# The Slotted Online One-Sided Crossing Minimization Problem on 2-Regular Graphs 

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#### Abstract

In the area of graph drawing, the One-Sided Crossing Minimization Problem (OSCM) is defined on a bipartite graph with both vertex sets aligned parallel to each other and all edges being drawn as straight lines. The task is to find a permutation of one of the node sets such that the total number of all edge-edge intersections, called crossings, is minimized. Usually, the degree of the nodes of one set is limited by some constant $k$, with the problem then abbreviated to OSCM- $k$.

In this work, we study an online variant of this problem, in which one of the node sets is already given. The other node set and the incident edges are revealed iteratively and each node has to be inserted into placeholders, which we call slots. The goal is again to minimize the number of crossings in the final graph. Minimizing crossings in an online way is related to the more empirical field of dynamic graph drawing. Note the slotted OSCM problem makes instances harder to solve for an online algorithm but in the offline case it is equivalent to the version without slots.

We show that the online slotted OSCM- $k$ is not competitive for any $k \geq 2$ and subsequently limit the graph class to that of 2-regular graphs, for which we show a lower bound of $4 / 3$ and an upper bound of 5 on the competitive ratio.


## 1 Introduction

Online algorithms were introduced by Sleator and Tarjan [16] to solve problems for which the instance is piecewise revealed to an algorithm, which must make some irrevocable decision before the next element of the instance is presented. Online algorithms are classically analyzed using competitive analysis, where the performance of an online algorithm is compared to that of an optimal offline algorithm working on the same instance. The worst case ratio between any online algorithm and the optimal offline solution is the competitive ratio of a problem. For a deeper introduction to online algorithms and competitive analysis we refer the reader to the reference books [2, 8].

In graph drawing problems, given a graph, one usually wants to embed the graph into some space with limited dimensions. The most common and practical examples are on the

Euclidean plane. It is also usual to try to embed such graphs in a way that minimizes the number of edges that cross each other, i.e., their depictions overlap in a point that is not occupied by a vertex. If a graph can be embedded in the Euclidean plane without any crossings, we say the graph is planar. A survey on graph drawing and crossing minimization can be found in [1, 14].

One common way to depict bipartite graphs is by arranging the vertices in each partition on a straight (horizontal) line, making the lines for the two partition sides parallel. In this scenario, the edges are drawn vertically from one side of the partition to the other as straight segments. The problem of minimizing the crossings in this scenario is reduced, thus, to properly ordering the vertices in each partition. However, in some practical applications it is enough to restrict ourselves to ordering one set of the partition (the free side), while the other set remains fixed (the fixed side). It is also usual to restrict the degree of the vertices in the free side [10, 9] This (one sided) problem is formally defined as follows.

Definition 1. Given a bipartite Graph $G=(S \cup \dot{V}, E)$. Let the nodes of $S$ and $V$ be aligned in some ordering on straight lines parallel to each other, where $S$ is on the top line and $V$ on the bottom line. Let the edges $E$ be drawn as straight lines only. Let the degree of the nodes of $S$ be bound by some $k \in \mathbf{N}$. The One-Sided Crossing Minimization Problem (OSCM-k) is defined as the problem of finding a total ordering of the nodes of $S$ such that the number of resulting edge crossings in the graph is minimized.

We will assume that the ordering of $V$ is part of the instance and fixed, such that we can label and reference the nodes of $V$ with ascending natural numbers, starting from the "left". If $|S|=|V|$, we sometimes speak of nodes "above" and "below" one another, by assuming that the nodes on both lines are drawn equidistantly.

### 1.1 Related Work

The OSCM problem has already been extensively studied in the past under different names, such as bipartite crossing number [6, 14], crossing problem [4], fixed-layer bipartite crossing minimization [9] and others. Eades and Wormald [4] showed that the OSCM problem is NP-complete for dense graphs, while Muñoz et al. [10] showed NP-completeness for sparse graphs. Muñoz et al. also introduced the OSCM- $k$ and showed that the OSCM-2 can be solved optimally using the barycenter heuristic.

Li and Stallmann [9] showed that the approximation ratio of the barycenter heuristic is in $\Omega(\sqrt{n})$ on general bipartite graphs and also proved that OSCM- $k$ admits a tight $k-1$ approximation. Nagamochi presented a randomized approximation algorithm for general graphs [11] and another approximation algorithm for bipartite graphs of large degree [12].

Further researching the complexity, Dujmović and Whitesides 3 first showed that OSCM is fixed parameter tractable, i.e., it can be solved in $f(k) n^{\mathcal{O}(1)}$, where the parameter $k$ is the number of crossings. The currently best known FPT running time is $\mathcal{O}\left(3^{\sqrt{2 k}+n}\right)$ and was given by Kobayashi and Tamaki [7].

To the best of our knowledge, the field of online analysis on crossing minimization is hardly researched. A closely related problem arises in the field of graph drawing, called
dynamic graph drawing. Here, the task is to visually arrange a graph that is iteratively expanded over time. The visualization follows certain empirical criteria to make the data comprehensible, where crossing minimization is one of these criteria. For a survey regarding dynamic graph drawing, see [15]. Dynamic graph drawing has many applications, for instance, Frishman and Tal [5] present an algorithm to compute online layouts for a sequence of graphs and its application in discussion thread visualization and social network visualization. In another example, North and Woodhull [13] focus on hierarchical graph drawing, a more restricted graph class that needs to be visualized in a tree-like fashion, which overlaps with our topic regarding applications. While one of the most mentioned applications of the offline OSCM is wire crossing minimization in VLSI this is arguably less applicable when looking at an online version of the problem. However, the results of an online analysis can be helpful for the application fields of graph drawing, e.g., software visualization, decision support systems and interactive graph editors.

While dynamic graph drawing and online graph problems are similar in that parts of the graph are revealed in an iterative fashion and not previously known, a central difference is that in dynamic graph drawing the manipulation of previous decisions is usually allowed. This is not the case in the classical online model. Thus, while theory and practice are looking at similar problems, and are following the same goal of aesthetic graph drawings, the methods to achieve this goal are different.

### 1.2 Our Contribution

In this paper, we look at the online version of the OSCM- $k$ problem. Observe, that the online version of OSCM- $k$ can be defined in two different ways. The first version is the online free OSCM- $k$, where given a bipartite graph $(S \cup \dot{V}, E)$, an algorithm initially sees a fixed set of vertices $V$, and then, in each step a request appears for a subset of vertices $R_{i} \subseteq V$, which must be made adjacent to a vertex in $S$. Thus, after the arrival of the request $R_{i}$, one has to place a vertex $s_{i} \in S$ on the top line and adjacent to the vertices in $R_{i}$. In this version, one chooses the partial ordering of $s_{i}$ with respect to the other vertices already present in $S$.

The online free OSCM- $k$ problem is solvable with a competitive ratio of at most $k-1$, using the same barycenter algorithm as in the offline case 9].

In this paper, we focus on a different version of this problem, which we call the online slotted OSCM- $k$, which is formally defined as follows.

Definition 2. Given a vertex set $V$, a request sequence for online slotted OSCM- $k$ is a sequence $R_{1}, \ldots, R_{n}$ of subsets of $V$, each of size $k$. The set of vertices $S$ is initiated as $S=\left\{s_{1}, \ldots, s_{n}\right\}$. Initially there are no edges between $S$ and $V$. Once a request $R_{j} \subseteq V$ arrives, an online algorithm solving online slotted OSCM- $k$ chooses a vertex $s_{i}$ without any edges, and places an edge between $s_{i}$ and every vertex in $R_{j}$. The goal is to minimize total number of crossings.

The slotted OSCM- $k$ is a model that follows the aesthetic paradigms of the area of dynamic graph drawing, where the so-called mental map and human readability is sustained.

The term mental map describes the goal to make current visualization of the graph recognizable in later iterations of the graph. Compared to the free OSCM- $k$ no upper or lower bound on the competitive ratio is known.

We call the vertices $s_{i} \in S$ slots moving forward. If a request $R_{j} \subseteq V$ is fulfilled by adding edges between every vertex in $R_{j}$ and slot $s_{i}$ we say that request $R_{j}$ is assigned to slot $s_{i}$. Moreover, we call a slot $s_{i}$ unfulfilled or free if no request has been satisfied using this slot, thus the slot has no edges yet. Correspondingly, a fulfilled slot $s_{i}$ is a slot in $S$ with edges to a subset $R_{j} \subseteq V$.

Online slotted OSCM- $k$ has the advantage of knowing in advance the number of requests. However, one has the distinct constraint that, once two consecutive slots are fulfilled, the algorithm will not be able to assign any request to a vertex between the fulfilled slots, as such a vertex does not exist.

We prove, that online slotted OSCM- $k$ is not competitive for any $k \geq 2$ in general graph classes. However, if we focus on 2-regular graphs, we prove that this problem has a constant competitive ratio. In particular, we prove a lower bound of $4 / 3$ in this case, and then present an algorithm with a competitive ratio of at most 5 as an upper bound.

## 2 Lower Bounds on General Graphs

We begin by looking at online slotted OSCM- $k$ on general graphs, and show that for every non-trivial value of $k$, i.e., $k \geq 2$, there is no algorithm with a constant competitive ratio.
Theorem 1. There is no online algorithm with a constant competitive ratio for online slotted OSCM- $k$, for any $k \geq 2$.


Figure 1: Theorem 1; An algorithm is presented the requests colored in blue first. Some slot has to be left open for which the request associated with the red edges is given.

Proof. Let us consider an algorithm $A$ solving online slotted OSCM- $k$. Given the initial sets of vertices $V=\left\{v_{1}, \ldots, v_{n}\right\}$ and slots $S=\left\{s_{1}, \ldots, s_{n}\right\}, A$ is presented the following request sequence: $\left\{v_{1}, v_{2}\right\},\left\{v_{2}, v_{3}\right\}, \ldots,\left\{v_{n-1}, v_{n}\right\}$. Assume without loss of generality that $A$ has assigned these requests to slots in $S$ without producing a single crossing. Since we have $n$ requests to fill $n$ slots with, and $A$ has only one unfulfilled slot $s_{i}$ for some $i \in\{1, \ldots, n\}$, the last request will be assigned to $s_{i}$. We assume, without loss of generality, that $i \leq\left\lceil\frac{n}{2}\right\rceil$. The adversary now presents the request $\left\{v_{n-1}, v_{n}\right\}$ as the last request of the input. This results in at least $2 \cdot 2 \cdot\left(\frac{n}{2}-1\right)$ crossings as opposed to the optimal solution, which only results in a single crossing as depicted in Figure 1. The competitive ratio is thus at least $\frac{2 \cdot 2 \cdot\left(\frac{n}{2}-1\right)}{1}=2 n-4$ and therefore not bounded by any fixed constant $c$.

If we closely look at the proof, we see that the proof relies on the adversary being able to freely choose the degrees of the vertices in $V$. If we would require the degree of the vertices in $V$ to be defined in advance, the same strategy would not work. Thus, it makes sense to look at graph classes where the degree of the vertices in the graph is fixed, in particular, regular graphs.

In what follows we focus, on online slotted OSCM-2 on 2-regular graphs, as this particular case is already hard to analyze, and we prove that the competitive ratio is within the range between $4 / 3$ and 5 .

We conjecture that for any higher degree, online slotted OSCM- $k$ on $k$-regular graphs would also have a constant competitive ratio, with the constant depending on $k$. One can observe, that a higher vertex degree means that even optimal solutions must have a lot of crossings. Thus, even when an online algorithm makes a sub-optimal choice, the number of crossings of the optimal solution that it is compared to should compensate for the mistakes.

## 3 Lower Bound for 2-Regular Graphs

We begin by proving a lower bound for the competitive ratio of online slotted OSCM-2 on 2-regular graphs.

It is important to note that an offline algorithm can find an optimal solution in a greedy fashion, as we will see in Lemma [1. In the following lower bound, we prove that online algorithms cannot find an optimal solution, greedily or otherwise. The difficulty is that a request cannot be assigned in between two consecutive fulfilled slots. Thus, an online algorithm has to fulfill a request by assigning it to a sub-optimal slot. An example of such a situation is depicted in Figure 2. We can use this fact to construct a lower bound for online slotted OSCM-2 on 2-regular graphs as follows.


Figure 2: In this graph, a new request $R_{i}=\left\{x_{2}, y_{1}\right\}$ appears. This request cannot be fulfilled optimally. An assignment between $s_{x}$ and $s_{y}$ would be optimal, but there is no free slot between them.

Theorem 2. Every deterministic online algorithm, solving the slotted OSCM-2 on 2-regular graphs, has a competitive ratio of at least $4 / 3-\varepsilon$.

Proof. For every node, every algorithm only has a finite amount of slots to insert it into. Given an empty graph of size $n>6$, the adversary will repeat its strategy on the set of the


Figure 3: Theorem 2, All behaviors of any algorithm, when presented with request $\left\{v_{3}, v_{4}\right\}$, and the resulting competitive ratio of each decision branch, when confronted with this adversarial requests.
five leftmost free nodes, filling up the graph from left to right, until 6 or fewer free nodes are left in the graph. Given five free slots, the adversary will repeat the strategy depicted in Figure 3 which we will now describe in detail. For ease of notation, we will denote the five left-most free slots as $s_{1}, \ldots, s_{5}$ and the five left-most edge-free vertices as $v_{1}, \ldots, v_{5}$.

The adversary starts by presenting the request pair $\left\{v_{3}, v_{4}\right\}$. We consider two possibilities, either an algorithm places this request at $s_{3}$ or smaller, or $s_{4}$ or higher.

Case 1 (Algorithm assigns $\left\{v_{3}, v_{4}\right\}$ to $s_{3}$ or smaller): The adversary presents the request $\left\{v_{1}, v_{2}\right\}$. We assume, that any reasonable algorithm places the second request to the left of the first one (on a smaller slot). If an algorithm would place the second request to the right it directly incurs 4 crossings instead of none.

We branch on three possibilities depending on the free slots after the first two placements. So, the free slots are either $\left\{s_{1}, s_{4}, s_{5}\right\},\left\{s_{2}, s_{4}, s_{5}\right\}$, or $\left\{s_{3}, s_{4}, s_{5}\right\}$.

The adversary presents the request $\left\{v_{2}, v_{4}\right\}$ and subsequently the request $\left\{v_{1}, v_{3}\right\}$.
If the free slots are $\left\{s_{3}, s_{4}, s_{5}\right\}$ any assignment of $\left\{v_{2}, v_{4}\right\}$ and $\left\{v_{1}, v_{3}\right\}$ results in at least 7 crossings.

If the free slots are $\left\{s_{2}, s_{4}, s_{5}\right\}$ any assignment of $\left\{v_{2}, v_{4}\right\}$ and $\left\{v_{1}, v_{3}\right\}$ results in at least 4 crossings.

If the free slots are $\left\{s_{1}, s_{4}, s_{5}\right\}$ any assignment of $\left\{v_{2}, v_{4}\right\}$ and $\left\{v_{1}, v_{3}\right\}$ results in at least 5 crossings.

However, an assignment of $\left\{v_{3}, v_{4}\right\}$ to $s_{4},\left\{v_{1}, v_{2}\right\}$ to $s_{1},\left\{v_{2}, v_{4}\right\}$ to $s_{3}$ and $\left\{v_{1}, v_{3}\right\}$ to $s_{2}$ results in only 3 crossings, where the considered algorithms all have at least four crossings. Thus, the competitive ratio of any algorithm assigning request $\left\{v_{3}, v_{4}\right\}$ to $s_{3}$ or lower is at least $4 / 3$-competitive on these four requests.

Case 2 (Algorithm assigns $\left\{v_{3}, v_{4}\right\}$ to $s_{4}$ or larger): The adversary presents the request $\left\{v_{4}, v_{5}\right\}$ followed by $\left\{v_{3}, v_{5}\right\}$ and the two identical requests $\left\{v_{1}, v_{2}\right\}$.

The assignment of the last two requests to slots $s_{1}$ and $s_{2}$ is optimal and generates one crossing.

The optimal assignment places $\left\{v_{3}, v_{4}\right\}$ to $s_{3},\left\{v_{4}, v_{5}\right\}$ to $s_{5}$ and $\left\{v_{3}, v_{5}\right\}$ to $s_{4}$ resulting
in two crossings, 3 in total with the last two requests.
However, the algorithms we consider cannot assign $\left\{v_{3}, v_{4}\right\}$ to $s_{3}$, they assign it to $s_{4}$ or higher, these algorithms incur at least 3 crossings for the first three requests, and 4 crossings in total for the five requests. Thus, they have a competitive ratio of at least $4 / 3$ on these five requests too, as expected.

Note that in Case 1, the adversary presents only four nodes in total, while in Case 2, five nodes are used. Independent of which case is used, the adversary can now use the five left-most free slots and edge-free vertices to repeat this tactic. Once $r \leq 6$ slots are left, the remaining slots are filled up as follows. The adversary presents the following $r$ requests: $\left\{v_{n-r}, v_{n-r+1}\right\}, \ldots,\left\{v_{n-1}, v_{n}\right\}$. One slot is still free after presenting these requests, and the last request is $\left\{v_{n-r}, v_{n}\right\}$. This results in $r-1 \leq 5$ additional, unavoidable crossings.

From the case distinction above and the argument to fill up the rest of the graph, one can easily verify that the competitive ratio of every algorithm tends to $4 / 3$ for growing $n$.

This lower bound proves that no online algorithm for online slotted OSCM-2 on 2-regular graphs can perform optimally on all instances. In the following, we introduce some notions that are used to prove an upper bound for the competitive ratio in the same setting.

## 4 Preliminaries and Notation

In order to prove upper bounds for online slotted OSCM-2 on 2-regular graphs, we need to first extract some structural properties of this problem. First, we introduce the notion of propagation arrows, which helps us to lower bound the total number of crossings of the remaining graph if we only have a partial request sequence. Then, we observe that finding an optimal placement, involves only looking at the placement of every pair of requests relative to each other.

The number of crossings of an optimal assignment for a request sequence is the number of unavoidable crossings of the request sequence. The difference between the number of crossings incurred by an algorithm $A$, and the number of unavoidable crossings is consequently the number of avoidable crossings of $A$ on that request sequence.

Consider a 2-regular instance for online slotted OSCM- $k$ with slots $S=\left\{s_{1}, \ldots, s_{n}\right\}$ and vertices $V=\left\{v_{1}, \ldots, v_{n}\right\}$, and a request sequence $R_{1}, \ldots, R_{n}$. Let us assume that at some point after the $k$-th request has been fulfilled by algorithm $A$, there are fulfilled slots, and the vertices in $V$ have degree 2,1 or 0 , depending on how many times these vertices have appeared in requests. Because we know that the final graph will be 2-regular, for those vertices in $V$ with degree less than two we are still expecting a request that contains the vertex, and for any unfulfilled slot, there will be a request which will be fulfilled using this slot.

Intuitively, we use propagation arrows to greedily match unfulfilled vertices to available slots in a way that minimizes the number of crossings. We can see this through an illustration in Figure 4. For instance, in an empty graph every vertex $v_{i}$ in $V$ will have two propagation arrows to the slot $s_{i}$, but once some slots are occupied we take the leftmost vertex with


Figure 4: Propagation arrows before the first instance and after part of the instance is fulfilled.
degree less than two and assign a propagation arrow to the left-most unfulfilled slot. We know that the instance is 2-regular, so for every pair of missing edges of vertices in $V$ there must be an empty slot. We can define the propagation arrows formally as follows.

First, we know that after $k$ requests for a 2-regular graph, there are $n-k$ unfulfilled requests, which corresponds to $2(n-k)$ missing edges. We will double count the missing edges with the following two lists.

The list of unfulfilled vertices $L_{V}$ of an instance after the $k$-th request, is an ordered list that contains every vertex $v_{i} \in V$ from smallest to largest at most twice. $L_{V}$ will contain no copies of a vertex $v_{i} \in V$ if it already has appeared twice in the request sequence $R_{1}, \ldots, R_{k}$, i.e., if $v_{i}$ has degree $2, L_{V}$ will contain $v_{i} \in V$ once if $v_{i}$ has appeared only once in $R_{1}, \ldots, R_{k}$, i.e., if $v_{i}$ has degree 1 in the partially fulfilled graph, finally, $L_{V}$ contains a vertex $v_{i}$ twice if $v_{i}$ does not appear in $R_{1}, \ldots, R_{k}$, and thus has degree 0 at that point.

We can, thus, analogously consider the list of unfulfilled slots $L_{S}$ as an ordered list that contains each unfulfilled slot twice, again from smallest to largest. From the previous observation it should be clear that $\left|L_{V}\right|=\left|L_{S}\right|$.

Definition 3. Consider a 2-regular instance for online slotted OSCM-k with slots $S=$ $\left\{s_{1}, \ldots, s_{n}\right\}$ and vertices $V=\left\{v_{1}, \ldots, v_{n}\right\}$, and a request sequence $R_{1}, \ldots, R_{n}$. Let $A$ be an algorithm that has fulfilled $k$ requests. Let us consider the corresponding $L_{V}$ and $L_{S}$ for this request. There is a propagation arrow from vertex $v$ to slot $s$ if both occupy the same place in the ordered lists $L_{V}$ and $L_{S}$, i.e., if $v$ is the $i$-th element of $L_{V}$ and $s$ is $i$-th element of $L_{S}$ for some $i \in[2(n-k)]$.

Observe that propagation arrows do not cross one another by construction. So, if we count the crossings of a partial graph including the crossings between graph edges and propagation arrows, we have a lower bound on the number of crossings that the graph will have after the request sequence is completely fulfilled.

In the following, we want to observe, that an instance is optimally solved if an only if, for every pair of requests, the relative order of their slot assignments is optimal, i.e., if the placement of these two requests is such that there are fewer crossings between them than otherwise. This basically means, that a crossing is unavoidable, if and only if, the relative order of the two requests involved in this crossing is optimal, regardless of any other placement of any other request within the graph. This provides us with a very powerful tool


Figure 5: Case distinction for step one of Lemma 1. Each case is depicted before and after the untangling. The request $s_{x}$ is drawn in red and $s_{y}$ in blue.
to analyze the performance of online algorithms solving online slotted OSCM-2 on 2-regular graphs.

In order to prove the aforementioned statement, we first need the following lemma.
Lemma 1. Given two requests $R_{x}=\left\{x_{1}, x_{2}\right\}$ and $R_{y}=\left\{y_{1}, y_{2}\right\}$ assigned to slots $s_{x}$ and $s_{y}$. Without loss of generality assume that $x_{1} \leq y_{1}$ and $x_{2} \leq y_{2}$. An assignment where $s_{x}<s_{y}$ generates fewer or equally many crossings in the final graph than an assignment where $s_{y}<s_{x}$ if every other assigned slot remains unchanged.

Proof. We separate this proof into two steps. In the first step, we show that the number of crossings between the two requests is always the same or smaller if $s_{x}<s_{y}$. This can be done with an exhaustive case distinction and is depicted in Figure 5,

Now, for the second step, we need to show that the number of crossings in an overall graph is still smaller or equal if $s_{x}<s_{y}$. We prove it by means of a contradiction. Let us consider a 2 -regular graph $G$ for which we have two such requests $\left\{x_{1}, x_{2}\right\}$ and $\left\{y_{1}, y_{2}\right\}$ and an assignment where $s_{y}<s_{x}$, with a total number of crossings $c_{G}$. And let us consider the graph $G^{\prime}$ which is the same as $G$ except that the placement of $\left\{x_{1}, x_{2}\right\}$ and $\left\{y_{1}, y_{2}\right\}$ is exchanged, making $s_{x}<s_{y}$, with a number of crossings $c_{G^{\prime}}$. Assume that $c_{G}<c_{G^{\prime}}$. We already know that this is not due to the number of crossings between edges to $s_{x}$ and $s_{y}$, as such a case would be covered by Figure 5. Without loss of generality assume thus that (one of) the extra crossing(s) in $G^{\prime}$ is between some edge ( $u, s_{u}$ ) and one of the modified edges $\left(x_{t}, s_{i}\right)$ with $t \in\{1,2\}$.

In order for this pair of edges to produce an extra crossing in $G^{\prime}$ compared to $G$ at all, we know that $u \notin\left[x_{1}, x_{2}\right]$, as otherwise this crossing is unavoidable and thus the same in $G$ and $G^{\prime}$. We make a case distinction over the remaining cases, which we depict in Figure 6 .

Assume now that $x_{2}<u$. Then, $s_{u}<s_{x}$ in order for the edges to cross at all. This positioning produces two crossings with $R_{x}$ in $G$ and possibly some crossings with $R_{y}$. However, since $R_{x}$ is only assigned further to the right in $G^{\prime}$, we get the exact same number of crossings between $\left(u, s_{u}\right)$ and the edges of $R_{x}$ and $R_{y}$ in $G^{\prime}$.

Assume finally that $u<x_{2}$. Then, $s_{x}<s_{u}$ in order for the edges to cross at all. This positioning produces two crossings with $R_{x}$ in $G$ and possibly some crossings with $R_{y}$. We do a case distinction whether $s_{u}<s_{y}$ or $s_{y}<s_{u}$.

If $s_{u}<s_{y}$, then $R_{y}$ and $R_{x}$ simply "change roles" in $G^{\prime}$ compared to $G$ and the number of crossings remains the same. If $s_{y}<s_{u}$, then $\left(u, s_{u}\right)$ crosses the edges of $R_{x}$ and of $R_{y}$


Figure 6: As shown in Lemma 1, there cannot be a crossing between $R_{x}$ and an edge ( $u, s_{u}$ ) that makes the ordering $s_{y}<s_{x}$ better than $s_{x}<s_{y}$.
completely in both $G$ and $G^{\prime}$.
Thus, by swapping the slot assignment in this way one cannot reduce the number of crossings in the overall graph.

Lemma 1 plainly states that for each pair of requests, the optimal ordering gives the left-most request a slot that is to the left of the slot assigned to the right-most request. The notion of left and right requests only means here, that if the requests are not for identical pairs of vertices, the left request contains the left-most distinguished vertex.

In order to find an upper bound on the competitive ratio, we only have to see that any pair of requests is either placed optimally or otherwise bound the number of crossings generated by that pair with the number of unavoidable crossings in the optimal solution.

## 5 Upper Bound for 2-Regular Graphs

With these structural properties we are ready to present the algorithm that will provide us with an upper bound of 5 for the competitive ratio.

Neglecting to take the state of the graph into account when making decisions regarding the insertion of requests seems to result in relatively bad upper bounds. As an example, we take the simple barycenter algorithm (Algorithm (1) proposed in [10], which optimally solves the offline OSCM-2. This algorithm, computes the average between the two requested

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Algorithm 1 Barycenter algorithm from [10], adjusted for the slotted case.
    free_slots \(=\{1, \ldots, n\}\);
    for \(\left\{x_{1}, x_{2}\right\}\) in input do
        \(s:=\left\lfloor\frac{x_{1}+x_{2}}{2}\right\rfloor ;\)
        while s.isUsed() do
            \(s:=\left\{t\left|\operatorname{argmin}_{\forall t \in S . \rightarrow t . i s U s e d}()(t-s)\right|\right\} / /\) take leftmost on tie
        Assign \(\left\{x_{1}, x_{2}\right\}\) to \(s\);
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Figure 7: The request sequence is $\left\{x_{n-1}, x_{n}\right\},\left\{x_{n-1}, x_{n}\right\},\left\{x_{n-3}, x_{n-2}\right\}$, $\left\{x_{n-3}, x_{n-2}\right\}, \ldots,\left\{x_{1}, x_{2}\right\},\left\{x_{1}, x_{2}\right\}$. The last node crosses all others, resulting in roughly $4 n$ crossings compared to $\frac{n}{2}$ crossings in the optimal case.
vertices and assigns it to this particular point. In the case of the slotted version of the problem, we have to adjust it to take the nearest free slot.

Algorithm 1 is no better than 8-competitive, as the following simple example illustrated in Figure 7 shows. If we request the sequence $\left\{x_{n-1}, x_{n}\right\},\left\{x_{n-1}, x_{n}\right\},\left\{x_{n-3}, x_{n-2}\right\}$, $\left\{x_{n-3}, x_{n-2}\right\}, \ldots,\left\{x_{1}, x_{2}\right\},\left\{x_{1}, x_{2}\right\}$, the two first requests are placed on slots $s_{n-1}, s_{n-2}$ and then consecutively, the following requests occupy slots to the left of those until the last request, which is assigned the only available slot $s_{n}$. The last pair of edges crosses all others, resulting in roughly $4 n$ crossings compared to $\frac{n}{2}$ crossings in the optimal case.

In order to achieve a good upper bound for the OSCM-2, we present Algorithm 2 that given a request, selects the slot which minimizes the total number of crossings - including crossings between edges and propagation arrows - among all available slots.

Note that analyzing an algorithm in this setting is not completely trivial. Our approach is to show that the types of crossings between two requests produced by our algorithm are good-natured. Specifically, we look at pairs of requests for which the crossings can be completely avoided if they are appropriately ordered, i.e., 3-0 or 4-0 crossings as depicted in Figure 5 (c) and (e) respectively. This type of crossing, then, is either not produced by Algorithm 2 or we can show that a number of unavoidable crossings is necessary to produce this configuration. With this, we can then upper bound the competitive ratio. Note that this relatively rough estimate is most likely an overestimation of the actual competitive ratio of the algorithm, but even such an estimate already requires a lot of structural analysis.

First, we present some lemmata outlining some relevant structural properties of assignments made by Algorithm 2, then we consider each type of critical crossing, 4-0 crossings and then 3-0 crossings and show that the competitive ratio is still bounded when these types

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Algorithm 2 Chooses in each step the insertion with the lowest number of additional edge-
edge and edge-propagation arrow crossings.
    free_slots \(=\{1, \ldots, n\} ;\)
    for element in input do
        least_crossings \(:=\infty\);
        best_slot :=0;
        for slot in free_slots do
            G.simulate_node_insertion(slot, element);
            new_crossings \(=G . e d g e_{-} e d g e_{-} c r o s s i n g s()+G . e d g e \_p r o p \_c r o s s i n g s() ;\)
            if new_crossings < least_crossings then
                    least_crossings \(=\) new_crossings;
                    best_slot \(=\) slot;
            G.revert_simulated_insertion(slot, element);
        G.insert_node(best_slot, element);
        free_slots \(:=\) free_slots \(\backslash\) best_slot;
```

of crossings appear.

### 5.1 Structural Properties

To start the analysis of Algorithm 2 we first make a few observations on the changes of the propagation arrows after a request is fulfilled.

Consider a request $\left\{x_{1}, x_{2}\right\}$, which is assigned to slot $s_{x}$ by some algorithm. Before this request arrived, there were two propagation arrows from vertices $y_{1}$ and $y_{2}$ going to slot $s_{x}$ (note that it is possible that $y_{1}=y_{2}$ ). After the request is assigned to $s_{x}$ the propagation arrows pointing to $s_{x}$ have to be shifted, as slot is not available anymore. Simultaneously, one propagation arrow of each $x_{1}$ and $x_{2}$ disappears as the request is fulfilled. The rest of the propagation arrows have to reflect this movement out of $s_{x}$ and into the two empty positions left by $x_{1}$ and $x_{2}$, and they do so in the following way.


Figure 8: Schematic diagram showing how propagation arrows shift after a placement.

Observation 1. Let $R=\left\{x_{1}, x_{2}\right\}$ be a request assigned to slot $s_{x}$. And let $y_{1} \leq y_{2}$ be the vertices (or vertex) whose propagation arrows point to $s_{x}$ before this request arrived. Only


Figure 9: Types of crossings avoided by Algorithm 2, mentioned in Lemma 2. The propagation arrows are drawn in blue and the edges already present in the graph are drawn in black.
propagation arrows connected to nodes between the leftmost vertex of $x_{1}$ and $y_{1}$ and the rightmost vertex of $x_{2}$ and $y_{2}$ will be shifted.

Observe that there are no propagation arrows connected to nodes between $y_{1}$ and $y_{2}$ as otherwise these would be connected to slots other than $s_{x}$ and produce crossings between propagation arrows, which is impossible by definition. The observation can be seen through Figure 8 .

Proof. Let $t$ be the amount of propagation arrows attached to nodes in the interval between the leftmost vertex of $x_{1}$ and $y_{1}$ and the rightmost vertex of $x_{2}$ and $y_{2}$ before $R$ is assigned to $s_{x}$. After the placement, the number of propagation arrows in $x_{1}$ and $x_{2}$ is reduced by 1. The number of slots that require two arrows has been reduced by 1 in $s_{x}$. If $t=2$, then $x_{1}=y_{1}, x_{2}=y_{2}$ and the interval has no remaining propagation arrows, after $R$ is placed. Otherwise, each available slot has to be matched to each available vertex and no additional propagation arrows from outside the interval are required because the placement of $R$ removes two propagation arrows and one slot.

While Observation 1 is not specific to Algorithm 2, we can use it in the proofs to come. We continue with a lemma that allows us to shorten a lot of case distinctions in the following proofs.

Lemma 2. There is no instance during which two propagation arrows connected to a slot $s_{2}$ cross both edges adjacent to a fulfilled slot $s_{1}$ when using Algorithm 园.

Alternatively, the situations depicted in Figure 9 will never occur if one uses Algorithm 2,
Proof. We prove the lemma by contradiction and assume that after Algorithm 2 fulfills a request $R_{x}=\left\{x_{1}, x_{2}\right\}$ there are two propagation arrows crossing edges ( $v_{3}, s_{2}$ ) and ( $v_{4}, s_{2}$ ). Figure 9 shows three different situations how these crossings can occur: (a) Either both propagation arrows are connected to a single node $v_{2}<v_{3}$ that cross the edges of $s_{1}$, (b) there is a propagation arrow from two nodes $v_{1}<v_{2}$ crossing the nodes of $s_{1}$ to a slot $s_{2}$ with $s_{1}<s_{2}$ and another edge from $v_{2}$ to a slot $s_{3}$ with $s_{1}<s_{2}<s_{3}$ or (c) there are the


Figure 10: Possible configuration before the request $R_{x}$ is added and the propagation arrows are shifted to $s_{2}$. The propagation arrows are drawn in blue and the edges already present in the graph are drawn in black.
edges of (b) without the additional edge $\left(v_{2}, s_{3}\right)$. Cases (a) and (b) are very similar, but in case (a) both propagation arrows from $v_{2}$ go to the same slot, whereas in case (b) they are split between $s_{2}$ and $s_{3}$. In case (c) the vertex $v_{2}$ is already assigned an edge, thus it only has one remaining propagation arrow. We ignore this already present edge, as its precise nature makes no difference for the following argumentation.

We assume that the propagation arrows from $v_{2}$ (and possibly $v_{1}$ ) are the first ones that cross the edges of $s_{1}$ as described in the lemma after request $R_{x}$ has been fulfilled. It is possible that there are vertices between $v_{1}$ and $v_{2}$ or between $v_{2}$ and $v_{3}$. However, if these vertices exist, they cannot have propagation arrows. Otherwise, $v_{2}$ (and possibly $v_{1}$ ) would not be responsible for the first two propagation arrows that cross $s_{1}$, but the propagation arrows of these other nodes. We look at the first request $R_{x}$ whose assignment results in such a structure and how the graph looked like before serving $R_{x}$.

Note that every slot has two propagation arrows pointing to it and after assigning a request to this slot, the propagation arrows pointing to that slot move to a neighboring free slot. Thus, there are four different configurations possible before the request $R_{x}$ is fulfilled, presented in Figure 10: (a) Both arrows are connected to a single node $v_{2}<v_{3}$ and a slot $s_{0}<s_{1}$, (b) the two arrows are from different nodes $v_{2}$ and $v_{1}$ and are connected to a slot $s_{0}<s_{1}$, (c) Both arrows are connected to $v_{2}$, one of them pointing to $s_{0}$ and one to $s_{2}$ with $s_{0}<s_{1}<s_{2}$ (d) the two arrows are from different nodes $v_{1}$ and $v_{2}$, one of them points to $s_{0}$ and one to $s_{2}$ with $s_{0}<s_{1}<s_{2}$.

We know also by using Observation 1, that the assignment of $R_{x}$ will only shift the propagation arrows around $s_{1}$ if these arrows are part of the affected interval between the vertices of $R_{x}$ and the propagation arrows pointing to the slot assigned to $R_{x}$.

Cases (a) and (b) have no previous arrows crossing with edges of $s_{1}$, thus, they require that two propagation arrows are shifted to the right hand side. As we saw in Observation (1, this can only happen if $R_{x}$ is assigned to $s_{0}$ or to the left hand side of it and the vertices $x_{1}$ and $x_{2}$ are both to the right of $v_{2}$. The propagation arrows of $x_{1}$ and $x_{2}$ point previously to a slot to the right of $s_{0}$, thus, assigning $R_{x}$ to $s_{0}$ will result in the two propagation arrows previously pointing to $s_{0}$ shifting to the right to fill up the slots left by the missing propagation arrows of $x_{1}$ and $x_{2}$. These gaps are filled from the left hand side, which results


Figure 11: Comparing the crossings of assigning $R_{x}$ to the left or right hand side of $s_{1}$ for the cases (a) and (b) from Figure 10. The propagation arrows are blue, already present edges are black and the newly introduced edges, adjacent to the recently fulfilled request $R_{x}$, are red.
in the two crossing propagation arrows shown in Figure 9 ,
Assume that Algorithm [2, given case 10(a) or (b) and a request $R_{x}$ with $v_{2}<x_{1}<x_{2}$, assigns $R_{x}$ to a free slot to the left hand side of $s_{1}$. Figure 11 shows that assigning $R_{x}$ more to the right results in fewer crossings, which is a contradiction to the procedure of the algorithm itself. If the vertices $x_{1}$ and $x_{2}$ do not coincide with $v_{3}$ and $v_{4}$, they are even more to the right hand side. If this is the case, we get even more crossings if $R_{x}$ is assigned to the left hand side of $s_{1}$.

For the cases (c) and (d) from Figure 10 only one propagation arrow needs to be pushed to the right hand side. Thus, w.l.o.g. only $x_{2}$ has to be to the right hand side of $v_{2}$ and the position of $x_{1}$ is arbitrary. Either $x_{1}$ is a vertex to the left hand side of $v_{2}$ (it is even possible that $x_{1}=v_{1}$ ) or it is to the right. The latter case is equivalent to the cases (a) and (b) in the sense that two more propagation arrows will cross over $s_{1}$ and a placement to the right of $s_{1}$ will result in less crossings as we saw in Figure 11 ,

In the first case, on the other hand, $x_{1}$ is to the left of $v_{2}$, and we push only one more propagation arrow to the right hand side of $s_{1}$. Figure 12 shows that choosing the position $s_{2}$ to the right of $s_{1}$ results in fewer crossings. Just as with cases (a) and (b) we can assume that $x_{2}$ is the leftmost possible vertex to the right of $v_{2}$ and otherwise the number of avoided crossings with the placement to the left of $s_{1}$ only grows.

Thus, Algorithm 2 will not place a request such that two edges of a slot are crossed by two propagation arrows.

Lemma 2 forbids specific configurations of the propagation arrows during the course of applying Algorithm 2 to a request sequence. The following lemma uses a counting argument to guarantee that a specific request between two (far apart) vertices must eventually appear in a specific setting. Such requests from vertices that are far apart, always guarantee the appearance of unavoidable crossings as depicted in Figure 5 (f). The appearance of such requests guarantees, in later proofs, the existence of such unavoidable crossings, which can be counted in a way that bounds the competitive ratio.

Lemma 3. Let there be two request $\left\{x_{1}, x_{2}\right\}$ and $\left\{y_{1}, y_{2}\right\}$ that are assigned to slots $s_{x}$ and $s_{y}$, with $x_{1}<x_{2}<y_{1}<y_{2}$ and no free slot between $s_{x}$ and $s_{y}$. If there are two neighboring


Figure 12: Comparing the crossings for assigning $R_{x}$ to the left or right hand side of $s_{1}$, for the cases (c) and (d) from Figure 10. The propagation arrows are blue, already present edges are black and the newly introduced edges, adjacent to $R_{x}$, are red.
vertices $u, v$, with $x_{2} \leq u<v \leq y_{1}$ and propagation arrows pointing to two different slots $s_{l}, s_{r}$, with $s_{l}<s_{x}<s_{y}<s_{r}$, and the request $\{u, v\}$ appears, then there must be a future request $\{a, b\}$, with $a \leq x_{2}$ and $y_{1} \leq b$, which unavoidably crosses all edges of $u$ and $v$.

Figure 13 depicts the situation described in the statement of Lemma 3,


Figure 13: Sketch of the situation described in the statement of Lemma 3.

Proof. Our proof is a simple counting argument. The request $\{u, v\}$ removes two propagation arrows. One points to the left of the filled block between $s_{x}$ and $s_{y}$ and the other one points to the right of it. The request, depending on its placement, pushes one propagation arrow from one side of the fulfilled block between $s_{x}$ and $s_{y}$ to the other one.
W.l.o.g. we assume that $\{u, v\}$ is placed on $s_{l}$. The second propagation arrow pointing to $s_{l}$ comes from $x_{2}$ (if $u \neq x_{2}$ ) or a vertex even more to the left. It is not possible that is comes from a vertex between $x_{2}$ and $v$ due to Lemma2. When the request $\{u, v\}$ is placed, it pushes this second propagation arrow to the slot $s_{r}$. This propagation arrow represents a mismatch between open slots and "open/remaining" edges. The number of "open/remaining" edges to the left of $u$ and to the right of $v$ is odd, but the slots always consume two of these "open/remaining" edges. This has to be compensated by some request $\{a, b\}$ that is placed right of $s_{y}$, where $a$ is to the left hand side of $u$ and $b$ is to the right hand side of $v$. This request crosses all edges of $u$ and $v$.

Where Lemmas 2 and 3 are applicable for specific configurations, the following lemma provides a tool that gives a set of edges or propagation arrows that are necessary to make a


Figure 14: Depending on the vertex $v_{i}$, the vertices are split into four sets, $A, B, C$ and $D$.
local configuration (e.g., a crossing of two requests) feasible in the context of the remaining graph.

Lemma 4. For every edge or propagation arrow, starting at a vertex $v_{i}$ of $V$ and pointing to a slot $s_{j}$ with $i<j$ (analogously $j<i$ ), there is one edge or propagation arrow pointing from $a$ vertex $v_{k}$ to a slot $s_{l}$ with $i<k$ and $l \leq i$ (analogously $k<i$ and $i \leq l$ ).

Proof. We use a simple handshake argument and count the already present edges and the propagation arrows in the graph to prove the statement.

At first, we separate the vertices into four sets, as depicted in Figure 14. The set $A$ contains the vertex $s_{l}$ and all vertices from $S$ that are to the left hand side of $s_{l}$. The set $B$ contains the vertices from $S$ that are to the right hand side of $s_{l}$. The set $C$ contains the vertex $v_{i}$ and all vertices from $V$ that are to the left hand side of $v_{i}$. The last set, called $D$, contains the vertices from $V$ that are to the right hand side of $v_{l}$.

The vertices in the set $A$ have two incident edges or two incident propagation arrows. These edges or propagation arrows start either at a vertex in $C$ or $D$. We denote the set of edges that connect a vertex from $A$ with a vertex from $C$ as $E_{A C}$. Analogously, we define the edge set $E_{A D}$. We also split the propagation arrows, starting at the vertices from $V$ and ending at a vertex in $A$, into two sets, $P_{A C}$ and $P_{A D}$. We can observe that

$$
\begin{equation*}
2|A|=E_{A C}+E_{A D}+P_{A C}+P_{A D} \tag{1}
\end{equation*}
$$

must always be true.
Additionally, the sum of the edges and propagation arrows starting at a vertex in $C$ must be $2 i$. Or more formal,

$$
\begin{equation*}
2 i=E_{A C}+E_{B C}+P_{A C}+P_{B C} \tag{2}
\end{equation*}
$$

The number of vertices in the set $A$ must be $i$, because we choose the vertex $v_{i}$ at position $i$ as a reference point to define the set $A$. Thus, we can combine Equation (11) and Equation (2) to obtain

$$
0=E_{A D}+P_{A D}-E_{B C}-P_{B C}
$$

Note, because propagation arrows never cross each other, either $P_{A D}$ or $P_{B C}$ is empty (it is also possible that both are empty). Thus, every edge or propagation arrow crossing from one side to the other is compensated by an edge, crossing into the other direction.

With our structural properties and observations regarding the propagation arrows we can now start to analyze the critical crossings depicted in Figure 5 (e) and (c). These crossings are critical in the sense that they have only avoidable crossings and no unavoidable ones. So, they decrease the performance of our algorithm and do not guarantee a constant competitive ratio like the other crossings depicted in Figure 5. In the following sections, we overcome this problem by showing that for each of these critical crossings there must exist some other request that unavoidably crosses one of the requests, involved in the critical crossing.

### 5.2 The 4-0 Crossings

Recall that, by Lemma 1, the optimal solution for a 2-regular instance of the online OSCM2 consists on minimizing crossings between every pair of requests. Thus, we can look at a pair of requests and exhaustively classify them as depicted in Figure 5, and analyze the competitive ratio of an algorithm depending on how many of these types of crossings appear. In particular, if no 3-0 crossings (Figure 5(c)) or 4-0 crossings (Figure5(e)) were produced by an algorithm, the algorithm would be 3-competitive at worst, as any sub-optimal placement would be trivially compensated by at least one unavoidable crossing. Thus, in order to analyze the competitive ratio of Algorithm 2, we only have to look at 3-0 and 4-0 crossings.

Using Lemma 2 we can now prove that Algorithm 2 will not make too many mistakes when producing $4-0$ crossings. First we prove that Algorithm 2 will never produce $4-0$ crossings with gaps, i.e., unfulfilled slots between the 2 slots generating the $4-0$ crossing as depicted for instance in Figure 15.


Figure 15: A 4-0 crossing with a slot in between. These types of crossings are forbidden by Lemma 5

Lemma 5. Algorithm圆 never generates 4-0 crossings with gaps in between. More precisely, for each pair $s_{i}, s_{j}$ with $i<j$ assigned by Algorithm 圆 that generate a 4-0 crossing, every $s_{k}$ with $i<k<j$ is already full.

Proof. Let us assume that there are no 4-0 crossings with gaps in the graph yet. We prove this lemma by means of a contradiction.

Let $\left\{v_{1}, v_{2}\right\}$ be the request assigned to slot $s_{i}$ by Algorithm 2, and a new request $R=$ $\left\{v_{3}, v_{4}\right\}$ is made where $v_{1}<v_{2}<v_{3}<v_{4}$ without loss of generality.

Let $s_{j}$ be a slot to the left of $s_{i}$ with the smallest crossing values for $R$ and let $s_{k}$ be the leftmost empty slot between $s_{j}$ and $s_{i}$.


Figure 16: Counting crossings in a $4-0$ crossing with a gap in Lemma 5 . Only edges like the one depicted in red on top will make a placement in $s_{k}$ more favorable than a placement in $s_{j}$. But for every red edge a green edge must exist or the graph is not free of $4-0$ crossings.

The only crossings that would make a placement in $s_{k}$ more unfavorable than a placement in $s_{j}$ are edges coming from the right of $v_{4}$ to a slot between $s_{j}$ and $s_{k}$ as depicted in the left of Figure 16. There cannot be any propagation arrows of this kind as we assume that all the slots between $s_{j}$ and $s_{k}$ are full.

For any edge coming from a vertex $v_{t_{1}}$ to the right of $v_{4}$ into slot $s_{t}$ with $j<t<k$ there must be another edge coming from a vertex $v_{t_{2}}$ to the left of (or directly from) the vertex $v_{2}$. Otherwise we would have a 4-0 crossing with an empty slot, namely $v_{1}<v_{2}<v_{t_{2}}<v_{t_{1}}$ and the slots $s_{t}<s_{k}<s_{i}$, which would be a contradiction to the assumption that this is the first occurrence, as we can see in Figure 16. Thus, this means that $v_{t_{2}} \leq v_{2}$. However, then the edge $v_{t_{2}}$ generates crossings only for the assignment of $R$ to $s_{j}$ and not for the assignment to $s_{k}$, which means that for every crossing counting for $s_{k}$ there is at least one crossing counting for $s_{j}$.

Finally we are only left to count the crossings for the propagation arrows going to $s_{j}$ with the placing in $s_{k}$ and vice-versa as depicted in the three drawings of Figure 17 ,


Figure 17: Lemma 5: At most one propagation arrow crosses from the left of $v_{2}$ to $s_{k}$ by Lemma 2,

Before we assign the request $\left\{v_{3}, v_{4}\right\}$, we know that by Lemma 2 only one propagation arrow can cross from the left of (or directly from) $v_{2}$ to the slot $s_{k}$. Thus, when assigning the request to the slot $s_{k}$ there are no extra crossings for the propagation arrows going to $s_{j}$. However, if we assign the request to slot $s_{j}$, the propagation arrows assigned to $s_{j}$ will now be transferred to $s_{k}$ as we saw in Observation 1, creating four new crossings between these propagation arrows and the new edges. This results in a contradiction, as we have just seen that the placement in $s_{j}$ generates more crossings than the placement in $s_{k}$ which contradicts our assumption that $s_{j}$ has the smallest crossing values.

We prove now that Algorithm 2 only generates 4-0 crossings when they are forced or in a very specific configuration. We will prove this in two different lemmas.

If we have a request for a pair of vertices, such that every available slot generates at least one $4-0$ crossing, we call it a forced $4-0$ crossing. Observe, that it is possible that more than one $4-0$ crossing is forced by the same request (See Figure 18).


Figure 18: More than one 4-0 crossing might be forced by the same request

Lemma 6. If Algorithm $0^{2}$ is used, for every forced $4-0$ crossing there is at least one uniquely identifiable and unavoidable crossing.

Proof. We will prove this using Lemma 3. If a request $\left\{v_{1}, v_{2}\right\}$ arrives in time step $t$ and every possible placement generates a 4-0 crossing, the propagation arrows of $v_{1}$ and $v_{2}$ have to point to two different slots before the request is served due to Lemma 2. We assume $v_{1}<v_{2}$ and call the slot on the left hand side $s_{\ell}$ and the other one $s_{r}$, as sketched in Figure 18,

We denote the set of edges that are crossed by the request $\left\{v_{1}, v_{2}\right\}$ when it is placed in $s_{\ell}$ with $L$ and analogously we define the set of crossed edges $R$ for the slot $s_{r}$. To be precise, the set $L$ contains the edges $\left(s_{j}, v_{i}\right)$ with $s_{\ell}<s_{j}<s_{r}$ and $v_{i}<v_{1}$ (See Figure 19). Our algorithm will always choose the slot which results in the least amount of crossings. Therefore, if our algorithm chooses (w.l.o.g.) the slot $s_{\ell}$, we know that positioning the request in slot $s_{r}$ results in at least the same number of crossings. Thus, we know that $|L| \leq|R|$ (or $|L| \leq|R|+1$ if the edge other connected to $v_{2}$ is placed between $s_{\ell}$ and $s_{r}$ but the edge connected to $v_{1}$ is not).


Figure 19: The set of edges $L$ is dashed, and the set of edges $R$ is dotted. The request $\left\{v_{1}, v_{2}\right\}$ will generate two crossings per dashed edge if positioned in $s_{\ell}$ and two crossings per dotted edge respectively if positioned in $s_{r}$. There is one unavoidable crossing for the edges going to vertices between $v_{1}$ and $v_{2}$ no matter the positioning.

Because we are in the situation of a forced 4-0 crossing there are at least two edges in each set $L$ and $R$ that belong to the same request. We look at the pair of edges $\left(s_{i}, v_{i_{1}}\right)$ and $\left(s_{i}, v_{i_{2}}\right)$ in $L$, with $v_{i_{1}}<v_{i_{2}}$ with the smallest possible $v_{i_{2}}$, respectively the pair of edges $\left(s_{j}, v_{j_{1}}\right)$ and $\left(s_{j}, v_{j_{2}}\right)$ in $R$, with $v_{j_{1}}<v_{j_{2}}$ and the largest possible $v_{j_{1}}$. Applying now Lemma 3, we know that there will be a future request between at least $v_{i_{2}}$ and $v_{j_{1}}$, meaning that a future housing request will cross at least one of the edges - in $L$ and $R$ respectively - of every pair generating a 4-0 crossing except at most one. Observe also, that by Lemma 2 there cannot be any available edge slot between $v_{i_{2}}$ and $v_{j_{1}}$ other than $v_{1}$ and $v_{2}$, this will mather further in the proof.

Note that every request, from which only one edge is in $L$ (or $R$ ), unavoidably crosses the request $\left\{v_{1}, v_{2}\right\}$ anyway. Thus, at least $\frac{|L|}{2}+\frac{|R|}{2}(+1)$ edges are unavoidably crossed after the request in Lemma 3. Here, the unavoidable crossing between the "housing request" and $v_{1}$ and $v_{2}$ compensate for the potentially missing crossings with the edges from $v_{i_{2}}$ and $v_{j_{1}}$. Or in other words, for every slot between $s_{\ell}$ and $s_{r}$ there is at least one edge which is crossed unavoidably, except for the aforementioned exceptions.

The request $\left\{v_{1}, v_{2}\right\}$ crosses all edges in $L$ twice, if it is placed in $s_{\ell}$. Thus, the number of avoidable crossings is $2|L|$ which is at most twice as large as the number of unavoidable crossings $\frac{|L|}{2}+\frac{|R|}{2}(+1) \geq|L|$.

The argument above works if there is only one forced $4-0$ crossing for a set of requests before the housing request from Lemma 3 appears. In the following, we discuss why we can assign, for each set of potentially overlapping forced 4-0 crossings, a uniquely identifiable unavoidable crossing. Overlapping 4-0 crossings appear, when both involved requests in a 4-0 crossing are again completely crossed by another third request (See Figure 20).

We call the request that generates the first forced 4-0 crossing after its placement $\left\{v_{1}, v_{2}\right\}$. The algorithm had the decision to place it in the left slot $s_{\ell}$, crossing $|L|$ edges or in the right slot $s_{r}$, crossing $|R|$ edges.

Without loss of generality assume that the request $\left\{v_{1}, v_{2}\right\}$ was placed in $s_{\ell}$. In order to have an overlapping 4-0 request, we assume that the request of the second forced 4-0 crossing $\left\{v_{3}, v_{4}\right\}$ is to the left of the request $\left\{v_{1}, v_{2}\right\}$. Because this request is also a forced $4-0$ crossing, it can be placed in a slot to the left $s_{\ell}^{\prime}<s_{\ell}$ or a slot to the right $s_{r}^{\prime} \geq s_{r}$. The slots must be


Figure 20: Two sets of 4-0 crossings overlap each other before a housing request appears. The first one is drawn with a dotted pair of edges and the second one is dashed. This particular scenario is not completely realistic for Algorithm 2 but could happen if the overall graph is larger.


Figure 21: If there is no available slot without a 4-0 crossings, the propagation arrows point to different sides, and a request $u_{1}, u_{2}$ must eventually exist.
more to the left (respectively, more to the right) because when the request $\left\{v_{1}, v_{2}\right\}$ arrives, all of the other vertices between $v_{i_{2}}$ and $v_{j_{1}}$ must be filled, as we already argued, thus $v_{3}$ and $v_{4}$ are to the left of $v_{i_{2}}$. Together with the assumption that the second 4-0 crossing is forced, we get the restricted position for $v_{3}$ and $v_{4}$.

Analogous to the previous case, let $L^{\prime}$ be the set of edges $(v, s)$ with $v<v_{3}$ and $s_{\ell}^{\prime}<s<s_{r}^{\prime}$ and let $R^{\prime}$ be the set of edges $(v, s)$ with $v>v_{4}$ and $s_{\ell}^{\prime}<s<s_{r}^{\prime}$ Note that, the edges of the first 4-0 request are now part of $R^{\prime}$. Like already explained above, if the algorithm decides to place the request in $s_{r}^{\prime}$, this implies that $\left|R^{\prime}\right| \leq\left|L^{\prime}\right|(+1)$ holds. Moreover, $\left|R^{\prime}\right| \geq \frac{|L|}{2}+|L|$ holds, because $v_{4}<v_{i_{2}}$, which by definition means that half of the edges of $R$ are to the right of $v_{4}$ and, thus, part of $R^{\prime}$. Thus, applying again Lemma 3, we know that there will be a future request that unavoidably crosses at least

$$
\frac{\left|L^{\prime}\right|}{2}(+1)+\frac{\left|R^{\prime}\right|}{2} \geq \frac{\left|R^{\prime}\right|}{2}+\frac{\left|R^{\prime}\right|}{2} \geq \frac{\left|R^{\prime}\right|}{2}+\frac{|R|}{4}+\frac{|L|}{2}
$$

edges. The number of avoidable crossings is $2|L|+2\left|R^{\prime}\right| \leq|L|+|R|+2\left|R^{\prime}\right|$, which if divided by 4 , for each possible $4-0$ crossing, means that

$$
\frac{|L|}{4}+\frac{|R|}{4}+\frac{\left|R^{\prime}\right|}{2} \leq \frac{\left|R^{\prime}\right|}{2}+\frac{|R|}{4}+\frac{|L|}{2} .
$$

Thus, we have for each of the avoidable $4-0$ crossings at least one uniquely identifiable unavoidable crossing. Observe that we can iterate this argument for every possible overlapping $4-0$ crossing. Moreover, if in the second case, the request $\left\{v_{3}, v_{4}\right\}$ was placed in $s_{\ell}^{\prime}$ the analogous counting argument still holds.

We just proved that forced 4-0 when using Algorithm 2, incur in one additional unavoidable crossing, this means that we can consider 4-0 crossings as if they were, in a sense 5-1 crossings instead, with a competitive ratio of 5 instead of being unbounded. However, this is not enough, there can be $4-0$ crossings produced by Algorithm 2 that are not forced. In the following lemma we prove that non-forced 4-0 crossings are only produced by Algorithm 2 in a very specific configuration. Then we will proceed to look at the number of uniquely identifiable unavoidable crossings of that configuration.

Lemma 7. Given a request for a pair of vertices in a graph, whose 4-0 crossings have either been forced (Figure 21) or were served because any alternative placement would result in two


Figure 22：If there is more than one slot positioned like the red ones（between the slots $s_{i}$ and $s_{k}$ with one vertex between $v_{3}$ and $v_{4}$ ，and one to the right of $v_{4}$ each），Algorithm 2 may choose slot $s_{j}$ generating a $4-0$ crossing．

3－1 crossings as sketched in Figure 2．2．If a slot is available which will not generate any 4－0 crossings this slot will be selected by Algorithm $⿴ 囗 ⿱ 一 𧰨 殳$ over any slot which will generate a 4－0 crossing，unless there are two additional requests resulting in two 3－1 crossings for the alternative placement as depicted in Figure 22．

Proof．Let us assume that we have a graph with the only 4－0 crossings appearing in the configurations of Figures 21］and 22．Let $\left\{v_{1}, v_{2}\right\}$ be a request assigned to slot $s_{i}$ ．Let $\left\{v_{3}, v_{4}\right\}$ be a new request with $v_{2}<v_{3}$ without loss of generality．The new request can be assigned to a slot $s_{k}$ right of $s_{i}$ without generating new $4-0$ crossings or to a slot $s_{j}$ to the left of $s_{i}$ as depicted in the first drawing of Figure［23．We can assume by Lemma 5 that $s_{j}$ is the rightmost available slot after $s_{i}$ ．

As we did in Lemma 5，we first count edge crossings and then count the propagation arrow crossings．

In order to do this，we divide the relevant slots into two subsets．The subset $X$ contains the slots between $s_{j}$ and $s_{i}$ ．Recall that all the slots in this area are filled．The subset $Y$ contains the slots between $s_{i}$ and $s_{k}$ ，all of them are filled too．We also divide the vertices into three subsets．Any vertex to the left of $v_{3}$ belongs to subset $A$ ．Vertices between $v_{3}$ and $v_{4}$ belong to subset $B$ and vertices to the right of $v_{4}$ belong to subset $C$ ．This division is depicted in Figure 23，

Only edges to slots in $X$ or $Y$ will generate crossings that count only for one of the two placements．In particular any edge from $C$ to $X$ or $Y$ will generate two additional crossings for the placement in $s_{k}$ with respect to the placement in $s_{j}$ ，those edges are depicted in red in the second drawing of Figure 23，On the other hand any edge from $A$ to a slot in $X$ or $Y$ generates two additional crossings for for the placement in $s_{j}$ with respect to the placement in $s_{k}$ ．Those edges are depicted in green in the second drawing of Figure 23．Finally，the edges from $B$ to $X$ or $Y$ are neutral with respect to both placements．This means that we only need to analyze previously placed requests in $X$ or $Y$ with one endpoint in $C$ ，as these are the only ones that will make a placement in $s_{j}$ more likely with respect to a placement in $s_{j}$ ．

We now analyze all possible requests in $X$ with at least one endpoint in $C$ ．Recall that we assume that there is always a slot that does not force a $4-0$ crossing．This means that there cannot be a pair of edges from $C$ connected to the same slot in $X$ or $Y$ ．On the other


Figure 23: We have two possible placements for the request $v_{3}, v_{4}$ red edges contribute extra crossings to the placement in $s_{k}$ and green edges contribute extra crossings to the placement in $s_{j}$. For the slots in $Y$ if a slot has both endpoints in $C$ it is a forced 4-0 crossing, but it can happen that one endpoint is in $B$.
hand if a pair of edges from $C$ and $B$ respectively go to a slot $X$ (we call this request $C X B$ ), we have a previous 4-0 crossing in the graph. This means that either the request placed in $s_{i}$ generated a 4-0 crossing, or the request $C X B$ did, we distinguish these two cases.

If the placement in $s_{i}$ generated the $4-0$ crossing we argue that there was a situation like in Figure 23. If the $4-0$ crossing was forced when $s_{i}$ was placed, this means that there was a request to the right of $v_{1}$, and it was fulfilled by a slot in $X$, but between this request and $C X B$ there were at least 3 propagation arrows, in particular, from $v_{1}, v_{2}$ and $v_{3}$, so this situation is forbidden by Lemma 2. If there was a situation Figure 23 involving the request $C X B$ and $v_{1}, v_{2}$, then there must be two requests between $v_{1}$ and $v_{2}$ are in region $A$ and are fulfilled in the region of $X$. These two requests will completely counteract the crossing contributions of the request $C X B$.

If the request $C X B$ generated the $4-0$ crossing, it also could not have been forced, as the propagation arrow from $v_{4}$ is between the two propagation arrows of the request and it also generates a situation forbidden by Lemma 2. This means that also in this case there must have been a situation like in Figure 22 involving the request $C X B$. In this case, depicted in Figure 24, the two requests contributing to the situation in Figure 22 will already be present.

Finally, if a request has one endpoint in $C$ and one in $A$, this means that their crossings for the placements in $s_{j}$ and $s_{k}$ compensate.

We thus only care for slots in $Y$ with at least one endpoint in $C$. If a slot in $Y$ with one


Figure 24: If we have a request with one endpoint in $C$ and one in $B$ placed in $X$, a 4-0 request was already present in the graph. This means that we had a situation like Figure 22 already with respect to that placement, and we either have a situation like Figure 22 with respect to $v_{3}$ and $v_{4}$ too (left picture) or we have a forced $4-0$ crossing (right picture), contradicting the assumption of Lemma 7 .
endpoint in $C$ has the other endpoint in $A$, the number of edge crossings will be higher for the placement in $s_{j}$ already. Moreover, there can not be a slot in $Y$ with two edges directed to $C$, as in the case for slots in $X$, this would contradict the assumption that we are not in the case of a forced $4-0$ crossing, as depicted with two red edges in the third drawing of Figure 23. We are only left with one case, if there is a fulfilled request in $Y$ with a vertex in $B$ and a vertex in $C$, as depicted in the fourth drawing of Figure 23. This type of request generates two extra crossings for the placement in $s_{k}$ with respect to the placement in $s_{j}$. This is still not a problem if there is only one such request, as these crossings would still be offset by the 4 extra crossings of the placement in $s_{j}$. Moreover, if there is more than one such request we are in the case of Figure 22, where a $4-0$ placement is allowed.

Finally, we are left to count propagation arrow crossings. As depicted in Figure 25, the


Figure 25: Only one propagation arrow might cross $s_{i}$ due to Lemma 2, and at the rightmost it comes from $v_{3}$.
rightmost placement of the propagation arrows has the arrow from $v_{3}$ pointing to $s_{i}$ and only the leftmost arrow from $C$ pointing to $s_{k}$. In the second and third pictures we see what happens to these arrows after a possible $s_{j}$ and $s_{k}$ placement. The number of crossings
due to the propagation arrows stays the same. If the propagation arrows would be more to the left, the number of crossings in the 4-0 placement would only possibly increase, and the number of crossings for the $s_{k}$ placement would only possibly decrease. This means that if there is at most one slot in $Y$ with an endpoint in $B$ and an endpoint in $C$, a placement in $s_{k}$ is prefered.

We now prove that the $4-0$ crossings described in Lemma 7 , also have uniquely identifiable unavoidable crossings, just as we did in Lemma 6 for the forced 4-0 crossings.

Lemma 8. Any 4-0 crossing incurred by Algorithm 园, because any alternative placement would result in two 3-1 crossings as sketched in Figure 22, has two uniquely identifiable unavoidable crossings.

Proof. Observe, that if we only consider the crossings generated by the placement of the request generating the $4-0$ crossings we do not risk double counting unavoidable crossings in this case. In a configuration like depicted in Figure 22, where Algorithm 2 generates a 4-0 crossing, an optimal algorithm can place the same requests as depicted in the right side of Figure 26. The placement of the new request with a $4-0$ crossing by Algorithm 2 has 6 crossings with previously placed requests (Figure 26 left) while the optimal placement for this request has only 2 crossings with previously placed requests (Figure 26 right). These crossings are unavoidable.


Figure 26: Algorithm 2has generated a 4-0 crossing. We also depict the optimal configuration for such a situation

We can finally conclude, using Lemmas 6 and 8, that any 4-0 crossings incurred by Algorithm 2 have at least one unavoidable crossing.

Theorem 3. Forced and non-forced 4-0 crossings incurred by Algorithm 圆 have at least one unavoidable crossing.

### 5.3 The 3-0 Crossings

It remains to prove that Algorithm 2 only generates a 3-0 crossing - depicted in Figure 5 (c) - if there is at least one unavoidable crossing for one of the two requests that are responsible for the 3-0 crossing. In general the proofs use case distinction in a similar way to the proofs from the previous section, handling the $4-0$ crossings.

Similarly as in the 4-0 case, we start by proving that Algorithm 2 never produces a 3-0 crossing with a gap.

Lemma 9. Algorithm never generates 3-0 crossings with gaps in between. More precisely, for each pair $s_{j}, s_{i}$ assigned by Algorithm 圆 with $j<i$ that generate a 3-0 crossing, every slot $s_{k}$ with $j<k<i$ is already full.

Proof. Let us assume that $v_{1}$ and $v_{2}$ is the pair of vertices adjacent to a filled slot $s_{i}$. Let $v_{2}$ and $v_{3}$ be a pair of vertices from a new request, with $v_{1}<v_{2}<v_{3}$. If Algorithm 2 creates a $3-0$ crossing between the requests $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{2}, v_{3}\right\}$, it places the second request in a slot $s_{j}$ with $s_{j}<s_{i}$.

We can assume that $s_{j}$ is the rightmost available slot to the left of $s_{i}$, due to the following observations. If there exists a slot $s_{i}^{\prime}$ between $s_{j}$ and $s_{i}$, we observe that the propagation arrow of $v_{2}$ cannot point to $s_{j}$, when the request $\left\{v_{2}, v_{3}\right\}$ arrives. Because then, the two propagation arrows pointing to $s_{i}^{\prime}$ have to start at vertices to the right of $v_{2}$ and cross the edges of the request $\left\{v_{1}, v_{2}\right\}$ which violates Lemma 2. This means that the propagation arrow of $v_{2}$ must point to $s_{i}^{\prime}$ or to a slot to the right of $s_{i}$. But in this case, when the request $\left\{v_{2}, v_{3}\right\}$ is placed in $s_{j}$ it pushes the propagation arrows that pointed to $s_{j}$, which must come from vertices to the left of $v_{2}$, to a slot to the right of $s_{j}$, in this case $s_{i}^{\prime}$, crossing the edges of the newly placed request and violating, again, Lemma 2. Thus, $s_{j}$ must be the rightmost available slot to the left of $s_{i}$.

In the following lemma we explore the situation that the 3-0 crossing happens at the edge of the graph, that is, a placement on any remaining slot causes a 3-0 crossing.

Lemma 10. Given two requests $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{2}, v_{3}\right\}$ with $v_{1}<v_{2}<v_{3}$. Assume without loss of generality that Algorithm圆 creates a 3-0 crossing between these requests, with the first request for vertices $\left\{v_{1}, v_{2}\right\}$ being placed in slot $s_{i}$ and during the placement of the second request there is no available slot $s_{k}>s_{i}$. Then there is at least one uniquely identifiable unavoidable crossing with the request $\left\{v_{2}, v_{3}\right\}$.

Proof. Let us assume that $v_{1}$ and $v_{2}$ is the pair of vertices adjacent to a filled slot $s_{i}$. Let $v_{2}$ and $v_{3}$ be a pair of vertices from a new request, with $v_{1}<v_{2}<v_{3}$. If Algorithm 2 creates a $3-0$ crossing between the requests $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{2}, v_{3}\right\}$, it places the second request in a slot $s_{j}$ with $s_{j}<s_{i}$. Recall that by Lemma 9, $s_{j}$ must be the rightmost available slot to the left of $s_{i}$.

If there is no free slot $s_{k}$ with $s_{k}>s_{i}$, then all slots to the right of $s_{j}$ are filled. Moreover the propagation arrows from $v_{2}$ and $v_{3}$ are the two right most propagation arrows and both point to $s_{j}$, when the request $\left\{v_{2}, v_{3}\right\}$ arrives. Let the number of fulfilled slots to the right of $s_{j}$ be $t$ and the number of vertices that are to the right of $v_{2}$ be $b$. If the number of fulfilled slots is larger than the number of vertices on the bottom line $(t>b)$, there are at least two edges from vertices that are to the left of $v_{2}$ pointing to fulfilled slots that are to the right of $s_{j}$. It is not possible that a fulfilled slot right of $s_{j}$ is adjacent to two vertices that are to the left of $v_{2}$ due to Lemma 2, because the propagation arrows of $v_{2}$ and $v_{3}$ would completely cross it. Thus, the second adjacent vertex must be to the right of $v_{2}$, resulting in at least one unavoidable crossing for the request $\left\{v_{2}, v_{3}\right\}$. If $t \leq b$, the slot $s_{j}$ is above the vertex $v_{2}$ or to the right of it. In this case, the edge $\left\{v_{1}, s_{i}\right\}$ must be compensated by an edge that starts to
the right of $v_{2}$ and points to a slot to the left of $s_{j}$. It is not possible that the second edge of this slot comes from a vertex to the right of $v_{2}$, too (see Lemma 4 or Lemma (5). Thus, when no slot $s_{k}$ exists, there must exist an edge that unavoidably crosses the request $\left\{v_{2}, v_{3}\right\}$.

What remains is an exhaustive case distinction analogous to the analysis done for the 4-0 crossings.

Theorem 4. If Algorithm creates a 3-0 crossing between two requests $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{2}, v_{3}\right\}$, there is at least one uniquely identifiable unavoidable crossing for at least one of the two requests.

Proof. Let us assume that $v_{1}$ and $v_{2}$ is the pair of vertices adjacent to a filled slot $s_{i}$. Let $v_{2}$ and $v_{3}$ be a pair of vertices from a new request, with $v_{1}<v_{2}<v_{3}$. If Algorithm 2 creates a 3-0 crossing between the requests $\left\{v_{1}, v_{2}\right\}$ and $\left\{v_{2}, v_{3}\right\}$, it places the second request in a slot $s_{j}$ with $s_{j}<s_{i}$. Recall that, by Lemma 9, $s_{j}$ must be the rightmost available slot to the left of $s_{i}$. Moreover, the case where there is no available slot $s_{i}<s_{k}$ is already covered by Lemma 10.

Thus, in the following we assume our algorithm can choose between the slot $s_{j}$, resulting in a 3-0 crossing, and the slot $s_{k}$ which does not generate a 3-0 crossing $\left(s_{j}<s_{i}<s_{k}\right)$. This situation is depicted in Figure 27 (a). We look at the cases where Algorithm 2 prefers a placement on $s_{j}$. In the following we investigate which edges must exits to make the slot $s_{j}$ more preferable.

To count the crossings we divide the relevant slots into two subsets. The subset $X$ contains the slots between $s_{j}$ and $s_{i}$ and the subset $Y$ contains the slots between $s_{i}$ and $s_{k}$. We do not need to consider the area to the left of $s_{j}$ or to the right of $s_{k}$, because the number of crossings with edges incident to a slot in one of these areas is independent of the position of the request $\left\{v_{2}, v_{3}\right\}$. We also divide the vertices on the bottom line into three different subsets. The vertices to the left of $v_{2}$ form the set $A$. The vertices between $v_{2}$ and $v_{3}$ are in the set $B$ and the vertex $v_{3}$ and all vertices to its right form the set $C$.

Figure 27 (b) shows which edges or propagation arrows cross the new request only if placed in the the slot $s_{j}$ (in green) and the ones that cross the new request only if placed in slot $s_{k}$ (in red). An edge that is incident to a vertex between $v_{2}$ and $v_{3}$ (area $B$ ) does not favor a particular slot, because it crosses the request $\left\{v_{2}, v_{3}\right\}$ once, independent of its placement.

There must be edges or propagation arrows in our graph such that, avoiding the 3-0 crossing results in at least three crossings in order to make a placement in $s_{j}$ favorable. Thus, requests like in Figure 27 (c) or (d) must be present in our graph in order to make the 3-0 crossing a feasible choice.

But, if a vertex between $v_{2}$ and $v_{3}$ exists (area $B$ is not empty), its edges unavoidably cross the request $\left\{v_{2}, v_{3}\right\}$ two times, fulfilling the statement of our lemma. So, in the following, we can assume that there is no vertex between $v_{2}$ and $v_{3}$. Therefore, the edges that make the 3-0 crossing more favorable must be incident to vertices from $C$. The corresponding slot for these edges can be either in $X$ or $Y$.


Figure 27: We have two possible placements for the request $v_{2}, v_{3}$. Red edges contribute extra crossings to the placement in $s_{k}$ and green edges contribute extra crossings to the placement in $s_{j}$. The blue edge cross with one edge of the request independent of its placement.

We start with the case that there is a filled slot $s_{x} \in X$ with both edges incident to the set $C$ and analyze it in more detail. Assume that the request $\left\{v_{2}, v_{3}\right\}$ arrives in the time step $t$. Thus, at the end of time step $t-1$ the slots $s_{i}$ and $s_{x}$ are filled. If the propagation arrows of $v_{2}$ and $v_{3}$ point to two different slots ( $s_{j}$ and $s_{k}$ ), we can apply Lemma 3 and know that eventually there will be two unavoidable crossings between the request $\left\{v_{2}, v_{3}\right\}$ and the overarching request. To see this more clearly, we point out explicitly how we can apply Lemma 3: The request $\left\{v_{2}, v_{3}\right\}$ is the request $\{u, v\}$, the free slots $s_{j}$ and $s_{k}$ are the slots $s_{l}$ and $s_{r}$, the filled slots $s_{i}$ and $s_{x} \in X$ correspond to the slots $s_{x}$ and $s_{y}$.

If the propagation arrows of $v_{2}$ and $v_{3}$ point to the same slot Lemma 2 prohibits that the propagation arrows point towards $s_{k}$. Therefore, the only possible configuration at the end of time step $t-1$ is that both propagation arrows already point towards $s_{j}$. This indicates that the placement of a previous request from a time step $t^{\prime}<t$ lead to this situation.

In the following we look at the time step $t^{\prime}$, in which the request arrived that pushed the propagation arrows - starting at $v_{2}$ and $v_{3}$ - to the slot $s_{j}$ for the last time. Note that the request $\left\{v_{1}, v_{2}\right\}$, placed in $s_{i}$, cannot push the propagation arrows to $s_{j}$, because positioning it onto $s_{j}$ instead of $s_{i}$ creates fewer crossings. To be precise, the three crossings between the propagation arrows of $v_{2}$ and $v_{3}$ and the edges of the request $\left\{v_{1}, v_{2}\right\}$ can be avoided by placing it in $s_{j}$ instead of $s_{i}$. This means, that when the request $\left\{v_{1}, v_{2}\right\}$ arrived, the propagation arrow configuration must have been different. Thus, the slot $s_{i}$ must be filled
and there must be a different request that is responsible for the last push of the propagation arrows in time step $t^{\prime}$.

So, at the beginning of time step $t^{\prime}$, at least one of the propagation arrows of $v_{2}$ and $v_{3}$ points to a free slot left of $s_{j}$ or to a free slot right of $s_{j}$. But, if the propagation arrow of $v_{2}$ points to a free slot left of $s_{j}$, there must be another propagation arrow, starting to the right of $v_{2}$, that points to $s_{j}$. Thus, there are two propagation arrows, starting at a vertex to the right of $v_{2}$ that cross the edges incident to $s_{i}$, violating Lemma 2. Thus, the request in time step $t^{\prime}$ cannot push the propagation arrows from left to right; it must push the propagation arrows from a free slot to the right of $s_{j}$ to the left, onto $s_{j}$. Note, that this implies that all of the slots $s_{x} \in X$ are also already filled in the time step $t^{\prime}$, because otherwise propagation arrows, coming from the right of $v_{2}$, would point to $s_{x} \in X$ and cross both edges incident to $s_{i}$, violating again Lemma 2.

To push the propagation arrows to the left, it is necessary that at least one vertex of the request is to the left of $v_{3}$. And because there is no vertex between $v_{2}$ and $v_{3}$, it must also be to the left of $v_{2}$. We differentiate between two different cases. The other vertex of this request can be to the left of $v_{2}$ or to the right of $v_{3}$.

If the other vertex of the request is also to the left of $v_{2}$, the propagation arrows of $v_{2}$ and $v_{3}$ cross both edges of this request at the end of $t^{\prime}$, violating Lemma 2. If the second vertex is to the right of $v_{3}$ we have a request that unavoidably crosses all edges of $v_{2}$ and $v_{3}$. Thus, for every feasible configuration for the time step $t^{\prime}$, we have unavoidable crossings.

Now, we consider the case that there is a slot $s_{y} \in Y$ with two incident edges to vertices from $C$ and analyze it in more detail. Assume that the request $\left\{v_{2}, v_{3}\right\}$ arrives in the time step $t$. Thus, at the end of time step $t-1$ the slot $s_{i}$ and $s_{y}$ are filled. We differentiate between two cases. Either the propagation arrows of $v_{2}$ and $v_{3}$ point to different slots, $s_{j}$ and $s_{k}$ respectively, or they point to the same slot.

If they point to different slots, we can apply Lemma 3 just as in the previous case and know that there will be two unavoidable crossings between the edges from the request $\left\{v_{2}, v_{3}\right\}$ and the future overarching request.

In the following, we assume that the propagation arrows from $v_{2}$ and $v_{3}$ point to the same slot, $s_{j}$ or $s_{k}$. Note, if both point to $s_{k}$, the vertex $v_{3}$ must be adjacent to $s_{y}$, otherwise Lemma 2 is violated. But in this case, the configuration in which the propagation arrows point to the same slot becomes symmetric. Thus, in the following, we look w.l.o.g. at the case that both propagation arrows point to $s_{j}$.

Just as in the previous case, we look at the last time step in which the propagation arrows are pushed to the slot $s_{j}$ and call it $t^{\prime}$. At the start of time step $t^{\prime}$, the request $\left\{v_{1}, v_{2}\right\}$ must already be placed in the slot $s_{i}$, by the same argument as before. It is, again, also not possible that the last time the propagation arrows are pushed is from left to right: If the propagation arrow of $v_{2}$ points to a slot left of $s_{j}$, the propagation arrows pointing to $s_{j}$ start at a vertex to the right of $v_{2}$ and violate Lemma 2. Thus, at least one propagation arrow from $v_{3}$ is pointing to a slot right of $s_{j}$, at the start of $t^{\prime}$. To push the propagation arrows to the left, it is necessary that at least one vertex of the request is to the left of $v_{3}$. The other vertex can be to the left of $v_{2}$ too or to the right of $v_{3}$ again.

If the other vertex of the request is also to the left of $v_{2}$, by the same argument as before, the propagation arrows of $v_{2}$ and $v_{3}$ cross both edges at the end of $t^{\prime}$, violating Lemma 2 , If the second vertex is to the right of $v_{3}$ it unavoidably crosses two times with the request $\left\{v_{2}, v_{3}\right\}$.

Thus, we have proven, finally that for every feasible configuration leading to a 3-0 crossing there must be, at least by the end of the request sequence, two unavoidable crossings with the edges of the request $\left\{v_{2}, v_{3}\right\}$.

This theorem shows that 3-0 crossings incurred by Algorithm 2 only happen in conjunction with two extra unavoidable crossings with the request generating the 3-0 crossing, this means, that any $3-0$ crossing is in effect a 5-2 crossing, which would be better than 5-competitive.

### 5.4 The Upper Bound

We can finally put all results together to conclude with an upper bound for the competitive ratio of Algorithm 2 to solve online slotted OSCM-2 on 2-regular graphs.

Theorem 5. Algorithm 圆 solves the online slotted OSCM-2 on 2-regular graphs with a competitive ratio of at most 5 .
Proof. In order to calculate the competitive ratio of Algorithm 2 we simply compare for every pair of requests, what the optimal placement compared to the placement chosen by Algorithm 2 would be.

We exhaustively look at possible placements of pairs of requests, as depicted in Figure 5. Observe, that except for the 3-0 crossings and 4-0 crossings, the rest of possible request pairs are no worse than 3 -competitive regardless of the algorithm used. Moreover, Theorem 3 ensures that for every 4-0 crossing incurred by Algorithm 2, there is at least one uniquely identifiable unavoidable crossing, meaning that the number of crossings incurred by Algorithm 2 is 5 , but optimally there must be at least 1 unavoidable crossing. Finally, Theorem 4 guarantees that there are also two uniquely identifiable unavoidable crossings for every occurrence of a 3-0 crossing. Thus, Algorithm 2 is at most 5-competitive.

## 6 Conclusion

In this work we have shown that the general slotted OSCM- $k$ is not competitive for any $k \geq 2$, which led us to analyze the case of the slotted OSCM- $k$ on 2-regular graphs. On this graph class, we have given a construction which proved a lower bound on the competitive ratio of $4 / 3$. Algorithm 2, which utilizes the information of the remaining space and unavoidable crossings in the graph in the form of our so-called propagation arrows, was proven to be at most 5 -competitive. This was done by limiting the number of total crossings generated by pairs of requests that do not cross one another in an optimal solution.

There are several open questions which we were not able to answer in the scope of this work. First, there is still a considerable gap between the lower and upper bound of the
competitive ratio that we have given. We assume that Algorithm 2 performs better than analyzed and that the upper bound can be made tighter.

While Theorem 1 proves non-competitiveness on general graphs for any $k \geq 2$, the case of regular graphs with degree 3 or higher is still open. We suggest to analyze this graph class further.

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