Decidable and undecidable problems for first-order definability and modal definability

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Abstract. The core of this paper is Chagrova's Theorems about first-order definability of given modal formulas and modal definability of given elementary conditions. We consider classes of frames for which modal definability is decidable and classes of frames for which first-order definability is trivial. We give a new proof of Chagrova's Theorem about modal definability and sketches of proofs of new variants of Chagrova's Theorem about modal definability.

Keywords: First-order definability. Modal definability. Chagrova's Theorems.

1 Introduction

The question of the correspondence between elementary conditions and modal formulas is concomitant with the creation of the relational semantics of modal logic, frames serving as interpretation structures both for first-order formulas in the signature with one binary predicate and equality and for propositional modal formulas in the language with one box. Kripke [22] already observed that some elementary conditions possess a modal correspondent: transitivity vs $\Box p \to \Box\Box p$, symmetry vs $p \to \Box\Diamond p$, etc. Less than 20 years have elapsed between Kripke's observation and the development of Correspondence Theory culminating in the publication of the book "Modal Logic and Classical Logic" [5]: in 1975, Sahlqvist [26] isolated a large set of modal formulas which guarantee completeness with respect to first-order definable classes of frames whereas van Benthem [4] and Goldblatt [17] independently noticed that McKinsey formula $\Box\Diamond p \to \Diamond\Box p$ has no first-order correspondent.

Since the first-order conditions corresponding to Sahlqvist formulas are effectively computable [6, Section 3.6], it is natural to ask whether Sahlqvist fragment contains all modal formulas possessing first-order correspondents. This question has received a negative answer, the conjunction $(\Box\Diamond p \to \Diamond\Box p) \wedge (\Box p \to \Box\Box p)$ possessing a first-order correspondent while not being equivalent to a Sahlqvist formula. See [6, Example 3.57 and Exercise 3.7.1] for details. See also [18] for an extension of the Sahlqvist set of modal formulas. Hence, owing to the significance of Correspondence Theory, it is natural to ask whether the following problems are decidable:

First-order definability: determine whether a given modal formula possesses a first-order correspondent,

Modal definability: determine whether a given first-order sentence possesses a modal correspondent.

This question has received a negative answer, the limitative results in this topic having been firstly obtained by Chagrova in her doctoral thesis [11] and then further developed in [7–9, 12]. Chagrova's results (henceforth called Chagrova's Theorems) have been obtained by reductions from accessibility problems in Minsky machines and by the use of the frames presented in [8, Figures 1 and 2].

In Chagrova's Theorems, when we are talking about first-order sentences corresponding to modal formulas, we mean that they correspond with respect to the class of all frames. Thus, immediately, there is the question whether Chagrova's Theorems still hold if one consider restricted classes of frames. Giving rise to the modal logic $\mathbf{S5}$, the class of all partitions is perhaps the most simple class of frames that one may conceive of. The simple character of the class of all partitions also appears within the context of first-order definability: every modal formula being equivalent in this class to a modal formula of degree at most 1, it follows from a remark of van Benthem [5, Lemma 9.7] that the class of all partitions gives rise to a trivial first-order definability problem. As for the modal definability problem, Balbiani and Tinchev [2] have proved that it is $\mathbf{PSPACE}$-complete with respect to the class of all partitions when the modal language is extended by the universal modality.

Other classes of frames of simple character are the classes giving rise to the modal logics $\mathbf{KD45}$ (the class of all serial, transitive and Euclidean frames) and $\mathbf{K45}$ (the class of all transitive and Euclidean frames). As for the class of all partitions and for the same reason, Georgiev [15, 16] has proved that the first-order definability problem is trivial with respect to these classes whereas the modal definability problem is $\mathbf{PSPACE}$-complete. The most important computational property shared by the modal logics $\mathbf{S5}$, $\mathbf{KD45}$ and $\mathbf{K45}$ is the $\mathbf{NP}$-completeness of the satisfiability problem. The satisfiability problem of $\mathbf{K5}$ is $\mathbf{NP}$-complete too and this modal logic shares many computational properties with the modal logics $\mathbf{S5}$, $\mathbf{KD45}$ and $\mathbf{K45}$ as well, for instance the polysize model property. Nevertheless, with respect to the class of all $\mathbf{K5}$-frames (the class of all Euclidean frames), although the first-order definability problem is still trivial, the modal definability problem becomes undecidable [1].

The core of this paper will be Chagrova's Theorems about first-order definability and modal definability. In Section 3, we will consider classes of frames for which modal definability is decidable. In particular, we will demonstrate a new result — namely, Theorem 1 — saying that the problem of deciding modal definability of first-order sentences with respect to the class of all partitions is $\mathbf{PSPACE}$-complete. In Section 4, we will consider classes of frames for which first-order definability is trivial. In particular, we will demonstrate a new result — namely, Theorem 2 — saying that the problem of deciding first-order definability of modal formulas with respect to the class of all reflexive, transitive and connected frames with finitely many clusters is trivial. In

Section 5, using standard methods in model theory such as relativization of first-order formulas and reduct of frames, we will give a new proof of Chagrova's Theorem about modal definability and we will give sketches of proofs of new variants of Chagrova's Theorem about modal definability. We assume the reader is at home with the basic tools and techniques in model theory and modal logics. For more on them, see [14, 19] and [6, 10, 21].

2 Preliminaries

We introduce a handful of definitions that will be useful throughout the paper.

2.1 Frames

For all sets E, $\|E\|$ will denote the cardinality of E. A *frame* is a structure $\mathcal{F}{=}(W, R)$ where W is a nonempty set of *states* and R is a binary relation on W. For all frames $\mathcal{F}{=}(W, R)$, for all s in $\mathcal{F}$ and for all subsets S of $\mathcal{F}$, let $R(s){=}\{t \in W : sRt\}$ and $R(S) = \bigcup\{R(s) : s \in S\}$. For all frames $\mathcal{F}{=}(W, R)$ and for all s in $\mathcal{F}$, let $R^\star(s){=}\bigcup\{R^n(s) : n \in \mathbb{N}\}$ where $R^0(s){=}\{s\}$ and for all $n \geq 1$, $R^n(s){=} R(R^{n-1}(s))$. For all frames $\mathcal{F}{=}(W, R)$, we say $\mathcal{F}$ is *rooted* if there exists s in $\mathcal{F}$ such that $R^\star(s){=}W$. In that case, we say s is a *root* of $\mathcal{F}$. For all frames $\mathcal{F}{=}(W, R)$ and for all s in $\mathcal{F}$, the *subframe of $\mathcal{F}$ generated from s* is the frame $\mathcal{F}_s{=}(W_s, R_s)$ where $W_s{=}R^\star(s)$ and R_s is the restriction of R to W_s. Obviously, s is a root of $\mathcal{F}_s$. In a frame $\mathcal{F}{=}(W, R)$, we will say that

- R is *reflexive* if for all s in $\mathcal{F}$, sRs,
- R is *serial* if for all s in $\mathcal{F}$, there exists t in $\mathcal{F}$ such that sRt,
- R is *symmetric* if for all s, t in $\mathcal{F}$, if sRt then tRs,
- R is *transitive* if for all s, t, u in $\mathcal{F}$, if sRt and tRu then sRu,
- R is *Euclidean* if for all s, t, u in $\mathcal{F}$, if sRt and sRu then tRu and uRt,
- R is *connected* if for all s, t, u in $\mathcal{F}$, if sRt and sRu then either tRu, or uRt.

The frame $\mathcal{F}{=}(W, R)$ is *reflexive* (respectively *serial, symmetric, transitive, Euclidean, connected)* if R is reflexive (respectively serial, symmetric, transitive, Euclidean, connected). The frame $\mathcal{F}{=}(W, R)$ is a *partition* if R is reflexive, symmetric and transitive. The partition $\mathcal{F}{=}(W, R)$ is *bounded* if there exists a positive integer n such that for all s in $\mathcal{F}$, $\|R(s)\|{\leq}n$. For all bounded partitions $\mathcal{F}{=}(W, R)$, let $n_\mathcal{F}$ be the least positive integer n such that for all s in $\mathcal{F}$, $\|R(s)\|{\leq}n$. The partition $\mathcal{F}{=}(W, R)$ is *small* if there exists a positive integer π such that for all s in $\mathcal{F}$, $\|\{R(t) : t{\in}W$ and $\|R(s)\|{=}\|R(t)\|\}\|{\leq}\pi$. For all small partitions $\mathcal{F}{=}(W, R)$, let $\pi_\mathcal{F}$ be the least positive integer π such that for all s in $\mathcal{F}$, $\|\{R(t) : t{\in}W$ and $\|R(s)\|{=}\|R(t)\|\}\|{\leq}\pi$. In this paper, we will consider the following classes of frames: the class $\mathcal{C}_{all}$ of all frames, the class $\mathcal{C}_E$ of all Euclidean frames, the class $\mathcal{C}_{sE}$ of all serial and Euclidean frames, the class $\mathcal{C}_{tE}$ of all transitive and Euclidean frames, the class $\mathcal{C}_{stE}$ of all serial, transitive and Euclidean frames and the class $\mathcal{C}_{par}$ of all partitions. We will also consider other classes of frames: the class $\mathcal{C}_{rtc}$ of all reflexive, transitive and connected frames and the class $\mathcal{C}_{rtc}^\omega$ of all reflexive, transitive and connected frames $\mathcal{F}$ such that for all s in $\mathcal{F}$, $\mathcal{F}_s$

contains finitely many clusters. Remind that for all reflexive, transitive and connected frames $\mathcal{F}=(W,R)$, a *cluster* is an equivalence class modulo the equivalence relation $\simeq_{\mathcal{F}}$ on $\mathcal{F}$ such that for all s,t in $\mathcal{F}$, $s \simeq_{\mathcal{F}} t$ iff sRt and tRs.

2.2 Modal language and truth

Modal language Let us consider a countable set $\mathbf{PVAR}$ of *propositional variables* (denoted $p, q, \ldots$). The set $\mathcal{L}_{\mathbf{MF}}$ of all *modal formulas* (denoted $\varphi, \psi, \ldots$) is inductively defined as follows:

- $\varphi, \psi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$,

where p ranges over $\mathbf{PVAR}$. We define the other Boolean constructs as usual. The modal formula $\Diamond\phi$ is obtained as the well-known abbreviation: $\Diamond\phi ::= \neg\Box\neg\phi$. We adopt the standard rules for omission of the parentheses. The *degree* of the modal formula φ (in symbols $\mathtt{deg}(\varphi)$) is the nonnegative integer inductively defined as usual [6, Definition 2.28]. The set of all *subformulas* of the modal formula φ (in symbols $\mathtt{sf}(\varphi)$) is the set of modal formulas inductively defined as follows:

- $\mathtt{sf}(p)=\{p\}$,
- $\mathtt{sf}(\bot)=\{\bot\}$,
- $\mathtt{sf}(\neg\varphi)=\{\neg\varphi\} \cup \mathtt{sf}(\varphi)$,
- $\mathtt{sf}(\varphi \vee \psi)=\{\varphi \vee \psi\} \cup \mathtt{sf}(\varphi) \cup \mathtt{sf}(\psi)$,
- $\mathtt{sf}(\Box\varphi)=\{\Box\varphi\} \cup \mathtt{sf}(\varphi)$.

The set of all *boxed subformulas* of the modal formula φ (in symbols $\mathtt{sf}^{\Box}(\varphi)$) is the set of modal formulas inductively defined as follows:

- $\mathtt{sf}^{\Box}(p)=\emptyset$,
- $\mathtt{sf}^{\Box}(\bot)=\emptyset$,
- $\mathtt{sf}^{\Box}(\neg\varphi)=\mathtt{sf}^{\Box}(\varphi)$,
- $\mathtt{sf}^{\Box}(\varphi \vee \psi)=\mathtt{sf}^{\Box}(\varphi) \cup \mathtt{sf}^{\Box}(\psi)$,
- $\mathtt{sf}^{\Box}(\Box\varphi)=\{\Box\varphi\} \cup \mathtt{sf}^{\Box}(\varphi)$.

As is well-known, for all modal formulas φ, $\|\mathtt{sf}^{\Box}(\varphi)\| + 1 \leq \|\mathtt{sf}(\varphi)\|$.

Truth A *valuation* on a frame $\mathcal{F}=(W,R)$ is a function V assigning to each propositional variable p a subset $V(p)$ of W. The *satisfiability* of a modal formula φ at a state s with respect to a valuation V in a frame $\mathcal{F}=(W,R)$ (in symbols $\mathcal{F}, V, s \models \varphi$) is inductively defined as follows:

- $\mathcal{F}, V, s \models p$ iff $s \in V(p)$,
- $\mathcal{F}, V, s \not\models \bot$,
- $\mathcal{F}, V, s \models \neg\varphi$ iff $\mathcal{F}, V, s \not\models \varphi$,
- $\mathcal{F}, V, s \models \varphi \vee \psi$ iff either $\mathcal{F}, V, s \models \varphi$, or $\mathcal{F}, V, s \models \psi$,
- $\mathcal{F}, V, s \models \Box\varphi$ iff for all states t in $\mathcal{F}$, if sRt then $\mathcal{F}, V, t \models \varphi$.

As a result, $\mathcal{F}, V, s \models \Diamond\varphi$ iff there exists a state t in $\mathcal{F}$ such that sRt and $\mathcal{F}, V, t \models \varphi$. A modal formula φ is *true* with respect to a valuation V in a frame $\mathcal{F}$ (in symbols $\mathcal{F}, V \models \varphi$) if φ is satisfied at all states with respect to V in $\mathcal{F}$. A modal formula φ is *valid* in a frame $\mathcal{F}$ (in symbols $\mathcal{F} \models \varphi$) if φ is true with respect to all valuations on $\mathcal{F}$. A modal formula φ is *valid* in a class $\mathcal{C}$ of frames (in symbols $\mathcal{C} \models \varphi$) if φ is valid in all frames in $\mathcal{C}$. A frame $\mathcal{F}$ is *weaker* than a frame $\mathcal{F}'$ (in symbols $\mathcal{F} \preceq \mathcal{F}'$) if for all modal formulas φ, if $\mathcal{F} \models \varphi$ then $\mathcal{F}' \models \varphi$. For all positive integers n, let

- $\psi_n ::= \bigwedge\{\Diamond p_i : 0 \leq i \leq n\} \to \bigvee\{\Diamond(p_i \wedge p_j) : 0 \leq i < j \leq n\}$.

It is a well-known fact that for all positive integers n and for all partitions $\mathcal{F}$, $\mathcal{F} \models \psi_n$ iff $\mathcal{F}$ is bounded and $n_{\mathcal{F}} \leq n$.

Generated subframes A frame $\mathcal{F}'=(W', R')$ is a *generated subframe* of a frame $\mathcal{F}=(W, R)$ (in symbols $\mathcal{F} \rightarrowtail \mathcal{F}'$) if $W' \subseteq W$ and

- for all s', t' in $\mathcal{F}'$, if $s'R't'$ then $s'Rt'$,
- for all s' in $\mathcal{F}'$ and for all t in $\mathcal{F}$, if $s'Rt$ then t is in $\mathcal{F}'$ and $s'R't$.

The least generated subframe of a frame $\mathcal{F}=(W, R)$ generated by a state s in $\mathcal{F}$ is the frame $\mathcal{F}_s = (W_s, R_s)$ where $W_s = R^\star(s)$ and R_s is the restriction of R to W_s. Generated subframes give rise to the following results:

Proposition 1 (Generated subframes Theorem). *If the frame $\mathcal{F}'$ is a generated subframe of the frame $\mathcal{F}$ then $\mathcal{F} \preceq \mathcal{F}'$.*

Proof. See [6, Theorem 3.14 (ii)].

Proposition 2. *Let $\mathcal{F}=(W, R)$ be a frame, s be a state in $\mathcal{F}$, V be a valuation on $\mathcal{F}$ and V_s be the restriction of V to W_s. For all modal formulas φ and for all t in W_s, $\mathcal{F}, V, t \models \varphi$ iff $\mathcal{F}_s, V_s, t \models \varphi$.*

Proof. By induction on φ.

Disjoint unions The frame $\mathcal{F}'=(W', R')$ is the *disjoint union* of a family of frames $\mathcal{F}_i=(W_i, R_i)$ where i ranges over a nonempty set I if for all $i, j \in I$, if $i \neq j$ then $W_i \cap W_j = \emptyset$, $W' = \bigcup\{W_i : i \in I\}$ and $R' = \bigcup\{R_i : i \in I\}$. Disjoint unions give rise to the following result:

Proposition 3 (Disjoint unions Theorem). *If the frame $\mathcal{F}'$ is the disjoint union of a family of frames $\mathcal{F}_i$ where i ranges over a nonempty set I then for all $i \in I$, $\mathcal{F}' \preceq \mathcal{F}_i$.*

Proof. Suppose the frame $\mathcal{F}'$ is the disjoint union of a family of frames $\mathcal{F}_i$ where i ranges over a nonempty set I. Let $i \in I$. Obviously, $\mathcal{F}_i$ is a generated subframe of $\mathcal{F}'$. Hence, by Proposition 1, $\mathcal{F}' \preceq \mathcal{F}_i$

Bounded morphic images A frame $\mathcal{F}'=(W',R')$ is a *bounded morphic image* of a frame $\mathcal{F}=(W,R)$ (in symbols $\mathcal{F} \twoheadrightarrow \mathcal{F}'$) if there exists a function f assigning to each state s in $\mathcal{F}$ a state $f(s)$ in $\mathcal{F}'$ such that

- f is surjective,
- for all s,t in $\mathcal{F}$, if sRt then $f(s)R'f(t)$,
- for all s in $\mathcal{F}$ and for all t' in $\mathcal{F}'$, if $f(s)R't'$ then there exists t in $\mathcal{F}$ such that sRt and $f(t)=t'$.

In that case, the function f is a *surjective bounded morphism.* Bounded morphic images give rise to the following result:

Proposition 4 (Bounded morphic images Theorem). *If the frame $\mathcal{F}'$ is a bounded morphic image of the frame $\mathcal{F}$ then $\mathcal{F} \preceq \mathcal{F}'$.*

Proof. See [6, Theorem 3.14 (iii)].

2.3 First-order language and truth

First-order language Let us consider a countable set $\mathbf{IVAR}$ of *individual variables* (denoted x, y, $\ldots$). The set $\mathcal{L}_{\mathbf{FOF}}$ of all *first-order formulas* (denoted A, B, $\ldots$) is inductively defined as follows:

- $A, B ::= \mathbf{R}(x,y) \mid x = y \mid \neg A \mid (A \vee B) \mid \forall x A,$

where x and y range over $\mathbf{IVAR}$. We define the other Boolean constructs as usual. The first-order formula $\exists x A$ is obtained as the well-known abbreviation: $\exists x A ::= \neg\forall x \neg A$. We adopt the standard rules for omission of the parentheses. For all first-order formulas A, let $\mathtt{fiv}(A)$ be the set of all free individual variables occurring in A. A first-order formula A is a *sentence* if $\mathtt{fiv}(A)=\emptyset$. The *quantifier rank* of the first-order formula A (in symbols $\mathtt{qr}(A)$) is the nonnegative integer inductively defined as usual [14, Chapter 1]. The *relativization* of a first-order formula C with respect to a first-order formula A and an individual variable x (in symbols $(C)^A_x$) is inductively defined as follows:

- $(\mathbf{R}(y,z))^A_x$ is $\mathbf{R}(y,z)$,
- $(y = z)^A_x$ is $y = z$,
- $(\neg C)^A_x$ is $\neg(C)^A_x$,
- $(C \vee D)^A_x$ is $(C)^A_x \vee (D)^A_x$,
- $(\forall y C)^A_x$ is $\forall y (A[x/y] \to (C)^A_x)$.

In the above definition, $A[x/y]$ denotes the first-order formula obtained from the first-order formula A by replacing every free occurrence of the individual variable x in A by the individual variable y. From now on, when we write $(C)^A_x$, we will always assume that the sets of individual variables occurring in A and C are disjoint. The reader may easily verify by induction on the first-order formula C that $\mathtt{fiv}((C)^A_x) \subseteq (\mathtt{fiv}(A) \setminus \{x\}) \cup \mathtt{fiv}(C)$. Hence, if C is a sentence then $\mathtt{fiv}((C)^A_x) \subseteq \mathtt{fiv}(A) \setminus \{x\}$.

Truth An *assignment* on a frame $\mathcal{F}$ is a function g assigning to each individual variable x a state $g(x)$ in $\mathcal{F}$. The *update* of an assignment g on a frame $\mathcal{F}$ with respect to a state s in $\mathcal{F}$ and an individual variable x (in symbols g_s^x) is the assignment g_s^x on $\mathcal{F}$ such that $g_s^x(x){=}s$ and for all individual variables $y{\neq}x$, $g_s^x(y){=}g(y)$. Given a frame $\mathcal{F}$, for all nonnegative integers n, for all states $s_1, \ldots, s_n$ in $\mathcal{F}$ and for all individual variables $x_1, \ldots, x_n$, $g_{s_1 \ldots s_n}^{x_1 \ldots x_n}$ is the assignment g' on $\mathcal{F}$ inductively defined as follows

- if $n = 0$ then $g' = g$,
- if $n \geq 1$ then $g' = (g_{s_1 \ldots s_{n-1}}^{x_1 \ldots x_{n-1}})_{s_n}^{x_n}$.

The *satisfiability* of a first-order formula A with respect to an assignment g in a frame $\mathcal{F}{=}(W, R)$ (in symbols $\mathcal{F}, g \models A$) is inductively defined as follows:

- $\mathcal{F}, g \models \mathbf{R}(x, y)$ iff $g(x)Rg(y)$,
- $\mathcal{F}, g \models x = y$ iff $g(x){=}g(y)$,
- $\mathcal{F}, g \models \neg A$ iff $\mathcal{F}, g \not\models A$,
- $\mathcal{F}, g \models A \vee B$ iff either $\mathcal{F}, g \models A$, or $\mathcal{F}, g \models B$,
- $\mathcal{F}, g \models \forall x A$ iff for all states s in $\mathcal{F}$, $\mathcal{F}, g_s^x \models A$.

As a result, $\mathcal{F}, g \models \exists x A$ iff there exists a state s in $\mathcal{F}$ such that $\mathcal{F}, g_s^x \models A$. A first-order formula A is *valid* in a frame $\mathcal{F}$ (in symbols $\mathcal{F} \models A$) if A is satisfied with respect to all assignments in $\mathcal{F}$. A first-order formula A is *valid* in a class $\mathcal{C}$ of frames (in symbols $\mathcal{C} \models A$) if A is valid in all frames in $\mathcal{C}$. For all positive integers n, let

- $B_n ::= \forall x_0 \ldots \forall x_n (\bigwedge \{\mathbf{R}(x_i, x_j) :\ 0{\leq}i{<}j{\leq}n\} \rightarrow \bigvee \{x_i = x_j :\ 0{\leq}i{<}j{\leq}n\})$.

It is a well-known fact that for all positive integers n and for all partitions $\mathcal{F}$, $\mathcal{F} \models B_n$ iff $\mathcal{F}$ is bounded and $n_{\mathcal{F}}{\leq}n$.

Lemma 1. *Let A be a sentence. The following conditions are equivalent:*

1. $\mathcal{C}_{par} \models A$,
2. *for all small and bounded partitions $\mathcal{F}$, if $n_{\mathcal{F}}, \pi_{\mathcal{F}}{\leq}\mathtt{qr}(A)$ then $\mathcal{F} \models A$,*

Proof. $(1 \Rightarrow 2)$ Obvious.
$(2 \Rightarrow 1)$ Suppose $\mathcal{C}_{par} \not\models A$. Hence, there exists a partition $\mathcal{F}$ such that $\mathcal{F} \not\models A$. Let $\mathcal{F}'$ be the bounded partition obtained from $\mathcal{F}$ by eliminating in all equivalence classes, as many states as it is needed so that the size of each equivalence class becomes at most equal to $\mathtt{qr}(A)$. Obviously, $n_{\mathcal{F}'}{\leq}\mathtt{qr}(A)$. Moreover, Duplicator wins the Ehrenfeucht-Fraïssé game $G_{\mathtt{qr}(A)}(\mathcal{F}, \mathcal{F}')$[1]. Thus, for all sentences B, if $\mathtt{qr}(B){\leq}\mathtt{qr}(A)$ then $\mathcal{F} \models B$ iff $\mathcal{F}' \models B$. Since $\mathcal{F} \not\models A$, $\mathcal{F}' \not\models A$. Let $\mathcal{F}''$ be the small and bounded partition obtained from $\mathcal{F}'$ by eliminating for all positive integers π, as many equivalence classes as it is needed so that the number of equivalence classes of size π becomes at most equal to $\mathtt{qr}(A)$. Obviously, $n_{\mathcal{F}''}, \pi_{\mathcal{F}''}{\leq}\mathtt{qr}(A)$. Moreover, Duplicator wins the Ehrenfeucht-Fraïssé game $G_{\mathtt{qr}(A)}(\mathcal{F}', \mathcal{F}'')$. Consequently, for all sentences B, if $\mathtt{qr}(B){\leq}\mathtt{qr}(A)$ then $\mathcal{F}' \models B$ iff $\mathcal{F}'' \models B$. Since $\mathcal{F}' \not\models A$, $\mathcal{F}'' \not\models A$.

[1] Ehrenfeucht-Fraïssé games constitute a useful tool for characterizing frames modulo elementary equivalence. See [14, Chapter 2] for a general introduction.

Lemma 2. *The problem of deciding the $\mathcal{C}_{par}$-validity of $\mathcal{L}_{\mathbf{FOF}}$-formulas is* $\mathbf{PSPACE}$*-complete.*

Proof. By Lemma 1, a sentence A is not $\mathcal{C}_{par}$-valid iff there exists a small and bounded partition $\mathcal{F}$ such that $n_{\mathcal{F}}, \pi_{\mathcal{F}} \leq \mathtt{qr}(A)$ and $\mathcal{F} \not\models A$. Hence, in order to determine whether a given sentence A is $\mathcal{C}_{par}$-valid, it suffices to execute the following procedure:

procedure $\mathtt{val}(A)$
begin
for all small and bounded partitions $\mathcal{F}$ such that $n_{\mathcal{F}}, \pi_{\mathcal{F}} \leq \mathtt{qr}(A)$, call $\mathtt{MC}(\mathcal{F}, A)$;
if all these calls are accepting then accept;
otherwise, reject;
end

where the call $\mathtt{MC}(\mathcal{F}, A)$ is accepting iff $\mathcal{F} \models A$. Obviously, the call $\mathtt{val}(A)$ is accepting iff A is $\mathcal{C}_{par}$-valid. Since the procedure $\mathtt{MC}$ can be implemented in polynomial space [27, 30], the procedure $\mathtt{val}$ can be implemented in polynomial space. Thus, the problem of deciding the $\mathcal{C}_{par}$-validity of $\mathcal{L}_{\mathbf{FOF}}$-formulas is in $\mathbf{PSPACE}$. As for the $\mathbf{PSPACE}$-hardness of the problem of deciding the $\mathcal{C}_{par}$-validity of $\mathcal{L}_{\mathbf{FOF}}$-formulas, it immediately follows from the $\mathbf{PSPACE}$-hardness of the membership problem in the first-order theory of pure equality [28].

Relativization Let $\mathcal{F}, \mathcal{F}'$ be frames. $\mathcal{F}'$ is a *relativized reduct* of $\mathcal{F}$ if there exists a first-order formula A, there exists an individual variable x and there exists an assignment g on $\mathcal{F}$ such that $\mathcal{F}'$ is the restriction of $\mathcal{F}$ to the set of all states s in $\mathcal{F}$ such that $\mathcal{F}, g^x_s \models A$. In that case, we say $\mathcal{F}'$ is the *relativized reduct* of $\mathcal{F}$ with respect to A, x and g. Relativized reducts give rise to the following result:

Proposition 5 (Relativization Theorem). *Let $\mathcal{F}$, $\mathcal{F}'$ be frames, A be a first-order formula, x be an individual variable and g be an assignment on $\mathcal{F}$. If $\mathcal{F}'$ is the relativized reduct of $\mathcal{F}$ with respect to A, x and g then for all first-order formulas $C(y_1, \ldots, y_n)$ and for all assignments g' on $\mathcal{F}'$, $\mathcal{F}, g^{y_1 \ldots y_n}_{g'(y_1) \ldots g'(y_n)} \models (C(y_1, \ldots, y_n))^A_x$ iff $\mathcal{F}', g' \models C(y_1, \ldots, y_n)$.*

Proof. See [19, Theorem 5.1.1].

2.4 Modal definability and first-order definability

Let $\mathcal{C}$ be a class of frames. A sentence A is *modally definable* with respect to $\mathcal{C}$ if there exists a modal formula φ such that for all frames $\mathcal{F}$ in $\mathcal{C}$, $\mathcal{F} \models A$ iff $\mathcal{F} \models \varphi$. In that case, we say φ is a *modal definition* of A with respect to $\mathcal{C}$. A modal formula φ is *first-order definable* with respect to $\mathcal{C}$ if there exists a first-order sentence A such that for all frames $\mathcal{F}$ in $\mathcal{C}$, $\mathcal{F} \models \varphi$ iff $\mathcal{F} \models A$. In that case, we say A is a *first-order definition* of φ with respect to $\mathcal{C}$. Table 1 contain examples of the correspondence between modal formulas and sentences.

φ	A
$p \to \Diamond p$	"**R** is reflexive"
$\Diamond\Diamond p \to \Diamond p$	"**R** is transitive"
$p \to \Box\Diamond p$	"**R** is symmetric"
$\Diamond\top$	"**R** is serial"
$\Diamond p \to \Box\Diamond p$	"**R** is Euclidean"

Table 1. Examples of the correspondence between modal formulas and sentences.

Proposition 6. *Let $\mathcal{C}$ be a class of frames. For all modal formulas φ, if there exists a modal formula ψ such that $\mathcal{C} \models \varphi \leftrightarrow \psi$ and $\mathtt{deg}(\psi) \leq 1$ then φ is first-order definable with respect to $\mathcal{C}$.*

Proof. See [5, Lemma 9.7].

3 Modal definability: decidable cases

We consider classes of frames for which modal definability is decidable: $\mathcal{C}_{tE}$, $\mathcal{C}_{stE}$ and $\mathcal{C}_{par}$. For the purpose of proving the decidability of modal definability with respect to $\mathcal{C}_{par}$, we need to consider the following lemmas.

Lemma 3. *Let $\mathcal{F}{=}(W, R), \mathcal{F}'{=}(W', R')$ be bounded partitions. If $n_{\mathcal{F}} {\geq} n_{\mathcal{F}'}$ then for all modal formulas φ, if $\mathcal{F} \models \varphi$ then $\mathcal{F}' \models \varphi$.*

Proof. Suppose $n_{\mathcal{F}} {\geq} n_{\mathcal{F}'}$. Let φ be a modal formula. Suppose $\mathcal{F} \models \varphi$ and $\mathcal{F}' \not\models \varphi$. Hence, there exists a valuation V' on $\mathcal{F}'$ and there exists a state s' in $\mathcal{F}'$ such that $\mathcal{F}', V', s' \not\models \varphi$. Thus, by Proposition 2, $\mathcal{F}'_{s'}, V'_{s'}, s' \not\models \varphi$ where $V'_{s'}$ is the restriction of V' to $W_{s'}$. Consequently, $\mathcal{F}'_{s'} \not\models \varphi$. Obviously, $n_{\mathcal{F}'} {\geq} n_{\mathcal{F}'_{s'}}$. Since $n_{\mathcal{F}} {\geq} n_{\mathcal{F}'}$, $n_{\mathcal{F}} {\geq} n_{\mathcal{F}'_{s'}}$. Hence, let s be a state in $\mathcal{F}$ such that $\|R(s)\| {\geq} \|R'_{s'}(s')\|$. Since $\mathcal{F} \models \varphi$, by Proposition 1, $\mathcal{F}_s \models \varphi$. Moreover, $\mathcal{F}_s \twoheadrightarrow \mathcal{F}'_{s'}$. Thus, by Proposition 4, $\mathcal{F}'_{s'} \models \varphi$: a contradiction.

Lemma 4. *Let $\mathcal{F}, \mathcal{F}'$ be bounded partitions such that $n_{\mathcal{F}} {\geq} n_{\mathcal{F}'}$. For all sentences A, if A is modally definable with respect to $\mathcal{C}_{par}$ and $\mathcal{F} \models A$ then $\mathcal{F}' \models A$.*

Proof. Let A be a sentence. Suppose A is modally definable with respect to $\mathcal{C}_{par}$ and $\mathcal{F} \models A$. Hence, there exists a modal formula φ such that for all partitions $\mathcal{F}''$, $\mathcal{F}'' \models A$ iff $\mathcal{F}'' \models \varphi$. Since $\mathcal{F} \models A$, $\mathcal{F} \models \varphi$. Since $n_{\mathcal{F}} {\geq} n_{\mathcal{F}'}$, by Lemma 3, $\mathcal{F}' \models \varphi$. Since for all partitions $\mathcal{F}''$, $\mathcal{F}'' \models A$ iff $\mathcal{F}'' \models \varphi$, $\mathcal{F}' \models A$.

Lemma 5. *Let A be a sentence. If $\mathcal{C}_{par} \not\models A$ and $\mathcal{C}_{par} \not\models \neg A$ then A is modally definable with respect to $\mathcal{C}_{par}$ iff there exists a positive integer n such that $n {<} \mathtt{qr}(A)$ and for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $n {\geq} n_{\mathcal{F}}$.*

Proof. Suppose $\mathcal{C}_{par} \not\models A$ and $\mathcal{C}_{par} \not\models \neg A$.
($\Rightarrow$) Suppose A is modally definable with respect to $\mathcal{C}_{par}$. Let $N{=}\{n_{\mathcal{F}} : \mathcal{F}$ is a bounded partition such that $\mathcal{F} \models A\}$. Since $\mathcal{C}_{par} \not\models A$ and $\mathcal{C}_{par} \not\models \neg A$, by Lemma 1, there exists bounded partitions $\mathcal{G}'$ and $\mathcal{G}''$ such that $n_{\mathcal{G}'}{\leq}\mathtt{qr}(A)$, $\mathcal{G}' \not\models A$ and $\mathcal{G}'' \not\models \neg A$. Hence, $\mathcal{G}'' \models A$. Since A is modally definable with respect to $\mathcal{C}_{par}$, by Lemma 4, $n_{\mathcal{G}'}$ is strictly greater than all positive integers in N. Moreover, $n_{\mathcal{G}''}{\in}N$. Thus, $N{\neq}\emptyset$. Since $n_{\mathcal{G}'}$ is strictly greater than all positive integers in N, N possesses a maximal element. Let $n{=}\max N$. Since $n_{\mathcal{G}'}{\leq}\mathtt{qr}(A)$ and $n_{\mathcal{G}'}$ is strictly greater than all positive integers in N, $n{<}\mathtt{qr}(A)$. For the sake of the contradiction, suppose there exists a bounded partition $\mathcal{H}$ such that either $\mathcal{H} \models A$ and $n{<}n_{\mathcal{H}}$, or $\mathcal{H} \not\models A$ and $n{\geq}n_{\mathcal{H}}$. In the former case, $n_{\mathcal{H}}$ is in N. Consequently, $n{\geq}n_{\mathcal{H}}$: a contradiction. In the latter case, by Lemma 4, $n_{\mathcal{H}}$ is strictly greater than all positive integers in N. Hence, $n{<}n_{\mathcal{H}}$: a contradiction.
($\Leftarrow$) Suppose there exists a positive integer n such that $n{<}\mathtt{qr}(A)$ and for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $n{\geq}n_{\mathcal{F}}$. For the sake of the contradiction, suppose there exists a partition $\mathcal{G} = (W, R)$ such that either $\mathcal{G} \models A$ and $\mathcal{G} \not\models \psi_n$, or $\mathcal{G} \not\models A$ and $\mathcal{G} \models \psi_n$, ψ_n being the modal formula defined in Section 2.2. In the former case, since for all partitions $\mathcal{F}$, $\mathcal{F} \models \psi_n$ iff $\mathcal{F}$ is bounded and $n_{\mathcal{F}}{\leq}n$, if $\mathcal{G}$ is bounded then $n_{\mathcal{G}}{>}n$. Thus, there exists a state s in $\mathcal{G}$ such that $\|R(s)\|{\geq}n+1$. Since $n{<}\mathtt{qr}(A)$, $n+1{\leq}\mathtt{qr}(A)$. Let $\mathcal{G}'$ be the bounded partition obtained from $\mathcal{G}$ by eliminating in all equivalence classes, as many states as it is needed so that the size of each equivalence class becomes at most equal to $\mathtt{qr}(A)$. As the reader can check, Duplicator wins the Ehrenfeucht-Fraïssé game $G_{\mathtt{qr}(A)}(\mathcal{G}, \mathcal{G}')$. Consequently, for all sentences B, if $\mathtt{qr}(B){\leq}\mathtt{qr}(A)$ then $\mathcal{G} \models B$ iff $\mathcal{G}' \models B$. Since $\mathcal{G} \models A$, $\mathcal{G}' \models A$. Since there exists a state s in $\mathcal{G}$ such that $\|R(s)\|{\geq}n{+}1$ and $n+1{\leq}\mathtt{qr}(A)$, $n_{\mathcal{G}'}{\geq}n+1$. Hence, $n_{\mathcal{G}'}{>}n$. Since for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $n{\geq}n_{\mathcal{F}}$, $\mathcal{G}' \not\models A$: a contradiction. In the latter case, since for all partitions $\mathcal{F}$, $\mathcal{F} \models \psi_n$ iff $\mathcal{F}$ is bounded and $n_{\mathcal{F}}{\leq}n$, $\mathcal{G}$ is bounded and $n_{\mathcal{G}}{\leq}n$. Since for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $n{\geq}n_{\mathcal{F}}$, $\mathcal{G} \models A$: a contradiction. As a result, we obtain that for all partitions $\mathcal{G}$, $\mathcal{G} \models A$ iff $\mathcal{G} \models \psi_n$. Thus, A is modally definable with respect to $\mathcal{C}_{par}$.

Lemma 6. *Let A be a sentence. The following conditions are equivalent:*

- *A is modally definable with respect to $\mathcal{C}_{par}$,*
- *one of the following conditions holds:*
  - *$\mathcal{C}_{par} \models A$,*
  - *$\mathcal{C}_{par} \models \neg A$,*
  - *there exists a positive integer n such that $n{<}\mathtt{qr}(A)$ and for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $n{\geq}n_{\mathcal{F}}$.*

Proof. By Lemma 5, using the fact that if $\mathcal{C}_{par} \models A$ then A corresponds to the modal formula $\top$ with respect to $\mathcal{C}_{par}$ and if $\mathcal{C}_{par} \models \neg A$ then A corresponds to the modal formula $\bot$ with respect to $\mathcal{C}_{par}$.

Lemma 7. *Let A be a sentence. If $\mathcal{C}_{par} \not\models A$ and $\mathcal{C}_{par} \not\models \neg A$ then A is modally definable with respect to $\mathcal{C}_{par}$ iff there exists a positive integer n such that $n{<}\mathtt{qr}(A)$ and $\mathcal{C}_{par} \models A \leftrightarrow B_n$, B_n being the sentence defined in Section 2.3.*

Proof. Suppose $\mathcal{C}_{par} \not\models A$ and $\mathcal{C}_{par} \not\models \neg A$.
($\Rightarrow$) Suppose A is modally definable with respect to $\mathcal{C}_{par}$. Since $\mathcal{C}_{par} \not\models A$ and $\mathcal{C}_{par} \not\models$

$\neg A$, by Lemma 6, there exists a positive integer n such that $n<\texttt{qr}(A)$ and for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $n \geq n_{\mathcal{F}}$. Hence, for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $\mathcal{F} \models B_n$. Thus, for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A \leftrightarrow B_n$. Consequently, by Lemma 1, $\mathcal{C}_{par} \models A \leftrightarrow B_n$.
($\Leftarrow$) Suppose there exists a positive integer n such that $n<\texttt{qr}(A)$ and $\mathcal{C}_{par} \models A \leftrightarrow B_n$. Hence, for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A \leftrightarrow B_n$. Thus, for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $\mathcal{F} \models B_n$. Consequently, for all bounded partitions $\mathcal{F}$, $\mathcal{F} \models A$ iff $n \geq n_{\mathcal{F}}$. Since $n<\texttt{qr}(A)$, by Lemma 6, A is modally definable with respect to $\mathcal{C}_{par}$.

Lemma 8. *Let A be a sentence. The following conditions are equivalent:*

- *A is modally definable with respect to $\mathcal{C}_{par}$,*
- *one of the following conditions holds:*
  - *$\mathcal{C}_{par} \models A$,*
  - *$\mathcal{C}_{par} \models \neg A$,*
  - *there exists a positive integer n such that $n<\texttt{qr}(A)$ and $\mathcal{C}_{par} \models A \leftrightarrow B_n$.*

Proof. By Lemma 7, using the fact that if $\mathcal{C}_{par} \models A$ then A corresponds to the modal formula $\top$ with respect to $\mathcal{C}_{par}$ and if $\mathcal{C}_{par} \models \neg A$ then A corresponds to the modal formula $\bot$ with respect to $\mathcal{C}_{par}$.

Lemma 9. *Let A be a sentence. The following conditions are equivalent:*

- *$\mathcal{C}_{par} \models A$,*
- *$B_{\texttt{qr}(A)} \rightarrow A$ is modally definable with respect to $\mathcal{C}_{par}$.*

Proof. ($\Rightarrow$) Suppose $\mathcal{C}_{par} \models A$. Hence, $\mathcal{C}_{par} \models B_{\texttt{qr}(A)} \rightarrow A$. Thus, $B_{\texttt{qr}(A)} \rightarrow A$ corresponds to the modal formula $\top$ with respect to $\mathcal{C}_{par}$. Consequently, $B_{\texttt{qr}(A)} \rightarrow A$ is modally definable with respect to $\mathcal{C}_{par}$.
($\Leftarrow$) Suppose $B_{\texttt{qr}(A)} \rightarrow A$ is modally definable with respect to $\mathcal{C}_{par}$. For the sake of the contradiction, suppose $\mathcal{C}_{par} \not\models A$. Hence, by Lemma 1, there exists a bounded partition $\mathcal{F}$ such that $n_{\mathcal{F}} \leq \texttt{qr}(A)$ and $\mathcal{F} \not\models A$. Thus, $\mathcal{F} \models B_{\texttt{qr}(A)}$. Since $\mathcal{F} \not\models A$, $\mathcal{F} \not\models B_{\texttt{qr}(A)} \rightarrow A$. Consequently, $\mathcal{C}_{par} \not\models B_{\texttt{qr}(A)} \rightarrow A$. Moreover, obviously, $\mathcal{C}_{par} \not\models \neg(B_{\texttt{qr}(A)} \rightarrow A)$. Since $B_{\texttt{qr}(A)} \rightarrow A$ is modally definable with respect to $\mathcal{C}_{par}$, by Lemma 6, there exists a positive integer n such that $n<\texttt{qr}(B_{\texttt{qr}(A)} \rightarrow A)$ and for all bounded partitions $\mathcal{G}$, $\mathcal{G} \models B_{\texttt{qr}(A)} \rightarrow A$ iff $n \geq n_{\mathcal{G}}$. Hence, $n \leq \texttt{qr}(A)$. Let $\mathcal{F}'$ be a bounded partition such that $n_{\mathcal{F}'} > \texttt{qr}(A)$. Thus, $\mathcal{F}' \not\models B_{\texttt{qr}(A)}$. Consequently, $\mathcal{F}' \models B_{\texttt{qr}(A)} \rightarrow A$. Since for all bounded partitions $\mathcal{G}$, $\mathcal{G} \models B_{\texttt{qr}(A)} \rightarrow A$ iff $n \geq n_{\mathcal{G}}$, $n \geq n_{\mathcal{F}'}$. Since $n_{\mathcal{F}'} > \texttt{qr}(A)$, $n > \texttt{qr}(A)$: a contradiction.

As a result,

Theorem 1. *The problem of deciding the modal definability with respect to $\mathcal{C}_{par}$ of $\mathcal{L}_{\mathbf{FOF}}$-formulas is* $\mathbf{PSPACE}$*-complete.*

Proof. By Lemma 8, a sentence A is modally definable with respect to $\mathcal{C}_{par}$ iff either $\mathcal{C}_{par} \models A$, or $\mathcal{C}_{par} \models \neg A$, or there exists a positive integer n such that $n<\texttt{qr}(A)$ and $\mathcal{C}_{par} \models A \leftrightarrow B_n$. Hence, in order to determine whether a given sentence A is modally definable with respect to $\mathcal{C}_{par}$, it suffices to execute the following procedure:

```
procedure MD(A)
begin
call val(A);
if this call is accepting then accept;
otherwise, call val(¬A);
if this call is accepting then accept;
otherwise, for all positive integers n such that n<qr(A), call val(A ↔
B_n);
if one of these calls is accepting then accept;
otherwise, reject;
end
```

Obviously, the call $\mathtt{MD}(A)$ is accepting iff A is modally definable with respect to $\mathcal{C}_{par}$. Since the procedure `val` can be implemented in polynomial space, the procedure `MD` can be implemented in polynomial space. Thus, the problem of deciding the modal definability with respect to $\mathcal{C}_{par}$ of $\mathcal{L}_{\mathbf{FOF}}$-formulas is in $\mathbf{PSPACE}$. As for the $\mathbf{PSPACE}$-hardness of the problem of deciding the modal definability with respect to $\mathcal{C}_{par}$ of $\mathcal{L}_{\mathbf{FOF}}$-formulas, it immediately follows from Lemmas 2 and 9.

An interesting question is the following: when the ordinary language of modal logic is extended either with the universal modality, or with the difference modality, is the problem of deciding the modal definability with respect to $\mathcal{C}_{par}$, $\mathcal{C}_{tE}$ and $\mathcal{C}_{stE}$ of $\mathcal{L}_{\mathbf{FOF}}$-formulas still decidable ? If the answer is "yes", is this problem still $\mathbf{PSPACE}$-complete ? The answers to these questions have been given in [2, 15, 16].

Proposition 7. *When the ordinary language of modal logic is extended with the universal modality, the problem of deciding the modal definability with respect to $\mathcal{C}_{par}$, $\mathcal{C}_{tE}$ and $\mathcal{C}_{stE}$ of $\mathcal{L}_{\mathbf{FOF}}$-formulas is $\mathbf{PSPACE}$-complete.*

4 First-order definability: trivial cases

In this section, we consider classes of frames for which first-order definability is trivial: $\mathcal{C}_{tE}$, $\mathcal{C}_{stE}$ and $\mathcal{C}_{par}$. We take as well a special interest in $\mathcal{C}_{E}$, $\mathcal{C}_{sE}$ and $\mathcal{C}_{rtc}^{\omega}$ and we prove that they give rise to a trivial first-order definability problem too. It is a well-known fact that with respect to $\mathcal{C}_{tE}$, $\mathcal{C}_{stE}$ and $\mathcal{C}_{par}$, every modal formula is equivalent to a modal formula of degree less than or equal to 1. As a result,

Proposition 8. *The problem of deciding first-order definability with respect to $\mathcal{C}_{tE}$, $\mathcal{C}_{stE}$ and $\mathcal{C}_{par}$ is trivial: every modal formula is first-order definable with respect to $\mathcal{C}_{tE}$, $\mathcal{C}_{stE}$ and $\mathcal{C}_{par}$.*

Proof. By Proposition 6.

The reader may ask whether there exists classes of frames with respect to which the problem of deciding first-order definability is trivial and there exists modal formulas equivalent to no modal formula of degree less than or equal to 1. It is a well-known fact that with respect to $\mathcal{C}_E$ and $\mathcal{C}_{sE}$, every modal formula is equivalent to a modal formula of degree less than or equal to 2 but some modal formula is equivalent to no modal formula of degree less than or equal to 1. Nevertheless,

Proposition 9. *The problem of deciding first-order definability with respect to $\mathcal{C}_E$ and $\mathcal{C}_{sE}$ is trivial: every modal formula is first-order definable with respect to $\mathcal{C}_E$ and $\mathcal{C}_{sE}$.*

Proof. Since $\mathcal{C}_E$ contains $\mathcal{C}_{sE}$, it suffices to prove that (Π) every modal formula is first-order definable with respect to $\mathcal{C}_E$. The proof of (Π) has been presented by Balbiani *et al.* [1]. It is based on the following line of reasoning. For all frames $\mathcal{F}{=}(W, R)$ in $\mathcal{C}_E$ and for all states s in $\mathcal{F}$, exactly one of the following conditions holds:

- $R_s{=}\emptyset$,
- $R_s = W_s \times W_s$,
- $R_s = (\{s\} \times S) \cup (T \times T)$ for some nonempty subsets S and T of $W_s \setminus \{s\}$ such that $S \subseteq T$.

When $\mathcal{F}$ is finite, for all states s in $\mathcal{F}$, $\mathcal{F}_s$ can be exactly characterized by a triple $\sigma{=}(\sigma_1, \sigma_2, \sigma_3)$ in $\{0, 1\} \times \mathbb{N}^2$: σ_1 will be the number of irreflexive states in $\mathcal{F}_s$; σ_2 will be the number of states accessible from s in 1 step; σ_3 will be the number of states accessible from s either in 1 step, or in 2 steps. When $R_s{=}\emptyset$, this triple will be such that $\sigma_1 = 1$, $\sigma_2 = 0$ and $\sigma_3 = 0$. When $R_s = W_s \times W_s$, this triple will be such that $\sigma_1 = 0$, $\sigma_2 \geq 1$ and $\sigma_3 = \sigma_2$. When $R_s = (\{s\} \times S) \cup (T \times T)$ for some nonempty subsets S and T of $W_s \setminus \{s\}$ such that $S \subseteq T$, this triple will be such that $\sigma_1 = 1$, $\sigma_2 \geq 1$ and $\sigma_3 \geq \sigma_2$. A *type* is a triple $\sigma{=}(\sigma_1, \sigma_2, \sigma_3)$ in $\{0, 1\} \times \mathbb{N}^2$ such that one of the following conditions holds:

- $\sigma_1 = 1$, $\sigma_2 = 0$ and $\sigma_3 = 0$,
- $\sigma_1 = 0$, $\sigma_2 \geq 1$ and $\sigma_3 = \sigma_2$,
- $\sigma_1 = 1$, $\sigma_2 \geq 1$ and $\sigma_3 \geq \sigma_2$.

Obviously, for all types $\sigma{=}(\sigma_1, \sigma_2, \sigma_3)$, one can construct a finite rooted frame $\mathcal{F}_\sigma{=}(W_\sigma, R_\sigma)$ in $\mathcal{C}_E$ which is characterized by σ. Moreover, for all types $\sigma{=}(\sigma_1, \sigma_2, \sigma_3)$, one can write a first-order formula $A_\sigma(x)$ such that for all assignments g on F_σ, if $g(x)$ is equal to the root of $\mathcal{F}_\sigma$ then $F_\sigma, g \models A_\sigma(x)$. For all types $\sigma{=}(\sigma_1, \sigma_2, \sigma_3)$, x is the only individual variable freely occurring in the first-order formula $A_\sigma(x)$ associated to it. Given a type $\sigma{=}(\sigma_1, \sigma_2, \sigma_3)$, how is constructed the finite rooted frame $\mathcal{F}_\sigma{=}(W_\sigma, R_\sigma)$ and how is written the first-order formula $A_\sigma(x)$? We will answer later in this section to a similar question within the context of the first-order definability problem with respect to $\mathcal{C}^\omega_{rtc}$. Now, for all modal formulas φ, let $\Delta(\varphi) = \{\sigma : \sigma{=}(\sigma_1, \sigma_2, \sigma_3)$ is a type such that $\mathcal{F}_\sigma \not\models \varphi$ and $\sigma_3 {\leq} \|\mathtt{sf}(\varphi)\|\}$. Obviously, for all modal formulas φ, $\Delta(\varphi)$ is finite. The finite rooted frame $\mathcal{F}_\sigma{=}(W_\sigma, R_\sigma)$ and the first-order formula $A_\sigma(x)$ associated to a given type $\sigma{=}(\sigma_1, \sigma_2, \sigma_3)$ possess interesting properties. For example[2],

[2] Lemmas 10, 11 and 12 assert the properties that are needed for proving Proposition 9. Their proofs have been given with full details in [1]. Similar properties needed for proving Theorem 2 below are asserted in Lemmas 14, 15 and 16. Their proofs are given with full details below.

Lemma 10. *For all types σ=$(\sigma_1, \sigma_2, \sigma_3)$ and for all assignments g on $\mathcal{F}_\sigma$, if $g(x)$ is the root of $\mathcal{F}_\sigma$ then $\mathcal{F}_\sigma, g \models A_\sigma(x)$.*

Lemma 11. *Let $\mathcal{F}$ be a frame in $\mathcal{C}_E$ and g be an assignment on $\mathcal{F}$. For all types σ=$(\sigma_1, \sigma_2, \sigma_3)$, if $\mathcal{F}, g \models A_\sigma(x)$ then there exists a surjective bounded morphism $f : \mathcal{F}_{g(x)} \twoheadrightarrow \mathcal{F}_\sigma$ such that $f(g(x))$ is the root of $\mathcal{F}_\sigma$.*

Lemma 12. *Let φ be a modal formula. For all frames $\mathcal{F}$ in $\mathcal{C}_E$, if $\mathcal{F} \not\models \varphi$ then there exists a type σ=$(\sigma_1, \sigma_2, \sigma_3)$ such that $\mathcal{F}_\sigma \not\models \varphi$, $\sigma_3 \leq \|\mathtt{sf}(\varphi)\|$ and $\mathcal{F} \models \exists x\ A_\sigma(x)$.*

In Lemmas 10, 11 and 12, $\mathcal{F}_\sigma$ denotes the finite rooted frame in $\mathcal{C}_E$ associated to σ and $\mathcal{F}_{g(x)}$ denotes the subframe of $\mathcal{F}$ generated from $g(x)$. For all modal formulas φ, let A_φ be the first-order formula $\neg\exists x \bigvee\{A_\sigma(x) :\ \sigma \in \Delta(\varphi)\}$. Notice that for all modal formulas φ, A_φ is a sentence. Given a modal formula φ, the reason for our interest in the sentence A_φ is the following result:

Lemma 13. *Let φ be a modal formula. For all frames $\mathcal{F}$ in $\mathcal{C}_E$, the following conditions are equivalent:*

- $\mathcal{F} \models \varphi$,
- $\mathcal{F} \models A_\varphi$.

This ends the proof of Proposition 9.

The reader may ask whether there exists classes of frames with respect to which every modal formula is first-order definable and for all $n \in \mathbb{N}$, there exists modal formulas equivalent to no modal formula of degree less than or equal to n. It is a well-known fact that with respect to $\mathcal{C}^\omega_{rtc}$, for all $n \in \mathbb{N}$, some modal formula is equivalent to no modal formula of degree less than or equal to n. Nevertheless,

Theorem 2. *The problem of deciding first-order definability with respect to $\mathcal{C}^\omega_{rtc}$ is trivial: every modal formula is first-order definable with respect to $\mathcal{C}^\omega_{rtc}$.*

Proof. We will follow a line of reasoning similar to the line of reasoning sketched in the proof of Proposition 9. For all frames $\mathcal{F}$ in $\mathcal{C}^\omega_{rtc}$ and for all states s in $\mathcal{F}$, $\mathcal{F}_s$ contains finitely many clusters. When $\mathcal{F}$ is finite, for all states s in $\mathcal{F}$, $\mathcal{F}_s$ can be exactly characterized by a finite nonempty sequence σ=$(\sigma_1, \ldots, \sigma_a)$ of positive integers. In this proof, a *type* is a finite nonempty sequence σ=$(\sigma_1, \ldots, \sigma_a)$ of positive integers. For all types σ=$(\sigma_1, \ldots, \sigma_a)$, let $\|\sigma\|$=$\sigma_1 + \ldots + \sigma_a$. For all types σ=$(\sigma_1, \ldots, \sigma_a)$, let $\mathcal{F}_\sigma$=(W_σ, R_σ) be the $\mathcal{C}^\omega_{rtc}$-frame such that W_σ=$\{(i, k) :\ 1 \leq i \leq a$ and $1 \leq k \leq \sigma_i\}$ and R_σ is the binary relation on W_σ such that for all $(i, k), (j, l)$ in W_σ, $(i, k) R_\sigma (j, l)$ iff $i \leq j$. For all types σ=$(\sigma_1, \ldots, \sigma_a)$, let $A_\sigma(x)$ be the first-order formula $\exists x_{1,1} \ldots \exists x_{1,\sigma_1} \ldots \exists x_{a,1} \ldots \exists x_{a,\sigma_a} B_\sigma$ where B_σ is the conjunction of the following formulas:

- $x = x_{1,1} \vee \ldots \vee x = x_{1,\sigma_1}$,
- $x_{i,k} \neq x_{j,l}$ for all $(i, k), (j, l)$ in W_σ such that either $i \neq j$, or $k \neq l$,
- $\mathbf{R}(x_{i,k}, x_{j,l})$ for all $(i, k), (j, l)$ in W_σ such that $i \leq j$,
- $\neg\mathbf{R}(x_{j,l}, x_{i,k})$ for all $(i, k), (j, l)$ in W_σ such that $i < j$,

- $\forall y(\mathbf{R}(x,y) \to \bigvee\{\mathbf{R}(y,x_{i,k}) : (i,k) \text{ is in } W_\sigma\})$.

Notice that for all types $\sigma{=}(\sigma_1,\ldots,\sigma_a)$, x is the only individual variable freely occurring in $A_\sigma(x)$. Now, for all modal formulas φ, let $\Delta(\varphi) = \{\sigma : \sigma{=}(\sigma_1,\ldots,\sigma_a)$ is a type such that $\mathcal{F}_\sigma \not\models \varphi$ and $\|\sigma\|{\leq}3.\|\mathtt{sf}(\varphi)\|\}$. Obviously, for all modal formulas φ, $\Delta(\varphi)$ is finite. The finite rooted frame $\mathcal{F}_\sigma{=}(W_\sigma,R_\sigma)$ and the first-order formula $A_\sigma(x)$ associated to a given type $\sigma{=}(\sigma_1,\ldots,\sigma_a)$ possess interesting properties. The following result will play in this proof the role played by Lemma 10 in the proof of Proposition 9.

Lemma 14. *For all types $\sigma{=}(\sigma_1,\ldots,\sigma_a)$ and for all assignments g on $\mathcal{F}_\sigma$, if $g(x)$ is in $\{(1,1),\ldots,(1,\sigma_1)\}$ then $\mathcal{F}_\sigma, g \models A_\sigma(x)$.*

Proof. Let $\sigma{=}(\sigma_1,\ldots,\sigma_a)$ be a type and g be an assignment on $\mathcal{F}_\sigma$. Suppose $g(x)$ is in $\{(1,1),\ldots,(1,\sigma_1)\}$. Let g' be the assignment on $\mathcal{F}_\sigma$ such that

- $g'(x){=}g(x)$,
- $g'(x_{i,k}){=}(i,k)$ for all (i,k) in W_σ,
- for all individual variables $z{\neq}x$, if $z{\neq}x_{i,k}$ for all (i,k) in W_k then $g'(z){=}g(z)$.

Since $g(x)$ is in $\{(1,1),\ldots,(1,\sigma_1)\}$,

- either $g'(x){=}g'(x_{1,1})$, ..., or $g'(x){=}g'(x_{1,\sigma_1})$,
- $g'(x_{i,k}){\neq}g'(x_{j,l})$ for all $(i,k),(j,l)$ in W_σ such that either $i{\neq}j$, or $k{\neq}l$,
- $R_\sigma(g'(x_{i,k}),g'(x_{j,l}))$ for all $(i,k),(j,l)$ in W_σ such that $i{\leq}j$,
- not $R_\sigma(g'(x_{j,l}),g'(x_{i,k}))$ for all $(i,k),(j,l)$ in W_σ such that $i<j$,
- for all (j,l) in W_σ, if $R_\sigma(g'(x),(j,l))$ then there exists (i,k) in W_σ such that $R_\sigma((j,l),g'(x_{i,k}))$.

Hence, $\mathcal{F}_\sigma, g' \models B_\sigma$. Since g' is an assignment on $\mathcal{F}_\sigma$ such that $g'(x){=}g(x)$ and for all individual variables $z{\neq}x$, if $z{\neq}x_{i,k}$ for all (i,k) in W_k then $g'(z){=}g(z)$, $\mathcal{F}_\sigma, g \models A_\sigma(x)$.

The following result will play in this proof the role played by Lemma 11 in the proof of Proposition 9.

Lemma 15. *Let $\mathcal{F}{=}(W,R)$ be a frame in $\mathcal{C}^\omega_{rtc}$ and g be an assignment on $\mathcal{F}$. For all types $\sigma{=}(\sigma_1,\ldots,\sigma_a)$, if $\mathcal{F}, g \models A_\sigma(x)$ then there exists a surjective bounded morphism $f : \mathcal{F}_{g(x)} \twoheadrightarrow \mathcal{F}_\sigma$ such that $f(g(x))$ is in $\{(1,1),\ldots,(1,\sigma_1)\}$.*

Proof. Let $\sigma{=}(\sigma_1,\ldots,\sigma_a)$ be a type. Suppose $\mathcal{F}, g \models A_\sigma(x)$. Let g' be an assignment on $\mathcal{F}$ such that

- $g'(x){=}g(x)$,
- for all individual variables $z{\neq}x$, if $z{\neq}x_{i,k}$ for all (i,k) in W_k then $g'(z){=}g(z)$,
- $\mathcal{F}, g' \models B_\sigma$.

Hence,

- either $g'(x){=}g'(x_{1,1})$, ..., or $g'(x){=}g'(x_{1,\sigma_1})$,
- $g'(x_{i,k}){\neq}g'(x_{j,l})$ for all $(i,k),(j,l)$ in W_σ such that either $i{\neq}j$, or $k{\neq}l$,
- $R(g'(x_{i,k}),g'(x_{j,l}))$ for all $(i,k),(j,l)$ in W_σ such that $i{\leq}j$,

– not $R(g'(x_{j,l}), g'(x_{i,k}))$ for all $(i, k), (j, l)$ in W_σ such that $i < j$,
– for all states t in $\mathcal{F}$, if $R(g'(x), t)$ then there exists (i, k) in W_σ such that $R(t, g'(x_{i,k}))$.

Let C_1 be the cluster of $g'(x_{1,1}), \ldots, g'(x_{1,\sigma_1})$ in $\mathcal{F}_{g'(x)}$, ..., C_a be the cluster of $g'(x_{a,1}), \ldots, g'(x_{a,\sigma_a})$ in $\mathcal{F}_{g'(x)}$. By the above 5 itemized conditions,

– $g'(x)$ is in C_1,
– $\|C_i\| \geq \sigma_i$ for all i in $\{1, \ldots, a\}$,
– $C_i \ll C_j$ for all i, j in $\{1, \ldots, a\}$ such that $i \leq j$,
– not $C_j \ll C_i$ for all i, j in $\{1, \ldots, a\}$ such that $i < j$,
– for all states t in $W_{g'(x)}$, there exists a least element i in $\{1, \ldots, a\}$ such that $tR_{g'(x)}g'(x_{i,1})$, ..., $tR_{g'(x)}g'(x_{i,\sigma_i})$,

where $\ll$ is the reflexive, antisymmetric, transitive and connected relation between $\mathcal{F}$'s clusters such that for all $\mathcal{F}$'s clusters C, D, $C \ll D$ iff there exists states t, u in $\mathcal{F}$ such that $t \in C$, $u \in D$ and tRu. Let $f : W_{g'(x)} \longrightarrow W_\sigma$ be such that

– either $f(g'(x)) = (1, 1)$, ..., or $f(g'(x)) = (1, \sigma_1)$,
– for all i in $\{1, \ldots, a\}$, $f_{|C_i}$ is a surjective function from C_i to $\{(i, k) : 1 \leq k \leq \sigma_i\}$,
– for all states t in $W_{g'(x)} \setminus (C_1 \cup \ldots \cup C_a)$, $f(t)$ is in $\{(i, k) : 1 \leq k \leq \sigma_i\}$ where i is the least element in $\{1, \ldots, a\}$ such that $tR_{g'(x)}g'(x_{i,1})$, ..., $tR_{g'(x)}g'(x_{i,\sigma_i})$.

Obviously, $f : \mathcal{F}_{g(x)} \twoheadrightarrow \mathcal{F}_\sigma$ is a surjective bounded morphism. Moreover, since $g'(x) = g(x)$, $f(g(x))$ is in $\{(1, 1), \ldots, (1, \sigma_1)\}$.

The following result will play in this proof the role played by Lemma 12 in the proof of Proposition 9.

Lemma 16. *Let φ be a modal formula. For all frames $\mathcal{F}$ in $\mathcal{C}^\omega_{rtc}$, if $\mathcal{F} \not\models \varphi$ then there exists a type $\sigma = (\sigma_1, \ldots, \sigma_a)$ such that $\mathcal{F}_\sigma \not\models \varphi$, $\|\sigma\| \leq 3.\|\mathtt{sf}(\varphi)\|$ and $\mathcal{F} \models \exists x A_\sigma(x)$.*

Proof. Let $\mathcal{F} = (W, R)$ be a frame in $\mathcal{C}^\omega_{rtc}$. Suppose $\mathcal{F} \not\models \varphi$. Hence, there exists a valuation V on $\mathcal{F}$ and there exists a state s in $\mathcal{F}$ such that $\mathcal{F}, V, s \not\models \varphi$. Since $\mathcal{F}$ is a $\mathcal{C}^\omega_{rtc}$-frame, $\mathcal{F}_s$ contains finitely many clusters. Moreover, s belongs to the first cluster of $\mathcal{F}_s$. For all states t in $\mathcal{F}_s$, let $B(t) = \{\Box\psi \in \mathtt{sf}^\Box(\varphi) : \mathcal{F}_s, V_s, t \models \Box\psi\}$ where V_s is the restriction of V to W_s. Notice that for all states t, u in $\mathcal{F}_s$, if tR_su then $B(t) \subseteq B(u)$. Let $n \geq 1$ and $t_1, \ldots, t_n$ be states in $\mathcal{F}_s$ such that

– for all states t in $\mathcal{F}_s$, there exists i in $\{1, \ldots, n\}$ such that $B(t) = B(t_i)$,
– for all i, j in $\{1, \ldots, n\}$, if $i < j$ then $B(t_i)$ is strictly contained in $B(t_j)$.

Notice that $n \leq \|\mathtt{sf}^\Box(\varphi)\| + 1$. Thus, $n \leq \|\mathtt{sf}(\varphi)\|$. Moreover, for all i, j in $\{1, \ldots, n\}$, if $i < j$ then $t_iR_st_j$ and not $t_jR_st_i$. For all i in $\{1, \ldots, n\}$, let $CB(t_i) = \{C(u) : u$ is a state in $\mathcal{F}_s$ such that $B(u) = B(t_i)\}$. Obviously, for all i in $\{1, \ldots, n\}$, $C(t_i) \in CB(t_i)$. For all i in $\{1, \ldots, n\}$, let u_i be a state in the last cluster of $CB(t_i)$. For all i in $\{1, \ldots, n\}$, let $\alpha_i \geq 0$ and $\Box\psi_{i,1}, \ldots, \Box\psi_{i,\alpha_i}$ be a list of $\mathtt{sf}^\Box(\varphi) \setminus B(t_i)$ when $i = n$ and a list of $B(t_{i+1}) \setminus B(t_i)$ otherwise. Obviously, $\alpha_1 + \ldots + \alpha_n \leq \|\mathtt{sf}^\Box(\varphi)\|$. Consequently, $\alpha_1 + \ldots + \alpha_n + 1 \leq \|\mathtt{sf}(\varphi)\|$. For all i in $\{1, \ldots, n\}$ and for all j in $\{1, \ldots, \alpha_i\}$, let $v_{i,j}$ in $C(u_i)$ be such that $\mathcal{F}_s, V_s, v_{i,j} \not\models \psi_{i,j}$. For all i in $\{1, \ldots, n\}$, let τ_i be the cardinality of $\{s, u_i\} \cup \{v_{i,1}, \ldots, v_{i,\alpha_i}\}$ when s is in $C(u_i)$ and the cardinality of $\{u_i\} \cup \{v_{i,1}, \ldots, v_{i,\alpha_i}\}$ otherwise. Obviously, for all i in $\{1, \ldots, n\}$, $\tau_i \leq \alpha_i + 2$. Let σ be $(\tau_1, \ldots, \tau_n)$ when s is in $C(u_1)$ and $(1, \tau_1, \ldots, \tau_n)$ otherwise. Obviously, $\|\sigma\| \leq \tau_1 + \ldots + \tau_n + 1$. Since for all i in $\{1, \ldots, n\}$, $\tau_i \leq \alpha_i + 2$, $\|\sigma\| \leq \alpha_1 + \ldots + \alpha_n + 2.n + 1$. Since $n \leq \|\mathtt{sf}(\varphi)\|$ and $\alpha_1 + \ldots + \alpha_n + 1 \leq \|\mathtt{sf}(\varphi)\|$, $\|\sigma\| \leq 3.\|\mathtt{sf}(\varphi)\|$. Moreover, by construction of σ, $\mathcal{F}$ obviously satisfies the sentence $\exists x A_\sigma(x)$. In the end, let us notice that $\mathcal{F}_\sigma$ is isomorphic to $\mathcal{F}' = (W', R')$ where $W' = \{s, u_1, \ldots, u_n\} \cup \{v_{1,1}, \ldots, v_{1,\alpha_1}, \ldots, v_{n,1}, \ldots, v_{n,\alpha_n}\}$ and R' is the restriction of R to W'. More important is that, as the reader can prove it by induction on ψ, for all $\psi \in \mathtt{sf}(\varphi)$ and for all $w' \in W'$, $\mathcal{F}', V', w' \models \psi$ iff $\mathcal{F}, V, w \models \psi$ where V' is the restriction of V to W'. Since $\mathcal{F}, V, s \not\models \varphi$, $\mathcal{F}', V', s \not\models \varphi$. Hence, $\mathcal{F}' \not\models \varphi$. Since $\mathcal{F}_\sigma$ is isomorphic to $\mathcal{F}'$, $\mathcal{F}_\sigma \not\models \varphi$.

For all modal formulas φ, let A_φ be the first-order formula $\neg \exists x \bigvee \{A_\sigma(x) : \sigma \in \Delta(\varphi)\}$. Notice that for all modal formulas φ, A_φ is a sentence. Given a modal formula φ, the reason for our interest in the sentence A_φ is the following result:

Lemma 17. *Let φ be a modal formula. For all frames $\mathcal{F}$ in $\mathcal{C}^\omega_{rtc}$, the following conditions are equivalent:*

- $\mathcal{F} \models \varphi$,
- $\mathcal{F} \models A_\varphi$.

Proof. Let $\mathcal{F} = (W, R)$ be a frame in $\mathcal{C}^\omega_{rtc}$.
($\Rightarrow$) Suppose $\mathcal{F} \models \varphi$ and $\mathcal{F} \not\models A_\varphi$. Hence, there exists an assignment g on $\mathcal{F}$ such that $\mathcal{F}, g \models \exists x \bigvee \{A_\sigma(x) : \sigma \in \Delta(\varphi)\}$. Thus, there exists a state s in $\mathcal{F}$ such that $\mathcal{F}, g^x_s \models \bigvee \{A_\sigma(x) : \sigma \in \Delta(\varphi)\}$. Consequently, there exists $\sigma \in \Delta(\varphi)$ such that $\mathcal{F}, g^x_s \models A_\sigma(x)$. Hence, $\mathcal{F}_\sigma \not\models \varphi$. Moreover, by Lemma 15, $\mathcal{F}_s \twoheadrightarrow \mathcal{F}_\sigma$. Since $\mathcal{F} \models \varphi$, by Proposition 1, $\mathcal{F}_s \models \varphi$. Since $\mathcal{F}_s \twoheadrightarrow \mathcal{F}_\sigma$, by Proposition 4, $\mathcal{F}_\sigma \models \varphi$: a contradiction.
($\Leftarrow$) Suppose $\mathcal{F} \models A_\varphi$ and $\mathcal{F} \not\models \varphi$. Thus, by Lemma 16, there exists a type τ such that $\mathcal{F}_\tau \not\models \varphi$, $\|\tau\| \leq \|3.\mathtt{sf}(\varphi)\|$ and $\mathcal{F} \models \exists x\, A_\tau(x)$. Consequently, τ is in $\Delta(\varphi)$. Let g be an assignment on $\mathcal{F}$. Since $\mathcal{F} \models \exists x\, A_\tau(x)$, $\mathcal{F}, g \models \exists x\, A_\tau(x)$. Hence, there exists $s \in W$ such that $\mathcal{F}, g^x_s \models A_\tau(x)$. Since τ is in $\Delta(\varphi)$, $\mathcal{F}, g^x_s \models \bigvee \{A_\sigma(x) : \sigma \in \Delta(\varphi)\}$. Thus, $\mathcal{F}, g \models \exists x \bigvee \{A_\sigma(x) : \sigma \in \Delta(\varphi)\}$. Consequently, $\mathcal{F}, g \not\models A_\varphi$. Hence, $\mathcal{F} \not\models A_\varphi$: a contradiction.

This ends the proof of Theorem 2.

5 Chagrova's Theorem about modal definability

In this section, we give a new proof of Chagrova's Theorem about modal definability and we give sketches of proofs of new variants of Chagrova's Theorem about modal definability.

5.1 A new proof of Chagrova's Theorem about modal definability

Firstly, we give a new proof of Chagrova's Theorem about modal definability. Our strategy will be as follows:

– remind the reduction of Kalmár [20] of the problem of deciding the validity in $\mathcal{C}_{all}$ of sentences from an arbitrary first-order language to the problem of deciding the validity in $\mathcal{C}_{all}$ of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$,
– prove that the problem of deciding the validity in $\mathcal{C}_{all}$ of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$ is reducible to the problem of deciding the modal definability with respect to $\mathcal{C}_{all}$.

Proposition 10. *The problem of deciding the validity in $\mathcal{C}_{all}$ of sentences from an arbitrary first-order language is reducible to the problem of deciding the validity in $\mathcal{C}_{all}$ of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$.*

Proof. See [20].

Proposition 11. *The problem of deciding the validity in $\mathcal{C}_{all}$ of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$ is reducible to the problem of deciding modal definability with respect to $\mathcal{C}_{all}$.*

Proof. Let C be a sentence from the first-order language $\mathcal{L}_{\mathbf{FOF}}$. Let D be the sentence $\exists y\ (\exists x\ y \neq x \wedge \neg(C)_x^{y\neq x})$. We demonstrate $\mathcal{C}_{all} \models C$ iff D is modally definable with respect to $\mathcal{C}_{all}$.
($\Rightarrow$) Suppose $\mathcal{C}_{all} \models C$. For the sake of the contradiction, suppose D is not modally definable with respect to $\mathcal{C}_{all}$. We have to consider 2 cases.
1st case: $\mathcal{C}_{all} \models \neg D$**.** Hence, D corresponds to the modal formula $\bot$ with respect to $\mathcal{C}_{all}$. Thus, D is modally definable with respect to $\mathcal{C}_{all}$: a contradiction.
2nd case: $\mathcal{C}_{all} \not\models \neg D$**.** Consequently, there exists a frame $\mathcal{F}$ such that $\mathcal{F} \not\models \neg D$. Hence, $\mathcal{F} \models D$. Let g be an assignment on $\mathcal{F}$. Since $\mathcal{F} \models D$, $\mathcal{F}, g \models D$. Thus, there exists a state s in $\mathcal{F}$ such that $\mathcal{F}, g_s^y \models \exists x\ y \neq x$ and $\mathcal{F}, g_s^y \not\models (C)_x^{y\neq x}$. Consequently, $\mathcal{F}$ possesses a relativized reduct $\mathcal{F}'$ with respect to $y \neq x$, x and g_s^y. Hence, by Proposition 5, $\mathcal{F}, g_s^y \models (C)_x^{y\neq x}$ iff $\mathcal{F}', g \models C$. Since $\mathcal{F}, g_s^y \not\models (C)_x^{y\neq x}$, $\mathcal{F}', g \not\models C$. Thus, $\mathcal{F}' \not\models C$. Consequently, $\mathcal{C}_{all} \not\models C$: a contradiction.
($\Leftarrow$) Suppose D is modally definable with respect to $\mathcal{C}_{all}$. Hence, there exists a modal formula φ such that for all frames $\mathcal{G}$, $\mathcal{G} \models D$ iff $\mathcal{G} \models \varphi$. For the sake of the contradiction, suppose $\mathcal{C}_{all} \not\models C$. Thus, there exists a frame $\mathcal{F}_0$ such that $\mathcal{F}_0 \not\models C$. Let g be an assignment on $\mathcal{F}_0$. Since $\mathcal{F}_0 \not\models C$, $\mathcal{F}_0, g \not\models C$. Let $\mathcal{F}{=}(W, R)$ be the frame defined by $W{=}\{s\}$ and $R{=}\emptyset$ where s is a new state. Let $\mathcal{F}'$ be the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}$. Obviously, $\mathcal{F}_0$ is the relativized reduct of $\mathcal{F}'$ with respect to $y \neq x$, x and g_s^y. Consequently, by Proposition 5, $\mathcal{F}', g_s^y \models (C)_x^{y\neq x}$ iff $\mathcal{F}_0, g \models C$. Since $\mathcal{F}_0, g \not\models C$, $\mathcal{F}', g_s^y \not\models (C)_x^{y\neq x}$. Since $\mathcal{F}$ consists of a single state, $\mathcal{F} \not\models D$. Since $\mathcal{F}'$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}$, $\mathcal{F}', g_s^y \models \exists x\ y \neq x$. Since $\mathcal{F}', g_s^y \not\models (C)_x^{y\neq x}$, $\mathcal{F}', g \models D$. Hence, $\mathcal{F}' \models D$. Since for all frames $\mathcal{G}$, $\mathcal{G} \models D$ iff $\mathcal{G} \models \varphi$, $\mathcal{F}' \models \varphi$. Since $\mathcal{F}'$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}$, by Proposition 3, $\mathcal{F} \models \varphi$. Since φ is a modal definition of D with respect to $\mathcal{C}_{all}$, $\mathcal{F} \models D$: a contradiction.

This tight relationship between the problem of deciding the validity in $\mathcal{C}_{all}$ of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$ and the problem of deciding modal definability with respect to $\mathcal{C}_{all}$ constitutes the key result of our method. Notice that there are 2 modal-related constraints in the proof of Proposition 11. The 1st constraint is that the modal language contains a formula like $\perp$ which is valid in no frame. We have used this constraint at the beginning of the ($\Rightarrow$) part of the proof. The 2nd constraint is that the modal language does not contain modalities like the universal modality and the difference modality which prevent from using the Disjoint unions Theorem. We have used this constraint at the end of the ($\Leftarrow$) part of the proof. Now, we infer the following result:

Corollary 1 (Chagrova's Theorem about modal definability). *The problem of deciding modal definability with respect to $\mathcal{C}_{all}$ is undecidable.*

Proof. By Propositions 10 and 11.

5.2 Proofs of new variants of Chagrova's Theorem about modal definability

Secondly, we give sketches of proofs of new variants of Chagrova's Theorem about modal definability. In the proof of Proposition 11, the unique occurrences of the subformulas $\exists x\ y \neq x$ and $\neg(C)_x^{y\neq x}$ in the sentence D associated to the given sentence C play specific roles. More precisely, in the ($\Rightarrow$) direction of the proof of Proposition 11, $\exists x\ y \neq x$ is used to show the existence of some relativized reduct $\mathcal{F}'$ of $\mathcal{F}$ whereas $\neg(C)_x^{y\neq x}$ is used to infer that C does not hold in $\mathcal{F}'$ by means of the Relativization Theorem between $\mathcal{F}$ and $\mathcal{F}'$. The truth is that in this direction of the proof of Proposition 11, the Relativization Theorem is used to infer some information about $\mathcal{F}'$, namely $\mathcal{F}', g \not\models C$, from some other information about $\mathcal{F}$, namely $\mathcal{F}, g_s^y \not\models (C)_x^{y\neq x}$. As for the ($\Leftarrow$) direction of the proof of Proposition 11, the Relativization Theorem is used to infer some information about $\mathcal{F}'$, namely $\mathcal{F}', g_s^y \not\models (C)_x^{y\neq x}$, from some other information about $\mathcal{F}_0$, namely $\mathcal{F}_0, g \not\models C$. This use of the Relativization Theorem is possible and leads to a contradiction with the assumption that D is modally definable with respect to $\mathcal{C}_{all}$ because $\mathcal{F}'$ has been constructed from $\mathcal{F}_0$ in such a way that

- $\mathcal{F}_0$ is the relativized reduct of $\mathcal{F}'$ with respect to appropriate syntactic and semantics elements,
- $\mathcal{F}'$ is the disjoint union of $\mathcal{F}_0$ and some other frame.

In [3], the above line of reasoning has been generalized to restricted classes of frames such as the class of all reflexive frames, the class of all symmetric frames, etc. The common property of these classes of frames is their *stability* where a class $\mathcal{C}$ of frames is *stable* if there exists a first-order formula A, there exists an individual variable x and there exists a sentence B such that

(a) for all frames $\mathcal{F}$ in $\mathcal{C}$, for all assignments g on $\mathcal{F}$ and for all frames $\mathcal{F}'$, if $\mathcal{F}'$ is the relativized reduct of $\mathcal{F}$ with respect to A, x and g then $\mathcal{F}'$ is in $\mathcal{C}$,
(b) for all frames $\mathcal{F}_0$ in $\mathcal{C}$, there exists frames $\mathcal{F}, \mathcal{F}'$ in $\mathcal{C}$ and there exists an assignment g on $\mathcal{F}$ such that $\mathcal{F}_0$ is the relativized reduct of $\mathcal{F}$ with respect to A, x and g, $\mathcal{F} \models B$, $\mathcal{F}' \not\models B$ and $\mathcal{F} \preceq \mathcal{F}'$.

In this case, (A, x, B) is a *witness of the stability of* $\mathcal{C}$. The following result proved in [3] states that if $\mathcal{C}$ is stable then the problem of deciding the modal definability of sentences with respect to $\mathcal{C}$ is at least as difficult as the problem of deciding the validity of sentences in $\mathcal{C}$.

Proposition 12. *If $\mathcal{C}$ is stable then the problem of deciding the validity of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$ in $\mathcal{C}$ is reducible to the problem of deciding the modal definability of sentences with respect to $\mathcal{C}$.*

As a result, if one wants to show that the problem of deciding the modal definability of sentences with respect to a class $\mathcal{C}$ of frames is undecidable, a possible strategy is the following:

- prove that the problem of deciding the validity of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$ in $\mathcal{C}$ is undecidable,
- find a first-order formula A, an individual variable x and a sentence B such that (A, x, B) is a witness of the stability of $\mathcal{C}$.

Obviously, if $\mathcal{C}$ is the class of all frames satisfying a universal elementary condition then $\mathcal{C}$ satisfies the condition (a) defining stability with respect to any first-order formula A, any individual variable x and any sentence B. It is quite easy to see why. Suppose $\mathcal{C}$ is the class of all frames satisfying a universal elementary condition. Let $\mathcal{F}$ be a frame in $\mathcal{C}$, g be an assignment on $\mathcal{F}$ and $\mathcal{F}'$ be a frame. Suppose $\mathcal{F}'$ is the relativized reduct of $\mathcal{F}$ with respect to A, x and g. This means that $\mathcal{F}'$ is the restriction of $\mathcal{F}$ to the set of all states s in $\mathcal{F}$ such that $\mathcal{F}, g_s^x \models A$. Since $\mathcal{C}$ is the class of all frames satisfying a universal elementary condition and $\mathcal{F}$ is in $\mathcal{C}$, $\mathcal{F}'$ is in $\mathcal{C}$. In other respect, if $\mathcal{C}$ is closed under taking disjoint unions, generated subframes and bounded morphic images then $\mathcal{C}$ satisfies the condition (b) defining stability with respect to the first-order formula $A := x_1 \neq x$, the individual variable x and the sentence $B := \exists y \exists z y \neq z$. It is quite easy to see why. Suppose $\mathcal{C}$ is closed under taking disjoint unions, generated subframes and bounded morphic images. Let $\mathcal{F}_0$ be a frame in $\mathcal{C}$. We have to consider 2 cases.
1st case: $\mathcal{F}_0$ is serial. Let $\mathcal{F}' = (W', R')$ be the frame defined by $W' = \{s'\}$ and $R' = \{(s', s')\}$ where s' is a new state. Since $\mathcal{F}_0$ is serial, obviously, $\mathcal{F}'$ is a bounded morphic image of $\mathcal{F}_0$. Since $\mathcal{C}$ is closed under taking bounded morphic images and $\mathcal{F}_0$ is in $\mathcal{C}$, $\mathcal{F}'$ is in $\mathcal{C}$. Let $\mathcal{F}$ be the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}'$. Since $\mathcal{C}$ is closed under taking disjoint unions, $\mathcal{F}_0$ is in $\mathcal{C}$ and $\mathcal{F}'$ is in $\mathcal{C}$, $\mathcal{F}$ is in $\mathcal{C}$. Since $\mathcal{F}'$ consists of a single state, $\mathcal{F}' \not\models B$. Since $\mathcal{F}$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}'$, $\mathcal{F} \models B$. Let g be an assignment on $\mathcal{F}$ such that $g(x_1) = s'$. Obviously, $\mathcal{F}_0$ is the relativized reduct of $\mathcal{F}$ with respect to A, x and g. Finally, since $\mathcal{F}$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}'$, $\mathcal{F} \preceq \mathcal{F}'$.
2nd case: $\mathcal{F}_0$ is not serial. Let $\mathcal{F}' = (W', R')$ be the frame defined by $W' = \{s'\}$ and $R' = \emptyset$ where s' is a new state. Since $\mathcal{F}_0$ is not serial, obviously, $\mathcal{F}'$ is isomorphic to a generated subframe of $\mathcal{F}_0$. Since $\mathcal{C}$ is closed under taking generated subframes and $\mathcal{F}_0$ is in $\mathcal{C}$, $\mathcal{F}'$ is in $\mathcal{C}$. Let $\mathcal{F}$ be the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}'$. Since $\mathcal{C}$ is closed under taking disjoint unions, $\mathcal{F}_0$ is in $\mathcal{C}$ and $\mathcal{F}'$ is in $\mathcal{C}$, $\mathcal{F}$ is in $\mathcal{C}$. Since $\mathcal{F}'$ consists of a single state, $\mathcal{F}' \not\models B$. Since $\mathcal{F}$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}'$, $\mathcal{F} \models B$. Let g be an assignment on $\mathcal{F}$ such that $g(x_1) = s'$. Obviously, $\mathcal{F}_0$ is the relativized reduct of $\mathcal{F}$ with respect to A, x and g. Finally, since $\mathcal{F}$ is the disjoint union of $\mathcal{F}_0$ and $\mathcal{F}'$, $\mathcal{F} \preceq \mathcal{F}'$.
The above remarks immediately show that $\mathcal{C}_{all}$ is stable. The truth is that

Proposition 13. *The following classes of frames are stable as well: $\mathcal{C}_E$, the class of all reflexive frames, the class of all transitive frames, the class of all reflexive transitive frames, the class of all strict partial orders, the class of all partial orders, the class of* *all lattices, the class of all symmetric frames and the class of all reflexive symmetric frames.*

Proof. See [1, 3] for details.

Gathering results from [13, 23–25, 29], one can prove that

Proposition 14. *The validity of sentences from the first-order language $\mathcal{L}_{\mathbf{FOF}}$ is undecidable in each of the following classes of frames: $\mathcal{C}_E$, the class of all reflexive frames, the class of all transitive frames, the class of all reflexive transitive frames, the class of all strict partial orders, the class of all partial orders, the class of all lattices, the class of all symmetric frames and the class of all reflexive symmetric frames.*

Proof. See [1, 3] for details.

As a corollary, one obtain the following variants of Chagrova's Theorem about modal definability.

Corollary 2 (Variants of Chagrova's Theorem about modal definability). *The problem of deciding modal definability with respect to the following classes of frames is undecidable: $\mathcal{C}_E$, the class of all reflexive frames, the class of all transitive frames, the class of all reflexive transitive frames, the class of all strict partial orders, the class of all partial orders, the class of all lattices, the class of all symmetric frames and the class of all reflexive symmetric frames.*

Proof. By Propositions 13 and 14.

6 Conclusion

The core of this paper has been Chagrova's Theorems about first-order definability of given modal formulas and modal definability of given elementary conditions. We have analyzed Chagrova's Theorems and we have tried to understand why their proofs cannot be easily repeated for proving the undecidability of first-order definability and modal definability with respect to restricted classes of frames. We have considered classes of frames for which modal definability is decidable, for instance $\mathcal{C}_{par}$, $\mathcal{C}_{tE}$ and $\mathcal{C}_{stE}$. We have considered classes of frames for which first-order definability is trivial, for instance $\mathcal{C}_{par}$, $\mathcal{C}_{tE}$ and $\mathcal{C}_{stE}$, but also $\mathcal{C}^{\omega}_{rtc}$. Using standard methods in model theory such as relativization of first-order formulas and reduct of frames, we have given a new proof of Chagrova's Theorem about modal definability and we have given sketches of proofs of new variants of Chagrova's Theorem about modal definability. Much remains to be done.

An obvious question is whether there exists other classes of frames for which modal definability is decidable. Is modal definability with respect to $\mathcal{C}_{rtc}$ decidable ? What about first-order definability with respect to $\mathcal{C}_{rtc}$? Another question is whether there exists other classes of frames for which first-order definability is trivial. It is also of interest to consider restrictions or extensions of the ordinary language of modal logic.

For example, one can consider the implication restriction of $\mathcal{L}_{\mathbf{MF}}$ based on the connectives $\rightarrow$ and $\Box$ or the tense extension of $\mathcal{L}_{\mathbf{MF}}$ based on the Boolean connectives and the modal connectives $\Box$ and $\Box^{-1}$. For such restrictions or extensions of $\mathcal{L}_{\mathbf{MF}}$, what is the computability of first-order definability and modal definability ? And in the end, there is the question whether there exists classes of frames for which modal definability is decidable and first-order definability is undecidable.

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