Definitional Quantifiers Realise Semantic Reasoning for Proof by Induction

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Abstract. Proof assistants offer tactics to apply proof by induction, but these tactics rely on inputs given by human engineers. To automate this laborious process, we developed SeLFiE, a boolean query language to represent experienced users' knowledge on how to apply the induct tactic in Isabelle/HOL: when we apply an induction heuristic written in SeLFiE to an inductive problem and arguments to the induct tactic, the SeLFiE interpreter judges whether the arguments are plausible for that problem according to the heuristic by examining both the syntactic structure of the problem and definitions of the relevant constants. To examine the intricate interaction between syntactic analysis and analysis of constant definitions, we introduce definitional quantifiers. For evaluation based on 347 inductive problems shows that our new prover achieves $1.4 \cdot 10^3\%$ improvement over the corresponding baseline prover for 1.0 second of timeout and the median value of speedup is 4.48x.

1 Introduction

The automation of proof by induction is a long standing challenge in Computer Science. Conventionally, human researchers manually investigate both inductive problems and relevant definitions to decide how to apply proof by induction. To mechanise such analysis, this paper introduces *definitional quantifiers*: quantifiers that range over the defining clauses of relevant constants to capture semantic properties of inductive problems.

1.1 Motivating Example

Consider the following two ways to define a reverse function for lists presented in a tutorial of Isabelle/HOL [39]:

2 Yutaka Nagashima | rev1 (x # xs) = rev1 xs @ [x] rev2 :: α list $\Rightarrow \alpha$ list $\Rightarrow \alpha$ list rev2 [] ys = ys | rev2 (x # xs) ys = rev2 xs (x # ys)

where # is the list constructor, [x] is a syntactic sugar for x # [], and @ is the infix operator to append two lists into one. How do you prove the following equivalence lemma?

lemma "rev2 xs ys = rev1 xs @ ys"

Since both reverse functions are defined recursively, it is natural to guess we can tackle this problem with proof by induction. But how do you apply proof by induction to this inductive problem? In this paper, we present SeLFiE, a boolean query language to encode induction heuristics in a declarative form, and its fast interpreter developed from scratch. SeLFiE is embedded in Isabelle/ML, the implementation language of Isabelle/HOL, and implemented for Isabelle2020. The key idea behind SeLFiE is *definitional quantifiers*: new kinds of quantifiers that allow for definitional reasoning in a domain-agnostic style.

1.2 Background

A prominent proof automation approach for proof assistants is the so-called hammer-style tools, such as HOL(y) Hammer [19] for HOL-light [13], CoqHammer [9] for Coq, and Sledgehammer [2] for Isabelle/HOL [39]. Sledgehammer, for example, translates proof goals in the polymorphic higher-order logic of Isabelle/HOL to monomorphic first-order logic and attempts to prove the translated goals using various external automated provers. Even though Sledgehammer brought powerful automation to Isabelle/HOL [3]; when it comes to inductive theorem proving the essence of inductive problems is lost in the translation, severely impairing the performance of Sledgehammer.

This is unfortunate: most analyses of programs and programming languages involve reasoning about recursive data structures and procedures containing recursion or iteration [6], and inductive problems are essential to these analyses.

We address this long standing challenge with SeLFiE. SeLFiE stands for semantic-aware logical feature extraction. SeLFiE has two main features: *definitional quantifiers*, and *domain-agnosticism*. Domain-agnosticism allows users to encode induction heuristics that can transcend problem domains, whereas definitional quantifiers allow SeLFiE heuristics to examine not only the syntactic structures of inductive problems but also the definitions of relevant constants.

Our implementation, available at GitHub [30], is specific to Isabelle/HOL: we implemented our system as an Isabelle theory for smooth user experience. However, the underlying concept of definitional reasoning is transferable to other proof assistants, such as Coq, Lean [29], and HOL[46]: no matter which proof assistant we use, we need to reason over not only the syntactic structure of proof

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goals but also definitions relevant to the goals to decide how to apply proof by induction.

The rest of the paper is organized as follows. Section 2 shows how to apply proof by induction in Isabelle using the example from Section 1.1 and clarifies the need for reliable heuristics. Section 3 gives an overview of what we mean by encoding induction heuristics and applying them to inductive problems in Isabelle/HOL. Since it is still a new approach to reason over inductive problems using a boolean query language, Section 4 reviews LiFtEr [31], an existing framework developed to encode syntax-based heuristics for Isabelle/HOL. In particular, we observe how LiFtEr's quantifiers allow us to write heuristics in a domain-agnostic style. Then, we identify what induction heuristics we can not encode in LiFtEr. In Section 5, we present SeLFiE and its fast interpreter developed from scratch. In addition to the domain-agnosticism given by LiFtEr's quantifiers, SeLFiE enables definitional reasoning using new language constructs that allow for the reasoning about both the syntactic structure of proof goals and the definitions of relevant constants. In Section, 6 we introduce a recommendation system for the induct tactic as a use case of SeLFiE and build a fast automatic inductive prover using this recommendation system, and we discuss how much performance gain SeLFiE brought to inductive theorem proving in Isabelle/HOL.

2 Proof by Induction in Isabelle/HOL

Modern proof assistants come with *tactics* to facilitate proof by induction. For example, Isabelle/HOL offers the induct tactic. The user-interface of the induct tactic allows for an intuitive application of proof by induction. For example, Nipkow *et al.* [39] proved our motivating example as follows:

```
lemma model_proof: "rev2 xs ys = rev1 xs @ ys"
apply(induct xs arbitrary: ys) by auto
```

That is to say, they firstly applied structural induction on **xs** while generalizing **ys**. Since **xs** is a list of any type, this application of structural induction resulted in the following two sub-goals:

```
    ∀ys. rev2 [] ys = rev1 [] @ ys
    ∀a xs ys. (∀ys. rev2 xs ys = rev1 xs @ ys) ⇒
rev2 (a # xs) ys = rev1 (a # xs) @ ys
```

where \forall and \implies represent the universal quantifier and implication of Isabelle's underlying logic respectively. The first sub-goal is the base case for the structural induction, whereas the second sub-goal is the step case where we are asked to prove that this conjecture holds for (a **#** xs) and ys, assuming that the conjecture holds for the same xs and an *arbitrary* ys. Then, they proved the remaining sub-goals using the general purpose tactic, **auto**. For the step case, **auto** rewrote the left-hand side of the meta-conclusion as follows:

<u>د</u>

rev2 (a # xs) ys	using	the	second	clause	defining	rev2
\hookrightarrow rev2 xs (a # ys)						

whereas auto rewrote the right-hand side as follows:

rev1 (a # xs) @ ys	using the second clause defining rev1
\hookrightarrow (rev1 xs @ [a]) @ ys	using the associative property of $@$
\hookrightarrow rev1 xs @ ([a] @ ys)	using the second clause defining $@$
\hookrightarrow rev1 xs @ (a # ([] @ ys))	using the first clause defining $@$
\hookrightarrow rev1 xs @ (a # ys)	

Applying such rewriting, **auto** internally transformed the step case to the following intermediate goal:

 $\forall a xs ys. (\forall ys. rev2 xs ys = rev1 xs @ ys) \implies$ rev2 xs (a # ys) = rev1 xs @ (a # ys)

Since ys was generalized in the induction hypothesis, auto proved rev2 xs (a # ys) = rev1 xs @ (a # ys) by considering it as a concrete case of the induction hypothesis. If Nipkow *et al.* had not passed ys to the arbitrary field, the induct tactic would have produced the following sub-goals:

 rev2 [] ys = rev1 [] @ ys
 ∀a xs ys. (rev2 xs ys = rev1 xs @ ys) ⇒ rev2 (a # xs) ys = rev1 (a # xs) @ ys

This step case requests us to prove that the original goal holds for (a **# xs**) and **ys**, assuming that it holds for the same **xs** and the *same* **ys** that appear in the induction hypothesis. If we apply **auto** to these sub-goals, **auto** proves the base case, but it leaves the step case as follows:

```
\forall a xs. rev2 xs ys = rev1 xs @ ys \implies
rev2 xs (a # ys) = rev1 xs @ (a # ys)
```

That is, **auto** is unable to complete the proof attempt because **ys** is shared both in the conclusion and induction hypothesis, illustrating the importance of variable generalization.

Note that we did not have to develop induction principles manually for model_proof since the induct tactic found out how to apply structural induction from the arguments passed by Nipkow *et al.* In fact, for most of the time Isabelle users do not have to develop induction principles manually, but they only have to pass the right arguments to the induct tactic.

Furthermore, there are often multiple equally appropriate ways to prove one theorem. For example, we could have proved our running example with the following script: apply (induct xs ys rule: rev2.induct) by auto. This script applies computation induction using the auxiliary lemma, rev2.induct, in the rule field. Fortunately, in many cases Isabelle automatically creates such auxiliary lemmas when defining relevant constants. In our case, Isabelle derived rev2.induct automatically when defining rev2. This way, the induct tactic reduces the problem of how to apply induction to the following three questions:

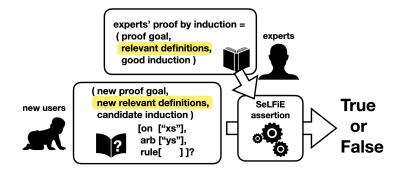


Fig. 1: The overview of SelFiE.

- On which terms do we apply induction?
- Which variables do we pass to the arbitrary field to generalize them?
- Which rule do we pass to the rule field?

However, answering these questions is a well-known challenge, which used to require hard-won expertise. We developed SeLFiE to encode such expertise.

3 Overview of SelFiE

Figure 1 shows how SeLFiE transfers such experienced users' knowledge to new users: when experienced users tackle inductive problems of their own, they encode their expertise about how they use the induct tactic as SeLFiE heuristics. Each SeLFiE heuristic is an assertion that takes a triple of a proof goal, relevant constant definitions, and arguments passed to the induct tactic. A well-written SeLFiE assertion should return True if the arguments to the induct tactic are likely to be useful to prove the problem, whereas it should return False if the combination is not likely to be useful to prove the problem. When new users want to know if their use of the induct tactic is appropriate or not, they apply the assertion written by an expert to their own problem and learn if their choice of arguments is compatible with the induction heuristic encoded by the expert. Note that we highlighted parts of Figure 1 to emphasize the main differences from the SeLFiE's predecessor, LiFtEr, developed for a similar purpose.

Originally, we developed SeLFiE's interpreter as an interactive tool to test a choice of proof by induction in terms of experts' heuristics. However, we can also use SeLFiE to build fully automated inductive provers as shown in Section 6. In the following, we review LiFtEr and explain why we need a reasoning framework that can take relevant definitions into account to encode reliable heuristics.

```
Syntax 1 The abstract syntax of LiFtEr / SeLFiE in one. The language components unique to SeLFiE are highlighted.
```

```
argument := term | number
literal := term_occ | rule | argument | ...
assertion := atomic | literal | connective | quantifier | ( assertion )
            | \lambda \text{ assertions. assertion } | \text{ assertion assertions} |
type := term | term_occ | rule | number
modifier := induction | arbitrary | rule
quantifier := \exists x : type. assertion \mid \forall x : type. assertion
             | \exists x : term \in modifier. assertion | \forall x : term \in modifier. assertion
             | \exists x : term\_occ \in y : term. assertion
             | \forall x : term\_occ \in y : term. assertion
             \exists_D (term , \lambda arguments. assertion , arguments )
             \forall_D ( term , \lambda arguments. assertion , arguments )
connective := True | False | assertion \lor assertion | assertion \land assertion
 | assertion \rightarrow assertion | \neg assertion
atomic := term_is_free ( term )
         | are_same_term ( term , term )
         is_nth_argument_of ( term_occ, number, term_occ )
         | is_nth_argument_in ( term_occ, number, term_occ )
         | are_of_same_term ( term_occ , term_occ ) | ...
```

4 Syntactic Reasoning in LiFtEr

4.1 LiFtEr: Logical Feature Extraction

LiFtEr is the first framework designed to describe how to use the induct tactic without relying on domain-specific constructs. Syntax 1 outlines LiFtEr's syntax, which resembles that of first-order logic. When reading Syntax 1, we ignore highlighted parts, which we discuss in Section 5.1.

As shown in Syntax 1, LiFtEr offers four primitive variable types: natural numbers, induction rules, terms, and term occurrences. An induction rule is an auxiliary lemma passed to the rule field of the induct tactic. The domain of terms is the set of all sub-terms appearing in the inductive problem at hand, whereas the domain of term occurrences is the set of all occurrences of such sub-terms. LiFtEr distinguishes terms and term occurrences explicitly because we often have multiple distinct occurrences of the same term in a syntax tree and have to analyze the locations of such occurrences. For instance, the variable ys appears twice in our theorem about list reversal. But what matters when deciding which variables to generalize is the occurrence of ys on the left-hand side and its location relative to the only occurrence of rev2, as we shall see in Section 5.2. Quantifiers over terms can be restricted to those terms that appear as arguments to the induct tactic under consideration.

Program 1 Naive generalization heuristic in LiFtEr

∀ free_var : term.
 term_is_free (free_var)
 ^
 ¬ ∃ induct : induction. are_same_terms (free_var, induct)
 →
 ∃ generalized : arbitrary. are_same_terms (free_var, generalized)

4.2 Naive Generalization Heuristic in LiFtEr

As we saw in Section 2, the key to the successful application of the induct tactic for our motivating example is the generalization of ys using the arbitrary field. When explaining why they decided to generalize ys, Nipkow *et al.* introduced the following generalization heuristic [38]:

Generalize induction by generalizing all free variables (except the induction variable itself).

We can encode this generalization heuristic in LiFtEr as shown in Program 1. In plain English, Program 1 reads as follows:

For any term, *free_var*, in a proof goal, if *free_var* is a free variable but not passed to the **induct** tactic as an induction term, there exists a term, *generalized*, in the **arbitrary** field such that *free_var* and *generalized* are the same term.

If we evaluate this heuristic for our ongoing example and its model proof by Nipkow *et al.*, the LiFtEr interpreter returns True, approving the generalization of ys. But this heuristic seems too coarse to produce reliable recommendations. In fact, Nipkow *et al.* articulate the limitation of this heuristic:

However, it (this generalization heuristic) should not be applied blindly. It is not always required, and the additional quantifiers can complicate matters in some cases. The variables that need to be quantified are typically those that change in recursive calls.

Unfortunately, it is not possible to encode this provision in LiFtEr because it involves reasoning on the structure of the syntax tree representing the definition of a constant appearing in a proof goal, which is rev2 in this particular case. In other words, LiFtEr heuristics can describe the structures of proof goals in a domain-independent style, but they cannot describe the structures of relevant constants' definitions. What is much needed is a framework to reason about both arbitrary proof goals and their relevant definitions in terms of the arguments passed to the induct tactic in a domain-agnostic style. And this is the main challenge addressed by SelFiE.

5 Semantic Reasoning in SelFiE

5.1 Semantics-Aware Logical Feature Extraction

We designed SeLFiE to overcome LiFtEr's limitation while preserving its capability to transcend problem domains. Syntax 1 presents the abstract syntax of SeLFiE. Since SeLFiE inherits design choices from LiFtEr, we re-use Syntax 1; however, we now include the highlighted constructs into our consideration.

Compared to LiFtEr, which resembles first-order logic, SeLFiE adopts lambda abstractions and function applications to support the *definitional quantifiers*, \exists_D and \forall_D . These new quantifiers range over definitions of constants, so that we can handle constant definitions abstractly to develop semantic-aware induction heuristics that can transcend problem domains, whereas the conventional quantifiers from LiFtEr range over terms and term occurrences, so that we can handle terms and their occurrences abstractly to develop syntax-based induction heuristics in a domain-agnostic style.

More specifically, each definitional quantifier takes a triple of:

- a term whose defining clauses are to be examined,
- a lambda function, which examines the relevant definitions, and
- a list of arguments, each of which is either a term or natural number. They are passed to the aforementioned lambda function to bridge the gap between the analysis of a proof goal and the analysis of relevant definitions.

For example, \exists_D (const, $\lambda xs. f xs, as$) returns True if $\lambda xs. f xs$ returns True when applied to as for *at least one* clause that defines const. Similarly, \forall_D (const, $\lambda xs. f xs, as$) returns True if $\lambda xs. f xs$ returns True when applied to as for *all* clauses that define const.

The conventional quantifiers outside and inside definitional quantifiers behave differently: inside the lambda function passed as the second argument to definitional quantifiers, conventional quantifiers' domains are based on the relevant definitions under consideration. For example, a quantifier over terms inside a definitional quantifier ranges over terms that appear in the relevant defining clause under consideration.

In the following we focus on the operational aspect of definitional quantifiers, so that readers can grasp their nature using a concrete example in Section 5.2.

Figure 2 illustrates the overall workflow of the SeLFiE interpreter when applied to an inductive problem and arguments of the induct tactic. In this figure, we assume that the SeLFiE assertion has only one definitional quantifier for a simpler explanation; however, in general, a SeLFiE heuristic may contain multiple definitional quantifiers. The small square, labelled as inner part, represents the lambda function passed as the second argument to this definitional quantifier, whereas outer part represents everything else in the SeLFiE assertion. Now based on this figure we explain how the SeLFiE interpreter works using the following eight steps from S1 to S8.

S1. Firstly, the SelFiE interpreter takes a SelFiE heuristic.

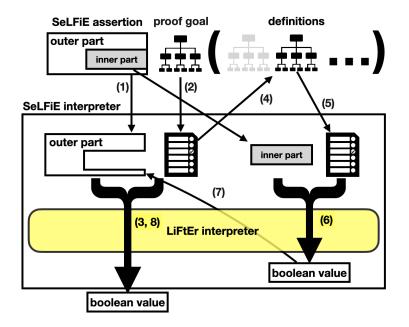


Fig. 2: The workflow of the SeLFiE interpreter.

- S2. Then, the preprocessor of SeLFiE transforms the syntax tree representing the inductive problem into a look-up table. This look-up table replaces slow traversals in the syntax tree with quick accesses to term occurrences using their paths from the root node.
- S3. The SelFiE interpreter processes the outer part of the assertion using the newly implemented LiFtEr interpreter.
- S4. When the SeLFiE interpreter reaches the definitional quantifier, it extracts the clauses that define the first argument of the definitional quantifier from the underlying proof context.
- S5. The interpreter transforms the syntax tree representing the relevant definitions into look-up tables.
- S6. The LiFtEr interpreter applies the inner part of the assertion, which is the lambda function passed as the second argument of the definitional quantifier, to the list of arguments, which is the third argument of the definitional quantifier, based on the look-up tables produced in S5.
- S7. The result of S6 is then returned to the LiFtEr interpreter.
- S8. The LiFtEr interpreter continues to evaluate the remaining outer part using the return value from the inner part.

We named our language SeLFiE partly because we extended LiFtEr, so that LiFtEr can call it<u>self</u> to support definitional quantifiers, but also because SeLFiE heuristics can attain the <u>semantics</u> of inductive problems using definitional quantifiers. Our motto is that:

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```
{\bf Program\ 2}\ {\rm Syntactic\ analysis\ of\ more\ reliable\ generalization\ heuristic\ in\ {\tt SelFiE}}
```

```
∀ arb_term : term ∈ arbitrary.
∃ f_term : term.
∃ f_occ : term_occ ∈ f_term.
∃ arb_occ ∈ arb_term.
∃ generalize_nth : number.
is_nth_argument_of (arb_occ, generalize_nth, f_occ)
∧
∃_D(f_term, generalize_nth_argument_of, [generalize_nth, f_term])
```

Program 3 Definitional analysis of a generalization heuristic in SeLFiE generalize_nth_argument_of :=

```
λ [generalize_nth, f_term].
∃ lhs_occ : term_occ. is_left_hand_side (lhs_occ)
^
∃ nth_param_on_lhs : term_occ.
is_nth_argument_in (nth_param_on_lhs, generalize_nth, lhs_occ)
^
∃ nth_param_on_rhs : term_occ.
¬ are_of_same_term (nth_param_on_rhs, nth_param_on_lhs)
^
∃ f_occ_on_rhs : term_occ ∈ f_term.
is_nth_argument_of (nth_param_on_rhs, generalize_nth, f_occ_on_rhs)
```

We analyze inductive problems *semantically* by analyzing their relevant definitions *syntactically*.

5.2 Semantics-Aware Generalization Heuristic

We now improve the naive generalization heuristic from Section 4.2 in SeLFiE. More specifically, we encode the provision to the generalization heuristic discussed in Section 4.2 as Program 2 and Program 3. Intuitively, when applied to model_proof, these programs realise the following dialogue:

- Program 2 asks "Should we generalize ys, which appears as the second argument of rev2?"
- Program 3 answers "Yes, because the second argument changes from the left-hand side to the right-hand side in the second clause defining rev2."

Keeping this dialogue in mind, we examine how the SeLFiE interpreter formally processes this heuristic for our running example.

- S1. We pass Program 2 and 3, and model_proof to the SelFiE interpreter.
- S2. The interpreter transforms the syntax tree representing the proof goal into a look-up table for faster processing.

S3. The SeLFiE interpreter processes the outer part for the syntax tree representing the proof goal itself. Note that the domains of quantifiers over terms and term occurrences are based on those terms and their occurrences within the proof goal itself.

In model_proof, only one variable, ys, is generalized in the arbitrary field. Therefore, for model_proof to satisfy this generalization heuristic we only have to satisfy inner existential quantifiers when *arb_term* is ys. Thus, we instantiate each existentially quantified variable in Program 2 as follows:

- *f_term* with rev2,
- *f_occ* with the sole occurrence of **rev2** in the proof goal,
- arb_occ with the occurrence of ys on the left-hand side in the goal, and
 generalize_nth with 2.

Then, is_nth_argument_of returns True since ys on the left-hand side is the second argument to rev2 in the goal.

- S4. When the interpreter hits \exists_D with f_term being rev2, it extracts the two syntax trees defining rev2 from the proof context. Since \exists_D is an existential quantifier, we only have to show that Program 3 returns True for one of the two equations defining rev2. In the following, we focus on the second clause, rev2 (x # xs) ys = rev2 xs (x # ys).
- S5. The interpreter transforms each syntax tree representing a clause defining rev2 into a look-up table for faster processing.
- S6. The interpreter evaluates Program 3 with 2 as generalize_nth and rev2 as f_term, since they are passed from Program 2. Note that the domains of quantifiers over terms and term occurrences are now all terms and term occurrences in rev2 (x # xs) ys = rev2 xs (x # ys). To satisfy Program 3 we instantiate existentially quantified variables as follows:
 - *lhs_occ* with the left-hand side of the equation, rev2 (x # xs) ys,
 - *nth_param_on_lhs* with the occurrence of ys, which appears as the second argument on the left-hand side,
 - *f_occ_on_rhs* with the sole occurrence of **rev2** on the right-hand side, and
 - nth_param_on_rhs with the sole occurrence of x # ys, which is the second argument to rev2 bound by f_occ_on_rhs.

Since x # ys and ys are not the same term, the interpreter evaluates Program 3 to True for the second clause defining rev2, which is tantamount to say we generalize the second argument of rev2 because the second argument of rev2 changes in a recursive call in a domain-agnostic style.

- S7. Program 3 returns **True** to Program 2.
- S8. With this returned value, the interpreter evaluates Program 2 to True.

This is how Program 3 encodes the provision to the generalization heuristic discussed in Section 4.2. Note that the interaction between the two programs involves natural numbers, terms, and boolean values only: more complex reasoning, such as quantification over natural numbers, terms, and term occurrences, happens only within each program because each module has its own domains for terms and term occurrences. Furthermore, it is not allowed to pass term occurrences from a syntactic analysis to a definitional analysis through definitional

tool	top 1	top 3	top 5	top 10	tool $0.2 \ 0.5 \ 1.0 \ 2.0 \ 5.0$
sem_ind	38.2	59.3	64.5	72.7	sem_ind 8.8 24.7 47.8 69.8 86.8
\mathtt{smart}_{induct}	20.1	42.8	48.5	55.3	<pre>smart_induct 0.0 3.5 16.9 38.3 70.2</pre>
(a) Coincidence rates [%]					(b) Return rates [%] within timeout

Table 1: Coincidence rates and return rates

quantifiers. Therefore, we discuss relative locations of certain term occurrences across syntax trees, by passing natural numbers and terms from the syntax level to the definition level, as is done in this example. This clear separation between syntactic and definitional reasoning improves the readability of this heuristic.

In this particular example, we demonstrated two-level analysis of syntax trees using two SeLFiE programs. However, SeLFiE's definitional quantifiers can orchestrate reasoning on arbitrary number of levels.

6 Case Studies and Evaluations

6.1 Interactive Recommendation System

Using SeLFiE, we previously developed sem_ind, an interactive recommendation system for proof by induction in Isabelle/HOL [33]. Given an inductive problem, sem_ind produces a number of induction candidates and applies 44 SeLFiE heuristics to these candidates. Each heuristic is tagged with a certain point, representing the weight of each heuristic. Based on the sum of these points, sem_ind ranks the candidates and presents the 10 most promising ones to its users.

Nagashima evaluated sem_ind against 1,095 inductive proofs from the Archive of Formal Proofs (AFP) [21] and compared sem_ind against its predecessor, smart_induct [32], which is written in LiFtEr.

Table 1a summarizes how often sem_ind's recommendations coincide with the choices of human engineers. For example, Table 1a shows 38.2% for "sem_ind" at "top 1". This means when considering only the top one candidate recommended by sem_ind, sem_ind's recommendations coincide with the choices of human engineers for 38.2% of proof goals in the dataset. This is a 90.0% improvement compared to smart_induct, which reported 20.1 % for "top 1".

Table 1b, on the other hand, summarizes how long it takes for sem_ind to produce recommendations. For example, Table 1b shows 8.8% for "sem_ind" at "0.2". This means sem_ind managed to produce recommendations for 8.8% of proof goals in the dataset within 0.2 seconds of timeout. Furthermore, Nagashima also reported that the median value of the execution time of sem_ind is 1.06 seconds, while that of smart_induct is 2.79 seconds, which is a 2.63x speedup.

6.2 Automatic Proof Search using SelFiE

We integrated sem_ind into an automatic inductive prover written in PSL [35] and measured how SeLFiE improved PSL's automatic proof search. PSL is a

Program 4 Automatic inductive prover without SeLFiE
Auto_Solve = Thens[Auto, Solved]
PSL_W0_SeLFiE =
Ors[Auto_Solve,
PThenOne[Dynamic (Induct), Auto_Solve]
<pre>PThenOne[Dynamic(Induct), Thens[Auto, RepeatN(Hammer), Solved]]]</pre>

Program 5 Automatic inductive prover with SeLFiE					
PSL_W_SeLFiE =					
Ors[Auto_Solve,					
PThenOne[<mark>Semantic_Induct</mark> , Auto_Se	olve]				
PThenOne[<mark>Semantic_Induct</mark> , Thens[Auto, RepeatN(Hammer), Solved]]]				

domain-specific language to describe rough ideas about how to find a proof using backtracking search over tactics in Isabelle/HOL. In the following, we focus on PSL's constructs used in our evaluation leaving out irrelevant details of PSL.

Program 4 shows an example automatic inductive prover written in PSL, which we use as the baseline prover in this evaluation. The strategy is called PSL_WO_SeLFiE, and it combines three sub-strategies using the deterministic combinator Ors: it first tries the first sub-strategy, Auto_Solve, and proceeds to the second sub-strategy only if the first sub-strategy fails, and so on. Thens used in Auto_Solve is the sequential combinator, which combines Auto and Solved sequentially, and Auto in PSL corresponds to the auto tactic in Isabelle, while the following Solved checks if all sub-goals are proved by auto. Hammer represents the invocation of Sledgehammer, which is wrapped in RepeatN in Program 4. This means "repeat applying Sledgehammer to the remaining sub-goals n times where n is the number of sub-goals before applying Sledgehammer". PThenOne is the sequential parallel combinator: PThenOne takes exactly two sub-strategies and applies the second sub-strategy to the results of the first sub-strategy in parallel until at least one of them succeeds.

Dynamic (Induct) creates variants of the induct tactics with different arguments based on the given goal and combine such variants non-deterministically. However, when the interpreter produces such variants of the induct tactics using Dynamic (Induct), it does not know which one would be the most suitable induction. Therefore, the interpreter naively combines variables and arguments appearing in the proof goal to produce candidate induct tactics. In PSL, it is the subsequent sub-strategies that are to identify the right arguments for the induct tactic: PThenOne [Dynamic (Induct), Auto_Solve], for example, keeps applying auto to sub-goals emerging after applying the induct tactic with various sequences of arguments until it finds a sequence that results in sub-goals that are all proved by auto.

The drawback of this approach is that PSL's interpreter cannot identify the appropriate arguments for the induct tactic if it cannot complete a proof search:

¹⁴ Yutaka Nagashima

timeouts	Program 5	Program 4	speedup [times] occur	rence
0.3[s]	11.0%	1.2%	x < 1.0 3 (2	.4%)
1.0[s]	25.6%	1.7%	$1.0 \le x < 5.0$ 64 (5)	0.8%)
3.0[s]	28.2%	21.9%	$5.0 \le x < 10.0$ 44 (3-	4.9%)
10.0[s]	34.9%	28.0%	$10.0 \le x < 15.0$ 9 (7)	.1%)
30.0[s]	45.8%	38.3%	$15.0 \le x < 6 (4)$.8%)
(a	a) Success r	ates	(b) Speedup of execution	on time

Table 2: Success rates and speedup

for difficult inductive problems, the interpreter often fails to complete a proof search within a realistic timeout because Dynamic (Induct) tends to produce a large number of induction candidates and the necessary proof steps after applying the induct tactic tend to be complicated. What was lacking was the mechanism to identify promising induction candidates without relying on a proof search, so that PSL's interpreter can spend limited computational resources for a small number of promising candidates to complete a proof search. For this reason, we integrated sem_ind into PSL, and we counted how many goals are proved within each timeout.

Program 5 shows the new automatic prover. Here, Semantic_Induct represents sem_ind integrated into PSL's environment. We highlighted the differences in Program 5 from Program 4 to clarify that we are using almost the same PSL strategy for a fair comparison except for the introduction of Semantic_Induct.

For our evaluation, we used 12 Isabelle theory files from 8 projects about various topics in the AFP, which in total include 347 proofs by induction. These projects are about the depth-first search [40], binomial heaps [25], a boolean expression checker [37], multi-dimensional binary search trees [42], the priority search tree [23], linear temporal logic [45], imperative programming language Simpl [44], and program verification competition [24]. We conducted this evaluation on a MacBook Pro (15-inch, 2019) with 2.6 GHz Intel Core i7 6-core memory 32 GB 2400 MHz DDR4, and the reported execution times are based on elapsed real time.

Table 2a shows how many inductive problems were proved by each program within each timeout. For example, the timeout of 0.3[s] for Program 5 has 11.0%. This means Program 5 proved 11.0% inductive problems in the dataset within 0.3 seconds. For a fair comparison we included not only the time spent by tactics for proof search but also the time spent by sem_ind when measuring the execution time of each proof search. As shown in Table 2a, PSL enhanced with sem_ind proved more inductive problems than PSL without sem_ind for various timeouts. For 30.0 seconds of timeout, PSL with sem_ind proved 159 inductive problems, while PSL without sem_ind proved 133 problems only. 126 problems were proved by both provers within this timeout. For each problem proved by both programs within 30.0 seconds, we computed the speedup of execution time spent to complete each proof search. For example, Program 5 spent 0.325 sec-

onds and Program 4 spent 2.171 seconds to prove a lemma named nexts_set in DFS.thy. Therefore, the speedup of execution time for this lemma is (2.171 / 0.325) = 6.68.

Table 2b shows the distribution of speedup observed among such problems. For example, the second row reads $1.0 \leq x < 5.0$ and 64.0 (50.8%), and this means that Program 5 achieved between 1.0x to 5.0x speedup compared to Program 4 for 64 inductive problems proved by both provers. As shown in this table, we confirmed that Program 5 achieved speedups over Program 4 except for 3 cases, which constitutes 2.4% of problems proved by both provers within 30.0 seconds of timeout. The median value for speedup is 4.48x.

7 Conclusion

We presented SeLFiE, a boolean-query language to encode induction heuristics. The abstraction brought by definitional quantifiers allow SeLFiE to transcend problem domains while analysing not only the syntactic structures of inductive problems but also definitions of relevant constants in a modular style.

Our conservative extension to LiFtEr's syntax allows us to take advantage of LiFtEr's domain-agnosticism, while adding the capability to reason on the semantics of proof goals. To realise such extension, we implemented SeLFiE's interpreter from scratch: since LiFtEr's original interpreter was not designed with definitional reasoning in mind, it did not support even lambda abstraction or function application, and suffered from poor performance, incremental improvement was not realistic.

Nagashima implemented sem_ind in SeLFiE, and we integrated sem_ind into PSL and built an automatic inductive prover. Our experiment showed that compared to the baseline prover our inductive prover based on SeLFiE achieves $1.4 \cdot 10^3\%$ improvement of success rate for 1.0 second of timeout as well as a 4.48x speedup as the median value.

The final goal of this project is to build a strong inductive prover. It remains our future work to further strengthen the automatic prover introduced in Section 6, by incorporating two conjecturing mechanisms, top-down conjecturing [36] and bottom-up conjecturing [17], into our system.

8 Related Work

A well-known approach for inductive theorem proving is the Boyer-Moore waterfall model [26], which was invented for a first-order logic on Common Lisp [18]. In the original waterfall model, a prover tries to apply any of the six techniques, including simplification, generalization and induction. If any of these techniques works, the prover stores the resulting sub-goals in a pool and continues to apply the techniques until it empties the pool.

ACL2 [27] is the latest incarnation of this line of work with industrial applications [20]. To decide how to apply induction, ACL2 estimates how good

each induction scheme is by computing a score, called *hitting ratio*, based on a fixed formula [4,28], and it proceeds with the induction scheme with the highest hitting ratio. Heras *et al.* used ML4PG learning method to find patterns to generalize and transfer inductive proofs from one domain to another in ACL2 [14]. Instead of computing a hitting ratio, we provide SeLFiE as a language, so that Isabelle experts can encode their expertise as assertions.

There are ongoing attempts to extend saturation-based superposition provers with induction: Cruanes presented an extension of typed superposition that can perform structural induction [8], while Reger *et al.* incorporated lightweight automated induction [43] to the Vampire prover [22] and Hajdú *et al.* extended it to cover induction with generalization [15]. Contrary to their work, our approach to proof by induction uses Isabelle's default **induct** tactic, which we can use for arbitrary data types.

For more expressive logics, Jiang *et al.* employed multiple waterfalls [16] in HOL Light [13]. However, to decide induction variables, they naively picked the first free variable with recursive type and left the selection of promising induction variables as future work. Passmore *et al.* developed the Imandra automated reasoning system [41], which also uses the waterfall model for its typed higher-order setting. For Isabelle/HOL, Dixon *et al.* developed IsaPlanner [11], a generic framework to encode proof plans [5]. IsaPlanner can incorporate reasoning techniques, such as rippling [7], for proof by induction. For generalization, however, IsaPlanner naively generalizes all non-induction variables [10].

Machine learning tools for tactic-based theorem proving mainly focus on tactic recommendations and premise selections, leaving the problem of arguments selection for tactics as an open question when arguments are terms [12,34,1]. Instead of relying on machine learning algorithms, we developed a language, in which one can explicitly encode heuristics. We plan to use SeLFiE as a feature extractor for machine learning algorithms: by applying SeLFiE heuristics to inductive problems, we can convert each pair of an inductive problem and induction arguments to an array of boolean values, which is amenable for machine learning algorithms. The application of SeLFiE as a preprocessor for machine learning algorithms remains as our future work.

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