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Local planar domination revisited

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Abstract. We show how to compute a 20-approximation of a minimum dominating set in a planar graph in a constant number of rounds in the LOCAL model of distributed computing. This improves on the previously best known approximation factor of 52, which was achieved by an elegant and simple algorithm of Lenzen et al. Our algorithm combines ideas from the algorithm of Lenzen et al. with recent work of Czygrinow et al. and Kublenz et al. to reduce to the case of bounded degree graphs, where we can simulate a distributed version of the classical greedy algorithm.

Keywords: Dominating set · LOCAL algorithms · Planar graphs

1 Introduction

A dominating set in an undirected and simple graph G is a set $D \subseteq V(G)$ such that every vertex $v \in V(G)$ either belongs to D or has a neighbor in D. The dominating set problem is a classical NP-complete problem [9] with many applications in theory and practice, see e.g. [7,15]. In this paper we study the distributed time complexity of finding dominating sets in planar graphs in the classical LOCAL model of distributed computing. In this model, a distributed system is modeled by an undirected (planar) graph G. Every vertex represents a computational entity and the vertices communicate through the edges of G. The vertices are equipped with unique identifiers and initially, every vertex is only aware of its own identity. A computation then proceeds in synchronous rounds. In every round, every vertex sends messages to its neighbors, receives messages from its neighbors and performs an arbitrary computation. The complexity of a LOCAL algorithm is the number of rounds until all vertices return their answer, in our case, whether they belong to a dominating set or not.

The problem of approximating dominating sets in the LOCAL model has received considerable attention in the literature. Since in general graphs it is not possible to compute a constant factor approximation in a constant number of rounds [11], much effort has been invested to improve the ratio between approximation factor and number of rounds on special graph classes. A very successful line of structural analysis of graph properties that can lead to improved algorithms was started by the influential paper of Lenzen et al. [12], who in particular proved that on planar graphs a 130-approximation of a minimum dominating set can be computed in a constant number of rounds. A careful analysis of Wawrzyniak [18] later showed that the algorithm computes in fact a 52-approximation. In terms of lower bounds, Hilke et al. [8] showed that there is no deterministic local algorithm (constant-time distributed graph algorithm) that finds a $(7-\epsilon)$-approximation of a minimum dominating set on planar graphs, for any positive constant ϵ. Better approximation ratios are known for some special cases, e.g. 32 if the planar graph is triangle-free [1, Theorem 2.1], 18 if the planar graph has girth five [2] and 5 if the graph is outerplanar (and this bound is tight) [4, Theorem 1].

In this work we tighten the gap between the best-known lower bound of 7 and the best-known upper bound of 52 on planar graphs by providing a new approximation algorithm computing a 20-approximation.

Our algorithm proceeds in three phases. The first phase is a preprocessing phase that was similarly employed in the algorithm of Lenzen et al. [12]. In a key lemma, Lenzen et al. proved that there are only few vertices whose open neighborhood cannot be dominated by at most six vertices. We improve this lemma and show that there are only slightly more vertices whose open neighborhood cannot be dominated by *three* other vertices. All these vertices are selected into an initial partial dominating set and as a consequence the open neighborhoods of all remaining vertices can be dominated be at most three vertices.

By defining the notion of *pseudo-covers*, Czygrinow et al. [5] provided a tool to carry out a fine grained analysis of the vertices that can potentially dominate the remaining neighborhoods. Using ideas of [10] and [16] we provide an even finer analysis for planar graphs on which we base the second phase of our distributed algorithm and compute a second partial dominating set.

We prove that after the second phase we are left with a graph where every vertex has at most 30 non-dominated neighbors. In particular, every vertex from a minimum dominating set D can dominate at most 30 non-dominated vertices, hence, we could at this point pick all non-dominated vertices to add at most $31|D|$ vertices (each vertex dominates its neighbors and itself). We could also apply a general algorithm of Lenzen and Wattenhofer that computes in a graph of arboricity a and maximum degree Δ a $16a \log \Delta$-approximation in $6 \lceil \log \Delta + 1 \rceil$ rounds [13]. Planar graphs have arboricity 3 and $\log 30 \approx 4.907$, hence, in our situation $16a \log \Delta \approx 235$ and this would not yield an improvement. Of course, Lenzen and Wattenhofer optimized not only towards minimizing the approximation factor, which they could have easily improved, but also towards minimizing the number of rounds with respect to Δ. This is well-motivated as in general graphs the maximum degree can be large, however, in our algorithm we always arrive at this fixed constant degree so we can now proceed in a constant number of rounds.

We proceed in a greedy manner in 30 rounds as follows. Call the number of non-dominated neighbors of a vertex v the *residual degree* of v. In the first round, if a non-dominated vertex has a neighbor of residual degree 30, it elects one such neighbor into the dominating set (or if it has residual degree 30 itself, it may choose itself). The neighbors of the chosen elements are marked as dominated and the residual degrees are updated. Note that all non-dominated neighbors of a vertex of residual degree 30 in this round choose a dominator, hence, the residual degrees of all vertices of residual degree 30 are decreased to 0, hence, after this round there are no vertices of residual degree 30 left. In the second round, if a non-dominated vertex has a neighbor of residual degree 29, it elects one such vertex into the dominating set, and so on, until after 30 rounds in the final round every vertex chooses a dominator. Unlike in the general case, where nodes cannot learn the current maximum residual degree in a constant number of rounds, by establishing an upper bound on the maximum residual degree and proceeding in exactly this number of rounds, we ensure that we iteratively exactly choose the vertices of maximum residual degree. It remains to analyze the performance of this algorithm.

A simple density argument shows that there cannot be too many vertices of degree $i \geqslant 6$ in a planar graph. At a first glance it seems that the algorithm would perform worst when in every of the 30 rounds it would pick as many vertices as possible, as the constructed dominating set would grow as much as possible. However, this is not the case, as picking many high degree vertices at the same time makes the largest progress towards dominating the whole graph. It turns out that there is a delicate balance between the vertices that we pick in round i and the remaining non-dominated vertices that leads to the worst case. We formulate these conditions as a linear program and solve the linear program. In total, this leads to the claimed 20-approximation (Theorem 1).

We then analyze our algorithm on more restricted graphs classes, and prove that it computes approximations of factors: 14 for triangle-free planar graphs, 13 for bipartite planar graphs, 7 for planar graphs of girth 5, and 12 for outerplanar graphs (Theorems 2 to 5). This improves the currently best known approximation ratios of 32 and 18 for triangle-free planar graphs and planar graphs of girth 5, respectively, while our algorithm fails short of achieving the optimal approximation ratio of 5 on outerplanar graphs.

2 Preliminaries

In this paper we study the distributed time complexity of finding dominating sets in planar graphs in the classical LOCAL model of distributed computing. We assume familiarity with this model and refer to the survey [17] for extensive background on distributed computing and the LOCAL model.

We use standard notation from graph theory and refer to the textbook [6] for extensive background. All graphs in this paper are undirected and simple. We write $V(G)$ for the vertex set of a graph G and $E(G)$ for its edge set. The *girth* of a graph G is the length of a shortest cycle in G. A graph is called *triangle-free* if it does not contain a triangle, that is, a cycle of length three as a subgraph. Equivalently, a triangle-free graph is a graph of girth at least four.

A graph is *bipartite* if its vertex set can be partitioned into two parts such that all its edges are incident with two vertices from different parts. More generally, the *chromatic number* $\chi(G)$ of a graph G is the minimum number k such that the vertices of G can be partitioned into k parts such that all edges are incident with two vertices from different parts. Hence, the bipartite graphs are exactly the graphs with chromatic number two. A set A is *independent* if all two distinct vertices $u, v \in A$ are non-adjacent. Every graph G contains an independent set of size at least $\lceil |V(G)|/\chi(G) \rceil$.

A graph is *planar* if it can be embedded in the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. By the famous theorem of Wagner, planar graphs can be characterized as those graphs that exclude the complete graph K_5 on five vertices and the complete bipartite $K_{3,3}$ with parts of size three as a minor. A graph H is a *minor* of a graph G, written $H \preceq G$, if there is a set $\{G_v : v \in V(H)\}$ of pairwise vertex disjoint and connected subgraphs $G_v \subseteq G$ such that if $\{u,v\} \in E(H)$, then there is an edge between a vertex of G_u and a vertex of G_v. We call $V(G_v)$ the *branch set* of v and say that it is *contracted* to the vertex v. We call H a 1-*shallow minor*, written $H \preceq_1 G$, if $H \preceq G$ and there is a minor model $\{G_v : v \in V(H)\}$ witnessing this, such that all branch sets G_v have radius at most 1, that is, in each G_v there exists w adjacent to all other vertices of G_v. In other words, $H \preceq_1 G$ if H is obtained from G by deleting some vertices and edges and then contracting a set of pairwise disjoint stars. We refer to [14] for an in-depth study of the theory of sparsity based on shallow minors.

A graph is *outerplanar* if it has an embedding in the plane such that all vertices belong to the unbounded face of the embedding. Equivalently, a graph is outerplanar if it does not contain the complete graph K_4 on four vertices and the complete bipartite graph $K_{2,3}$ with parts of size 2 and 3, respectively, as a minor. If $J \preceq H$ and $H \preceq G$, then $J \preceq G$, hence a minor of a planar graph is again planar and a minor of an outerplanar graph is again outerplanar.

By Euler's formula, planar graphs are sparse: every planar n-vertex graph ($n \geqslant 3$) has at most $3n-6$ edges (and a graph with at most two vertices has at most one edge). The ratio $|E(G)|/|V(G)|$ is called the *edge density* of G. In particular, every planar graph G has edge density strictly smaller than three.

Lemma 1. *Let G be a planar graph. Then the edge density of G is strictly smaller than 3 and $\chi(G) \leqslant 4$. Furthermore,*

1. *if G is bipartite, then the edge density of G is strictly smaller than 2 and $\chi(G) \leqslant 2$,*
2. *if G is triangle-free or outerplanar, then the edge density of G is strictly smaller than 2 and $\chi(G) \leqslant 3$.*

For a graph G and $v \in V(G)$ we write $N(v) = \{u : \{u,v\} \in E(G)\}$ for the *open neighborhood* of v and $N[v] = N(v) \cup \{v\}$ for the *closed neighborhood* of v. For a set $A \subseteq V(G)$ let $N[A] = \bigcup_{v \in A} N[v]$. A *dominating set* in a graph G is a set $D \subseteq V(G)$ such that $N[D] = V(G)$. We write $\gamma(G)$ for the size of a minimum dominating set of G. For $W \subseteq V(G)$ we say that a set $Z \subseteq V(G)$ *dominates* W if $W \subseteq N[Z]$.

In the following we mark important definitions and assumptions about our input graph in gray boxes and steps of the algorithm in red boxes.

We fix a planar graph G and a minimum dominating set D of G with $\gamma := |D| = \gamma(G)$.

3 Preprocessing

As outlined in the introduction, our algorithm works in three phases. In phase i for $1 \leqslant i \leqslant 3$ we select a partial dominating set D_i and estimate its size in comparison to D. In the end we will return $D_1 \cup D_2 \cup D_3$. We will call vertices have been selected into a set D_i *green*, vertices that are dominated by a green vertex but are not green themselves are called *yellow* and all vertices that still need to be dominated are called *red*. In the beginning, all vertices are marked red.

The first phase of our algorithm is similar to the first phase of the algorithm of Lenzen et al. [12]. It is a preprocessing step that leaves us with only vertices whose neighborhoods can be dominated by a few other vertices. Lenzen et al. proved that there exist less than 3γ many vertices v such that the open neighborhood $N(v)$ of v cannot be dominated by 6 vertices of $V(G)\setminus\{v\}$ [12, Lemma 6.3]. The lemma can be generalized to more general graphs, see [3]). We prove the following lemma, which is stronger in the sense that the number of vertices required to dominate the open neighborhoods is smaller than 6, at the cost of having slightly more vertices with that property.

We define D_1 as the set of all vertices whose neighborhood cannot be dominated by 3 other vertices.

$$D_1 := \{v \in V(G) : \text{ for all sets } A \subseteq V(G)\setminus\{v\} \text{ with } N(v) \subseteq N[A] \text{ we have } |A| > 3\}.$$

We prove a very general lemma that can be applied also for more general graph classes, even though we will apply it only for planar graphs. Hence, in the following lemma, G can be an arbitrary graph, while in the following lemmas G will again be the planar graph that we fixed in the beginning.

Lemma 2. *Let G be a graph, let D be a minimum dominating set of G of size γ and let ∇ be an integer strictly larger than the edge density of a densest bipartite 1-shallow minor of G. Let $\hat{D}$ be the set of vertices $v \in V(G)$ whose neighborhood cannot be dominated by $(2\nabla - 1)$ vertices of D other than v, that is,*

$$\hat{D} := \{v \in V(G) : \textit{for all sets } A \subseteq D\backslash\{v\} \textit{ with } N(v) \subseteq N[A] \textit{ we have } |A| > (2\nabla - 1)\}.$$

Then $|\hat{D}\backslash D| < \chi(G) \cdot \gamma$.

Recall that minors of planar graphs are again planar, hence, the maximum edge density of a bipartite 1-shallow minor of a planar graph is smaller than 2 and hence we can choose $\nabla = 2$ for the case of planar graphs and we note the following corollary.

Corollary 3. *Let $\hat{D}$ be the set of vertices v whose neighborhood cannot be dominated by 3 vertices of D other than v, that is,*

$$\hat{D} := \{v \in V(G) : \textit{for all sets } A \subseteq D\backslash\{v\} \textit{ with } N(v) \subseteq N[A] \textit{ we have } |A| > 3\}.$$

Then $|\hat{D}\backslash D| < 4\gamma$.

Proof (of Lemma 2). Assume $D = \{b_1, \ldots, b_\gamma\}$. Assume that there are $\chi(G) \cdot \gamma$ vertices $a_1, \ldots, a_{\chi(G)\gamma} \notin D$ satisfying the above condition. As the chromatic number is monotone over subgraphs, the subgraph induced by the a_is is also $\chi(G)$-chromatic, so we find an independent subset of the a_is of size γ. We can hence assume that $a_1, \ldots, a_\gamma$ are not connected by an edge. We proceed towards a contradiction.

We construct a bipartite 1-shallow minor H of G with the following 2γ branch sets. For every $i \leqslant \gamma$ we have a branch set $A_i = \{a_i\}$ and a branch set $B_i = N[b_i]\backslash(\{a_1, \ldots, a_\gamma\} \cup \bigcup_{j<i} N[b_j] \cup \{b_{i+1}, \ldots, b_\gamma\})$. Note that the B_i are vertex disjoint and hence we define proper branch sets. Intuitively, for each vertex $v \in N(a_i)$ we mark the smallest b_j that dominates v as its dominator. We then contract the vertices that mark b_j as a dominator together with b_j into a single vertex. Note that because the a_i are independent, the vertices a_i themselves are not associated to a dominator as no a_j lies in $N(a_i)$ for $i \neq j$. Denote by $a'_1, \ldots, a'_\gamma, b'_1, \ldots, b'_\gamma$ the associated vertices of H. Denote by A the set of the a'_is and by B the set of the b'_js. We delete all edges between vertices of B. The vertices of A are independent by construction. Hence, H is a bipartite 1-shallow minor of G. By the assumption that $N(a_i)$ cannot be dominated by $2\nabla - 1$ elements of D, we associate at least 2∇ different dominators with the vertices of $N(a_i)$. Note that this would not necessarily be true if A was not an independent set, as all $a_j \in N(a_i)$ would not be associated a dominator.

Since $\{b_1, \ldots, b_\gamma\}$ is a dominating set of G and by assumption on $N(a_i)$, we have that in H, every a'_i has at least 2∇ neighbors in B. Hence, $|E(H)| \geqslant 2\nabla|V(A)| = 2\nabla\gamma$. As $|V(H)| = 2\gamma$ we conclude $|E(H)| \geqslant \nabla|V(H)|$. This however is a contradiction, as ∇ is strictly larger than the edge density of a densest bipartite 1-shallow minor of G.

Let us fix the set $\hat{D}$ for our graph G.

$$\hat{D} := \{v \in V(G) : \text{ for all sets } A \subseteq D\backslash\{v\} \text{ with } N(v) \subseteq N[A] \text{ we have } |A| > 3\}.$$

Note that $\hat{D}$ cannot be computed by a local algorithm as we do not know the set D. It will only serve as an auxiliary set in our analysis.

The first phase of the algorithm is to compute the set D_1, which can be done in 2 rounds of communication. Obviously, if the open neighborhood of a vertex v cannot be dominated by 3 vertices from $V(G)\backslash\{v\}$, then in particular it cannot be dominated by 3 vertices from $D\backslash\{v\}$. Hence $D_1 \subseteq \hat{D}$ and we can bound the size of D_1 by that of $\hat{D}$.

Lemma 4. *We have $D_1 \subseteq \hat{D}$, $|\hat{D}\backslash D| < 4\gamma$, and $|\hat{D}| < 5\gamma$.*

Proof. Lemma 4 follows the observation above together with Corollary 3.

From Lemma 4 we can conclude that $|D_1| < 5\gamma$. However, it is intuitively clear that every vertex that we pick from the minimum dominating set D is optimal progress towards dominating the whole graph. We will later show that this intuition is indeed true for our algorithm, that is, our algorithm performs worst when $D_1 \cap D = \varnothing$, which will later in fact allow us to estimate $|D_1| < 4\gamma$.

We mark the vertices of D_1 that we add to the dominating set in the first phase of the algorithm as green, the neighbors of D_1 as yellow and leave all other vertices red. Denote the set of red vertices by R, that is, $R = V(G)\backslash N[D_1]$. For $v \in V(G)$ let $N_R(v) := N(v) \cap R$ and $\delta_R(v) := |N_R(v)|$ be the *residual degree* of v, that is, the number of neighbors of v that still need to be dominated.

By definition of D_1, the neighborhood of every non-green vertex can be dominated by at most 3 other vertices. This holds true as well for the subset $N_R(v)$ of neighbors that still need to be dominated. Let us fix such a small dominating set for the red neighborhood of every non-green vertex.

For every $v \in V(G)\backslash D_1$, we fix $A_v \subseteq V(G)\backslash\{v\}$ such that:

$$N_R(v) \subseteq N[A_v] \text{ and } |A_v| \leqslant 3.$$

There are potentially many such sets A_v – we fix one such set arbitrarily. Let us stress that even though we could compute the sets A_v in a local algorithm (making decisions based on vertex ids), we only use these sets for our further argumentation and do not need to compute them.

4 Analyzing the local dominators

The second phase of our algorithm is inspired by results of Czygrinow et al. [5] and the greedy domination algorithm for biclique-free graphs of [16]. Czygrinow et al. [5] defined the notion of *pseudo-covers*, which provide a tool to carry out a fine grained analysis of vertices that can potentially belong to the sets A_v used to dominate the red neighborhood $N_R(v)$ of a vertex v. This tool can in fact be applied to much more general graphs than planar graphs, namely, to all graphs that exclude some complete bipartite graph $K_{t,t}$. A refined analysis for classes of bounded expansion was provided by Kublenz et al. [10]. We provide an even finer analysis for planar graphs on which we base a second phase of our distributed algorithm.

We first describe what our algorithm computes, and then provide bounds on the number of selected vertices. Intuitively, we select every pair of vertices with sufficiently many neighbors in common.

- For $v \in V(G)$ let $B_v := \{z \in V(G)\backslash\{v\} : |N_R(v) \cap N_R(z)| \geqslant 10\}$.
- Let W be the set of vertices $v \in V(G)$ such that $B_v \neq \varnothing$.
- Let $D_2 := \bigcup_{v\in W} (\{v\} \cup B_v)$.

Once D_1 has been computed in the previous phase, 2 more rounds of communication are enough to compute the sets B_v and D_2. Before we update the residual degrees, let us analyze the sets B_v and D_2. First note that the definition is symmetric: since $N_R(v) \cap N_R(z) = N_R(z) \cap N_R(v)$ we have for all $v, z \in V(G)$ if $z \in B_v$, then $v \in B_z$. In particular, if $v \in D_1$ or $z \in D_1$, then $N_R(v) \cap N_R(z) = \varnothing$, which immediately implies the following lemma.

Lemma 5. *We have $W \cap D_1 = \varnothing$ and for every $v \in V(G)$ we have $B_v \cap D_1 = \varnothing$.*

Now we prove that for every $v \in W$, the set B_v cannot be too big, and has nice properties.

Lemma 6. *For all vertices $v \in W$ we have*

- *$B_v \subseteq A_v$ (hence $|B_v| \leqslant 3$), and*
- *if $v \notin \hat{D}$, then $B_v \subseteq D$.*

Proof. Assume $A_v = \{v_1, v_2, v_3\}$ (a set of possibly not distinct vertices) and assume there exists $z \in V(G)\backslash\{v, v_1, v_2, v_3\}$ with $|N_R(v) \cap N_R(z)| \geqslant 10$. As v_1, v_2, v_3 dominate $N_R(v)$, and hence also $N_R(v) \cap N_R(z)$, there must be some v_i, $1 \leqslant i \leqslant 3$, with $|N_R(v) \cap N_R(z) \cap N[v_i]| \geqslant \lceil 10/3 \rceil \geqslant 4$. Therefore, $|N_R(v) \cap N_R(z) \cap N(v_i)| \geqslant 3$, which shows that $K_{3,3}$ is a subgraph of G, contradicting the assumption that G is planar.

If furthermore $v \notin \hat{D}$, by definition of $\hat{D}$, we can find w_1, w_2, w_3 from D that dominate $N(v)$, and in particular $N_R(v)$. If $z \in V(G)\backslash\{v, w_1, w_2, w_3\}$ with $|N_R(v) \cap N_R(z)| \geqslant 10$ we can argue as above to obtain a contradiction.

Let us now analyze the size of D_2. For this we refine the set D_2 and define

1. $D_2^1 := \bigcup_{v \in W \cap D}(\{v\} \cup B_v)$,
2. $D_2^2 := \bigcup_{v \in W \cap (\hat{D} \setminus D)}(\{v\} \cup B_v)$, and
3. $D_2^3 := \bigcup_{v \in W \setminus (D \cup \hat{D})}(\{v\} \cup B_v)$.

Obviously $D_2 = D_2^1 \cup D_2^2 \cup D_2^3$. We now bound the size of the refined sets D_2^1, D_2^2 and D_2^3.

Lemma 7. $|D_2^1 \setminus D| \leqslant 3\gamma$.

Proof. We have

$$|D_2^1 \setminus D| = |\bigcup_{v \in W \cap D}(\{v\} \cup B_v) \setminus D| \leqslant |\bigcup_{v \in W \cap D} B_v| \leqslant \sum_{v \in W \cap D} |B_v|.$$

By Lemma 6 we have $|B_v| \leqslant 3$ for all $v \in W$ and as we sum over $v \in W \cap D$ we conclude that the last term has order at most 3γ.

Lemma 8. *$D_2^2 \subseteq \hat{D}$ and therefore $|D_2^2 \setminus D| < 4\gamma$.*

Proof. Let $v \in \hat{D} \setminus D$ and let $z \in B_v$. By symmetry, $v \in B_z$ and according to Lemma 6, if $z \notin \hat{D}$, then $v \in D$. Since this is not the case, we conclude that $z \in \hat{D}$. Hence $B_v \subseteq \hat{D}$ and, more generally, $D_2^2 \subseteq \hat{D}$. Finally, according to Lemma 4 we have $|\hat{D} \setminus D| < 4\gamma$.

Finally, the set D_2^3, which appears largest at first glance, was actually already counted, as shown in the next lemma.

Lemma 9. $D_2^3 \subseteq D_2^1$.

Proof. If $v \notin \hat{D}$, then $B_v \subseteq D$ by Lemma 6. Hence $v \in B_z$ for some $z \in D$, and $v \in D_2^1$.

Again, it is intuitively clear that the situation when the sets D_2^i are large does not lead to the worst case for the overall algorithm. For example, when D_2^1 is large we have added many vertices of the optimum dominating set D. For a formal analysis, we analyze the number of vertices of D that have been selected so far.

Let $\epsilon \in [0, 1]$ be such that $|(D_1 \cup D_2) \cap D| = \epsilon\gamma$.

Lemma 10. *We have $|D_1 \cup D_2| < 4\gamma + 4\epsilon\gamma$.*

Proof. By Lemma 9 we have $D_2^3 \subseteq D_2^1$, hence, $D_1 \cup D_2 = D_1 \cup D_2^1 \cup D_2^2$. By Lemma 4 we have $D_1 \subseteq \hat{D}$ and by Lemma 8 we also have $D_2^2 \subseteq \hat{D}$, hence $D_1 \cup D_2^2 \subseteq \hat{D}$. Again by Lemma 4, $|\hat{D} \setminus D| < 4\gamma$ and therefore $|(D_1 \cup D_2^2) \setminus D| < 4\gamma$.

We have $W \cap D \subseteq D_2^1 \cap D$, hence with Lemma 6 we conclude that

$$|D_2^1 \setminus D| \leqslant \Big| \bigcup_{v \in D \cap D_2^1} B_v \Big| \leqslant \sum_{v \in D \cap D_2^1} |B_v| \leqslant 3\epsilon\gamma,$$

hence $(D_1 \cup D_2) \setminus D < 4\gamma + 3\epsilon\gamma$. Finally, $D_1 \cup D_2 = (D_1 \cup D_2) \setminus D \cup ((D_1 \cup D_2) \cap D)$ and with the definition of ϵ we conclude $|D_1 \cup D_2| < 4\gamma + 4\epsilon\gamma$.

The analysis of the next and final step of the algorithm will actually show that the worst case is obtained when $\epsilon = 0$.

We now update the residual degrees, that is, we update R as $V(G) \setminus N[D_1 \cup D_2]$ and for every vertex the number $\delta_R(v) = N(v) \cap R$ accordingly.

5 Greedy domination in planar graphs of maximum residual degree

We will show next that after the first two phases of the algorithm we are in the very nice situation where all residual degrees are small. This will allow us to proceed in a greedy manner.

Lemma 11. *For all $v \in V(G)$ we have $\delta_R(v) \leqslant 30$.*

Proof. First, every vertex of $D_1 \cup D_2$ has residual degree 0. Assume that there is a vertex v of residual degree at least 31. As v is not in D_1, its 31 non-dominated neighbors are dominated by a set A_v of at most 3 vertices. Hence there is a vertex z (not in D_1 nor D_2) with $|N_R(v) \cap N_R[z]| \geqslant \lceil 31/3 \rceil = 11$, hence, $|N_R(v) \cap N_R(z)| \geqslant 10$. This contradicts that v is not in D_2.

In the light of Lemma 11, we could now simply choose D_3 as the set of elements not in $N[D_1 \cup D_2]$. We would get a constant factor approximation, but not a very good one. Instead, we now start to simulate the classical greedy algorithm, which in each round selects a vertex of maximum residual degree. Here, we let all non-dominated vertices that have a neighbor of maximum residual degree choose such a neighbor as its dominator (or if they have maximum residual degree themselves, they may choose themselves). In general this is not possible for a LOCAL algorithm, however, as we established a bound on the maximum degree we can proceed as follows. We let $i = 30$. Every red vertex that has at least one neighbor of residual degree 30 arbitrarily picks one of them and elects it to the dominating set. Then every vertex recomputes its residual degree and i is set to 29. We continue until i reaches 0 when all vertices are dominated. More formally, we define several sets as follows.

For $30 \geqslant i \geqslant 0$, for every $v \in R$ in parallel:

if there is some u with $\delta_R(u) = i$ and ($\{u, v\} \in E(G)$ or $u = v$), then
$dom_i(v) := \{u\}$ (pick one such u arbitrarily),
$dom_i(v) := \varnothing$ otherwise.

- $R_i := R$ — *What currently remains to be dominated*
- $\Delta_i := \bigcup_{v \in R} dom_i(v)$ — *What we pick in this step*
- $R := R \setminus N[\Delta_i]$ — *Update red vertices*

Finally, $D_3 := \bigcup_{1 \leqslant i \leqslant 30} \Delta_i$.

Let us first prove that the algorithm in fact computes a dominating set.

Lemma 12. *When the algorithm has finished the iteration with parameter $i \geqslant 1$, then all vertices have residual degree at most $i - 1$.*

In particular, after finishing the iteration with parameter 1, there is no vertex with residual degree 1 left and in the final round all non-dominated vertices choose themselves into the dominating set. Hence, the algorithm computes a dominating set of G.

Proof. By induction, before the iteration with parameter i, all vertices have residual at most i. Assume v has residual degree i before the iteration with parameter i. In that iteration, all non-dominated neighbors of v choose a dominator (possibly v, then the statement is trivial), hence, are removed from R. It follows that the residual degree of v after the iteration is 0. Hence, after this iteration and before the iteration with parameter $i - 1$, we are left with vertices of residual degree at most $i - 1$.

We now analyze the sizes of the sets Δ_i and R_i. The first lemma follows from the fact that every vertex chooses at most one dominator.

Lemma 13. *For every $i \leqslant 30$, $\sum_{j \leqslant i} |\Delta_j| \leqslant |R_i|$.*

Proof. The vertices of R_i are those that remain to be dominated in the last i rounds of the algorithm. As every vertex that remains to be dominated chooses at most one dominator in one of the rounds $j \leqslant i$, the statement follows.

As the vertices of D that still dominate non-dominated vertices also have bounded residual degree, we can conclude that not too many vertices remain to be dominated.

Lemma 14. *For every* $i \leqslant 30$, $|R_i| \leqslant (i+1)(1-\epsilon)\gamma$.

Proof. First note that for every i, $D\backslash(D_1 \cup D_2 \cup \bigcup_{j>i} \Delta_j)$ is a dominating set for R_i; additionally each vertex in this set has residual degree at most i. And finally, this set is a subset of $D\backslash(D_1 \cup D_2)$. Hence by the definition of ϵ, we get that $|D\backslash(D_1 \cup D_2 \cup \bigcup_{j>i} \Delta_j)| \leqslant (1-\epsilon)\gamma$. As every vertex dominates its residual neighbors and itself, we conclude $|R_i| \leqslant (i+1)(1-\epsilon)\gamma$.

The next lemma shows that we cannot pick too many vertices of high residual degree. This follows from the fact that planar graphs have bounded edge density.

Lemma 15. *For every* $7 \leqslant i \leqslant 30$, $|\Delta_i| \leqslant \frac{3|R_i|}{i-6}$.

Proof. Let $7 \leqslant i \leqslant 30$ be an integer. We bound the size of Δ_i by a counting argument, using that G (as well as each of its subgraphs) is planar, and can therefore not have to many edges.

Let $J := G[\Delta_i]$ be the subgraph of G induced by the vertices of Δ_i, which all have residual degree i. Let $K := G[\Delta_i \cup (N[\Delta_i] \cap R_i)]$ be the subgraph of G induced by the vertices of Δ_i together with the red neighbors that these vertices dominate.

As J is planar, $|E(J)| < 3|V(J)| = 3|\Delta_i|$. As every vertex of J has residual degree exactly i, we get $|E(K)| \geqslant i\Delta_i - |E(J)| > (i-3)|\Delta_i|$ (we have to subtract $|E(J)|$ to not count twice the edges of K that are between two vertices of J). We also have that $|V(K)| \leqslant |V(J)| + |R_i|$. We finally apply Euler's formula again to K and get that $|E_K| < 3|V_K|$ hence $(i-3)|\Delta_i| < 3|\Delta_i| + 3|R_i|$. Therefore $|\Delta_i| < \frac{3|R_i|}{i-6}$.

Finally, we can give a lower bound on how many elements are newly dominated by the chosen elements of high residual degree.

Lemma 16. *For every* $1 \leqslant i \leqslant 29$, $|R_i| \leqslant |R_{i+1}| - \frac{(i-5)|\Delta_{i+1}|}{3}$.

Proof. Similarly to the proof of Lemma 15 (by replacing i by $i+1$), we define $J := G[\Delta_{i+1}]$ and $K := G[\Delta_{i+1} \cup (N[\Delta_{i+1}] \cap R_{i+1})]$.

We then replace the bound $|V(K)| \leqslant |V(J)| + |R_{i+1}|$ by $|V(K)| \leqslant |V(J)| + |N[\Delta_{i+1}] \cap R_{i+1}|$.

We then get:

$$|E_K| \leqslant 3|V_K|,$$

$$(i+1)|\Delta_{i+1}| - 3|\Delta_{i+1}| \leqslant 3(|\Delta_{i+1}| + |N[\Delta_{i+1}] \cap R_{i+1}|), \text{ and}$$

$$|N[\Delta_{i+1}] \cap R_{i+1}| \geqslant \frac{(i+1-6)|\Delta_{i+1}|}{3}.$$

Now, as $R_i = R_{i+1} \backslash N[\Delta_{i+1}]$, we have $|R_i| \leqslant |R_{i+1}| - |N[\Delta_{i+1}] \cap R_{i+1}| \leqslant |R_{i+1}| - \frac{(i+1-6)|\Delta_{i+1}|}{3}$.

We now formulate (and present in Appendix A) a linear program to maximize $|D_3|$ under these constraints. As a result we conclude the following lemma.

Lemma 17. $|D_3| \leqslant 15.9(1-\epsilon)\gamma$.

6 Summarizing the planar case

We already noted that the definition of D_3 implies that $D_1 \cup D_2 \cup D_3$ is a dominating set of G. We now conclude the analysis of the size of this computed set. First, by Lemma 10 we have $|D_1 \cup D_2| < 4\gamma + 4\epsilon\gamma$. Then, by Lemma 17 we have $|D_3| \leqslant 15.9(1-\epsilon)\gamma$. Therefore $|D_1 \cup D_2 \cup D_3| < 19.9\gamma - 11.9\epsilon\gamma$. As $\epsilon \in [0,1]$, this is maximized when $\epsilon = 0$. Hence $|D_1 \cup D_2 \cup D_3| < 19.9\gamma$.

Theorem 1. *There exists a distributed LOCAL algorithm that, for every planar graph G, computes in a constant number of rounds a dominating set of size at most $20\gamma(G)$.*

7 Restricted classes of planar graphs

In this section we further restrict the input graphs, requiring e.g. planarity together with a lower bound on the girth. Our algorithm works exactly as before, however, using different parameters. From the different edge densities and chromatic numbers of the restricted graphs we will then derive different constants and as a result a better approximation factor. Throughout this section we use the same notation as in the first part of the paper and state in the adapted lemmas with the same numbers as in the first part the adapted sizes of the respective sets.

As in the general case in the first phase we begin by computing the set D_1 and analyzing it in terms of the auxiliary set $\hat{D}$.

Adapted Corollary 3.

1. *If G is bipartite, then $|\hat{D}\backslash D| < 2\gamma$.*
2. *If G is triangle-free, outerplanar, or has girth 5, then $|\hat{D}\backslash D| < 3\gamma$.*

Proof. This is immediate from Lemma 1 and Lemma 2.

The inclusion $D_1 \subseteq \hat{D}$ continues to hold and the bound on the sizes as stated in the next lemma is again a direct consequence of the corollary.

Adapted Lemma 4. *We have $D_1 \subseteq \hat{D}$, and*

1. *if G is bipartite, then $|\hat{D}\backslash D| < 2\gamma$ and $|\hat{D}| < 3\gamma$.*
2. *if G is triangle-free, outerplanar, or has girth 5, then $|\hat{D}\backslash D| < 3\gamma$ and $|\hat{D}| < 4\gamma$.*

In case of triangle-free planar graphs (in particular in the case of bipartite planar graphs) we proceed with the second phase exactly as in the second phase of the general algorithm (Section 4), however, the parameter 10 is replaced by the parameter 7. In case of planar graphs of girth at least five or outerplanar graphs, we simply set $D_2 = \varnothing$.

If G is triangle-free:

- For $v \in V(G)$ let $B_v := \{z \in V(G)\backslash\{v\} : |N_R(v) \cap N_R(z)| \geqslant 7\}$.
- Let W be the set of vertices $v \in V(G)$ such that $B_v \neq \varnothing$.
- Let $D_2 := \bigcup_{v \in W} (\{v\} \cup B_v)$.

If G has girth at least 5 or G is outerplanar, let $D_2 = \varnothing$.

Lemma 5 is based only on the definition of B_v and W and does not use particular properties of planar graphs, hence, it also holds in the restricted case and we recall it for convenience.

Lemma 5. *We have $W \cap D_1 = \varnothing$ and for every $v \in V(G)$ we have $B_v \cap D_1 = \varnothing$.*

The next lemma uses the triangle-free property.

Adapted Lemma 6. *If G is triangle-free, then for all vertices $v \in W$ we have*

- *$B_v \subseteq A_v$ (hence $|B_v| \leqslant 3$), and*
- *if $v \notin \hat{D}$, then $B_v \subseteq D$.*

Proof. Assume $A_v = \{v_1, v_2, v_3\}$ and assume there is $z \in V(G)\backslash\{v, v_1, v_2, v_3\}$ with $|N_R(v) \cap N_R(z)| \geqslant 7$. As the vertices v_1, v_2, v_3 dominate $N_R(v)$, and hence $N_R(v) \cap N_R(z)$, there must be some v_i, $1 \leqslant i \leqslant 3$, with $|N_R(v) \cap N_R(z) \cap N[v_i]| \geqslant \lceil 7/3 \rceil \geqslant 3$. Then on of the following holds: either $|N_R(v) \cap N_R(z) \cap N(v_i)| \geqslant 3$, or $|N_R(v) \cap N_R(z) \cap N(v_i)| = 2$. The first case shows that $K_{3,3}$ is a subgraph of G contradicting the assumption that G is planar. The second case implies that $v_i \in N_R(v)$. In this situation, by picking $w \in N_R(v) \cap N_R(z) \cap N(v_i)$, we get that (v, v_i, w) is a triangle, hence we also reach a contradiction.

If furthermore $v \notin \hat{D}$, by definition of $\hat{D}$, we can find w_1, w_2, w_3 from D that dominate $N(v)$, and in particular $N_R(v)$. If $z \in V(G)\backslash\{v, w_1, w_2, w_3\}$ with $|N_R(v) \cap N_R(z)| \geqslant 7$ we can argue as above to obtain a contradiction.

For our analysis we again split D_2 into three sets D_2^1, D_2^2 and D_2^3. The next lemmas hold also for the restricted cases. We repeat them for convenience.

Adapted Lemma 7 *If G is triangle-free, then $|D_2^1 \backslash D| \leqslant 3\gamma$.*

Adapted Lemma 8 *If G is triangle-free, then $D_2^2 \subseteq \hat{D}$ and therefore $|D_2^2 \backslash D| < 3\gamma$.*

Adapted Lemma 9 *If G is triangle-free, then $D_2^3 \subseteq D_2^1$.*

Again, for a fine analysis, we analyze the number of vertices of D that have been selected so far and let $\epsilon \in [0,1]$ be such that $|(D_1 \cup D_2) \cap D| = \epsilon\gamma$.

Adapted Lemma 10

1. *If G is bipartite, then $|D_1 \cup D_2| < 2\gamma + 4\epsilon\gamma$.*
2. *If G is triangle-free, then $|D_1 \cup D_2| < 3\gamma + 4\epsilon\gamma$.*
3. *If G has girth at least 5 or is outerplanar, then $|D_1 \cup D_2| < 3\gamma + \epsilon\gamma$.*

Proof. If G is outerplanar or G has girth at least 5, then $D_2 = \varnothing$. By Adapted Lemma 4 we have $D_1 \subseteq \hat{D}$ and $|\hat{D} \backslash D| < 3\gamma$, hence $(D_1 \cup D_2) \backslash D < 3\gamma$.

If G is triangle-free, by Adapted Lemma 9 we have $D_2^3 \subseteq D_2^1$, hence, $D_1 \cup D_2 = D_1 \cup D_2^1 \cup D_2^2$. By Adapted Lemma 4 we have $D_1 \subseteq \hat{D}$ and by Adapted Lemma 8 we also have $D_2^2 \subseteq \hat{D}$, hence $D_1 \cup D_2^2 \subseteq \hat{D}$. Again by Adapted Lemma 4, if G is bipartite, then $|\hat{D} \backslash D| < 2\gamma$, therefore $|(D_1 \cup D_2^2) \backslash D| < 2\gamma$, and if G is triangle-free, then $|\hat{D} \backslash D| < 3\gamma$, therefore $|(D_1 \cup D_2^2) \backslash D| < 3\gamma$. We have $W \cap D \subseteq D_2^1 \cap D$, hence with Adapted Lemma 6 we conclude that

$$|D_2^1 \backslash D| \leqslant \Big| \bigcup_{v \in D \cap D_2^1} B_v \Big| \leqslant \sum_{v \in D \cap D_2^1} |B_v| \leqslant 3\epsilon\gamma,$$

hence $(D_1 \cup D_2) \backslash D < 2\gamma + 3\epsilon\gamma$ if G is bipartite and $(D_1 \cup D_2) \backslash D < 3\gamma + 3\epsilon\gamma$ if G is triangle-free.

Finally, $D_1 \cup D_2 = (D_1 \cup D_2) \backslash D \cup (D_1 \cup D_2) \cap D$ and with the definition of ϵ we conclude

1. $|D_1 \cup D_2| < 2\gamma + 4\epsilon\gamma$ if G is bipartite,
2. $|D_1 \cup D_2| < 3\gamma + 4\epsilon\gamma$ if G is triangle-free,
3. $|D_1 \cup D_2| < 3\gamma + \epsilon\gamma$ if G has girth at least 5 or is outerplanar.

Again, we now update the residual degrees, that is, we update R as $V(G) \backslash N[D_1 \cup D_2]$ and for every vertex the number $\delta_R(v) = N(v) \cap R$ accordingly and proceed with the third phase.

Adapted Lemma 11.

1. *If G is triangle-free, then for all $v \in V(G)$ we have $\delta_R(v) \leqslant 18$.*
2. *If G has girth at least 5, then for all $v \in V(G)$ we have $\delta_R(v) \leqslant 3$.*
3. *If G is outerplanar, then for all $v \in V(G)$ we have $\delta_R(v) \leqslant 9$.*

Proof. Every vertex of $D_1 \cup D_2$ has residual degree 0, hence, we need to consider only vertices that are not in D_1 or D_2.

First assume that the graph is triangle-free and assume that there is a vertex v of residual degree at least 19. As v is not in D_1, its 19 non-dominated neighbors are dominated by a set A_v of at most 3 vertices. Hence, there is vertex z (not in D_1 nor D_2) dominating at least $\lceil 19/3 \rceil = 7$ of them. Here, z cannot be one of these 7 vertices, otherwise it would be connected to v and there would be a triangle in the graph. Therefore we have $|N_R(v) \cap N_R(z)| \geqslant 7$, contradicting that v is not in D_2.

Now assume that G has girth at least 5 and assume that there is a vertex v of residual degree at least 4. As v is not in D_1, its 4 non-dominated neighbors are dominated by a set A_v of at most 3 vertices. Hence, there is vertex z (not in D_1 nor D_2) dominating at least $\lceil 4/3 \rceil = 2$ of them. Here, z cannot be one of these 2 vertices, otherwise it would be connected to v and there would be a triangle in the graph. However, z can also not be any other vertex, as otherwise we find a cycle of length 4, contradicting that G has girth at least 5.

Finally, assume that G is outerplanar and assume that there is a vertex v of residual degree at least 10. As v is not in D_1, its 10 non-dominated neighbors are dominated by a set A_v of at most 3 vertices. Hence, there is vertex z (not in D_1 nor D_2) dominating at least $\lceil 10/3 \rceil = 4$ of them. Therefore $|N(v) \cap N(z)| \geqslant 3$, and we find a $K_{2,3}$ as a subgraph, contradicting that G is outerplanar.

We proceed to compute a dominating set of the remaining vertices as before for the respective number of rounds.

Adapted Lemma 13 *If G is triangle-free or outerplanar, for every $1 \leqslant i$, $\sum_{j \leqslant i} |\Delta_j| \leqslant |R_i|$.*

Adapted Lemma 14 *If G is triangle-free or outerplanar, for every $1 \leqslant i$, $|R_i| \leqslant (i+1)(1-\epsilon)\gamma$.*

Adapted Lemma 15 *If G is triangle-free or outerplanar, for every $5 \leqslant i$, $|\Delta_i| \leqslant \frac{2|R_i|}{i-4}$.*

Adapted Lemma 16 *If G is triangle-free or outerplanar, for every $1 \leqslant i$, $|R_i| \leqslant |R_{i+1}| - \frac{(i-3)|\Delta_{i+1}|}{2}$.*

The proofs of Adapted Lemma 13 to 16 are copies of the ones for Lemmas 13 to 16, with the execption that the edge density of 3 for planar graphs if now replaced by 2 for triangle-free and outerplanar. Similarly to Lemma 17 we formulate (and present in Appendix A) a linear program to maximize $|D_3|$ under these constraints, yielding the following lemma.

Adapted Lemma 17.

1. *If G is triangle-free, then $|D_3| \leqslant 10.5(1-\epsilon)\gamma$.*
2. *If G has girth at least 5, then $|D_3| \leqslant 4(1-\epsilon)\gamma$.*
3. *If G is outerplanar, then $|D_3| \leqslant 8.6(1-\epsilon)\gamma$.*

Theorem 2. *There exists a distributed LOCAL algorithm that, for every triangle free planar graph G, computes in a constant number of rounds a dominating set of size at most $14\gamma(G)$.*

Proof. By Adapted Lemma 10 we have $|D_1 \cup D_2| < 3\gamma + 4\epsilon\gamma$. Then, by Adapted Lemma 17 we have $|D_3| \leqslant 10.5(1-\epsilon)\gamma$. Therefore $|D_1 \cup D_2 \cup D_3| < 13.5\gamma - 6.5\epsilon\gamma$. As $\epsilon \in [0,1]$, this is maximized when $\epsilon = 0$. Hence $|D_1 \cup D_2 \cup D_3| < 13.5\gamma$.

Theorem 3. *There exists a distributed LOCAL algorithm that, for every bipartite planar graph G, computes in a constant number of rounds a dominating set of size at most $13\gamma(G)$.*

Proof. By Adapted Lemma 10 we have $|D_1 \cup D_2| < 2\gamma + 4\epsilon\gamma$. Then, by Adapted Lemma 17 we have $|D_3| \leqslant 10.5(1-\epsilon)\gamma$. Therefore $|D_1 \cup D_2 \cup D_3| < 12.5\gamma - 6.5\epsilon\gamma$. As $\epsilon \in [0,1]$, this is maximized when $\epsilon = 0$. Hence $|D_1 \cup D_2 \cup D_3| < 12.5\gamma$.

Theorem 4. *There exists a distributed LOCAL algorithm that, for every planar graph G of girth at least 5, computes in a constant number of rounds a dominating set of size at most $7\gamma(G)$.*

Proof. By Adapted Lemma 10 we have $|D_1 \cup D_2| < 3\gamma + \epsilon\gamma$. Then, by Adapted Lemma 17 we have $|D_3| \leqslant 4(1-\epsilon)\gamma$. Therefore $|D_1 \cup D_2 \cup D_3| < 7\gamma - 3\epsilon\gamma$. As $\epsilon \in [0,1]$, this is maximized when $\epsilon = 0$. Hence $|D_1 \cup D_2 \cup D_3| < 7\gamma$.

Theorem 5. *There exists a distributed LOCAL algorithm that, for every outerplanar graph G, computes in a constant number of rounds a dominating set of size at most $12\gamma(G)$.*

Proof. By Adapted Lemma 10 we have $|D_1 \cup D_2| < 3\gamma + \epsilon\gamma$. Then, by Adapted Lemma 17 we have $|D_3| \leqslant 8.6(1-\epsilon)\gamma$. Therefore $|D_1 \cup D_2 \cup D_3| < 11.6\gamma - 7.6\epsilon\gamma$. As $\epsilon \in [0,1]$, this is maximized when $\epsilon = 0$. Hence $|D_1 \cup D_2 \cup D_3| < 11.6\gamma$.

8 Conclusion

We provided a new LOCAL algorithm that computes a 20-approximation of a minimum dominating set in a planar graph in a constant number of rounds. Started with different parameters, the algorithm works also for several restricted cases of planar graphs. We showed that it computes a 14-approximation for triangle-free planar graphs, a 13-approximation for bipartite planar graphs, a 7-approximation for planar graphs of girth 5 and a 12-approximation for outerplanar graphs. In all cases except for the outerplanar case, where an optimal bound of 5 was already known, our algorithm improves on the previously best known approximation factors. This improvement is most significant in the case of general planar graphs, where the previously best known factor was 52. While we could tighten the gap between the best known lower bound of 7 and upper bound of 52, there is still some room for improvement. We believe that the optimum approximation rate is much closer to 7 than to 20.

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A The linear program

In this final section we present our formulation of the constraints as a linear program as well as the resulting bounds on how many vertices of the specific residual degrees can be found in the worst case. We formulate the constraints of Lemmas 13 to 16 in a straight forward way and remove the $(1-\epsilon)\gamma$ factor, which is then added to the result. This reasoning is correct thanks to the fact that all constraints are linear equations; we formally prove it below.

Define $r_i := \frac{|R_i|}{(1-\epsilon)\gamma}$ and $d_i := \frac{|\Delta_i|}{(1-\epsilon)\gamma}$. Then the constraints of Lemmas 13 to 16 imply respectively:

- For every $0 \leqslant i \leqslant 30$: $r_i \geqslant \sum_{j \leqslant i} d_j$.
- For every $0 \leqslant i \leqslant 30$: $r_i \leqslant i + 1$.
- For every $7 \leqslant i \leqslant 30$: $d_i \leqslant \frac{3r_i}{i-6}$.
- For every $0 \leqslant i \leqslant 29$: $r_i \leqslant r_{i+1} - \frac{(i-5)d_{i+1}}{3}$.

We then run the linear program with these variables; finally we provide the bound for D_3 using:

$$|D_3| = \sum_{i \leqslant 30} |\Delta_i| = \sum_{i \leqslant 30} d_i(1-\epsilon)\gamma = (1-\epsilon)\gamma \sum_{i \leqslant 30} d_i.$$

Before showing the code and the results, we briefly explain what we expect as a result for these linear programs.

A.1 Interpretation of the results

In all four cases, our sets of equations yield similar looking results. The step 3 can roughly be decomposed into two.

First, for several values of i, we have very small $d[i]$. We exactly have $d[i]$ such that given $r[i] = i + 1$ we get $r[i-1] = i$. Intuitively, picking less element in $d[i]$ is not the worst case as $r[i-1]$ cannot be bigger than i by Lemma 14. So it is “free” to take at least that many vertices. It is also not the worst case if more elements are picked, because then $r[i]$ would shrink drastically, making the forthcoming $d[j]_{j<i}$ much smaller.

Second, there is a turning point. It occurs a little bit above the average degree of planar graphs; so the number of vertices of degree 9 for example is not so small. This is when Lemma 13 become predominant: “Overall, we do not take more dominators that there are vertices to dominate.” So in one round every vertex gets picked and the algorithm stops. This turning point is $i = 9$ for planar graphs.

We did not manage to formally prove this statement, but it was confirmed for these cases by the linear programs.

A.2 The linear program for planar graphs

```
//the ranges i can have
range I  = 1..30;
range I2 = 1..29;
range I3 = 7..30;

//decision variables as arrays
dvar float+ d[I];
dvar float+ r[I];

//maximize the sum of A_i
maximize sum(i in I) (d[i]);

// our equations
subject to
{
  // By lemma 13
  forall(i in I) r[i] >= sum(x in 1..i)d[x];

  // By lemma 14
  forall(i in I) r[i] <= i+1;

  // By lemma 15
  forall(i in I3) d[i] <= (3 * r[i])  /  ( i-6 );

  // By Lemma 16
  forall(i in I2) r[i] <=  r[i+1]  -  ((( i-5 ) * d[i+1]) / 3);
}
```

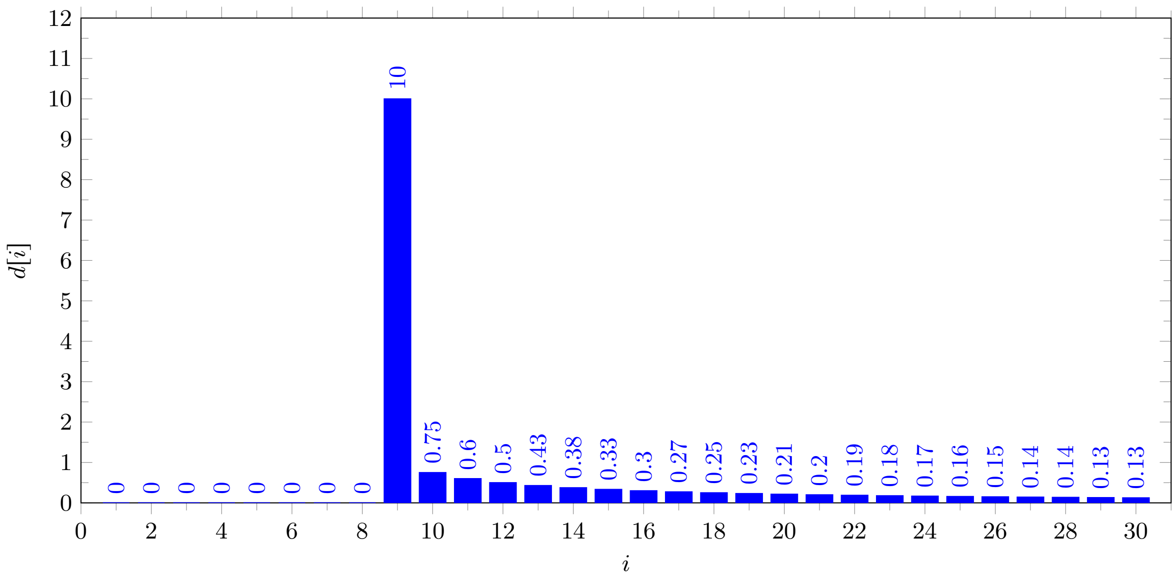

Fig. 1. The degree distribution in general planar graphs

A.3 The linear program for triangle-free planar graphs

```
//Tri-Free
  //the ranges i can have
  range I  = 1..18;
  range I2 = 1..17;
  range I3 = 5..18;

  //decision variables as arrays
  dvar float+ d[I];
  dvar float+ r[I];

  //maximize the sum of A_i
  maximize sum(i in I) (d[i]);

  // our equations
  subject to
  {
    // By Adapted lemma 13
    forall(i in I) r[i] >= sum(x in 1..i)d[x];

    // By Adapted lemma 14
    forall(i in I) r[i] <= i+1;

    // By Adapted lemma 15
    forall(i in I3) d[i] <= (2 * r[i])  /  ( i-4 );

    // By Adapted lemma 16
    forall(i in I2) r[i] <=  r[i+1]  -  ((( i-3 ) * d[i+1]) / 2);
  }
```

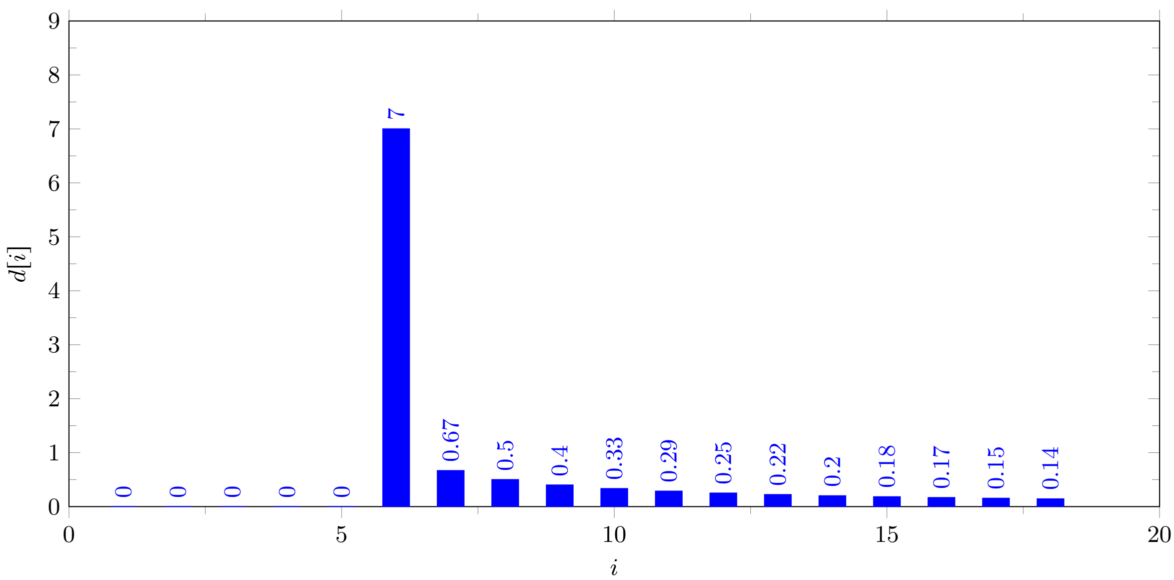

Fig. 2. The degree distribution in triangle-free planar graphs

A.4 The linear program for outerplanar graphs

```
//outerplanar

  //the ranges i can have
  range I  = 1..9;
  range I2 = 1..8;
  range I3 = 5..9;

  //decision variables as arrays
  dvar float+ d[I];
  dvar float+ r[I];

  //maximize the sum of A_i
  maximize sum(i in I) (d[i]);

  // our equations
  subject to
  {
    // By Adapted lemma 13
    forall(i in I) r[i] <= i+1;

    // By Adapted lemma 14
    forall(i in I2) r[i] <=  r[i+1]  -  ((( i-3 ) * d[i+1]) / 2);

    // By Adapted lemma 15
    forall(i in I3) d[i] <= (2 * r[i])  /  ( i-4 );

    // By Adapted lemma 16
    forall(i in I) r[i] >= sum(x in 1..i)d[x];
  }
```

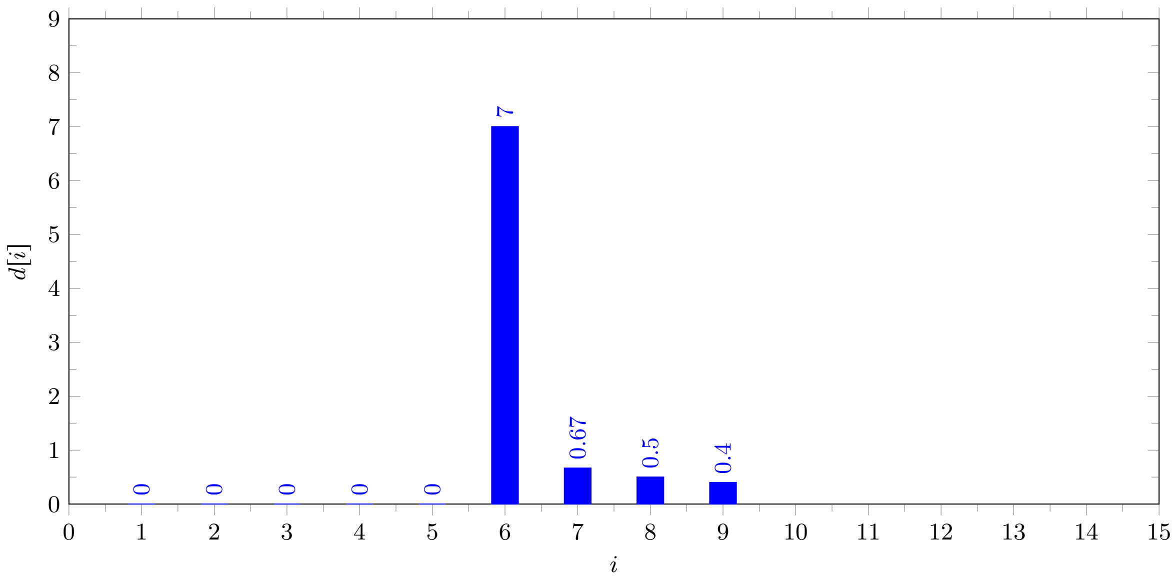

Fig. 3. The degree distribution in outerplanar graphs

A.5 The linear program for planar graphs of girth 5

In this case, the Adapted Lemma 13 to 16 can be slightly improved, as the edge density of planar graphs of girth 5 is at most 5/3. This is however not so useful. As shown below, the linear constraints do not yield something better than simply picking all 4γ non dominated vertices.

```
//girth5
  //the ranges i can have
  range I = 1..3;
  range I2 = 1..2;
  range I3 = 4..3;

  //decision variables as arrays
  dvar float+ d[I];
  dvar float+ r[I];

  //maximize the sum of A_i
  maximize sum(i in I) (d[i]);

  // our equations
  subject to
  {
    forall(i in I) r[i] >= sum(x in 1..i)d[x];

    forall(i in I) r[i] <= i+1;

    forall(i in I3) d[i] <= ((5 * r[i])  /  ( 3 * i -10 ));

    forall(i in I2) r[i] <=  r[i+1]  -  ((( 3*i-7 ) /5 )* d[i+1]);
  }
```

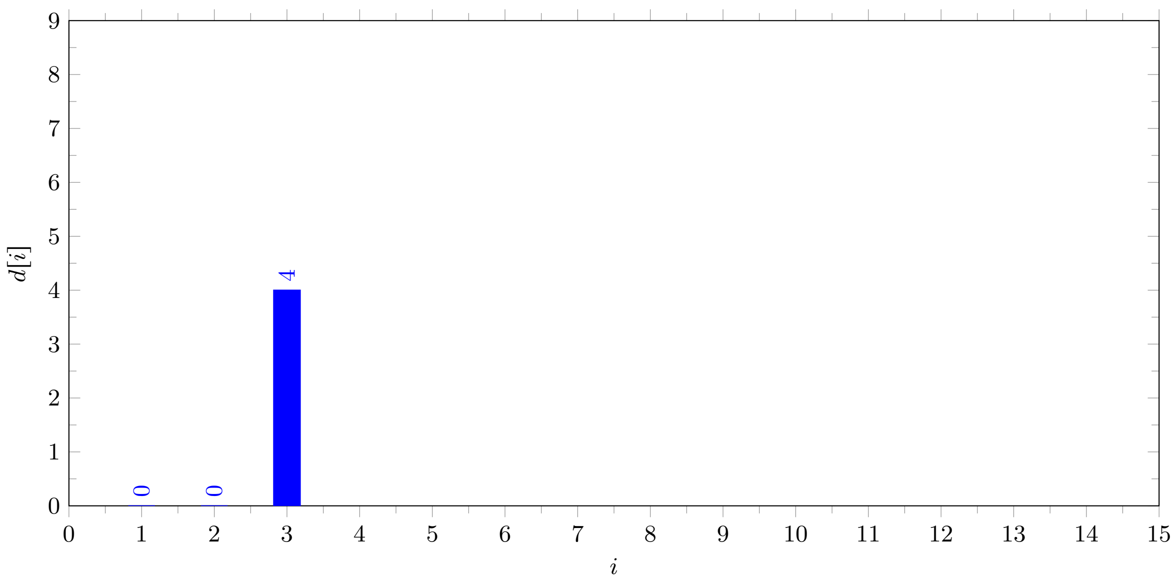

Fig. 4. The degree distribution in planar graphs of girth 5.