIB if and only if all the nodes in $X$ are labelled 1 and all the nodes in $Y(X)$ are labelled 0 , that is,

$$
X \subseteq V_{1} \text { and } Y(X) \cap V_{1}=\emptyset
$$

Indeed, in such a case the above procedure identifies $\mathrm{D}_{H}=X$ as a solution. This happens with probability $2^{-\left(\left|\mathrm{D}_{H}\right|+\left|Y\left(\mathrm{D}_{H}\right)\right|\right)} \geq 2^{-(k+\ell)}$. Hence, the algorithm requires time $2^{k+\ell} n^{O(1)}$.

A derandomization of the above process can be done using universal sets. A $(n, i)$-universal set is a collection of binary vectors of length $n$ such that for each set of $i$ indices, each of the $2^{i}$ possible combinations of values appears in some vector of the set. To run the algorithm, it suffices to try all labellings induced by a $(n, k+\ell)$-universal set. Naor et al. [18] give a construction of $(n, i)$-universal sets of size $2^{i} i^{O(\log i)} \log n$ that can be listed in linear time.

### 4.2 Parameters $k$ and $\zeta$

Theorem 7. IIB can be solved in time $O\left(\zeta^{3 k} n^{5}\right)$, where $\zeta=|\{v \in V \mid t(v)=0\}|$.
Proof. Let $\langle G, k, \ell\rangle$ be the input instance of IIB. Suppose $v_{1}, \ldots v_{\zeta}$ are the nodes in $G$ having threshold 0 and let $\Delta$ denote the maximum degree of a node in $G$. Consider the graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ obtained from $G$ by adding the internal nodes and the edges of a $\Delta$-ry tree whose leaves are $v_{1}, \ldots v_{\zeta}$. Assume $\langle G, k, \ell$, is a YES instance of IIB. We notice that in $G$, the solution set $X$ (cfr. (3)) can be disconnected but any of its connected components must include at least one node of threshold 0 . Hence, in $G^{\prime}$ the nodes in $X$ are now connected through a path in the $\Delta$-ry tree. This implies that there exists $X^{\prime} \subseteq V^{\prime}$ such that: $X \subseteq X^{\prime}$, $\left(X^{\prime}-X\right) \subseteq V^{\prime}-V$, and $G^{\prime}\left[X^{\prime}\right]$ is connected. In particular, if $s$ is the root of tree, we can assume that $s \in X^{\prime}$. In the worst case, all the paths within the $\Delta$-ry tree go through the root $s$, hence $\left|X^{\prime}\right| \leq|X| \log _{\Delta} \zeta+1$.

Let $k^{\prime}=k \log _{\Delta} \zeta+1$. We use the following result [29, Lemma 2]: There are at most $4^{k^{\prime}} \Delta^{k^{\prime}}$ connected subgraphs that contain $s$ and have order at most $k^{\prime}$. Furthermore, these subgraphs can be enumerated in $O\left(4^{k^{\prime}} \Delta^{k^{\prime}}\left(\left|V^{\prime}\right|+\left|E^{\prime}\right|\right)\right)$ time. We can then apply the result in [29] to enumerate all the connected subgraphs of $G^{\prime}$ of size up to $k^{\prime}$. For each candidate set $X^{\prime}$ (the node set of the current connected subgraph) one has to determine whether $X^{\prime} \cap V$ is a solution according to $\sqrt{3}$, which can be done in $O\left(n^{2}\right)$ time.

### 4.3 Parameters $k$ (or $\Delta$ ) and Treewidth

In this section we present a dynamic programming algorithm which exploiting the tree decomposition of a graph $G$ enables to solve a minimization version of IIB, namely the

Influence Diffusion Minimization (IDM): Given a graph $G=(V, E, t)$ and a budget $\ell$, find a set $Y$ such that $|Y| \leq \ell$ and $\left|\mathrm{D}_{G, Y}\right|$ is minimized.

We use the rooted tree decomposition named nice tree decomposition.

Definition 3. A tree decomposition $\left(T,\left\{W_{u}\right\}_{u \in V(T)}\right)$ is nice if conditions 1. and 2. hold:

1. $W_{r}=\emptyset$ for $r$ the root of $T$ and $W_{v}=\emptyset$ for every leaf $v$ of $T$.
2. Every non-leaf node of $T$ is of one of the following three types:

Introduce: a node $u$ with exactly one child $u^{\prime}$ such that $W_{u}=W_{u^{\prime}} \cup\{v\}$ for a node $v \notin W_{u^{\prime}}$.
Forget: a node $u$ with exactly one child $u^{\prime}$ such that $W_{u^{\prime}}=W_{u} \cup\{v\}$ for a node $v \notin W_{u}$.
Join: a node $u$ with two children $u_{1}, u_{2}$ such that $W_{u}=W_{u_{1}}=W_{u_{2}}$
Lemma 6. [17] If a graph $G$ admits a tree decomposition of width at most tw , then it admits a nice tree decomposition of width at most tw . Moreover, given a tree decomposition $\left(T,\left\{W_{u}\right\}_{u \in V(T)}\right)$ of $G$ of width at most tw , one can compute in time $O\left(\mathrm{tw}^{2} \max \{|V(T)|,|V(G)|\}\right)$ a nice tree decomposition of $G$ of width at most tw that has at most $O\left(\mathrm{tw}_{\mathrm{w}}|V(G)|\right)$ nodes.

Consider a graph $G=(V, E)$ with treewidth tw and nice tree decomposition $\left(T,\left\{W_{u}\right\}_{u \in V(T)}\right)$. Let $T$ be rooted at node $r$ and denote by $T(u)$ the subtree of $T$ rooted at $u$, for any node $u$ of $T$. Moreover, denote by $W(u)$ the union of all the bags in $T(u)$, i.e., $W(u)=\bigcup_{v \in T(u)} W_{v}$. We will denote by $s_{u}=\left|W_{u}\right|$ the size of $W_{u}$.

We are going to recursively compute the solution of IDM. The algorithm exploits a dynamic programming strategy and traverses the input tree $T$ in a breadth-first fashion. Moreover, in order to be able to recursively reconstruct the solution, we calculate optimal solutions under different hypothesis based on the following considerations:

- Fix a node $u$ in $T$, for each node $v \in W_{u}$ we have three cases: $v$ gets influenced, $v$ is immunized, or $v$ is safe. We are going to consider all the $3^{s_{u}}$ combinations of such states. We denote each combination with a vector $\mathcal{C}$ of size $s_{u}$ indexed by the elements of $W_{u}$, where the element indexed by $v \in W_{u}$ denotes the state influenced (0), immunized (1), safe (2) of node $v$. The configuration $\mathcal{C}=\emptyset$ denotes the vector of length 0 corresponding to an empty bag. We denote by $\mathbb{C}_{u}$ the family of all the $3^{s_{u}}$ possible state vectors of the $s_{u}$ nodes in $W_{u}$.
- Let $U$ be a subset of $V(G)$. Let us first notice that by 3 ) of Definition 2, all the edges between nodes in $V-W(u)$ and $W(u)$ connect a node in $V-W(u)$ with a node in $W_{u}$ (the bag corresponding to the root of $T(u)$ ). We are going to consider all the possible contribution to the diffusion process, of nodes in $V-W(u)$; that is, for each $v \in W_{u}$, we consider all the possible residual thresholds among $t(v), t(v)-1, \ldots, \max \{0, t(v)-k\}$ (recall that at most $k$ nodes belong to $X$ and can therefore reduce the threshold of $v$ ). We notice that, for each node $v$, it is possible to bound the number of residual thresholds by the value $\min \{t(v), k\}$. Moreover, since no node with $t(v)>d_{G}(v)$ can be influenced and can be then purged from $G$ in a preprocessing step, we can assume that in $G$ it holds $\left(\max _{v \in V} t(v)\right) \leq \Delta$. Hence, we will have up to $\mu^{s_{u}}$ threshold combinations, where $\mu=\min \{k, \Delta\}$. We will denote each possible threshold combination with a vector $\mathcal{T}$, indexed by the $s_{u}$ elements in $W_{u}$, where the element indexed by $v$ belongs to $\{\max \{0, t(v)-k\}, \ldots, t(v)\}$ and denotes the residual threshold of $v \in W_{u}$. The configuration $\mathcal{T}=\emptyset$ denotes the vector of length 0 corresponding to an empty bag. We denote by $\mathbb{T}_{u}$ the family of all the possible threshold combinations of nodes in $W_{u}$.

The following definition introduces the values that will be computed by the algorithm in order to keep track of all the above cases:

Definition 4. For each node $u \in T$, each $j=0, \ldots, \ell, \mathcal{C} \in \mathbb{C}_{u}$ and $\mathcal{T} \in \mathbb{T}_{u}$ we denote by $X_{u}(j, \mathcal{C}, \mathcal{T})$ the minimum number of influenced nodes one can attain in $G[W(u)]$ by immunizing at most $j$ nodes in $W(u)$, where the states and the thresholds of nodes in $W_{u}$ are given by $\mathcal{C}$ and $\mathcal{T}$.

Considering that the root $r$ of a nice tree decomposition has $W_{r}=\emptyset$, we have that the solution of the IDM instance $\langle G, \ell\rangle$ can be obtained by computing $X_{r}(\ell, \emptyset, \emptyset)$.
Claim 3. For each $u \in T$, the computation of $X_{u}(j, \mathcal{C}, \mathcal{T})$, for each $j \in\{0, \ldots, \ell\}$, state configuration $\mathcal{C} \in \mathbb{C}_{u}$, and threshold configuration $\mathcal{T} \in \mathbb{T}_{u}$ comprises $O\left(\ell 3^{\mathrm{tw}} \mu^{\mathrm{tw}}\right)$ values, where $\mu=\min \{k, \Delta\}$, each of which can be computed recursively in time $O\left(2^{\mathrm{tw}}+\ell\right)$.

Proof. We show now how use a bottom-up strategy to compute all the values of $X_{u}(j, \mathcal{C}, \mathcal{T})$, for each $u \in T, j=0, \ldots, \ell$, state configuration $\mathcal{C} \in \mathbb{C}_{u}$, and threshold configuration $\mathcal{T} \in \mathbb{T}_{u}$. By Definition 4 , we know that such values are $O\left(\ell 3^{\mathrm{tw}} \mu^{\mathrm{tw}}\right)$, where $\mu=\min \{k, \Delta\}$.
For each leaf $u \in T$ and for each $j=0, \ldots, \ell$ we have $X_{u}(j, \emptyset, \emptyset)=0$.
For any internal node $u$, we show how to compute each values $X_{u}(j, \mathcal{C}, \mathcal{T})$, for each $j=0, \ldots, \ell, \mathcal{C} \in \mathbb{C}_{u}$, and $\mathcal{T} \in \mathbb{T}_{u}$ in time $O\left(2^{\mathrm{tw}}+\ell\right)$.

We have three cases to consider according to the type of $u$ (cf. Definition 3):

1) $u$ is an introduce node: In this case $u$ has exactly one child $u^{\prime}$ and we have that $W_{u}=W_{u^{\prime}} \cup\{v\}$ for some node $v \notin W_{u^{\prime}}$. For a given node $u \in V(T)$ (introducing a node $v \in V$ ) and state configuration $\mathcal{C}$, we denote by $S_{u}(\mathcal{C})$ the set of influenced nodes (according to the configuration $\mathcal{C}$ ) that belongs to $W_{u} \cap \Gamma_{G}(v)$. Given a threshold configuration $\mathcal{T}$ associated to a set of nodes $W$, and a set of nodes $S \subseteq W$ we denote by $\mathcal{T}(S)$ the configuration obtained starting from $\mathcal{T}$ and decreasing by one the threshold of each node in $S$. In the following we assume w.l.o.g. that the element indexed by $v$ is the last element of the vectors $\mathcal{C}$ and $\mathcal{T}$. We have that for each $j=0, \ldots, \ell$, each $\mathcal{C} \in \mathbb{C}_{u}$ and each $\mathcal{T} \in \mathbb{T}_{u}$.

$$
X_{u}\left(j, \mathcal{C}=\left[\mathcal{C}^{\prime}, c\right], \mathcal{T}=\left[\mathcal{T}^{\prime}, t\right]\right)=\left\{\begin{array}{l}
\min _{S \subseteq S_{u}(\mathcal{C}),|S|=t}\left(X_{u^{\prime}}\left(j, \mathcal{C}^{\prime}, \mathcal{T}^{\prime}\left(S_{u}(\mathcal{C})-S\right)\right)\right)+1  \tag{7}\\
\text { if } c=0 \text { AND } t \leq\left|S_{u}(\mathcal{C})\right| \\
X_{u^{\prime}}\left(j-1, \mathcal{C}^{\prime}, \mathcal{T}^{\prime}\right), \\
\text { if } c=1 \text { AND } j>1 \\
X_{u^{\prime}}\left(j, \mathcal{C}^{\prime}, \mathcal{T}^{\prime}\right), \\
\quad \text { if } c=2 \text { AND } t>\left|S_{u}(\mathcal{C})\right| \\
+\infty, \quad \text { otherwise. }
\end{array}\right.
$$

It is worth to observe that the size of $S_{u}(C)$ is bounded by tw and for this reason the above value can be computed in time $O\left(2^{\mathrm{tw}}\right)$
2) $u$ is a forget node: In this case $u$ has exactly one child $u^{\prime}$ and we have that $W_{u^{\prime}}=W_{u} \cup\{v\}$ for some node $v \notin W_{u}$. We have for each $j=0, \ldots, \ell$, each $\mathcal{C} \in \mathbb{C}_{u}$, and each $\mathcal{T} \in \mathbb{T}_{u}$

$$
\begin{equation*}
X_{u}(j, \mathcal{C}, \mathcal{T})=\min _{c \in\{0,1,2\}}\left\{X_{u^{\prime}}\left(j, \mathcal{C}^{\prime}=[\mathcal{C}, c], \mathcal{T}^{\prime}=\left[\mathcal{T}, \max \left\{0, t(v)-\left|S_{u}(\mathcal{C})\right|\right\}\right]\right)\right\} \tag{8}
\end{equation*}
$$

```
Algorithm 1: \(\operatorname{IIB}-\mathrm{k}(G, k, \ell)\)
    Input: A graph \(G=(V, E, t)\), integers \(k, \ell\) and a type partition \(V_{1}, \ldots, V_{\text {nd }}\) of \(G\).
    foreach \(f=1, \ldots, k\) do
        foreach \(\mathbf{f}=\left(f_{1}, f_{2}, \ldots, f_{\mathrm{nd}}\right)\) such that \(\sum_{i=1}^{\mathrm{nd}} f_{i}=f\) do
            foreach \(i \in[\) nd \(]\) do let \(X_{i}=\left\{v_{i, 1}, \ldots, v_{i, f_{i}}\right\} \subseteq V_{i}\)
            Set \(X=\bigcup_{i=1}^{\text {nd }} X_{i}\)
            if \(|Y(X)| \leq \ell\) then return YES
    return No
```

3) $u$ is a join node: In this case $u$ has exactly two child $u_{1}, u_{2}$ such that $W_{u}=W_{u_{1}}=W_{u_{2}}$. We have for each $j=0, \ldots, \ell$, each $\mathcal{C} \in \mathbb{C}_{u}$, and each $\mathcal{T} \in \mathbb{T}_{u}$

$$
\begin{equation*}
X_{u}(j, \mathcal{C}, \mathcal{T})=\min _{0 \leq a \leq j-I(\mathcal{C})}\left\{X_{u_{1}}(a+I(\mathcal{C}), \mathcal{C}, \mathcal{T})+\left\{X_{u_{2}}(j-a, \mathcal{C}, \mathcal{T})\right\}\right. \tag{9}
\end{equation*}
$$

where $I(\mathcal{C})$ denotes the number of immunized nodes in the configuration state $\mathcal{C}$.
By induction on the tree, we can prove that the recursive formula presented in (7)-(9) coincides with the definition of $X_{u}(\cdot, \cdot, \cdot)$; hence, the algorithm is correct.

Hence, using [17, Lemma 18], we have that the desired value $\left.X_{r}(\ell, \emptyset, \emptyset)\right)$ can be computed in time $O\left(\mathrm{tw}|V|\left(2^{\mathrm{tw}}+\ell\right) \ell 3^{\mathrm{tw}} \mu^{\mathrm{tw}}\right)$. Standard backtracking techniques can be used to compute the optimal set $X$ and $Y(X)$ in the same time.
As a consequence we have that IDM is FPT with respect to tw and $\Delta$ or $k$.
Theorem 8. IDM is solvable in time $O\left(\mathrm{tw}|V|\left(2^{\mathrm{tw}}+\ell\right) \ell 3^{\mathrm{tw}} \mu^{\mathrm{tw}}\right)$, where $\mu=\min \{k, \Delta\}$.

### 4.4 Graphs of bounded neighborhood diversity

We present FPT algorithms for IIB with respect to both the pairs ( $k$, nd $)$ and ( $\ell$, nd $)$.
Let $\left\{V_{1}, V_{2}, \ldots, V_{\text {nd }}\right\}$ be the type partition of $G$. Below, we assume that the nodes of each $V_{i}=$ $\left\{v_{i, 1}, \ldots, v_{i,\left|V_{i}\right|}\right\}$ are sorted in non-decreasing order of thresholds, e.g. $t\left(v_{i, j}\right) \leq t\left(v_{i, j+1}\right)$.
Parameters nd and $k$. We consider all the nd-ples $\left(f_{1}, \ldots, f_{\text {nd }}\right)$ such that $\sum_{i=1}^{\text {nd }} f_{i} \leq k$. For each one, we construct a candidate set as detailed in Algorithm IIB-k below.

Theorem 9. Algorithm IIB-k solves IIB in time $O\left(n^{2} 2^{k+n d-1}\right)$
Proof. We first show Algorithm IIB-k outputs YES iff there exists $X$ satisfying (3).
If the output is YES then trivially the current set $X$ has $X \leq k$ and $|Y(X)| \leq \ell$.
Let now $\tilde{X}$ be a minimal set satisfying $\sqrt{3}$, that is, $\tilde{X}=\mathrm{D}_{G[\tilde{X}]},|X| \leq k$, and $|Y(X)| \leq \ell$. Let $\tilde{X}_{i}=\tilde{X} \cap V_{i}$ for each $i \in[\mathrm{nd}]$. Consider the iteration of the algorithm when $\mathbf{f}=\left(f_{1}, f_{2}, \ldots, f_{\text {nd }}\right)$ with $f_{i}=\left|\tilde{X}_{i}\right|$, for $i \in[\mathrm{nd}]$. The algorithm selects a set $X=\bigcup_{i=1}^{\text {nd }} X_{i}$ such that $\left|X_{i}\right|=f_{i}$ and $t(v) \leq t(w)$ for each $v \in X_{i}$ and $w \in V_{i}-X_{i}$, for each $i \in[$ nd $]$. We show that the algorithm outputs YES on $X$.

```
Algorithm 2: IIB- \(\ell(G, k, \ell)\)
    Input: A graph \(G=(V, E, t)\), integers \(k, \ell\) and a type partition \(V_{1}, \ldots, V_{\text {nd }}\) of \(G\).
    foreach \(h=1, \ldots, \ell\) do
        foreach \(\mathbf{h}=\left(h_{1}, h_{2}, \ldots, h_{\mathrm{nd}}\right)\) such that \(\sum_{i=1}^{\text {nd }} h_{i}=h\) do
            foreach \(i \in\left[\right.\) nd] do let \(Y_{i}=\left\{v_{i, 1}, \ldots, v_{i, h_{i}}\right\} \subseteq V_{i}\)
            Set \(Y=\bigcup_{i=1}^{\text {nd }} Y_{i}\)
            if \(\left|\mathrm{D}_{G, Y}\right| \leq k\) then return YES
    return NO
```

Fix any $i \in[$ nd $]$. Knowing that $\left|\tilde{X}_{i}\right|=\left|X_{i}\right|=f_{i}$, we have that if $\tilde{X}_{i} \neq X_{i}$, then there exists $u \in \tilde{X}_{i}-X_{i}$ and $v \in X_{i}-\tilde{X}_{i}$ such that $t(v) \leq t(u)$. W.l.o.g assume that $u$ is the node with maximum threshold in $\tilde{X}_{i}-X_{i}$. Since $\tilde{X}=\mathrm{D}_{G[\tilde{X}]}$, we have that $u$ has at least $t(u)$ neighbors in $\tilde{X}$. Furthermore, since $v, u \in V_{i}$ we have that $u$ and $v$ have the same neighbors. Hence, $v$ has at least $t(u) \geq t(v)$ neighbors in $\tilde{X}$. As a consequence, since $v \notin \tilde{X}$ we have $v \in Y(\tilde{X})$. Consider $\tilde{X}^{\prime}=\tilde{X}-\{u\} \cup\{v\}$. By (i) in Proposition 3 (see Appendix) we have that $\tilde{X}^{\prime}=\mathrm{D}_{G\left[\tilde{X}^{\prime}\right]}$ with $\left|\tilde{X}^{\prime}\right|=|\tilde{X}|$ and $\left|Y\left(\tilde{X}^{\prime}\right)\right|=|Y(\tilde{X})|$.

Hence, trading each node in $\tilde{X}_{i}-X_{i}$ for one in $X_{i}-\tilde{X}_{i}$, for each $i$ such that $\tilde{X}_{i} \neq X_{i}$, we can prove that $|Y(X)|=|Y(\tilde{X})| \leq \ell$. Therefore, the algorithm returns Yes.

We now evaluate the running time. Fix $f \in[k]$, for each $\left(f_{1}, \ldots, f_{\text {nd }}\right)$ with $\sum_{i=1}^{\text {nd }} f_{i}=f$, one needs time $O(f)$ to get $X$ and $O\left(n^{2}\right)$ to get $Y(X)$, moreover the number of all possible such nd-ple is $\binom{f+$ nd -1}{$f}$. Summing on all $f$ we get $\sum_{f \in[k]}(\underset{f}{f+\text { nd }-1})<2^{k+\text { nd-1 }}$ and the theorem holds.

Parameters nd and $\ell$. An idea similar to that in Algorithm 1 can be used to prove IIB is FPT with respect to nd and $\ell$.

Proposition 3. Fix $i \in[\mathrm{nd}]$.
(i) Let $X=\mathrm{D}_{G[X]}$ and $Y=Y(X)$ be its immunizing set. Set $u_{\text {max }}=\arg \max _{u \in X \cap V_{i}} t(u)$. If there exists $v \in Y \cap V_{i}$ such that $t(v) \leq t\left(u_{\max }\right)$ then $X^{\prime}=X-\left\{u_{\max }\right\} \cup\{v\}$ satisfies $X^{\prime}=\mathrm{D}_{G\left[X^{\prime}\right]}$ and $\left|Y\left(X^{\prime}\right)\right|=|Y|$.
(ii) Let $Y$ be an immunizing set. Set $v_{\max }=\arg \max _{v \in Y \cap V_{i}} t(v)$. If there exists $u \in \mathrm{D}_{G, Y} \cap V_{i}$ such that $t(u) \leq t\left(v_{\max }\right)$ then setting $Y^{\prime}=Y-\left\{v_{\max }\right\} \cup\{u\}$ it holds $\left|\mathrm{D}_{G, Y^{\prime}}\right| \leq\left|\mathrm{D}_{G, Y}\right|$.

Proof. Let us prove (i). Consider $X^{\prime}=X-\left\{u_{\max }\right\} \cup\{v\}$ and the diffusion process in $G\left[X^{\prime}\right]$. We have that $v$ is influenced at a round which is at most equal to that in which $u_{\max }$ is influenced during the diffusion process in $G[X]$ (recall $t(v) \leq t\left(u_{\max }\right)$ and that $v$ and $u_{\max }$ have the same neighbors). Furthermore, since all the neighbors of $v$ and $u_{\text {max }}$ have the same number of neighbors in $X^{\prime}$ as in $X$ we have that all the nodes in $X^{\prime}$ are influenced, that is $X^{\prime}=\mathrm{D}_{G\left[X^{\prime}\right]}$, and $u_{\max } \in Y\left(X^{\prime}\right)$. This allows to say that $\left|Y\left(X^{\prime}\right)\right|=|Y|$.
Let us prove now (ii). If we consider the diffusion process in $G\left[V-Y^{\prime}\right]$ we have that no node outside $\mathrm{D}_{G, Y}-\{u\}$, except eventually for node $v_{\max }$, can be influenced. Hence, $\mathrm{D}_{G, Y^{\prime}} \subseteq \mathrm{D}_{G, Y^{\prime}}-\{u\} \cup\left\{v_{\max }\right\}$.

Theorem 10. Algorithm IIB- $\ell$ solves IIB in time $O\left(n^{2} 2^{\ell+\text { nd-1 }}\right)$

Proof. Given $h \leq \ell$, Algorithm IIB- $\ell(G, k, \ell)$ considers all the possible nd-ples $\left(h_{1}, h_{2}, \ldots, h_{\text {nd }}\right)$ with $\sum_{i=1}^{\text {nd }} h_{i}=h$; for each $\mathbf{h}=\left(h_{1}, h_{2}, \ldots, h_{\text {nd }}\right)$ we construct the set $Y=\bigcup_{i=1}^{\text {nd }} Y_{i}$ where $Y_{i}$ consists of the first (e.g. with the smallest thresholds) $h_{i}$ nodes in $V_{i}$. We then consider the diffusion process in $G$ and the set $\mathrm{D}_{G, Y}$ of influenced nodes when the elements of $Y$ are immunized. If $\left|\mathrm{D}_{G, Y}\right| \leq k$ then we answer YES. In case no $\mathbf{h}$ gives a set $Y$ such that $\left|\mathrm{D}_{G, Y}\right| \leq k$, we answer no.

If Algorithm IIB-nd- $\ell$ returns YES then the set $Y$ constructed by algorithm IIB- $\ell$ has size at most $\ell$ and we know that $\left|\mathrm{D}_{G, Y}\right| \leq k$.

Assume now that there exists $\tilde{Y}$ such that $|\tilde{Y}|=h \leq \ell$ and $\left|\mathrm{D}_{G, \tilde{Y}}\right| \leq k$. Assume w.l.o.g. that no smaller solution exists, that is, for any $Y$ such that $\left|\mathrm{D}_{G, Y}\right| \leq k$ it holds $|Y| \geq h$.

Define $\tilde{Y}_{i}=Y(\tilde{X}) \cap V_{i}$ and let $\left|\tilde{Y}_{i}\right|=h_{i}$, for $i \in\left[\right.$ nd]. Clearly, $\sum_{i=1}^{\text {nd }} h_{i}=h$. Consider the nd-ple $\mathbf{h}=\left(h_{1}, h_{2}, \ldots, h_{\text {nd }}\right)$ and the set $Y=\bigcup_{i=1}^{\text {nd }} Y_{i}$ constructed at line 4 of algorithm IIB-nd- $\ell$. Recall that $\left|Y_{i}\right|=h_{i}$ and $t(v) \leq t(w)$ for each $v \in Y_{i}$ and $w \in V_{i}-Y_{i}$.
Since $\left|\tilde{Y}_{i}\right|=\left|Y_{i}\right|=h_{i}$, we have that if $\tilde{Y}_{i} \neq Y_{i}$, for some $i$, then there are $v \in \tilde{Y}_{i}-Y_{i}$ and $u \in Y_{i}-\tilde{Y}_{i}$ such that $t(u) \leq t(v)$. W.l.o.g select $u$ as the node with minimum threshold in $Y_{i}-\tilde{Y}_{i}$ and $v$ as the node with maximum threshold in $\tilde{Y}_{i}-Y_{i}$. By the fact that $v \in \tilde{Y}$ and $\tilde{Y}$ is minimal, we know that $v$ must have at least $t(v)$ neighbors in $\mathrm{D}_{G, \tilde{Y}}$ (otherwise, $\tilde{Y}-\{v\}$ would be a smaller solution). Furthermore, since $v, u \in V_{i}$ we have that they have the same neighbors. As a consequence, also $u$ has at least $t(v) \geq t(u)$ neighbors in $\mathrm{D}_{G, \tilde{Y}}$. Knowing that $u \notin \tilde{Y}$, we have that $u \in \mathrm{D}_{G, \tilde{Y}}$. Set $Y^{\prime}=\tilde{Y}-\{v\} \cup\{u\}$. By (ii) in Proposition 3 we have that $\mathrm{D}_{G, Y^{\prime}}$ satisfies $\mathrm{D}_{G, Y^{\prime}} \leq \mathrm{D}_{G, \tilde{Y}} \leq k$. Hence, $Y^{\prime}$ is also a solution.

Starting from $Y^{\prime}$, we then can repeat the above reasoning until we get $Y^{r}=Y$, the immunizing set considered in the algorithm for the tuple $\mathbf{h}$. Hence, $\left|\mathrm{D}_{G, Y}\right| \leq k$.

Now we evaluate the running time of the algorithm. For each fixed $h \in[\ell]$, the number of all the possible nd-ples $\left(h_{1}, h_{2}, \ldots, h_{\text {nd }}\right)$ such that $\sum_{i=1}^{\text {nd }} h_{i}=h$ is $\binom{h+$ nd-1 }{$h} \leq\binom{\ell+$ nd-1 }{$h}$. Noticing that for each choice of $\left(h_{1}, \ldots, h_{\text {nd }}\right)$, one needs time $O(h)$ to construct $Y$ and $O\left(n^{2}\right)$ to obtain $\mathrm{D}_{G, Y}$ and that

$$
\sum_{h \in[\ell]}\binom{\ell+\mathrm{nd}-1}{h}<2^{\ell+\mathrm{nd}-1}
$$

the desired result follows.

## 5 Conclusion

We introduced the influence immunization problem on networks under the threshold model and analyzed its parameterized complexity. We considered several parameters and showed that the problem remains intractable with respect to each one. We have also shown that for some pairs (e.g., $(\zeta, \ell)$ and $(\Delta, \ell)$ ) the problem remains intractable.
On the positive side, the problem was shown to be FPT for some other pairs: $(k, \ell),(k, \zeta),(k, \mathrm{tw}),(\Delta, \mathrm{tw}),(k, \mathrm{nd})$, and $(\ell, \mathrm{nd})$.
It would be interesting to asses the parameterized complexity of IIB with respect to the remaining pairs of parameters; in particular with respect to $k$ and $\Delta$.

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[^0]:    ${ }^{1}$ For a positive integer $a$, we use $[a]$ to denote the set of the first $a$ integers, that is $[a]=\{1,2, \ldots, a\}$.

