# Aspect Ratio Universal Rectangular Layouts* 

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#### Abstract

A generic rectangular layout (for short, layout) is a subdivision of an axis-aligned rectangle into axis-aligned rectangles, no four of which have a point in common. Such layouts are used in data visualization and in cartography. The contacts between the rectangles represent semantic or geographic relations. A layout is weakly (strongly) aspect ratio universal if any assignment of aspect ratios to rectangles can be realized by a weakly (strongly) equivalent layout. We give a combinatorial characterization for weakly and strongly aspect ratio universal layouts, respectively. Furthermore, we describe a quadratic-time algorithm that decides whether a given graph $G$ is the dual graph of a strongly aspect ratio universal layout, and finds such a layout if one exists.


## 1 Introduction

A rectangular layout (a.k.a. mosaic floorplan or rectangulation) is a subdivision of an axis-aligned rectangle into axis-aligned rectangle faces, it is generic if no four faces have a point in common. In the dual graph $G(\mathcal{L})$ of a layout $\mathcal{L}$, the nodes correspond to rectangular faces, and an edge corresponds to a pair of rectangles whose common boundary contains a line segment [6, 26, 27].

Two generic layouts are strongly equivalent if they have isomorphic dual graphs, and the corresponding line segments between rectangles have the same orientation (horizontal or vertical); see Fig. 1 for examples. Two generic layouts are weakly equivalent if there is a bijection between their horizontal and vertical segments, resp., such that the contact graphs of the segments are isomorphic plane graphs. Strong equivalence implies weak equivalence [9]; however, for example the brick layouts in Figs. 4a and 4b are weakly equivalent, but not strongly equivalent. The closures of weak (resp., strong) equivalence classes under the uniform norm extend to nongeneric layouts, and a nongeneric layout may belong to the closures of multiple equivalence classes.

Rectangular layouts have been studied for more than 40 years, originally motivated by VLSI design [19, 21, 33] and cartography [24], and more recently by data visualization [32]. The weak equivalence classes of layouts are in bijection with Baxter permutations [1, 25, 34].

An (abstract) graph is called a proper graph if it is the dual of a generic layout. Every proper graph is a near-triangulation (a plane graph where every bounded face is a triangle, but the outer face need not be a triangle). But not every near-triangulation is a proper graph [26, 27]. Ungar [31] gave a combinatorial characterization of proper graphs (see also [15, 29]); and they can be recognized in linear time [11, 20, 22, 23].

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Figure 1: (a-b) Two equivalent layouts. (c) Dual graph. (d) Another layout with the same dual graph. The layout in (d) is sliceable, none of them is one-sided.

In data visualization and cartography [32, 24], the rectangles correspond to entities (e.g., countries or geographic regions); adjacency between rectangles represents semantic or geographic relations, and the "shape" of a rectangle represent data associated with the entity. It is often desirable to use equivalent layouts to realize different statistics associated with the same entities. A generic layout $\mathcal{L}$ is weakly (strongly) area universal if any area assignment to the rectangles can be realized by a layout weakly (strongly) equivalent to $\mathcal{L}$. Wimer et al. [33] showed that every generic layout is weakly area universal (see also [9, Thm. 3]). Eppstein et al. [6] proved that a layout is strongly area universal if and only if it is one-sided (defined below). However, no polynomial-time algorithm is known for testing whether a given graph $G$ is the dual of some area-universal layout.

In some applications, the aspect ratios (rather than the areas) of the rectangles are specified. For example, in word clouds adapted to multiple languages, the aspect ratio of (the bounding box of) each word depends on the particular language. The aspect ratio of an axis-aligned rectangle $r$ is height $(r) / \operatorname{width}(r)$. A generic layout $\mathcal{L}$ is weakly (strongly) aspect ratio universal (ARU for short) if any assignment of aspect ratios to the rectangles can be realized by a layout weakly (strongly) equivalent to $\mathcal{L}$.

Our Results. We characterize strongly and weakly aspect ratio universal layouts.
Theorem 1. A generic layout is weakly aspect ratio universal if and only if it is sliceable.
Theorem 2. For a generic layout $\mathcal{L}$, the following properties are equivalent:
(i) $\mathcal{L}$ is strongly aspect ratio universal;
(ii) $\mathcal{L}$ is one-sided and sliceable;
(iii) the extended dual $G^{*}(\mathcal{L})$ of $\mathcal{L}$, admits a unique transversal structure.

The terms in Theorems 1-2 are defined below. It is not difficult to show that one-sided sliceable layouts are strongly aspect ratio universal; and admit a unique transversal structure. Proving the converses, however, is more involved.
Algorithmic Results. In some applications, the rectangular layout is not specified, and we are only given the dual graph of a layout (i.e., a proper graph). This raises the following problem: Given a proper graph $G$ with $n$ vertices, find a strongly (resp., weakly) ARU layout $\mathcal{L}$ such that $G \simeq G(\mathcal{L})$ or report that none exists. Using structural properties of one-sided sliceable layouts that we develop here, we present an algorithm for recognizing the duals of strongly ARU layouts.

Theorem 3. We can decide in $O\left(n^{2}\right)$ time whether a given graph $G$ with $n$ vertices is the dual of a one-sided sliceable layout.

Thomassen [29] gave a linear-time algorithm to recognize proper graphs if the nodes corresponding to corner rectangles are specified, using combinatorial characterizations of layouts [31]. Kant and $\mathrm{He}[12,14]$ described a linear-time algorithm to test whether a given graph $G^{*}$ is the extended dual of a layout, using transversal structures. Later, Rahman et al. [11, 20, 22, 23] showed that proper graphs can be recognized in linear time (without specifying the corners). However, a proper graph may have exponentially many nonequivalent realizations, and prior algorithms may not find a one-sided sliceable realization even if one exists. Currently, no polynomial-tile algorithm is known for recognizing the duals of sliceable layouts [4, 16, 35] (i.e., weakly ARU layouts); or one-sided layouts [6].

Background and Terminology. A rectangular layout (for short, layout) is a rectilinear graph in which each face is a rectangle, the outer face is also a rectangle, and the vertex degree is at most 3. A sublayout of a layout $\mathcal{L}$ is a subgraph of $\mathcal{L}$ which is a layout. A layout is irreducible if it does not contain any nontrivial sublayout. A rectangular arrangement is a 2 -connected subgraph of a layout in which bounded faces are rectangles (the outer face need not be a rectangle).

One-Sided Layouts. A segment of a layout $\mathcal{L}$ is a path of collinear inner edges of $\mathcal{L}$. A segment of $\mathcal{L}$ that is not contained in any other segment is maximal. In a one-sided layout, every maximal line segment $s$ must be a side of at least one rectangle $R$; in particular, any other segment orthogonal to $s$ with an endpoint in the interior of $s$ lies in a halfplane bounded by $s$, and points away from $R$.

Sliceable Layouts. A maximal line segment subdividing a rectangle or a rectangular union of rectangular faces is called a slice. A sliceable layout (a.k.a. slicing floorplan or guillotine rectangulation) is one that can be obtained through recursive subdivision with vertical or horizontal lines; see Fig 1(d). The recursive subdivision can be represented by a binary space partition tree (BSP-tree), which is a binary tree where each vertex is associated with either a rectangle with a slice, or just a rectangle if it is a leaf [5]. For a nonleaf vertex, the two subrectangles on each side of the slice are associated with the two children. The number of (equivalence classes of) sliceable layouts with $n$ rectangles is known to be the $n$th Schröder number [34]. One-sided sliceable layouts are in bijection with certain pattern-avoiding permutations, closed formulas for their number has been given by Asinowski and Mansour [2]; see also [18] and OEIS A078482 in the on-line encyclopedia of integer sequences (https://oeis.org/) for further references.

A windmill in a layout is a set of four pairwise noncrossing maximal line segments, called arms, which contain the sides of a central rectangle, and each arm has an endpoint on the interior of another (e.g., the maximal segments around $r_{3}$ or $r_{6}$ in Fig. 2 (a)). We orient each arm from the central rectangle to the other endpoint. A windmill is either clockwise or counterclockwise. It is well known that a layout is sliceable if and only if it does not contain a windmill [1].

Transversal Structure. The dual graph $G(\mathcal{L})$ of a layout $\mathcal{L}$ encodes adjacency between faces, but does not specify the relative positions between faces (above-below or left-right). The transversal structure (a.k.a. regular edge-labelling) was introduced by $\mathrm{He}[12,14]$ for the efficient recognition of proper graphs, and later used extensively for counting and enumerating (equivalence classes of)
layouts [10]. The extended dual graph $G^{*}(\mathcal{L})$ is the contact graph of the rectangular faces and the four edges of the bounding box of $\mathcal{L}$; it is a triangulation in an outer 4 -cycle without separating triangles; see Fig. 2.


Figure 2: (a) A layout $\mathcal{L}$ bounded by $e_{1}, \ldots, e_{4}$. (b) Extended dual graph $G^{*}(\mathcal{L})$ with an outer 4 -cycle $\left(e_{1}, \ldots, e_{4}\right)$. (c) Transversal structure.

A layout $\mathcal{L}$ is encoded by a transversal structure that comprises $G^{*}(\mathcal{L})$ and an orientation and bicoloring of the inner edges of $G^{*}(\mathcal{L})$, where red (resp., blue) edges correspond to above-below (resp., left-to-right) relation between two objects in contact. An (abstract) transversal structure is defined as a graph $G^{*}$, which is a 4 -connected triangulation of an outer 4-cycle ( $S, W, N, E$ ), together with a bicoloring and orientation of the inner edges of $G^{*}$ such that all the inner edges incident to $S, W, N$, and $E$, respectively, are outgoing red, outgoing blue, incoming red, and incoming blue; and at each inner vertex the counterclockwise rotation of incident edges consists of four nonempty blocks of outgoing red, outgoing blue, incoming red, and incoming blue edges; see Fig. 2(c).


Figure 3: A flip of an empty (left) and a nonempty (right) alternating cycle.

Flips and Alternating 4-Cycles. It is known that transversal structures are in bijection with the strong equivalence classes of generic layouts $[8,10,14]$. Furthermore, a sequence of flip operations can transform any transversal structure with $n$ inner vertices into any other [7, 10]. Each fip considers an alternating 4 -cycle $C$, which comprises red and blue edges alternatingly, and changes the color of every edge in the interior of $C$; see Fig. 3. If, in particular, there is no vertex in the interior of $C$, then the flip changes the color of the inner diagonal of $C$. Furthermore, every flip operation yields a valid transversal structure on $G^{*}(\mathcal{L})$, hence a new generic layout $\mathcal{L}^{\prime}$ that is strongly non-equivalent to $\mathcal{L}$. We can now establish a relation between geometric and combinatorial properties.

Lemma 4. A layout $\mathcal{L}$ is one-sided and sliceable if and only if $G^{*}(\mathcal{L})$ admits a unique transversal structure.

Proof. Assume that $\mathcal{L}$ is a layout where $G^{*}(\mathcal{L})$ admits two or more transversal structures. Consider a transversal structure of $G^{*}(\mathcal{L})$. Since any two transversal structures are connected by a sequence of flips, there exists an alternating 4-cycle. Any alternating 4-cycle with no interior vertex corresponds to a segment in $\mathcal{L}$ that is two-sided. Any alternating 4 -cycle with interior vertices corresponds to a windmill in $\mathcal{L}$. Consequently, $\mathcal{L}$ is not one-sided or not sliceable.

Conversely, if $\mathcal{L}$ is not one-sided (resp., sliceable), then the transversal structure of $G^{*}(\mathcal{L})$ contains an alternating 4 -cycle with no interior vertex (resp., with interior vertices). Consequently, we can perform a flip operation, and obtain another transversal structure for $G^{*}(\mathcal{L})$.

## 2 Aspect Ratio Universality

An aspect ratio assignment to a layout $\mathcal{L}$ is a function that maps a positive real to each rectangle in $\mathcal{L}$. An aspect ratio assignment to $\mathcal{L}$ is realizable if there exists an equivalent layout $\mathcal{L}^{\prime}$ with the required aspect ratios (a realization). A layout is aspect ratio universal (ARU) if every aspect ratio assignment is realizable. In this section, we characterize weakly and strongly ARU layouts (Theorems 1-2). We start with an easy observation about sliceable layouts.

Lemma 5. Let $\mathcal{L}$ be a sliceable layout. If an aspect ratio assignment for $\mathcal{L}$ is realizable, then there is a unique realization up to scaling and translation. Furthermore, for every $\alpha>0$ there exists a realizable aspect ratio assignment for which the bounding box of the realization has aspect ratio $\alpha$.

Proof. To prove the first claim, we proceed by induction on $k$, the height of the BSP-tree representing $\mathcal{L}$. Basis step: A layout of height 0 comprises a single rectangle, which is uniquely determined by its aspect ratio up to scaling and translation. Induction step: Assume, for induction, that every sublayout at height $k$ in the tree admits a unique realization in which all rectangles at the leaves of the BSP-tree have the required aspect ratios. A rectangle $r$ at height $k+1$ of the BSP-tree is composed of two rectangles at height $k$, say $r_{1}$ and $r_{2}$, that share an edge. Given a realization of $r_{1}$, there is a unique scaling and translation that attaches $r_{2}$ to $r_{1}$, and identifies their matching edges. Consequently, the sublayout at height $k+1$ has a unique realization up to scaling and translation.

The second claim follows trivially: Start with a bounding box of aspect ratio $\alpha$, subdivide it recursively into a layout equivalent to $\mathcal{L}$, and define an aspect ratio assignment using the aspect ratios of the resulting leaf rectangles.

Corollary 6. If $\mathcal{L}$ is one-sided and sliceable, then it is strongly $A R U$.
Proof. Let $\alpha$ be an aspect ratio assignment to a one-sided sliceable layout $\mathcal{L}$. A (unique) realization $\mathcal{L}^{\prime}$ can be constructed by the induction in the proof of Lemma 5 . Indeed, in the basis step, a layout with a single rectangle is strongly ARU. In the induction step, when a rectangle is composed of two rectangles $r=r_{1} \cup r_{2}$ separated by a segment $\ell$, then $r_{1}$ or $r_{2}$ is a leaf of the BSP-tree, since $\mathcal{L}$ is one-sided. Consequently, the pairs of adjacent rectangles on opposite sides of $\ell$ do not depend on the aspect ratio assignment, and the resulting layout $\mathcal{L}^{\prime}$ is equivalent to $\mathcal{L}$.

Corollary 7. If $\mathcal{L}$ is sliceable, then it is weakly ARU.

The proof of Corollary 7 is almost identical to the previous proof. In the proof of Corollary 6 we can glue $r_{1}$ and $r_{2}$ because of one-sidedness. In the case of weak equivalence we can glue $r_{1}$ and $r_{2}$ because adjacencies of rectangles on the two sides of $\ell$ are irrelevant,. Note, however, that the gluing may result in a cross of the layout, i.e., the layout $\mathcal{L}^{\prime}$ realizing $\alpha$ may be a nongeneric layout.

### 2.1 Sliceable and One-Sided Layouts

Next we show that any sliceable layout that is strongly ARU must be one-sided. We present two types of simple layouts that are not aspect ratio universal, and then show that all other layouts that are not one-sided or not sliceable can be reduced to these prototypes.

(a)

(b)

| $r_{1}$ | $r_{2}$ |
| :--- | :--- |
| $r_{3}$ | $r_{4}$ |

(c)

| $r_{1}$ |  |
| :---: | :---: |
| $r_{3}$ | $r_{4}$ |

(d)

(e)

(f)

Figure 4: Prototype layouts that are not aspect ratio universal: (a)-(d) brick layouts are sliceable but not one-sided; (e)-(f) windmills are one-sided but not sliceable.

Lemma 8. The brick layouts in Figs. $4 a-4 d$ are not strongly ARU; the windmill layouts in Figs. 4 e4f are neither strongly nor weakly ARU.

Proof. Suppose w.l.o.g. that a brick layout $\mathcal{L}_{0}$ in Fig. 4a is strongly ARU. Then there exists a strongly equivalent layout $\mathcal{L}$ for the aspect ratio assignment $\alpha\left(r_{2}\right)=\alpha\left(r_{3}\right)=1$ and $\alpha\left(r_{1}\right)=\alpha\left(r_{4}\right)=$ 2. Since $\operatorname{width}\left(r_{1}\right)=\operatorname{width}\left(r_{2}\right)$ and $\alpha\left(r_{1}\right)=2 \alpha\left(r_{2}\right)$, then height $\left(r_{1}\right)=2 \operatorname{height}\left(r_{2}\right)$, and the left horizontal slice is below the median of $r_{1} \cup r_{2}$. Similarly, width $\left(r_{3}\right)=\operatorname{width}\left(r_{4}\right)$ and $\alpha\left(r_{4}\right)=2 \alpha\left(r_{2}\right)$ imply that the right horizontal slice is above the median of $r_{3} \cup r_{4}$. Consequently, $r_{1}$ and $r_{4}$ are in contact, and $\mathcal{L}$ is not equivalent to $\mathcal{L}_{0}$, which is a contradiction.

Suppose w.l.o.g. that the windmill layout $\mathcal{L}_{1}$ in Fig. 4 e is weakly ARU. Then there exists a weakly equivalent layout $\mathcal{L}$ for the aspect ratio assignment $\alpha(c)=\alpha\left(r_{1}\right)=\alpha\left(r_{2}\right)=\alpha\left(r_{3}\right)=\alpha\left(r_{4}\right)=1$. In particular, $r_{1}, \ldots, r_{4}$ are squares; denote their side lengths by $s_{i}$, for $i=1, \ldots, 4$. Note that one side of $r_{i}$ strictly contains a side of $r_{i-1}$ for $i=1, \ldots, 4$ (with arithmetic modulo 4). Consequently, $s_{1}<s_{2}<s_{3}<s_{4}<s_{1}$, which is a contradiction.

Lemma 9. If a layout is sliceable but not one-sided, then it is not strongly ARU.
Proof. To show that a layout is not strongly ARU, it is sufficient to show that any of its sublayouts are not strongly ARU, because any nonrealizable aspect ratio assignment for a sublayout can be expanded arbitrarily to an aspect ratio assignment for the entire layout.

Let $\mathcal{L}$ be a sliceable but not one-sided layout. We claim that $\mathcal{L}$ contains a sublayout strongly equivalent to a layout in Figs. 4a-4d. Because $\mathcal{L}$ is not one-sided, it contains a maximal line segment $\ell$ which is not the side of any rectangle. Because $\mathcal{L}$ is sliceable, every maximal line segment in it subdivides a larger rectangle into two smaller rectangles. We may assume w.l.o.g. that $\ell$ is vertical. Because $\ell$ is not the side of any rectangle, the rectangles on the left and right of $\ell$ must be subdivided horizontally in the recursion. Let $\ell_{\text {left }}$ and $\ell_{\text {right }}$ be the first maximal horizontal line segments on the left and right of $\ell$, respectively. Assume that they each subdivide a rectangle adjacent to $\ell$ into
$r_{1}$ and $r_{2}$ (on the left) and $r_{3}$ and $r_{4}$ on the right. These rectangles comprise a layout equivalent to the one in Figs. 4a-4d; but they may be further subdivided recursively. By Lemma 8, there exists an aspect ratio assignment to $\mathcal{L}$ not realizable by a strongly equivalent layout.

In the remainder of this section, we prove that if a layout is not sliceable, then it contains a sublayout similar, in some sense, to a prototype in Figs. 4e-4f. In a nutshell, our proof goes as follows: Consider an arbitrary windmill in a nonslicable layout $\mathcal{L}$. We subdivide the exterior of the windmill into four quadrants, by extending the arms of the windmill into rays $\ell_{1}, \ldots, \ell_{4}$ to the bounding box; see Fig. 5. Each rectangle of $\mathcal{L}$ lies in a quadrant or in the union of two consecutive quadrants. We assign aspect ratios to the rectangles based on which quadrant(s) it lies in. If these aspect ratios can be realized by a layout $\mathcal{L}^{\prime}$ weakly equivalent to $\mathcal{L}$, then the rays $\ell_{1}, \ldots, \ell_{4}$ will be "deformed" into $x$ - or $y$-monotone paths that subdivide $\mathcal{L}^{\prime}$ into the center of the windmill and four arrangements of rectangles, each incident to a unique corner of the bounding box. We assign the aspect ratios for the rectangles in $\mathcal{L}^{\prime}$ so that these arrangements can play the same role as rectangles $r_{1}, \ldots, r_{4}$ in the prototype in Figs. 4e-4f. We continue with the details.


Figure 5: A rays $\ell_{1}, \ldots, \ell_{4}$ deform into monotone paths in an equivalent layout.
We clarify what we mean by a "deformation" of a (horizontal) ray $\ell$.
Lemma 10. Let a ray $\ell$ be the extension of a horizontal segment in a layout $\mathcal{L}$ such that $\ell$ does not contain any other segment and it intersects the rectangles $r_{1}, \ldots, r_{k}$ in this order. Suppose that in a weakly equivalent layout $\mathcal{L}^{\prime}$, the corresponding rectangles $r_{1}^{\prime}, \ldots, r_{k}^{\prime}$ are sliced by horizontal segments $s_{1}, \ldots, s_{k}$. Then there exists an $x$-monotone path comprised of horizontal edges $s_{1}, \ldots, s_{k}$, and vertical edges along vertical segment of the layout $\mathcal{L}^{\prime}$.

Proof. Assume w.l.o.g. that $\ell$ points to the right. Since $\ell$ does not contain any other segment and it intersects the rectangles $r_{1}, \ldots, r_{k}$ in this order, then $r_{i}$ and $r_{i+1}$ are on opposite sides of a vertical segment for $i=1, \ldots, k-1$. The same holds for $r_{i}^{\prime}$ and $r_{i+1}^{\prime}$ as $\mathcal{L}^{\prime}$ is weakly equivalent to $\mathcal{L}$. In particular, the right endpoint of $s_{i}$ and the left endpoint of $s_{i+1}$ are on the same vertical segment in $\mathcal{L}^{\prime}$, for all $i=1, \ldots k-1$.

The next lemma allows us to bound the aspect ratio of the bounding box of a rectangular arrangement in terms of the aspect ratios of individual rectangles.

Lemma 11. If every rectangle in a rectangular arrangement has aspect ratio $\alpha m$, where $m$ is the number of rectangles in the arrangement, then the aspect ratio of the bounding box of the arrangement is at least $\alpha$ and at most $\alpha m^{2}$.

Proof. Consider an arrangement $A$ with $m$ rectangles and a bounding box $R$. Let $w$ be the maximum width of a rectangle in $A$. Then, $\operatorname{width}(R) \leq m w$. A rectangle of width $w$ has height $\alpha m w$, and so height $(R) \geq \alpha m w$. The aspect ratio of $R$ is height $(R) / \operatorname{width}(R) \geq(\alpha m w) /(m w)=\alpha$.

Similarly, let $h$ be the maximum height of rectangle in $A$. Then height $(R) \leq m h$. A rectangle of height $h$ has width $\frac{h}{\alpha m}$, and so $\operatorname{width}(R) \geq \frac{h}{\alpha m}$. The aspect ratio of $R$ is height $(R) / \operatorname{width}(R) \leq$ $m h /\left(\frac{h}{\alpha m}\right)=\alpha m^{2}$, as claimed.

We can now complete the characterization of aspect ratio universal layouts.
Lemma 12. If a layout $\mathcal{L}$ is not sliceable, it is not weakly $A R U$.
Proof. Let $R$ be a nonslicable layout of $n$ rectangles in a bounding box of $\mathcal{L}$. We may assume that $\mathcal{L}$ is irreducible, otherwise we can choose a minimal nonsliceable sublayout $\mathcal{L}^{*}$ from $\mathcal{L}$, and replace each maximal sublayout of $\mathcal{L}^{*}$ with a rectangle to obtain an irreducible layout. By Lemma 5, a suitable aspect ratio assignment to each sliceable sublayout of $\mathcal{L}^{*}$ can generate any aspect ratio for the replacement rectangle.

In particular, $\mathcal{L}$ thus contains no slices, as any slice would create two smaller sublayouts. Every nonsliceable layout contains a windmill. Consider an arbitrary windmill in $\mathcal{L}$, assume w.l.o.g. that it is clockwise (cf. Fig. 4e). and let $c$ be its central rectangle. By extending the arms of the windmill into rays, $\ell_{1}, \ldots, \ell_{4}$, we subdivide $R \backslash c$ into four quadrants, denoted by $Q_{1}, \ldots, Q_{4}$ in counterclockwise order starting with the top-right quadrant.

Note that at most one ray intersects the interior of a rectangle in $\mathcal{L}$. Indeed, any two points in two different rays, $p_{i} \in \ell_{i}$ and $p_{j} \in \ell_{j}$, span an axis-parallel rectangle that intersects the interior of $c$. Consequently, $p_{i}$ and $p_{j}$ cannot be in the same rectangle in $R \backslash c$. It follows that every rectangle of $\mathcal{L}$ in $R \backslash c$ lies in one quadrant or in the union of two consecutive quadrants.

We define an aspect ratio assignment $\alpha$ as follows: Let $\alpha(c)=1$. If $r \subseteq Q_{1}$ or $r \subseteq Q_{3}$, let $\alpha(r)=6 n$; and if $r \subseteq Q_{2}$ or $r \subseteq Q_{4}$, let $\alpha(r)=\left(6 n^{2}\right)^{-1}$. For a rectangle $r$ split by a ray, we set $\alpha(r)=6 n+\left(6 n^{2}\right)^{-1}$ if $r$ is split by a horizontal ray $\ell_{1}$ or $\ell_{3}$; and $\alpha(r)=\left((6 n)^{-1}+\left(6 n^{2}\right)\right)^{-1}$ if split by a vertical ray $\ell_{2}$ or $\ell_{4}$.

Suppose that a layout $\mathcal{L}^{\prime}$ weakly equivalent to $\mathcal{L}$ realizes $\alpha$. Split every rectangle of aspect ratio $6 n+\left(6 n^{2}\right)^{-1}$ in $\mathcal{L}^{\prime}$ horizontally into two rectangles of aspect ratios $6 n$ and $\left(6 n^{2}\right)^{-1}$. Similarly, split every rectangle of aspect ratio $\left((6 n)^{-1}+\left(6 n^{2}\right)\right)^{-1}$ vertically into two rectangles of aspect ratios $6 n$ and $\left(6 n^{2}\right)^{-1}$; see Fig. 5b. By Lemma 10, there are four $x$ - or $y$-monotone paths $P_{1}, \ldots, P_{4}$ from the four arms of the windwill to four distinct sides of the bounding box that pass through the slitting segments. The paths $P_{1}, \ldots, P_{4}$ subdivide the exterior of the windmill into four arrangements of rectangles, $A_{1}, \ldots, A_{4}$ that each contain a unique corner of the bounding box. By construction, every rectangle in $A_{1}$ and $A_{3}$ has aspect ratio $6 n$, and every rectangle in $A_{2}$ and $A_{4}$ has aspect ratio $\left(6 n^{2}\right)^{-1}$.

Let $R_{1}, \ldots, R_{4}$ be the bounding boxes of $A_{1}, \ldots, A_{4}$, respectively. By Lemma 11, both $R_{1}$ and $R_{3}$ have aspect ratios at least 6 , and both $R_{2}$ and $R_{4}$ have aspect ratios at most $\frac{1}{6}$. By construction, the arrangements $A_{1}, \ldots, A_{4}$ each contain an arm of the windmill. This implies that $\operatorname{width}(c)<\min \left\{\operatorname{width}\left(R_{1}\right)\right.$, width$\left.\left(R_{3}\right)\right\}$ and height $(c)<\min \left\{\operatorname{height}\left(R_{2}\right)\right.$, height $\left.\left(R_{4}\right)\right\}$. Consider
the arrangement comprised of $A_{1}, c$, and $A_{3}$. It contains two opposite corners of $R$, and so its bounding box is $R$. Furthermore, $\operatorname{height}(R) \geq \max \left\{\operatorname{height}\left(R_{1}\right)\right.$, $\left.\operatorname{height}\left(R_{3}\right)\right\}$, and

$$
\begin{aligned}
\operatorname{width}(R) & \leq \operatorname{width}\left(R_{1}\right)+\operatorname{width}(c)+\operatorname{width}\left(R_{3}\right)<3 \max \left\{\operatorname{width}\left(R_{1}\right), \operatorname{width}\left(R_{3}\right)\right\} \\
& \leq 3 \max \left\{\frac{\operatorname{height}\left(R_{1}\right)}{6}, \frac{\operatorname{height}\left(R_{3}\right)}{6}\right\}=\frac{\max \left\{\operatorname{height}\left(R_{1}\right), \operatorname{height}\left(R_{3}\right)\right\}}{2}
\end{aligned}
$$

and so the aspect ratio of $R$ is at least 2. Similarly, the bounding box of the arrangement comprised of $A_{2}, c$, and $A_{3}$ is also $R$, and an analogous argument implies that its aspect ratio must be at most $\frac{1}{2}$. We have shown that the aspect ratio of $R$ is at least 2 and at most $\frac{1}{2}$, a contradiction. Thus the aspect ratio assignment $\alpha$ is not realizable, and so $\mathcal{L}$ is not weakly aspect ratio universal.

We are now ready to prove Theorems 1 and 2 . We restate both theorems for clarity.
Theorem 1. A generic layout is weakly aspect ratio universal if and only if it is sliceable.
Proof. Let $\mathcal{L}$ be a generic layout. If $\mathcal{L}$ is sliceable, then it is weakly ARU by Corollary 7. Otherwise, it is not weakly ARU by Lemma 12 .

Theorem 2. For a generic layout $\mathcal{L}$, the following properties are equivalent:
(i) $\mathcal{L}$ is strongly aspect ratio universal;
(ii) $\mathcal{L}$ is one-sided and sliceable;
(iii) the extended dual $G^{*}(\mathcal{L})$ of $\mathcal{L}$, admits a unique transversal structure.

Proof. Properties (ii) and (iii) are equivalent by Lemma 4. Property (ii) implies (i) by Corollary 6. For the converse, we prove the contrapositive. Let $\mathcal{L}$ be a generic layout that is not one-sided or not sliceable. If $\mathcal{L}$ is not sliceable, then it is not weakly ARU by Lemma 12 , hence not strongly ARU, either. Otherwise $\mathcal{L}$ is sliceable but not one-sided, and then $\mathcal{L}$ is not strongly ARU Lemma 9.

### 2.2 Unique Transversal Structure

Subdividing a square into squares has fascinated humanity for ages [3, 13, 30]. For example, a perfect square tiling is a tiling with squares with distinct integer side lengths. Schramm [28] (see also [17, Chap. 6]) proved that every near triangulation with an outer 4-cycle is the extended dual of a (possibly degenerate or nongeneric) subdivision of a rectangle into squares. The result generalizes to rectangular faces of arbitrary aspect ratios (rather than squares):

Theorem 13. (Schramm [28, Thm. 8.1]) Let $T=(V, E)$ be near triangulation with an outer 4cycle, and $\alpha: V^{*} \rightarrow \mathbb{R}^{+}$a function on the set $V^{*}$ of the inner vertices of $T$. Then there exists a unique (but possibly degenerate or nongeneric) layout $\mathcal{L}$ such that $G^{*}(\mathcal{L})=T$, and for every $v \in V^{*}$, the aspect ratio of the rectangle corresponding to $v$ is $\alpha(v)$.

The caveat in Schramm's result is that all rectangles in the interior of every separating 3cycle must degenerate to a point, and rectangles in the interior of some of the separating 4-cycles may also degenerate to a point. We only use the uniqueness claim under the assumption that a nondegenerate and generic realization exists for a given aspect ratio assignment.

Lemma 14. If a layout $\mathcal{L}$ is strongly $A R U$, then its extended dual $G^{*}(\mathcal{L})$ admits a unique transversal structure.

Proof. Consider the extended dual graph $T=G^{*}(\mathcal{L})$ of a strongly ARU layout $\mathcal{L}$. As noted above, $T$ is a 4-connected inner triangulation of a 4-cycle. If $T$ admits two different transversal structures, then there are two strongly nonequivalent layouts, $\mathcal{L}$ and $\mathcal{L}^{\prime}$, such that $T=G^{*}(\mathcal{L})=G^{*}\left(\mathcal{L}^{\prime}\right)$, which in turn yield two aspect ratio assignments, $\alpha$ and $\alpha^{\prime}$, on the inner vertices of $T$. By Theorem 13, the (nondegenerate) layouts $\mathcal{L}$ and $\mathcal{L}^{\prime}$, that realize $\alpha$ and $\alpha^{\prime}$, are unique. Consequently, neither of them can be strongly aspect ratio universal.

Lemma 14 readily shows that Theorem 2(i) implies Theorem 2(iii), and provides an alternative proof for the geometric arguments in Lemmata 9 and 12.

## 3 Recognizing Duals of Aspect Ratio Universal Layouts

In this section, we describe an algorithm that, for a given graph $G$, either finds a one-sided sliceable layout $\mathcal{L}$ whose dual graph is $G$, or reports that no such layout exists (Theorem 3).

Problem Formulation. The input of our recursive algorithm will be an instance $I=(G, C, P)$, where $G=(V, E)$ is a near-triangulation, $C: V(G) \rightarrow \mathbb{N}_{0}$ is a corner count, and $P$ is a set of ordered pairs $(u, v)$ of vertices on the outer face of $G$. An instance $I=(G, C, P)$ is realizable if there exists a one-sided sliceable layout $\mathcal{L}$ such that $G$ is the dual graph of $\mathcal{L}$, every vertex $v \in V$ corresponds to a rectangle in $\mathcal{L}$ incident to at least $C(v)$ corners of $\mathcal{L}$, and every pair $(a, b) \in P$ corresponds to a pair of rectangles in $\mathcal{L}$ incident to two ccw consecutive corners. When we have no information about corners, then $C(v)=0$ for all $v \in V$, and $P=\emptyset$. For convenience, we also maintain the total count $C(V)=\sum_{v \in V} C(v)$, and the set $K=\{v \in V(G): C(v)>0\}$ of vertices with positive corner count.

Structural Properties of One-Sided Sliceable Layouts. In this section, we use a default notation: If an instance $(G, C, P)$ is realizable by a one-sided sliceable layout $\mathcal{L}$, then $R$ denotes the bounding box of $\mathcal{L}$, and $r_{v}$ the rectangle in $\mathcal{L}$ corresponding to $v$. Note that any sublayout of a one-sided sliceable layout is also one-sided and sliceable (both properties are hierarchical).


Figure 6: (a-b) If $v$ is a cut vertex of $G$, then $r_{v}$ is bounded by two slices. (c-d) If there is no cut vertex, some rectangle $r_{v}$ is incident to two corners.

Lemma 15. Assume that $(G, C, P)$ admits a realization $\mathcal{L}$ and $|V(G)| \geq 2$. Then $G$ contains a vertex $v$ with one of the following (mutually exclusive) properties.
(I) Vertex $v$ is a cut vertex in $G$. Then $r_{v}$ is bounded by two parallel sides of $R$ and by two parallel slices; and $C(v)=0$. (See Fig. 6 (a-b).)
(II) Rectangle $r_{v}$ is bounded by three sides of $R$ and a slice; and $0 \leq C(v) \leq 2$. (See Fig. 6 ( $c-d$ ).)

Proof. Let $v$ be a cut vertex in $G$. Then $r_{v}$ intersects the boundary of $R$ in at least two disjoint arcs. Since both $r_{v}$ and $R$ are axis-parallel rectangles and $r_{v} \subset R$, their boundaries can intersect in at most two disjoint arcs, which are two parallel sides of $r_{v}$. The other two parallel sides of $r_{v}$ form slices. In particular, $r_{v}$ is bounded by two parallel sides of $R$ and two slices, and so it is not incident to any corner of $R$. In this case, $v$ has property (I).

Assume that $G$ does not have cut vertices. Since $\mathcal{L}$ is sliceable, it is subdivided by a slice $s$ which is a line segment between two opposite sides of $R$. Since $\mathcal{L}$ is one-sided, $s$ must be the side of a rectangle $r_{v}$ for some $v \in V(G)$. If both sides of $r_{v}$ parallel to $s$ are in the interior of $R$, then $r_{v}$ is bounded by two sides of $R$ and by two slices. Since the sublayouts of $\mathcal{L}$ on the opposite sides of these slices are disjoint, then $v$ is a cut vertex in $G$, contrarily to our assumption. Consequently, the other side of $r_{v}$ parallel to $s$ must be a side of $R$. Then $r_{v}$ is bounded by three sides of $R$ and by $s$. Clearly, $r_{v}$ is incident to precisely two corners of $R$, and so $v$ has property (II).

Based on property (II), a vertex $v$ of $G$ is a pivot if there exists a one-sided sliceable layout $\mathcal{L}$ with $G \simeq G(\mathcal{L})$ in which $r_{v}$ is bounded three sides of $R$ and a slice. If we find a cut vertex or a pivot $v$ in $G$, then at least one side of $r_{v}$ is a slice, so we can remove $v$ and recurse on the connected components of $G-v$.

Recursive calls. We define the subproblems created by $G-v$ in both cases:
(I) For a cut vertex $v$ of $G$ in an instance, we define the operation $\operatorname{Split}(G, C, P ; v)$. The graph $G-v$ must have precisely two components, $G_{1}$ and $G_{2}$. Let $\left(u_{1}, \ldots, w_{1}\right)$ and $\left(u_{2}, \ldots, w_{2}\right)$ be the sequence of neighbors of $v$ in $G_{1}$ and $G_{2}$, resp., in cw order. Initialize $C_{1}$ and $C_{2}$ as the restriction of $C$ to $V\left(G_{1}\right)$ and $V\left(G_{2}\right)$, respectively. Set $C_{i}\left(u_{i}\right) \leftarrow C_{i}\left(u_{i}\right)+1$ and $C_{i}\left(w_{i}\right) \leftarrow C_{i}\left(w_{i}\right)+1$ for $i=1,2$ (if $u_{i}=w_{i}$, we increment $C_{i}\left(u_{i}\right)$ by 2 ). For each pair $(a, b) \in P$, if both $a$ and $b$ are in $V\left(G_{i}\right)$ for some $i \in\{1,2\}$, then add $(a, b)$ to $P_{i}$. Otherwise, w.l.o.g., $a \in V\left(G_{1}\right)$ and $b \in V\left(G_{2}\right)$, and the ccw path $(a, b)$ contains either $u_{1}, v, w_{2}$ or $w_{1}, v, u_{2}$. The removal of $v$ splits the path into two subpaths, that we add into $P_{1}$ and $P_{2}$, accordingly. Finally we add $\left(u_{1}, w_{1}\right)$ to $P_{1}$ and ( $u_{2}, w_{2}$ ) to $P_{2}$. Return the instances $\left(G_{1}, C_{1}, P_{1}\right)$ and $\left(G_{2}, C_{2}, P_{2}\right)$.

Lemma 16. Let $v$ be a cut vertex of $G$. An instance $(G, C, P)$ is realizable iff both instances in $\operatorname{Split}(G, C, P ; v)$ are realizable.

Proof. First assume that $\mathcal{L}$ is a realization of instance $(G, C, P)$. The removal of rectangle $r_{v}$ splits $\mathcal{L}$ into two one-sided and sliceable sublayouts, $\mathcal{L}_{1}$ and $\mathcal{L}_{1}$. It is easily checked that they realize $\left(G_{1}, C_{1}, P_{1}\right)$ and ( $G_{2}, C_{2}, P_{2}$ ), respectively.

Conversely, if both $\left(G_{1}, C_{1}, P_{1}\right)$ and $\left(G_{2}, C_{2}, P_{2}\right)$ are realizable, then they are realized by some one-sided sliceable layouts $\mathcal{L}_{1}$ and $\mathcal{L}_{2}$, respectively. The union of a square $r_{v}$ and scaled copies of $\mathcal{L}_{1}$ and $\mathcal{L}_{2}$ attached to two opposite sides $r_{v}$ yields a one-sided sliceable layout $\mathcal{L}$ that realizes (G, C, P).
(II) Let $v$ be a pivot of $G$. We define the operation $\operatorname{Remove}(G, C, P ; v)$. Since $v$ is not a cut vertex, $G-v$ has precisely one component, denoted $G^{\prime}$. Let $(u, \ldots, w)$ be the sequence of neighbors of $v$ in $G^{\prime}$ in cw order. Initialize $C^{\prime}$ as the restriction of $C$ to $V\left(G^{\prime}\right)$, and then set $C^{\prime}(u) \leftarrow C^{\prime}(u)+1$
and $C^{\prime}(w) \leftarrow C^{\prime}(w)+1$. If for any $(a, b) \in P$, the ccw path from $a$ to $b$ in $G$ contains $v$, then return False. Otherwise, set $P^{\prime}=P$, and add $(u, w)$ to $P^{\prime}$. Return the instance ( $G^{\prime}, C^{\prime}, P^{\prime}$ ).

Lemma 17. Let $v$ be a vertex of the outer face of $G$, but not a cut vertex. Then an instance $(G, C, P)$ is realizable with pivot $v$ if and only if instance $\operatorname{Remove}(G, C, P ; v)$ is realizable.

Proof. Assume that $(G, C, P)$ is realized by a one-sided scliceable layout $\mathcal{L}$, and $v$ has property (II). The removal of rectangle $r_{v}$ from $\mathcal{L}$ creates a one-sided sliceable sublayouts $\mathcal{L}^{\prime}$. It is easily checked that $\mathcal{L}^{\prime}$ realizes the instance $\left(G^{\prime}, C^{\prime}, P^{\prime}\right)$.

Conversely, assume that $\left(G^{\prime}, C^{\prime}, P^{\prime}\right)$ is realized by a layout $\mathcal{L}^{\prime}$. Then we can attach a single rectangle $r_{v}$ to the bounding box of $\mathcal{L}^{\prime}$ between two consecutive corners incident to $r_{u}$ and $r_{w}$, and obtain a layout $\mathcal{L}$ that realizes $(G, C, P)$.

How to find a pivot. It is easy to find cut vertices in $G$, since $G$ is internally triangulated, then every cut vertex is incident to the outer face. In the absence of cut vertices, however, any vertex of the outer face of $G$ might be a pivot. We use partial information on the corners to narrow down the search for a pivot.

Lemma 18. Assume that an instance $(G, C, P)$ admits a realization $\mathcal{L}$ and $|V(G)| \geq 2$. If $C(v) \geq 2$ for some vertex $v \in V(G)$, then $v$ is a pivot.

Proof. The rectangle $r_{v}$ is incident to at least two corners of $R$. If $r_{v}$ is incident to two opposite corners of $R$, then $r_{v}=R$, contradicting the assumption that $G$ has two or more vertices. Hence $r_{v}$ is incident to two consecutive corners of $R$, and so it contains some side $s$ of $R$. The other side of $r_{v}$ parallel to $s$ is a maximal segment between two opposite sides of $R$, so it must be a slice.

Lemma 19. Let $\mathcal{L}$ be a one-sided sliceable layout such that $G(\mathcal{L})$ is 2-connected, and has a 2-cut $\{u, v\}$. Then there exists a one-sided sliceable layout $\mathcal{L}^{\prime}$ with the same dual graph such that the first slice separates the rectangles corresponding to $u$ and $v$. In particular, $u$ or $v$ is a pivot.

Furthermore, if there are two rectangles in $\mathcal{L}$ that are each incident to a single corner of $\mathcal{L}$, then the corresponding rectangles in $\mathcal{L}^{\prime}$ are also incident to some corners in $\mathcal{L}^{\prime}$, or there exists a one-sided sliceable layout $\mathcal{L}^{\prime \prime}$ in which one of these rectangles is a pivot.

Proof. Let $R$ be the bounding box of $\mathcal{L}$; let $r_{u}$ and $r_{v}$ denote the rectangles corresponding to $u$ and $v$, and let $s_{0}$ be the segment separating $r_{u}$ and $r_{v}$. If $s_{0}$ connects two opposite sides of $R$, the proof is complete with $\mathcal{L}^{\prime}=\mathcal{L}$, so we may assume otherwise. Because $\mathcal{L}$ is one-sided, whenever two rectangles are in contact, a side of one rectangle fully contains a side of the other. We distinguish between two cases:

Case 1: $r_{u}$ and $r_{v}$ contact opposite sides of $R$. We may assume w.l.o.g. that $r_{u}$ and $r-v$ contact the bottom and top side of $R$, respectively, and the bottom side of $r_{v}$ is contained in the top side of $r_{u}$, as in Fig. 7a. Since $\mathcal{L}$ is one-sided, $s_{0}$ is a side of some rectangle in $\mathcal{L}$, and we may assume that $s_{0}$ is the bottom side of $r_{v}$. The left (resp., right) side of $r_{v}$ lies either along $\partial R$ or in a vertical segments $s_{1}$ (resp., $s_{2}$ ). Since $s_{0}$ does not reach both left and right sides of $R$, then at least one of $s_{1}$ and $s_{2}$ exists. Assume w.l.o.g. that $s_{1}$ exists. Since the bottom-left corner of $r_{v}$ is the endpoint of $s_{0}$, it lies in the interior of segment $s_{1}$. The bottom endpoint of $s_{1}$ must be on the bottom side of $R$, or else the clockwise winding path starting with $s_{1}$ would create windmill (as it can cross neither $r_{u}$ nor $r_{v}$ ), contradicting the assumption that $\mathcal{L}$ is sliceable. As $\mathcal{L}$ is one-sided, $s_{1}$ is the side of a rectangle $r_{1}$, which is necessarily to the left of $s_{1}$. Rectangle $r_{1}$ contacts the top
and bottom sides of $R$. Since $G(\mathcal{L})$ is 2 -connected, is does not have a cut vertex, and $r_{1}$ is the only rectangle in $\mathcal{L}$ to the left of $s_{1}$.


Figure 7: (a) A one-sided sliceable layout $\mathcal{L}$ where $r_{u}$ and $r_{v}$ touch two opposite sides of the bounding box. (b) The modified layout $\mathcal{L}^{\prime}$.

We can now modify $\mathcal{L}$ by extending $s_{0}$ and $r_{v}$ horizontally to the right side of $R$, and clip both $s_{1}$ and $r_{1}$ to $s_{0}$, as in Fig. 7b. This modification changes the contacts between $r_{v}$ and $r_{1}$ from vertical to horizontal, but does not change any other contacts in the layout, so it does not change the dual graph. If segment $s_{2}$ exists, we can similarly extend $s_{0}$ to the right side of $R$ and clip $s_{2}$. We obtain a layout $\mathcal{L}^{\prime}$ with $G(\mathcal{L}) \simeq G\left(\mathcal{L}^{\prime}\right)$ in which the first slice $s_{0}^{\prime}$ separates $r_{u}^{\prime}$ and $r_{v}^{\prime}$, and rectangle $r_{v}^{\prime}$ is a pivot. Furthermore, every rectangle incident to a corner in $\mathcal{L}$ remains incident to some corner in $\mathcal{L}^{\prime}$.

Case 2: $r_{u}$ and $r_{v}$ do not contact opposite sides of $R$. Then they each contact a single side of $R$, and these sides are adjacent. We may assume w.l.o.g. that $r_{u}$ contacts the bottom side of $\mathcal{L}, r_{v}$ contacts the left side of $\mathcal{L}$, and the bottom side of $r_{u}$ contains the top side of $r_{v}$; refer to Fig. 8a. Because $\mathcal{L}$ is one-sided and $s_{0}$ is not the first slice, $s_{0}$ equals the bottom side of $r_{v}$.


Figure 8: The construction of $\mathcal{L}^{\prime}$ from $\mathcal{L}$.
We incrementally construct an $x$ - and $y$-monotone increasing directed path $P$ (staircase) starting with edge $s_{0}$, directed to its right endpoint $p_{0}$. Initially, let $P=\left\{s_{0}\right\}$ and $i:=0$. While $p_{i}$ is not in the top or right side of $R$, let $p_{i+1}$ be the top or right endpoint of segment $s_{i}$, append the edge
$p_{i} p_{i+1}$ to $P$, and let $s_{i+1}$ be the segment orthogonal to $s_{i}$ that contains $p_{i+1}$. Since the path $P$ is $x$ - and $y$-monotonically increases, it does not revisit any segment. Thus the recursion terminates, and $P$ reaches the top or right side of $R$.

Assume that $P$ is formed by the segments $s_{0}, s_{1}, \ldots, s_{k}$ of $\mathcal{L}$ for some $k \geq 1$. We claim that if $s_{i}$ is vertical, its bottom endpoint is on the bottom side of $R$, and if $s_{i}$ is horizontal, its right endpoint is on the right side of $R$. for $i=0$ since $s_{0}$ is the bottom side of $r_{v}$, which contacts the left side of $R$. Suppose for contradiction that the claim holds for $s_{i-1}$ but not for $s_{i}$. Then the the clockwise or counterclockwise winding path starting with $s_{i}$ would create windmill (as it can cross neither $s_{i-1}$ nor $s_{i-2}$, where $s_{-1}$ is the right side of $r_{u}$ ), contradicting the assumption that $\mathcal{L}$ is sliceable.

The segments $s_{0}, s_{1}, \ldots, s_{k}$ jointly form a one-sided sliceable layout, that is, they subdivide $R$ into $k+2$ rectangular regions, each of which contains a sublayout of $\mathcal{L}$. One of these regions is $r_{v}$. Label the remaining $k$ regions by $A_{0}, A_{1}, \ldots, A_{k}$ in the order in which they occur along $P$ (see Fig. 8a). In particular, we have $r_{u} \subset A_{0}$. For $i=1, \ldots, k$, region $A_{i}$ is bounded by $\partial R$ and segments $s_{i}, s_{i+1}$, and $s_{i+2}$ (if they exist); and $A_{k}$ is adjacent to the top-right corner of $R$. Because $\mathcal{L}$ is one-sided, segment $s_{i}$ is a side of a rectangle that we denote by $r_{i}$, for $i=1, \ldots, k$; and $r_{i} \subseteq A_{i}$ as the opposite side of $s_{i}$ is subdivided by segment $s_{i-1}$.

Furthermore, we claim that $A_{k}=r_{k}$. Indeed, $A_{k}$ is bounded by segment $s_{k}$ and three sides of $B$. If $A_{k} \neq r_{k}$, then $r_{k}$ separates the subarrangement $A \backslash r_{k}$ from $r$. This means that $v_{r_{k}}$ would be a cut vertex in $G(\mathcal{L})$, contradicting the assumption that $G(\mathcal{L})$ is 2-connected.

We recursively construct a one-sided sliceable $\mathcal{L}^{\prime}$ by placing rectangles and subarrangement corresponding to those in $\mathcal{L}$ such that $G(\mathcal{L}) \simeq G\left(\mathcal{L}^{\prime}\right)$; refer to Fig. 8 b. Let $R^{\prime}$ be the bounding box of $\mathcal{L}^{\prime}$. First subdivide $R^{\prime}$ by a horizontal segment $s_{0}^{\prime}$; and let $r_{v}^{\prime}$ be the rectangle below $s_{0}^{\prime}$. This ensures that $s_{0}^{\prime}$ is the first slice and $r_{v}^{\prime}$ is a pivot. Subdivide the region above $s_{0}^{\prime}$ by a vertical segment $s_{1}^{\prime}$ into two rectangular regions. Denote the right region by $A_{0}^{\prime}$, and subdivide the left region as follows: For $i=2, \ldots, k$, recursively subdivide the rectangle incident to the top-left corner of $R^{\prime}$ by a segment $s_{i}^{\prime}$ orthogonal to $s_{i}$.

Segments $s_{0}^{\prime}, \ldots, s_{k}^{\prime}$ jointly subdivide $R^{\prime}$ into $k+2$ rectangular regions: $r_{v}^{\prime}$ and $A_{0}^{\prime}, A_{1}^{\prime}, \ldots, A_{k}^{\prime}$ in the order in which they are created, where $A_{k}^{\prime}$ is incident to the top-left corner of $R^{\prime}$; and all other regions contact either the left or the top side of $R^{\prime}$. We insert a sublayout in each region $A_{i}^{\prime}$. First insert a $180^{\circ}$-rotated affine copy of $A_{0}$ into $A_{0}^{\prime}$. For $i=1, \ldots, k-1$, insert $r_{i}^{\prime}$ into $A_{i}^{\prime}$ such that its top or left side is $s_{i+1}^{\prime}$; and if $A_{i-2} \backslash r_{i-2}$ is nonempty, insert an affine copy of the sublayout $A_{i-1} \backslash r_{i-1}$ into $A_{k}^{\prime}$, as well. Finally, for $i=k$, subdivide $A_{k}^{\prime}$ into three rectangles by slices orthogonal to $s_{k}^{\prime}$ : If $A_{k-2} \backslash r_{k-2}$ or $A_{k-1} \backslash r_{k-1}$ is nonempty, insert an affine copy in the first and third rectangle in $A_{k}^{\prime}$; and fill all remaining space by $r_{k}^{\prime}$. This completes the construction of layout $\mathcal{L}^{\prime}$ (see Fig. 8b). By construction, we have $G(\mathcal{L}) \simeq G\left(\mathcal{L}^{\prime}\right)$.

It remains to track the rectangles incident to the corners of $\mathcal{L}$ and $\mathcal{L}^{\prime}$. In the original layout $\mathcal{L}$, rectangle $r_{k}$ is incident to two corners of $R$. Assume w.l.o.g. that $r_{k}$ is incident to the two top corners of $R$ (as in Fig. 8a), and two distinct rectangles $r_{\text {left }} \subset A_{0}$ and $r_{\text {right }} \subset A_{k-1}$ are incident to the bottom-left and bottom-right corners of $R$, respectively. The sublayout $A_{0}$ was inserted into $A_{0}^{\prime}$ after a $180^{\circ}$ rotation, and so $r_{\text {left }}^{\prime} \subset A_{0}^{\prime}$ is incident to the top-right corner in $\mathcal{L}^{\prime}$. If $r_{\text {right }} \subset A_{k-1} \backslash r_{k-1}$, then $A_{k-1} \backslash r_{k-1}$ is nonempty and it was inserted into the top third of $A_{k}^{\prime}$ after a $180^{\circ}$ rotation, and so $r_{\text {right }}^{\prime}$ is incident to the top-left corner in $\mathcal{L}^{\prime}$. Otherwise $A_{k-1}=r_{k-1}$, and then $r_{\text {right }}=r_{k-1}$. In this case $r_{k}^{\prime}$ is incident to the top-left corner in $\mathcal{L}^{\prime}$. However, we can modify $\mathcal{L}$ by extending $r_{k-1}$ and $s_{k-1}$ to the top side of $R$, and obtain a one-sided sliceable layout $\mathcal{L}^{\prime \prime}$ in which $r_{\text {right }}^{\prime \prime}=r_{k-1}^{\prime \prime}$ is a pivot. This completes the proof in Case 2.

Lemma 20. Assume that an instance $(G, C, P)$ is realizable; $G$ is 2-connected, it has 4 or more vertices; there exist two distinct vertices, $u$ and $v$, such that $C(u)=C(v)=1$, and $C(w)=0$ for all other vertices; and $P=\{(u, v)\}$. Then $u$ or $v$ is a pivot; or else $G$ has a 2-cut and a vertex of an arbitrary 2-cut can be taken to be a pivot.

Proof. Assume that $(G, C, P)$ is realized by a one-sided sliceable layout $\mathcal{L}$. The graph $G$ is 2 connected, so it has no cut vertices. Therefore, the pivot must correspond to a rectangle in $\mathcal{L}$ that contains two corners. So, if $u$ and $v$ correspond to rectangles containing opposite corners of $\mathcal{L}$, then one of them must also contain another corner, and thus be a pivot.

We will assume, then, that $u$ and $v$ correspond to rectangles $r_{u}$ and $r_{v}$ which contain adjacent corners of $\mathcal{L}$, which we may assume w.l.o.g. to be the top-left and bottom-left corners, respectively. If either spans the width of $\mathcal{L}$ and contains another corner, then it corresponds to a pivot and we are done.

If $r_{u}$ and $r_{v}$ each contain only one corner of $\mathcal{L}$, there must be some rectangle $r_{p}$ which contains the top-right and bottom-right corners of $\mathcal{L}$, or else there would be no pivot, contradicting Lemma 15. If $r_{u}$ contacts $r_{p}$, then we may reverse the contact between $r_{u}$ and $r_{p}$ by extending the bottom side of $r_{u}$ to the right side of $R$, and removing the segment of the left side of $r_{p}$ above the extended side. This reverses the contact between the two, but yields a layout with an equivalent contact graph in which $r_{u}$ contains two corners, and thus $u$ can be taken as a pivot. The same argument can be made for $v$ as a pivot if $r_{v}$ contacts $r_{p}$.

If neither $r_{u}$ nor $r_{v}$ contacts $r_{p}$, then they do not contact one another either, or else the line segment separating them would not be one-sided. The layout $\mathcal{L}$ is sliceable and $G$ is 2 -connected, so there must be at least one horizontal slice from the left side of $R$ to the left side of $r_{p}$. Let $s_{1}$ be the topmost such slice. As $\mathcal{L}$ is one-sided, then $s_{1}$ must be the side of some rectangle $r_{1}$. The rectangle $r_{1}$ can be neither $r_{u}$ nor $r_{v}$, since they do not contact $r_{p}$. The vertices in $G$ corresponding to $r_{1}$ and $r_{p}$ form a 2 -cut, so $G$ has a 2 -cut. Because rectangles $r_{u}$ and $r_{v}$ are each incident to a single corner of $\mathcal{L}$, Lemma 19 guarantees that $u$ or $v$ is a pivot; or for any 2-cut, there exists a one-sided sliceable layout $\mathcal{L}^{\prime}$ that realizes $(G, C, P)$ and has one of the vertices in the 2 -cut as a pivot.

Lemma 21. Assume that $(G, C, P)$ admits a realization $\mathcal{L}$ and $|V(G)| \geq 2$.

1. If $|K|=4$, then $G$ has a cut vertex.
2. If $|K|=3$, then $G$ has a cut vertex or some vertex $v \in K(G)$ is a pivot.

Proof. If $G$ has a cut vertex, the proof is complete. Assume otherwise. By Lemma 15, for every realization $\mathcal{L}$ of the instance $(G, C, P)$, there exists a pivot vertex $v$, and so $r_{v}$ is incident to two corners of $R$. As $R$ has only four corners, each of which is incident to a unique rectangle in $\mathcal{L}$, then at most two additional rectangles in $\mathcal{L}$ are incident to corners, hence $|K| \leq 3$.

Assume that $|K|=3$. Since $R$ has only four corners, each of which is incident to a unique rectangle in $\mathcal{L}$, one of the vertices in $K$ must be $v$.

We are now ready to prove the main result of this section.
Theorem 3. We can decide in $O\left(n^{2}\right)$ time whether a given graph $G$ with $n$ vertices is the dual of a one-sided sliceable layout.

Proof. Given a graph $G$, we can decide in $O(n)$ time whether $G$ is a proper graph [11, 20, 22, 23]. If $G$ is proper, then it is a connected plane graph in which all bounded faces are triangles. Let an initial instance be $I=(G, C, P)$, where $C(v)=0$ for all vertices $v$, and $P=\emptyset$. We run the following recursive algorithm.

```
Main}(G,C,P
begin
    if }|V(G)|=1 then
        return True
    else if }\existsv\inV(G):C(v)>2 then
        return False
    else if G has a cut vertex v then
        Split( }G,C,P;v)\mathrm{ yields ( }\mp@subsup{G}{1}{},\mp@subsup{C}{1}{},\mp@subsup{P}{1}{})\mathrm{ and ( }\mp@subsup{G}{2}{},\mp@subsup{C}{2}{},\mp@subsup{P}{2}{}
        return Main}(\mp@subsup{G}{1}{},\mp@subsup{C}{1}{},\mp@subsup{P}{1}{})\wedge\operatorname{Main}(\mp@subsup{G}{2}{},\mp@subsup{C}{2}{},\mp@subsup{P}{2}{}
    else if G has a vertex v with C(v)=2 then
            return Main(Remove( }G,C,P;v)
    else if P}={(u,v)}\mathrm{ with C(u)=C(v)=1 and K=2 then
        for all w\in{u,v} do
                if Main(Remove(G,C,P;w)) then
                return True
        for all vertices w\in{\mp@subsup{w}{1}{},\mp@subsup{w}{2}{}}\mathrm{ of an arbitrary 2-cut of G do}
            if Main(Remove(G,C,P;w)) then
                return True
    else if }|K|=3\mathrm{ then
        for all vertices v\inK do
            if Main(Remove(G,C,P;v)) then
                return True
    else if |K| =0 then
        for all vertices v in the outer face of G do
            if Main(Remove(G,C,P;v)) then
                return True
    return False
```

Correctness. We argue that algorithm Main $(G, C, P)$ correctly reports whether an instance ( $G, C, P$ ) is realizable.

Lines 3-4. A graph with only one vertex corresponds to a layout containing a single rectangle, which is clearly aspect ratio universal.

Lines 5-6. A rectangle that contains 3 or more corners of a layout must be the only rectangle in the layout. However, the algorithm only reaches this step if there are multiple vertices in the graph, so a vertex with a corner count of 3 or more is a contradiction.

Lines 7-9. The correctness of this step directly follows from Lemma 16.
Lines 10-26. In the absence of a cut vertex, we try to find a pivot. By Lemma 17, the instance
$(G, C, P)$ is realizable with a pivot $v$ if and only if the instance $\operatorname{Remove}(G, C, P ; v)$ is realizable.
(A) Lines 10-11. The correctness of this step follows from Lemma 18.
(B) Lines 12-18. The correctness of this step follows from Lemma 20.
(D) Lines 19-22. By Lemma 21, when $|K(G)|=3$, the pivot must be a vertex in $K$, or else the instance is not realizable.
(E) Lines 23-26. If we have no information about the corners and there is no cut vertex, one of the vertices in the outer face must correspond to a pivot by Lemma 15 , or else $(G, C, P)$ is not realizable.

Line 27. If we find neither a cut vertex nor a pivot, then the instance is not realizable by Lemma 17.

Runtime Analysis. Let $T$ be the recursion tree of the algorithm for some initial instance $(G, C, P)$. The number of vertices in $G$ strictly decreases along each descending path of $T$, and so the depth of the tree is $O(n)$.

We distinguish between two types of nodes in $T$ : If a step in Lines $8-9$ is executed, then the vertex set $V(G)$ is partitioned among the recursive subproblems; we call these partition nodes of $T$. In the steps in (B) Lines 12-18, (D) Lines 19-22, and (E) Lines 23-26, however, $|V(G)|-1$ vertices appear in all four, three, or $O(|V(G)|)$ recursive subproblems; we call these duplication nodes of $T$.

We first analyze the special case that $T$ does not have duplication nodes. Then $T$ is a binary tree with $O(n)$ nodes. The algorithm maintains the property that $G$ is a connected plane graph and all bounded faces are triangles; this in turn implies that any cut vertex of $G$ is incident to the outer face. Indeed, both operations $\operatorname{Split}(G, C, P ; v)$ and $\operatorname{Remove}(G, C, P ; v)$ remove a vertex from the outer face. Any new cut vertex is incident to a vertex that has been removed. Overall, the total time taken by maintaining the set of cut vertices and the annotation $C$ and $P$ is $O(n)$ over the entire algorithm.

Next, we analyze the impact of duplication nodes. We claim that steps (B), (D) and (E) are reached at most once. Note first that the total corner count $C(V)=\sum_{v \in V} C(v)$ monotonically increases along any descending path of $T$ : In the initial instance, we have $C(V)=0$. All Split and Remove operations produce subproblems with $C(V) \geq 2$. Furthermore, if $C(V)=4$, then we will never reach (A) or (B): Indeed, $C(V)=4$ implies $K \neq \emptyset$. If $|K|=2$ or 3 , there would be a vertex $v$ with $C(v)=2$; and $\operatorname{Remove}(G, C, P ; v)$ in Lines $13-14$ produces an instance with $C\left(V^{\prime}\right)=4$, as the corner counts are incremented by two. It follows that we can reach step (D) only if $C(V)=3$. Thus step (D) can only be reached once, when $C(V)=3$ (after which point, $C(V)$ will be 4 ).

Overall, the duplication steps increase the upper bound on the runtime by a factor of $12 n$, hence it is $O\left(n^{2}\right)$.

## 4 Conclusions

We have shown that a layout $\mathcal{L}$ is weakly (strongly) ARU if and only if $\mathcal{L}$ is sliceable (one-sided and sliceable); and we can decide in $O\left(n^{2}\right)$-time whether a given graph $G$ on $n$ vertices is the dual of a one-sided sliceable layout. An immediate open problem is whether the runtime can be improved. Recall that no polynomial-time algorithm is currently known for recognizing the duals of
sliceable layouts $[4,16,35]$ and one-sided layouts [6]. It remains an open to settle the computational complexity of these problems.

Cut vertices and 2-cuts play a crucial role in our algorithm. We can show (Proposition 22 below) that the duals of one-sided sliceable layouts have vertex cuts of size at most three. (In contrast, the minimum vertex cut in the duals of one-sided layouts (resp., sliceable layouts) is unbounded.) Perhaps 3-cuts can be utilized to speed up our algorithm.

Proposition 22. Let $G$ be the dual graph of a one-sided sliceable layout. If $G$ has 4 or more vertices, then it contains a vertex cut of size at most 3.

Proof. Let $\mathcal{L}$ be a one-sided sliceable layout with $n \geq 4$ rectangles in a bounding box $B$, and with dual graph $G=G(\mathcal{L})$. For a rectangle $r$ in $\mathcal{L}$, let $v(r)$ denote the corresponding vertex in $G$. If $G$ is outerplanar, then either $G$ has a cut vertex, or $G$ is a triangulated $n$-cycle, hence any diagonal forms a 2 -cut. We may assume that $G$ has an interior vertices.

Consider a sequence of segments that incrementally slice $B$ into the layout $\mathcal{L}$; and let us focus on the first step that created a rectangle $R_{0}$ that lies in the interior of $B$. We may assume w.l.o.g. that $R_{0}$ is bounded by the segments $s_{1}, \ldots, s_{4}$ in counterclockwise order, $s_{4}$ sliced a rectangle $R$ into $R_{0}$ and $R_{1}$, and $s_{4}$ is the bottom side of $R_{0}$. Since $\mathcal{L}$ is one-sided, then $s_{1}$ is the right side of some rectangle $r_{1}$, and then $s_{3}$ is the left side of some rectangle $r_{3}$ of $\mathcal{L}$; see Fig. 9a.


Figure 9: Schematic views of the arrangements in the proof of Proposition 22.
We claim that at most two horizontal segments in $\mathcal{L}$ intersect both $s_{1}$ and $s_{3}$. Indeed, if there are three or more such segments (Fig. 9b), let $s^{\prime}$ be one of them other than the lowest or highest. Then $s^{\prime}$ is a side of some rectangle $r^{\prime}$ in $\mathcal{L}$, which is adjacent to both $r_{1}$ and $r_{3}$, but not adjacent to the boundary of $B$, hence $\left\{v\left(r_{1}\right), v\left(r^{\prime}\right), v\left(r_{3}\right)\right\}$ is a 3 -cut in $G$. It follows that $s_{2}\left(s_{4}\right)$ is the highest (lowest) horizontal segment that intersects both $s_{1}$ and $s_{3}$. Since $\mathcal{L}$ is one-sided, then $s_{4}$ is the side of some rectangle $r_{4}$ in $\mathcal{L}$. If $s_{4}$ is the bottom side of a rectangle $r_{4}$ in $\mathcal{L}$, then $\left\{v\left(r_{1}\right), v\left(r_{2}\right), v\left(r_{3}\right)\right\}$ is a 3 -cut in $G$. We may assume that $s_{4}$ is the top side of a rectangle $r_{4}$ in $\mathcal{L}$. Since no segment intersects both $s_{1}$ and $s_{3}$ below $s_{4}$, then $r_{4}=R_{1}$ (as in Fig. 9c).

If $R_{0}$ is not sliced further recursively, then $\left\{v\left(r_{1}\right), v\left(R_{0}\right), v\left(r_{3}\right),\right\}$ is a 3-cut in $G$. Let $s_{5}$ be the first segment that slices $R_{0}$. Segment $s_{5}$ cannot be horizontal, as it would intersect both $s_{1}$ and $s_{3}$. So $s_{5}$ is vertical (Fig. 9c), and it is a side of some rectangle $r_{5}$ of $\mathcal{L}$, which is adjacent to $s_{2}$ and $s_{4}$. This further implies that $s_{2}$ is the bottom side of some rectangle $r_{2}$ of $\mathcal{L}$. If $r_{2}$ is adjacent to the boundary of $B$, then $\left\{v\left(v_{4}\right), v\left(r_{5}\right), v\left(r_{2}\right),\right\}$ is a 3 -cut; else $r_{2}$ lies in the interior of $B$ and $\left\{v\left(r_{1}\right), v\left(r_{2}\right), v\left(r_{3}\right)\right\}$ is a 3 -cut in $G$.

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