Saturated Kripke Structures as Vietoris Coalgebras

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Abstract. We show that the category of coalgebras for the compact Vietoris endofunctor \mathbb{V} on the category *Top* of topological spaces and continuous mappings is isomorphic to the category of all modally saturated Kripke structures. Extending a result of Bezhanishvili, Fontaine and Venema [4], we also show that Vietoris subcoalgebras as well as bisimulations admit topological closure and that the category of Vietoris coalgebras has a terminal object.

1 Introduction

The theory of coalgebras has provided Computer Science with a much needed general framework for dealing with all sorts of state based systems, with their structure theories and their logics. The varied types of systems, be they deterministic or nondeterministic automata, transition systems, probabilistic or weighted systems, neighborhood systems or the like, are fixed by the choice of an appropriate endofunctor F on the category of sets. From there on, with hardly any further assumptions, a mathematically pleasing structure theory and corresponding modal logics can be developed, see e.g. [20],[9],[10].

A particularly well behaved situation arises when choosing for F the finitepowerset functor $\mathbb{P}_{\omega}(-)$, perhaps augmented with a constant component $\mathbb{P}(\Phi)$ representing sets of atomic formulas. Coalgebras for the functor $\mathbb{P}_{\omega}(-) \times \mathbb{P}(\Phi)$ are precisely all image finite Kripke structures. Their logic is the standard modal logic based on the atomic formulae in Φ , and they possess a terminal coalgebra T, even though its description is always of an "indirect" nature (see [2,3,11]).

The well known Hennessy-Milner theorem[12], relating bisimulations and logical equivalence is a consequence of image finiteness and will not continue to hold for arbitrary Kripke structures, i.e. for coalgebras of type $\mathbb{P}(-) \times \mathbb{P}(\Phi)$, see [14].

The theory of modal logic knows of a class of Kripke structures, which lies between image finite structures and arbitrary Kripke structures and which continues to enjoy the Hennessy-Milner theorem. These structures are called *modally saturated*, or simply *m*-saturated[8]. Unfortunately, though, there seems to be no *Set*-functor *F*, somehow located in between $\mathbb{P}_{\omega}(-) \times \mathbb{P}(\Phi)$ and $\mathbb{P}(-) \times \mathbb{P}(\Phi)$, whose coalgebras would be just the *m*-saturated Kripke structures.

It is well known, that much of the theory of coalgebras can be generalized by turning to other categories than *Set*, provided they are co-complete and come with a reasonable factorization structure. Some of the examples studied in the literature replace the base category *Set* with the category *Rel* of sets and relations [15], with the category *Pos* of posets [1] or *Cpo* of complete partial orders, with the category *Meas* of measurable spaces [6][17], or the category *Stone* of Stone spaces. Relevant to this present work will be the works of Kupke, Kurz and Venema[16] as well as Bezhanishvili, Fontaine and Venema [4] regarding coalgebras for the Vietoris functor on the category of Stone spaces, i.e. compact zero-dimensional Hausdorff spaces with continuous mappings.

When extending the Vietoris functor from Stone spaces to arbitrary topological spaces \mathcal{X} , two natural choices offer themselves for the object map: either the collection of all closed subsets of \mathcal{X} or the collection of all compact subsets of \mathcal{X} , both equipped with appropriate topologies. Each of these choices yields a functor, generalizing the mentioned Vietoris functor on Stone spaces. Named the *lower Vietoris functor*, resp. the *compact Vietoris functor*, these endofunctors on the category *Top* of topological spaces with continuous functions were explored in recent work by Hofmann, Neves and Nora[13].

For our investigation of saturated Kripke structures, the *compact Vietoris* functor, which we denote by $\mathbb{V}(-)$, turns out to be appropriate. To model saturated Kripke-Structures, we choose the endofunctor $\mathbb{V}(-) \times \mathbb{P}(\Phi)$ on the category *Top* of topological spaces and continuous mappings, where the $\mathbb{P}(\Phi)$ -part is a constant component equipped with an appropriate topology, intuitively representing a set of atomic propositions, as above. We show that $\mathbb{V}(-) \times \mathbb{P}(\Phi)$ coalgebras precisely correspond to *m*-saturated Kripke structures, in fact there is an isomorphism of categories between the category of saturated Kripke structures and the category of all topological coalgebras for the compact Vietoris functor $\mathbb{V}(-) \times \mathbb{P}(\Phi)$.

This correspondence also yields a direct description of the terminal $\mathbb{V}(-) \times \mathbb{P}(\Phi)$ coalgebra, which seems to be simpler and more natural than the terminal $\mathbb{P}_{\omega}(-) \times \mathbb{P}(\Phi)$ coalgebra mentioned above: it is simply the Vietoris coalgebra corresponding to the canonical model of normal modal logic over Φ .

For Stone coalgebras we know from [4], that the topological closure \overline{R} of a bisimulation R is itself a bisimulation, again. We verify that their arguments carry over to the more general case of arbitrary Vietoris coalgebras, and we show also that a corresponding result holds true for subcoalgebras in place of bisimulations. For this we need to prepare some topological tools which may be interesting in their own right, relating convergence in the Vietoris space $\mathbb{V}(\mathcal{X})$ to convergence in the base space \mathcal{X} . In particular, topological nets $(\kappa_i)_{i \in I}$ converging to κ in the Vietoris space $\mathbb{V}(\mathcal{X})$ are shown to correspond, up to subnet formation, to nets $(a_i)_{i \in I}$ with $a_i \in \kappa_i$, converging in the base space \mathcal{X} to $a \in \kappa$, and conversely.

2 Preliminaries

For the remainder of this article, we shall fix a set Φ , the elements of which shall be called *propositional variables* or *atomic propositions*.

2.1 Kripke structures

Definition 1. A Kripke structure (also called Kripke model) $\mathcal{X} = (X, R, v)$ consists of a set X of states together with a relation $R \subseteq X \times X$, and a map $v : X \to \mathbb{P}(\Phi)$, where \mathbb{P} denotes the powerset functor.

In applications, X will typically be a set of possible states of a system, R is called the *transition relation*, describing the allowed transitions between states from X, and v is called the *valuation*, since v(x) consists of all atomic propositions true in state x. Instead of $(x, y) \in R$ we write $x \to y$ (or $x \to_R y$, if necessary). The idea is that $x \to y$ expresses that it is possible for the system to move from state x to state y. Instead of a relation, we can alternatively consider R as a map $R: X \to \mathbb{P}(X)$. This justifies the notation

$$R(x) := \{ y \in X \mid (x, y) \in R \},\$$

so R(x) denotes the successors of x, i.e. all states reachable from x in one step.

Definition 2. For subsets $U \subseteq X$ define $\langle R \rangle U := \{x \in X \mid \exists y \in U.(x,y) \in R\}$ and $[R]U := \{x \in X \mid \forall y \in X.(x,y) \in R \implies y \in U\}.$

Thus $x \in \langle R \rangle U$ if from x it is *possible* to reach an element of U in one step, and $x \in [R]U$ says that starting from x, each transition will *necessarily* take us to U. Obviously, $\langle R \rangle (X - U) = X - [R]U$, and $[R](X - U) = X - \langle R \rangle U$, so $\langle R \rangle$ and [R] are mutually expressible if complements are available.

2.2 Modal logic

Starting with the elements of Φ as *atomic formulae*, we obtain *modal formulae* by combining them with the standard boolean connectors \land , \lor , \neg or prefixing with the unary modal operator \Box . We also allow the usual shorthands $\bigvee_{i \in I_0} \phi_i$ and $\bigwedge_{i \in I_0} \phi_i$, whenever I_0 is a finite indexing set and each ϕ_i is a formula. Let \mathcal{L}_{Φ} be the set of all modal formulae so definable.

Validity $x \Vdash \phi$, is defined for $x \in X$ and $\phi \in \mathcal{L}_{\Phi}$ in the usual way (see [5]):

$$x \Vdash p : \iff p \in v(x), \quad \text{whenever } p \in \Phi$$
$$x \models \Box \phi : \iff \forall y \in X. \ (x \to y \implies y \models \phi).$$

For the boolean connectives \land, \lor, \neg , validity is defined as expected. We extend it to sets of formulas $\Sigma \subseteq \mathcal{L}_{\Phi}$, by

$$x \Vdash \Sigma : \iff \forall \phi \in \Sigma. \, x \Vdash \phi.$$

For any $x \in X$ we put $[\![x]\!] := \{\phi \in \mathcal{L}_{\Phi} \mid x \Vdash \phi\}$ and, similarly, for any $\phi \in \mathcal{L}_{\Phi}$ we set $[\![\phi]\!] := \{x \in X \mid x \Vdash \phi\}$. Two elements x, y from (possibly different) Kripke structures are called *logically equivalent* (in symbols $x \approx y$), if for each formula $\phi \in \mathcal{L}_{\Phi}$ we have $x \models \phi \iff y \models \phi$. Restricted to a single Kripke structure, \approx

is the kernel of the semantic map $x \mapsto [\![x]\!]$, and hence an equivalence relation. Similarly, two modal formulae ϕ, ψ are equivalent, and we write $\phi \equiv \psi$, if for each element x in any Kripke structure we have $x \Vdash \phi \iff x \Vdash \psi$.

Adding a further modality \diamond to our logical language by defining $\diamond \phi := \neg \Box \neg \phi$ provides more than only a convenient abbreviation. The resulting equivalences $\neg \Box \phi \equiv \diamond \neg \phi$ and $\neg \diamond \phi \equiv \Box \neg \phi$ allow one to push negations inside, just as the deMorgan laws permit to do so for \lor and \land , so that each modal formula becomes equivalent to a modal formula in *negation normal form* (*nnf*), where negations may only occur only in front of an atomic formula. We state this here for later reference:

Lemma 1. Every modal formula is equivalent to a modal formula in negation normal form (nnf).

2.3 Bisimulations

Definition 3. A bisimulation between two Kripke structures $\mathcal{X}_1 = (X_1, R_1, v_1)$ and $\mathcal{X}_2 = (X_2, R_2, v_2)$ is a relation $B \subseteq X_1 \times X_2$ such that for each $(x, y) \in B$:

1. $v_1(x) = v_2(y),$ 2. $\forall x' \in X_1. x \rightarrow_{R_1} x' \implies \exists y' \in X_2. y \rightarrow_{R_2} y' \land x'By',$ 3. $\forall y' \in X_2. y \rightarrow_{R_2} y' \implies \exists x' \in X_1. x \rightarrow_{R_1} x' \land x'By'.$

The empty relation $\emptyset \subseteq X_1 \times X_2$ is clearly a bisimulation, and the union of a family of bisimulations between \mathcal{X}_1 and \mathcal{X}_2 is again a bisimulation, hence there is a largest bisimulation between \mathcal{X}_1 and \mathcal{X}_2 , which we call $\sim_{\mathcal{X}_1, \mathcal{X}_2}$ or simply \sim , when \mathcal{X}_1 and \mathcal{X}_2 are clear from the context.

If $B_1 \subseteq X_1 \times X_2$ is a bisimulation between \mathcal{X}_1 and \mathcal{X}_2 , then the converse relation $B_1^{-1} \subseteq X_2 \times X_1$ is a bisimulation between \mathcal{X}_2 and \mathcal{X}_1 . Given another bisimulation B_2 between Kripke structures \mathcal{X}_2 and \mathcal{X}_3 then the relational composition $B_1 \circ B_2$ is a bisimulation between \mathcal{X}_1 and \mathcal{X}_3 .

A bisimulation on a Kripke structure $\mathcal{X} = (X, R, v)$ is a bisimulation between \mathcal{X} and itself. The identity $\Delta_X = \{(x, x) \mid x \in X\}$ is always a bisimulation on \mathcal{X} . Consequently, the largest bisimulation on \mathcal{X} is an equivalence relation, denoted by $\sim_{\mathcal{X}}$ or simply \sim . We say that two points $x \in X_1$ and $y \in X_2$ are bisimilar, if there exists a bisimulation B with x B y, which is the same as saying $x \sim y$. It is well known and easy to check by induction:

Lemma 2. Bisimilar points satisfy the same formulae $\phi \in \mathcal{L}_{\Phi}$.

A converse to this lemma was shown by Hennessy and Milner for the case of *image finite* Kripke structures. Here, an element x in a Kripke structure \mathcal{X} is called *image finite* if it has only finitely many successors, i.e. $\{x' \mid x \to x'\}$ is finite. \mathcal{X} is called image finite if each x from \mathcal{X} is image finite. Thus Hennessy and Milner proved in [12]:

Proposition 1. If $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ are image finite elements, then $x \sim y$ iff $x \approx y$.

2.4 Homomorphisms and congruences

Definition 4. A homomorphism $\varphi : \mathcal{X} \to \mathcal{Y}$ between Kripke structures $\mathcal{X} = (X, R_{\mathcal{X}}, v_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, v_{\mathcal{Y}})$ is a map whose graph

$$G(\varphi) := \{ (x, \varphi(x)) \mid x \in X \}$$

is a bisimulation.¹

We call \mathcal{X} a homomorphic preimage of \mathcal{Y} , and if φ is surjective (which we indicate by writing $\varphi : \mathcal{X} \to \mathcal{Y}$) then we call \mathcal{Y} a homomorphic image of \mathcal{X} . If $X \subseteq Y$ and the inclusion map $\iota : \mathcal{X} \to \mathcal{Y}$ is a homomorphism, then \mathcal{X} is called a Kripke substructure of \mathcal{Y} .

It is easy to check that a subset $X \subseteq Y$ with the restrictions of $R_{\mathcal{Y}}$ and $v_{\mathcal{Y}}$ to X is a substructure of \mathcal{Y} if only if $R_{\mathcal{X}}(x) \subseteq X$ for each $x \in X$. If $\varphi : \mathcal{X} \to \mathcal{Y}$ is a homomorphism, then its kernel

$$\ker \varphi := \{ (x, x') \in X \mid \varphi(x) = \varphi(x') \}$$

is called a *congruence relation*. This is clearly an equivalence relation and a bisimulation as well, since we can express it as a relation product of $G(\varphi)$, the graph of φ , with its converse $G(\varphi)^{-1}$ as

$$\ker \varphi = G(\varphi) \circ G(\varphi)^{-1}.$$

3 Saturated structures

The notion of *saturation* goes back to a similar concept of Fine in [7]. The terminology m-saturation (or modal saturation) was adopted by [5] and [8]:

Definition 5. An element x is m-saturated, if for each set Σ of formulas, such that each finite subset $\Sigma_0 \subseteq \Sigma$ is satisfied at some successor y_0 of x, there is a successor y of x satisfying all formulas in Σ . A Kripke structure is called m-saturated, if each of its elements is saturated.

In the following we shall find it convenient to informally use infinitary disjunctions $\bigvee_{i \in I} \phi_i$ – not as a logical expressions but as shorthands. In particular we write

$$x \Vdash \Box \bigvee_{i \in I} \phi_i$$

as an abbreviation for

$$\forall y. (x \to y \implies \exists i \in I. y \models \phi_i).$$

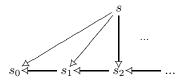
With this shorthand, the above definition can be reformulated:

¹ In the literature on Modal Logic (see e.g. [5],[8]), homomorphisms are usually called "bounded morphisms".

Lemma 3. An element x in a Kripke model $\mathcal{X} = (X, R, v)$ is m-saturated, if for each family $(\phi_i)_{i \in I}$ such that $x \Vdash \Box \bigvee_{i \in I} \phi_i$ there exists a finite subset $I_0 \subseteq I$ with $x \Vdash \Box \bigvee_{i \in I_0} \phi_i$.

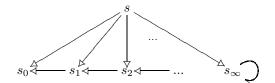
Image finite elements are clearly saturated, but they are not the only ones. Below, we consider two examples of Kripke structures. In both cases, we assume $v(x) := \emptyset$ for each x:

Example 1. On the set $S := \{s\} \cup \{s_i \mid i \in \mathbb{N}\}$ consider the relation $R = \{(s, s_i) \mid i \in \mathbb{N}\} \cup \{(s_{i+1}, s_i) \mid i \in \mathbb{N}\}$. Then for each s_i we have $s_i \Vdash \Box^{i+1} \bot$, but $s_i \nVdash \Box^{j} \bot$ for $j \leq i$. Therefore (S, R, v) is not saturated, since $s \Vdash \Box \bigvee_{i \in \mathbb{N}} (\Box^{i+1} \bot)$, but for no finite $I_0 \subseteq \mathbb{N}$ do we have $s \Vdash \Box \bigvee_{i \in I_0} (\Box^{i+1} \bot)$.



Next, we modify the above structure by adding a "point at infinity" s_{∞} together with a self-loop $s_{\infty} \rightarrow s_{\infty}$ to obtain the following structure:

Example 2.



The point at infinity changes the situation. We claim:

Lemma 4. The Kripke structure in Example 2 is saturated.

Proof. We first observe that for s_{∞} and any formula ϕ we have:

 $s_{\infty} \Vdash \Diamond \phi \iff s_{\infty} \Vdash \phi \iff s_{\infty} \Vdash \Box \phi.$

Next we prove for each formula ϕ :

Claim. If $s_{\infty} \Vdash \phi$, then there is some $k \in \mathbb{N}$ such that $s_i \Vdash \phi$ for each $i \geq k$.

We prove this claim by induction over the construction of nnf-formulae:

- For $\phi = \bot$ and $\phi = \top$, the claim is vacuously true. For $\phi = \phi_1 \land \phi_2$, from $s_{\infty} \Vdash \phi_1 \land \phi_2$, the hypothesis yields k_1 and k_2 such that $s_i \Vdash \phi_1$ for each $i \ge k_1$ and $s_i \Vdash \phi_2$ for each $i \ge k_2$. With $k = max(k_1, k_2)$ we obtain $s_i \Vdash \phi_1 \land \phi_2$ for $i \ge k$. For $\phi = \phi_1 \lor \phi_2$ we could similarly choose $k = min(k_1, k_2)$.
- For $\phi = \Box \phi_1$ we have $s_{\infty} \Vdash \phi \iff s_{\infty} \Vdash \phi_1$. By assumption, there is some k such that $s_i \Vdash \phi_1$ for each $i \ge k$. It follows that $s_i \Vdash \Box \phi_1$ for $i \ge k+1$. Similarly we argue for $\phi = \Diamond \phi_1$.

Now, to show that s in the structure of Example 2 is saturated, assume that $s \Vdash \Box \bigvee_{i \in I} \phi_i$, then there is some $i_{\infty} \in I$ such that $s_{\infty} \Vdash \phi_{i_{\infty}}$. The claim above provides a k such that for each $j \ge k$ we have $s_j \Vdash \phi_{i_{\infty}}$, and for each j < k there is some $i_j \in I$ with $s_j \Vdash \phi_{i_j}$. Altogether then with $I_0 := \{i_0, i_1, ..., i_{k-1}\} \cup \{i_{\infty}\}$ we have $s \Vdash \Box \bigvee_{i \in I_0} \phi_i$.

Thus s is saturated, and all other points in the structure are image finite, hence they are saturated, too.

We can extend Lemma 2 to "infinitary formulas" in the following sense:

Lemma 5. [Bisimulations preserve saturation] If $B \subseteq X_1 \times X_2$ is a bisimulation and $(x, y) \in B$, then x is saturated iff y is saturated.

Proof. Assume that x is saturated and $(x, y) \in B$. Suppose $y \Vdash \Box \bigvee_{i \in I} \phi_i$, then each y' with $y \to y'$ satisfies one of the formulas ϕ_i . Each x' with $x \to x'$ is bisimilar to some y' with $y \to y'$, so by Lemma 2 each x' satisfies one of the ϕ_i . This means that $x \Vdash \Box \bigvee_{i \in I} \phi_i$. By saturation of x there is a finite subset $I_0 \subseteq I$ with $x \Vdash \Box \bigvee_{i \in I_0} \phi_i$. The latter, being an honest modal formula, is preserved by bisimulation, so $y \Vdash \Box \bigvee_{i \in I_0} \phi_i$.

Lemma 2 implies that for each $x \in \mathcal{X}$ and each formula ϕ we have

$$x \Vdash \phi \iff \varphi(x) \Vdash \phi \tag{3.1}$$

and Lemma 5 tells us that x is saturated iff $\varphi(x)$ is saturated, which we might combine to:

Corollary 1. Homomorphisms preserve and reflect saturation.

On the level of Kripke structures, rather than elements, this translates to:

Corollary 2. Homomorphic images and homomorphic preimages of saturated Kripke structures are saturated.

Let \mathcal{X} and \mathcal{Y} be Kripke structures. Recall that for elements $x \in X$ and $y \in Y$ we write $x \approx y$, if they are logically equivalent, i.e. they satisfy the same modal formulae. The following generalization of the Hennessy-Milner theorem [12] is credited in [5] to unpublished notes of Alfred Visser:

Proposition 2. Let \mathcal{X} and \mathcal{Y} be saturated Kripke structures. Then elements $x \in X$ and $y \in Y$ are bisimilar if and only if they are logically equivalent. In short: $\sim_{\mathcal{X},\mathcal{Y}} = \approx_{\mathcal{X},\mathcal{Y}}$.

We shall next show that saturation allows us to describe the minimal homomorphic image of a Kripke structure:

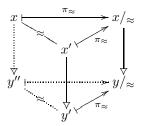
Lemma 6. If \mathcal{X} is saturated, then \approx is a congruence relation on \mathcal{X} .

Proof. Clearly, \approx is an equivalence relation and therefore it is the kernel of the map π_{\approx} sending arbitrary elements x to $x/_{\approx}$, which denotes the equivalence class of \approx containing x. To show that π_{\approx} is a homomorphism, we need to exhibit a coalgebra structure on $X/_{\approx}$, the factor set of X. Put

 $\begin{array}{l} x/_{\approx} \Vdash p : \iff \exists \ x' \approx x \, . \, x' \Vdash p. \\ x/_{\approx} \twoheadrightarrow y/_{\approx} \ : \iff \ \text{there exist} \ x' \approx x \ \text{and} \ y' \approx y \ \text{such that} \ x' \twoheadrightarrow y'. \end{array}$

We check that $\pi_{\approx} : \mathcal{X} \to \mathcal{X}/_{\approx}$ is indeed a Kripke homomorphism:

- Clearly, $x \Vdash p$ iff $x/_{\approx} \Vdash p$ by definition of \Vdash on $X/_{\approx}$, and
- if $x \to y$, then $x/_{\approx} \to y/_{\approx}$ is also immediate by definition. Conversely, given $\pi_{\approx}(x) = x/_{\approx} \to y/_{\approx}$ for some y, we must find a y'' with $x \to y''$ and $\pi_{\approx}(y'') = y/_{\approx}$. Since $x/_{\approx} \to y/_{\approx}$, we know that there are $x' \approx x$ and $y' \approx y$ with $x' \to y'$. By assumption, \approx is a bisimulation, so it follows that there is some y'' with $x \to y''$ and $y'' \approx y'$. Consequently, $x \to y''$ and $\pi_{\approx}(y'') = \pi_{\approx}(y') = y/_{\approx}$, as required.



Thus, π_{\approx} is a homomorphism with kernel \approx , which makes the latter a congruence relation.

Definition 6. A Kripke structure is called simple, if it does not have a proper homomorphic image.

Clearly, if $x \not\approx y$ then there cannot be a homomorphism φ with $\varphi(x) = \varphi(y)$, since $x \approx \varphi(x)$ and $y \approx \varphi(y)$. Thus, if \approx is a congruence, $\mathcal{X}/_{\approx}$ must be simple. It follows:

Theorem 1. A Kripke structure is saturated iff it has a simple and saturated homomorphic image.

Observe that Example 2 is a Kripke structure, which is saturated and simple, but not image finite. In particular it does not have a homomorphism to an image finite Kripke structure.

4 F-coalgebras

Given a category \mathscr{C} and an endofunctor $F : \mathscr{C} \to \mathscr{C}$, an F-coalgebra $\mathcal{A} = (A, \alpha)$ is an object A from \mathscr{C} together with a morphism $\alpha : A \to F(A)$. The object A is called the *base object* and α is called the *structure morphism* of the F-coalgebra $\mathcal{A} = (A, \alpha)$. Given a second coalgebra $\mathcal{B} = (B, \beta)$, a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is a \mathscr{C} -morphism $\varphi : \mathcal{A} \to \mathcal{B}$ which renders the following diagram commutative:

$$\begin{array}{c} A \xrightarrow{\varphi} B \\ \downarrow \\ f(A) \xrightarrow{\varphi} F(B) \end{array} \xrightarrow{\varphi} F(B)$$

F-coalgebras with homomorphisms, as defined above, form a category, which we shall call \mathscr{C}_F , or simply $Coalg_F$ when the base category is understood. When φ in the above figure is a monomorphism in the base category, then we call \mathcal{A} a *subcoalgebra* of \mathcal{B} .

Kripke structures are prime examples of coalgebras. Indeed, the successor relation $R \subseteq X \times X$ can be understood as a map $R : X \to \mathbb{P}(X)$ and the valuation v as a map $v : X \to \mathbb{P}(\Phi)$, where \mathbb{P} is the powerset functor and Φ is the fixed set of propositional atoms. Thus a Kripke structure $\mathcal{X} = (X, R, v)$ is simply an *F*-coalgebra for the combined functor $\mathbb{P}(-) \times \mathbb{P}(\Phi)$, that is a map

$$\alpha: X \to \mathbb{P}(X) \times \mathbb{P}(\Phi),$$

whose first component models the successor relation R and whose second component is the valuation v.

It is easy to check (see [19]), that a homomorphism of Kripke structures, as introduced earlier, is the same as a homomorphism of coalgebras when Kripke structures are understood as $\mathbb{P}(-) \times \mathbb{P}(\Phi)$ -coalgebras.

In this case a subcoalgebra \mathcal{U} of \mathcal{X} is uniquely determined by its base set U. To be precise, $U \subseteq X$ carries a subcoalgebra of the Kripke structure $\mathcal{X} = (X, \alpha)$ if and only if $R(U) \subseteq U$.

Choosing the finite-powerset functor $\mathbb{P}_{\omega}(-)$ instead of $\mathbb{P}(-)$, coalgebras for the functor $\mathbb{P}_{\omega}(-) \times \mathbb{P}(\Phi)$ are precisely the image finite Kripke structures.

Saturated Kripke structures, however, lying between image finite and arbitrary Kripke structures, do not seem to allow such a simple modelling by an appropriate *Set*-functor between $\mathbb{P}_{\omega(-)}$ and $\mathbb{P}(-)$. Instead, we shall have to pass to the category *Top* of topological spaces and continuous mappings and model them as coalgebras over *Top*.

5 Topological models

Definition 7. A topological model is a Kripke model $\mathcal{X} = (X, R, v)$ together with a topology τ on X, such that

1. $\forall x \in X. R(x) \text{ is compact}$ 2. $\forall O \in \tau. \langle R \rangle O \in \tau$ 3. $\forall O \in \tau. [R] O \in \tau$ 4. $\forall p \in \Phi. [p] \in \tau \text{ and } (X - [p]) \in \tau.$ A homomorphism $\varphi : \mathcal{X} \to \mathcal{Y}$ between topological models is simply a Kripkehomomorphism (see def. 4) which additionally is continuous with respect to the topologies on \mathcal{X} and \mathcal{Y} .

We need two simple technical lemmas:

Lemma 7. If C is closed, then so are $\langle R \rangle C$ and [R]C.

Proof. Let C = X - O where O is open, then $\langle R \rangle C = \langle R \rangle (X - O) = X - [R]O$ and $[R]C = [R](X - O) = X - \langle R \rangle O$.

Lemma 8. In every topological model the sets $\llbracket \phi \rrbracket$ where $\phi \in \mathcal{L}_{\Phi}$, are clopen (closed and open).

Proof. By induction on the construction of ϕ :

For $p \in \Phi$ the assertion is part of the definition. If the claim is true for ϕ, ϕ_1 and ϕ_2 , then it is obviously true for all boolean compositions, in particular for $\neg \phi$ and for $\phi_1 \wedge \phi_2$.

Lemma 7 and Definition 7 ensure that the claim remains true for $\Box \phi$ and $\Diamond \phi$, since $\llbracket \Box \phi \rrbracket = [R] \llbracket \phi \rrbracket$ and for $\llbracket \Diamond \phi \rrbracket = \langle R \rangle \llbracket \phi \rrbracket$.

Topological models with continuous Kripke-Homomorphisms obviously form a category which we shall call \mathscr{K}_{Top} .

6 The compact Vietoris-functor

Leopold Vietoris, in his 1922 paper [21], defined his *domains of second order* ("Bereiche zweiter Ordnung") as the collection of closed subsets of a compact Hausdorff space. Later several generalizations and modifications of this topology were introduced and studied under the heading of *hypertopology*.

In connection with Kripke structures, Bezhanishvili, Fontaine and Venema [4] consider the Vietoris functor and Vietoris coalgebras over Stone spaces, i.e. compact and totally disconnected Hausdorff spaces.

In compact Hausdorff spaces, all closed subsets are compact. Hence, when extending the Vietoris functor to act on arbitrary topological spaces $\mathcal{X} = (X, \tau)$, one has the choice to take as base set for $\mathbb{V}(\mathcal{X})$ all closed subsets or all compact subsets of X. In [13] the authors show that both choices lead to endofunctors on the category *Top* of topological spaces, the "lower" Vietoris functor, and the compact Vietoris functor. Here we shall only need to work with the latter, which for us then is "the" Vietoris functor:

Given a topological space $\mathcal{X} = (X, \tau)$, the Vietoris space $\mathbb{V}(\mathcal{X})$ takes as base set the collection of all compact subsets $K \subseteq X$. The Vietoris topology on $\mathbb{V}(\mathcal{X})$ is generated by a subbase consisting of all sets

where $O \in \tau$.

If $\mathcal{X} = (X, \tau)$ and $\mathcal{Y} = (Y, \rho)$ are topological spaces, then the Vietoris functor sends a continuous function $f : \mathcal{X} \to \mathcal{Y}$ to a map $(\mathbb{V}f) : \mathbb{V}(\mathcal{X}) \to \mathbb{V}(\mathcal{Y})$ by setting $(\mathbb{V}f)(K) := f(K)$. Recall that the image f(K) of a compact set K by a continuous map f is always compact. It is easy to calculate that $(\mathbb{V}f)^{-1}(\langle O \rangle) =$ $\langle f^{-1}(O) \rangle$ and $(\mathbb{V}f)^{-1}([O]) = [f^{-1}(O)]$, hence $(\mathbb{V}f)$ is continuous with respect to the Vietoris topologies. In fact, $(\mathbb{V}f)^{-1}$ takes the defining subbase of $\mathbb{V}(\mathcal{Y})$ to the defining subbase of $\mathbb{V}(\mathcal{X})$. This shows that \mathbb{V} is indeed an endofunctor on *Top*.

Let now $\mathbb{P}(\Phi)$ be the powerset of Φ , equipped with the topology having as a base the set of all

$$\uparrow p := \{ u \subseteq \Phi \mid p \in u \}$$

where $p \in \Phi$, together with their complements $\mathbb{P}(\Phi) - \uparrow p$. This topology, trivially, is Hausdorff, but in general not compact.

Definition 8. The product $\mathbb{V}(-) \times \mathbb{P}(\Phi)$ of the Vietoris functor \mathbb{V} with the constant functor of value $\mathbb{P}(\Phi)$, carrying the above topology, will be called the Φ -Vietoris functor, or simply the Vietoris functor, when Φ is clear.

The Vietoris functor is an endofunctor on the category Top of topological spaces with continuous maps. We can now define:

Vietoris coalgebras are coalgebras over Top for the Φ -Vietoris functor $\mathbb{V}(-) \times \mathbb{P}(\Phi)$, and the following result shows that they agree with our topological models:

Theorem 2. Vietoris coalgebras with coalgebra homomorphisms are the same as topological models with continuous Kripke-homomorphisms.

Proof. Given a topological model (X, R, v) with underlying space $\mathcal{X} = (X, \tau)$, we can consider it as a Vietoris coalgebra by defining the structure map $\alpha : \mathcal{X} \to \mathbb{V}(\mathcal{X}) \times \mathbb{P}(\Phi)$ as $\alpha(x) := (R(x), v(x))$. To show that α is continuous, we must verify that both components are continuous.

Continuity of (the map) $R : \mathcal{X} \to \mathbb{V}(\mathcal{X})$ needs to be tested only on the subbase for the Vietoris topology on $\mathbb{V}(\mathcal{X})$. Indeed, assume $O \in \tau$, then

$$R^{-1}([O]) = \{x \in X \mid R(x) \in [O]\}\$$

= $\{x \in X \mid R(x) \subseteq O\} = [R]O$

is open in τ and so is

$$R^{-1}(\langle O \rangle) = \{ x \in X \mid R(x) \in \langle O \rangle \}$$
$$= \{ x \in X \mid R(x) \cap O \neq \emptyset \} = \langle R \rangle O$$

To see that v also is continuous, let $\uparrow p \subseteq \mathbb{P}(\Phi)$ be given, then

$$v^{-1}(\uparrow p) = \{x \in X \mid p \in v(x)\} = [\![p]\!] \in \tau$$

as well as

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$$v^{-1}(\mathbb{P}(X) - \uparrow p) = \{x \in X \mid p \notin v(x)\} = (X - \llbracket p \rrbracket) \in \tau.$$

Conversely, let (X, α) be a Vietoris coalgebra, with $\mathcal{X} = (X, \tau)$ as base space and $\alpha : \mathcal{X} \to \mathbb{V}(\mathcal{X}) \times \mathbb{P}(\Phi)$ as structure morphism, then $\alpha = (R, v)$ with $R := \pi_1 \circ \alpha : \mathcal{X} \to \mathbb{V}(\mathcal{X})$ and $v := \pi_2 \circ \alpha : \mathcal{X} \to \mathbb{P}(\Phi)$, both of which are continuous. Since $R(x) \in \mathbb{V}(\mathcal{X})$, it is necessarily compact. If O is open in (X, τ) then [O] is open in $\mathbb{V}(\mathcal{X})$, hence $R^{-1}([O])$ must be open in (X, τ) , hence so is

$$[R]O = \{x \in X \mid R(x) \subseteq O\} = \{x \in X \mid R(x) \in [O]\} = R^{-1}([O]).$$

Similarly, for O open in (X, τ) we have $\langle O \rangle$ open in $\mathbb{V}(\mathcal{X})$, hence $R^{-1}(\langle O \rangle)$ is open in \mathcal{X} , which means that

$$\langle R \rangle O = \{ x \in X \mid R(x) \cap O \neq \emptyset \} = \{ x \in X \mid R(x) \in \langle O \rangle \} = R^{-1}(\langle O \rangle)$$

is open as well.

Finally, for $p \in \Phi$ we have $\uparrow p = \{u \subseteq \Phi \mid p \in u\}$ clopen in the topology on $\mathbb{P}(\Phi)$, so also $\llbracket p \rrbracket = \{x \in X \mid p \in v(x)\} = v^{-1}(\{u \subseteq \Phi \mid p \in u\}) = v^{-1}(\uparrow p)$ as well as its complement $X - \llbracket p \rrbracket$ are open in τ .

Coalgebra homomorphisms between Vietoris coalgebras, as coalgebras over Top, must be continuous and preserve both R and v which means they are the same as continuous Kripke homomorphisms between the corresponding topological models.

7 Characterization theorem

The following theorem shows that saturated Kripke structures arise precisely as the algebraic reducts of Vietoris coalgebras when forgetting the topology. Bezhanishvili, Fontaine and Venema [4], studying Vietoris coalgebras over Stone spaces, show that in this case the underlying Kripke structures are saturated. In contrast to their work, we consider the Vietoris functor over arbitrary topological spaces, which allows us to obtain an equivalence:

Theorem 3. For a Kripke structure \mathcal{X} the following are equivalent:

- 1. \mathcal{X} is saturated,
- 2. \mathcal{X} is the algebraic reduct of a topological model,
- 3. \mathcal{X} is the algebraic reduct of a Vietoris coalgebra.

Proof. "(1) \rightarrow (2)": Assuming that $\mathcal{X} = (X, R, v)$ is saturated, let τ be the topology on X generated by the sets $\llbracket \phi \rrbracket$ for $\phi \in \mathcal{L}_{\Phi}$. It follows that each $\llbracket \phi \rrbracket$ is clopen (*closed* and *open*), so each open set can be written as $O = \bigcup_{i \in I} \llbracket \phi_i \rrbracket$ and each closed set as $C = \bigcap_{i \in I} \llbracket \phi_i \rrbracket$.

To show that \mathcal{X} with this topology τ is a topological model, we show first, that R(x) is topologically compact. For that, assume $R(x) \subseteq \bigcup_{i \in I} O_i$, then $R(x) \subseteq \bigcup_{i \in I} \bigcup_{j \in J_i} \llbracket \phi_j \rrbracket$, i.e.

$$x \Vdash \Box \bigvee_{i \in I} \bigvee_{j \in J_i} \phi_j.$$

By saturation of \mathcal{X} , there are finitely many $j_{i_1} \in J_{i_1}, ..., j_{i_n} \in J_{i_n}$ with

$$x \Vdash \Box(\phi_{j_{i_1}} \lor \ldots \lor \phi_{j_{i_n}}),$$

so $R(x) \subseteq O_{i_1} \cup \ldots \cup O_{i_n}$. Next, to see that $\langle R \rangle O$ is open, we calculate

$$\begin{split} \langle R \rangle O &= \langle R \rangle (\bigcup_{i \in I} \llbracket \phi_i \rrbracket) \\ &= \bigcup_{i \in I} \langle R \rangle \llbracket \phi_i \rrbracket \\ &= \bigcup_{i \in I} \{ x \in X \mid x \models \Diamond \phi_i \} \\ &= \bigcup_{i \in I} \llbracket \Diamond \phi_i \rrbracket, \end{split}$$

which is open, and similarly

$$\begin{split} [R]O &= [R](\bigcup_{i \in I} \llbracket \phi_i \rrbracket) \\ &= \{ x \in X \mid R(x) \subseteq \bigcup_{i \in I} \llbracket \phi_i \rrbracket \} \\ &= \{ x \in X \mid R(x) \subseteq \bigcup_{i \in J_x} \llbracket \phi_i \rrbracket \text{ for some finite } J_x \subseteq I \} \\ &= \{ x \in X \mid x \models \Box \bigvee_{i \in J_x} \phi_i \text{ for some finite } J_x \subseteq I \} \\ &= \bigcup_{J \subseteq I, J \text{ finite }} \llbracket \Box \bigvee_{i \in J} \phi_i \rrbracket , \end{split}$$

which is open as well.

"(2) \leftrightarrow (3)" is Theorem 2.

"(2) \rightarrow (1)": Given a Kripke model \mathcal{X} which is the algebraic reduct of a topological model, assume $x \Vdash \Box \bigvee_{i \in I} \phi_i$, then $R(x) \subseteq \bigcup_{i \in I} \llbracket \phi_i \rrbracket$. By Lemma 8, the right hand side is a union of open sets, thus by compactness of R(x) there is a finite subset $I_0 \subseteq I$ with $R(x) \subseteq \bigcup_{i \in I_0} \phi_i$, which means $x \Vdash \Box \bigvee_{i \in I_0} \phi_i$.

Given a saturated Kripke-structure $\mathcal{X} = (X, R, v)$, let $F(\mathcal{X})$ denote the Vietoris coalgebra, as constructed above, and conversely, given a Vietoris coalgebra \mathcal{A} , let $G(\mathcal{A})$ be the corresponding saturated Kripke structure. On objects, F and G are clearly inverses to each other.

On morphisms, this is true as well, since a homomorphism $\varphi : \mathcal{X} \to \mathcal{Y}$ between saturated Kripke structures preserves (and reflects) modal formulae (see Lemma 2) and the topologies on $F(\mathcal{X})$ and $F(\mathcal{Y})$ are generated by validity sets of formulae. Conversely, a morphism between Vietoris coalgebras \mathcal{A} and \mathcal{B} is automatically a Kripke-homomorphism by forgetting continuity.

Corollary 3. Saturated Kripke structures, topological models, and Vietoris coalgebras are isomorphic as categories.

8 Closure of Vietoris structures

In those topological spaces where each point has a countable base for its neighbourhoods, such as, for instance, in metric spaces, continuity can be conveniently dealt with in terms of convergent sequences $(x_n)_{n \in \mathbb{N}}$. For general spaces $\mathcal{X} = (X, \tau)$, this intuitive approach is not sufficient, but its spirit and its power can be salvaged if one allows the linearly ordered set \mathbb{N} , indexing a sequence, to be replaced by arbitrary directed sets I indexing the elements $(x_i)_{i \in I}$ of a net. Often, a proof based on convergence of sequences can be easily generalized by replacing sequences with nets. Therefore net convergence can be considered more intuitive than the equally powerful notion of filter convergence. The following definitions and results on nets in general topological spaces will be needed. They can be found as a series of exercises in Munkres [18].

8.1 Nets and subnets

A partially ordered set (I, \leq) is called *directed*, if for each pair $i_1, i_2 \in I$ there is some $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$, i.e. i is an upper bound for $\{i_1, i_2\}$. It follows that each finite subset $I_0 \subseteq I$ has a common upper bound.

Definition 9. A subset $J \subseteq I$ is called cofinal in I, if for each $i \in I$ there is some $j \in J$ with $i \leq j$. A map $f : J \to I$ between ordered sets (J, \leq) and (I, \leq) is called cofinal if its image f[J] is cofinal in I.

Clearly, if J_1 is cofinal in J_2 and J_2 cofinal in I then J_1 is cofinal in I. Also, compositions of cofinal maps are cofinal.

Let $\mathcal{X} = (X, \tau)$ be a topological space and $x \in X$. By $\mathfrak{U}(x)$ we denote the collection of all open neighborhoods of x. Observe that $\mathfrak{U}(x)$, when ordered by reverse inclusion, is a directed set.

Definition 10. A net in X is a map $\sigma : I \to X$ from a directed set I to the set X.

If $\sigma(i) = x_i$, then one often denotes the net σ as $(x_i)_{i \in I}$ and if I is clear from the context one simply writes (x_i) .

The net $(x_i)_{i \in I}$ converges to $x \in X$ and we shall write $(x_i)_{i \in I} \longrightarrow x$, or when I is understood, simply $(x_i \longrightarrow x)$, provided that

$$\forall U \in \mathfrak{U}(x). \, \exists i_U \in I. \, \forall i \ge i_U. \, x_i \in U. \tag{8.1}$$

In this case, x is called a *limit point* of $(x_i)_{i \in I}$. Colloquially, condition 8.1 can be expressed as " (x_i) is eventually in every neighborhood of x".

Limit points need not exist, nor need they be unique, unless X is Hausdorff. In any case though, one has (see [18]):

Proposition 3. Let \mathcal{X} and \mathcal{Y} be arbitrary topological spaces.

1. A map $\varphi : \mathcal{X} \to \mathcal{Y}$ is continuous at x if and only if it "preserves convergence", i.e. for all nets $(x_i)_{i \in I}$ in X :

$$(x_i \longrightarrow x) \implies (\varphi(x_i) \longrightarrow \varphi(x))$$
.

- 2. Given a subset $A \subseteq X$, then $x \in X$ belongs to the topological closure \overline{A} of A if and only if some net (a_i) in A converges to x. Thus, A is closed iff it contains all limit points of nets in A.
- $x \in X$ is called an accumulation point of the net $(x_i)_{i \in I}$ if

$$\forall U \in \mathfrak{U}(x). \, \forall i \in I. \, \exists j \ge i. \, x_j \in U.$$

$$(8.2)$$

Condition (8.2) can be phrased as: " x_i is frequently in every neighborhood of x". A characterization of compactness using nets is ([18]):

Lemma 9. A subset $A \subseteq X$ is compact if and only if every net in A has an accumulation point in A.

Definition 11. A net $\lambda : J \to X$ is a subnet of $\sigma : I \to X$ if there is a monotonic and cofinal map $f : J \to I$ with $\lambda = \sigma \circ f$:



Thus, if $\sigma = (x_i)_{i \in I}$ then $\lambda = (x_{f(j)})_{j \in J}$. One easily checks that the subnet relation is reflexive and transitive, but mainly:

Lemma 10. If $(x_i)_{i \in I}$ converges to x then so does each subnet $(x_{f(j)})_{j \in J}$.

Lemma 11. $x \in X$ is an accumulation point of the net $\sigma : I \to X$ if and only if there is a subnet λ of σ converging to x.

Corollary 4. A subset $A \subseteq X$ is compact iff every net in A has a subnet converging to some $a \in A$.

8.2 Convergence in Vietoris spaces

In this section, we prepare our main result on net convergence in Vietoris spaces. Let $\mathcal{X} = (X, \tau)$ be a topological space. Recall that the *Vietoris space* $\mathbb{V}(\mathcal{X})$ over \mathcal{X} consists of all compact subsets $K \subseteq X$, with a topology generated by a subbase consisting of all sets

$$\langle O \rangle := \{ K \in \mathbb{V}(\mathcal{X}) \mid K \cap O \neq \emptyset \}$$
$$[O] := \{ K \in \mathbb{V}(\mathcal{X}) \mid K \subseteq O]$$

where O ranges over all open subsets of \mathcal{X} . The following results establish the relevant connections between convergence in $\mathbb{V}(\mathcal{X})$ and convergence in $\mathcal{X} = (X, \tau)$.

Lemma 12. Let $\kappa : I \to \mathbb{V}(\mathcal{X})$ be a net in the Vietoris space. If ($\kappa_i \longrightarrow K$) and $K \neq \emptyset$, then κ has a subnet, each member of which is nonempty.

Proof. Since $K \neq \emptyset$, we have $K \in \langle X \rangle$, so $\langle X \rangle$ is a neighborhood of K in $\mathbb{V}(\mathcal{X})$. As κ converges to K, there must be some $i_0 \in I$ such that $\forall i \geq i_0$. $\kappa_i \in \langle X \rangle$, i.e $\forall i \geq i_0$. $\kappa_i \neq \emptyset$. Put $J = (\{i \in I \mid i \geq i_0\}, \leq)$ and let $f : J \hookrightarrow I$ be the natural inclusion, then f is clearly monotonic and cofinal. Therefore $\tau := \kappa \circ f$ is a subnet of κ and $\tau_j = \kappa_{f(j)} = \kappa_j \neq \emptyset$, owing to $j \in J$.

Lemma 13. Given a net $\kappa : I \to \mathbb{V}(\mathcal{X})$ converging to $K \in \mathbb{V}(\mathcal{X})$ and $b_i \in \kappa_i$ for each $i \in I$. Then the net $(b_i)_{i \in I}$ has a subnet converging to some $b \in K$.

Proof. It is enough to show that $(b_i)_{i \in I}$ has an accumulation point $b \in K$. For then we obtain a subnet $(b_{f(j)})_{j \in J}$ converging to b. By Lemma 10, the subnet $(\kappa_{f(j)})_{j \in J}$ of κ still converges to K.

For every $x \in K$ which is not an accumulation point of $(b_i)_{i \in I}$, we obtain by negating (8.2) an open neighborhood U_x of x and an $i_x \in I$ such that for all $i \ge i_x$ we have $b_i \notin U_x$. Assuming that no $x \in K$ is an accumulation point, then the family $(U_x)_{x \in K}$ forms an open cover of K. By compactness, there is a finite subcover $U = U_{x_1} \cup \ldots \cup U_{x_n}$. Choose $i_U \ge i_{x_1}, \ldots, i_{x_n}$, then for every $i \ge i_U$ we have $b_i \notin U \supseteq K$.

But [U] is also an open neighborhood of K in $\mathbb{V}(\mathcal{X})$ and $(\kappa_i \longrightarrow K)$, so there exists $i_{[U]}$ with $\kappa_i \in [U]$, that is $b_i \in \kappa_i \subseteq U$ for $i \ge i_{[U]}$. For $i \ge \{i_U, i_{[U]}\}$ we enter the contradiction $b_i \in U$ and $b_i \notin U$.

Lemma 14. Given a net $\kappa : I \to \mathbb{V}(\mathcal{X})$ converging to $K \in \mathbb{V}(\mathcal{X})$ and $a \in K$. Then there is a subnet $(\kappa_i)_{i \in J}$ and elements $a_i \in \kappa_i$ converging to a.

Proof. By Lemma 12 and Lemma 10, we may assume that $\kappa_i \neq \emptyset$ for all $i \in \mathcal{I}$. For every $U \in \mathfrak{U}(a)$ we have $a \in U \cap K$, so $K \cap U \neq \emptyset$, which means that

 $K \in \langle U \rangle$, so $\langle U \rangle$ is an open neighborhood of K in $\mathbb{V}(\mathcal{X})$.

Since κ converges to K we have

$$\forall U \in \mathfrak{U}(a). \exists i_U \in I. \forall i \ge i_U. \kappa_i \in \langle U \rangle.$$

$$(8.3)$$

Consider a partial order on

$$J := \{ (i, U) \in I \times \mathfrak{U}(a) \mid \kappa_i \in \langle U \rangle \}$$

by defining:

$$(i_1, U_1) \leq (i_2, U_2) : \iff i_1 \leq i_2 \land U_1 \supseteq U_2.$$

To verify that $\mathcal{J} = (J, \leq)$ is directed, let arbitrary $j_1 = (i_1, U_1)$ and $j_2 = (i_2, U_2)$ be given. Pick $U = U_1 \cap U_2$ then by (8.3) there is an $i_U \in I$ with $\kappa_i \in \langle U \rangle$ for all $i \geq i_U$. It suffices to choose $i \geq i_1, i_2, i_U$, then $(i, U) \in J$ and $(i, U) \geq (i_1, U_1), (i_2, U_2)$.

The map $\pi_1 : J \to I$ given as $\pi_1(i, U) := i$ is clearly monotonic. For each $i \in I$ we have $(i, X) \in J$ since $\kappa_i \neq \emptyset$. Hence π_1 is cofinal. Therefore $\kappa \circ \pi_1 : J \to \mathbb{V}(\mathcal{X})$ is a subnet of κ and therefore also converges to K.

For each $(i, U) \in J$ we can pick some $a_{(i,U)} \in \kappa_i \cap U$. This defines a net $(a_i)_{i \in J}$ in X.

To show that $(a_j)_{j\in J}$ converges to a, let U be any open neighborhood of a. By 8.3 there exist some i_U such that in particular $j_U := (i_U, U) \in J$. We therefore have $a_{j_U} := a_{(i_U,U)} \in U$ and for each $j = (i,U') \ge (i_U,U) = j_U$, i.e. for $i \ge i_U$ and $U' \subseteq U$ we have $a_j = a_{(i,U')} \in \kappa_i \cap U' \subseteq U$.

We can combine the previous two lemmas to a theorem relating convergence in Vietoris spaces to convergence in their base spaces:

Theorem 4. Let $(\kappa_i)_{i \in I}$ converge to K in the Vietoris space $\mathbb{V}(\mathcal{X})$. Then

- 1. for each $a \in K$ there is a subnet $(\kappa_j)_{j \in J}$ and elements $a_j \in \kappa_j$ such that $(a_j \longrightarrow a)$, and
- 2. each net $(b_i)_{i \in I}$ with $b_i \in \kappa_i$ has a subnet $(b_i)_{i \in J}$ converging to some $b \in K$.

8.3 Closure of subcoalgebras and bisimulations

In this section we shall show that in topological Kripke structures, i.e. for Vietoris coalgebras, the topological closure of a substructure is again a substructure and the closure of a bisimulation is a bisimulation. The second of these results has previously been shown for Vietoris coalgebras over Stone spaces in [4], but now we work in the more general context of Vietoris coalgebras over arbitrary topological spaces, so we were forced to prepare our tools in the previous sections.

Theorem 5. Let $\mathcal{A} = (A, \alpha)$ be a Vietoris coalgebra. If $U \subseteq A$ is a Kripke substructure of \mathcal{A} , then so is its topological closure \overline{U} .

Proof. We may consider \mathcal{A} as a topological model (A, R, v_A) where $R(a) = (\pi_1 \circ \alpha)(a)$ for each $a \in A$ is the compact set of all successors of a. Assume that U is a subcoalgebra, i.e. a subset $U \subseteq X$ such that $R(u) \subseteq U$ for each $u \in U$. We need to show that the same property holds for \overline{U} .

Thus let $u \in \overline{U}$ be arbitrary and let v be a successor of u in the coalgebra \mathcal{X} , i.e. $v \in R(u)$. We need to show that $v \in \overline{U}$.

Due to Proposition 3, there is a net $(u_i)_{i \in I}$ converging to u with each $u_i \in U$. By continuity of α , the net $R(u_i)_{i \in I}$ converges to R(u) in the Vietoris topology.

As $v \in R(u)$, we may assume by Lemma 12, that each $R(u_i)$ is nonempty.² Next, we may assume by Theorem 4 that we can pick a v_i from each $R(u_i)$ so that the net $(v_i)_{i \in I}$ converges to v in \mathcal{X} .

Since U was a subcoalgebra, $R(u_i) \subseteq U$, so each v_i must belong to U. Therefore, we have found a net in U which converges to v, hence $v \in \overline{U}$.

Theorem 6. If S is a Kripke bisimulation between Vietoris coalgebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$, then so is its topological closure \overline{S} .

Proof. Again, we consider \mathcal{A} and \mathcal{B} as topological models with $\alpha = (R_A, v_A)$ and $\beta = (R_B, v_B)$. Given $(a, b) \in \overline{S}$, we need to show that

1. $v_A(a) = v_B(b)$ and

2. whenever $a \to u$ then there is some v with $b \to v$ and $(u, v) \in \overline{S}$.

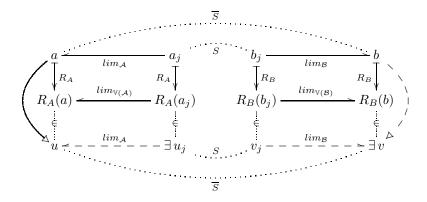
The third case of definition 3 will follow by a symmetric proof.

First note that by Theorem 3 there is a net $(a_i, b_i)_{i \in I}$ converging to (a, b) with each $(a_i, b_i) \in S$. The individual nets (a_i) , resp. (b_i) , converge to a, resp. to b, since the projection maps are continuous.

Also by continuity, $v_A(a_i)$ and $v_B(b_i)$ converge to $v_A(a)$ and $v_B(b) \in \mathbb{P}(\Phi)$. Since $(a_i, b_i) \in S$, we know $v_A(a_i) = v_B(b_i)$ for each $i \in I$. Since the topology on $\mathbb{P}(\Phi)$, the second component of the Vietoris functor, is Hausdorff, we get $v_A(a) = v_B(b)$ as required.

Next, assume $a \to u$, i.e. $u \in R_A(a)$, then we need to find some v with $b \to v$ and $(u, v) \in \overline{S}$.

By continuity of R_A and R_B , the nets $(R_A(a_i))_{i \in I}$ resp. $(R_B(b_i)_{i \in I})$, converge to $R_A(a)$, resp. to $R_B(b)$ in the Vietoris spaces $\mathbb{V}(\mathcal{A})$, resp. $\mathbb{V}(\mathcal{B})$.



² With the phrase "we may assume" we often hide the technicality that we might have to pass to a subnet, such as here to $(\alpha(u_{f(j)}))_{j \in J}$ and retroactively replace $(u_i)_{i \in I}$ by the subnet $(u_{f(j)})_{j \in J}$, which is always justified by Lemma 10.

In the sense mentioned previously, we may assume that the $R_A(a_i)$ are nonempty and further, using Theorem 4, and possibly passing to a subnet indexed by some J, we find $u_j \in R_A(a_j)$ with $(u_j \longrightarrow u)$.

Since S is a bisimulation and $a_j S b_j$ for each j and $u_j \in R_A(a_j)$ it follows that there are $v_j \in R_B(b_j)$ with $(u_j, v_j) \in S$ for each $j \in J$. Since $(b_j \longrightarrow b)$ it follows $(R_B(b_j) \longrightarrow R_B(b))$ by continuity of R_B . Therefore, Lemma 13 forces $(v_j)_{j \in J}$ to converge to some $v \in R_B(b)$.

Consequently, $((u_j, v_j) \longrightarrow (u, v))$ where $(u_j, v_j) \in S$ for each $j \in J$, hence by Lemma 3 $(u, v) \in \overline{S}$ as desired.

9 The terminal Vietoris coalgebra

To obtain the terminal Vietoris coalgebra, we utilize the equivalence with saturated Kripke structures and look for a terminal saturated Kripke structure instead. This will be found in the "canonical model".

Recall from [5], that the *canonical model* for a normal modal logic consists of all *maximally consistent* subsets of \mathcal{L}_{Φ} . Here $u \subseteq \mathcal{L}_{\Phi}$ is called maximally consistent, if

- no contradiction can be derived from the formulae in u, and
- for each formula $\phi \in \mathcal{L}_{\Phi}$, either $\phi \in u$ or $\neg \phi \in u$.

Typical sets of formulas which are maximally consistent arise as

$$\llbracket x \rrbracket := \{ \phi \mid x \Vdash \phi \},\$$

where x is any element of any Kripke structure. Moreover, any consistent set of formulas can be extended to a maximally consistent set.

It is also essential to know that a set u is consistent, if and only if every finite subset $u_0 \subseteq u$ is consistent, see [5].

The canonical model is now defined as $\mathcal{M} := (M, \rightarrow_{\mathcal{M}}, v_{\mathcal{M}})$ where M is the collection of all maximally consistent subsets of \mathcal{L}_{Φ} , and $\rightarrow_{\mathcal{M}}$ and $v_{\mathcal{M}}$ are defined as

$$u \to_{\mathcal{M}} v :\Leftrightarrow \forall \phi. (\Box \phi \in u \implies \phi \in v), \tag{9.1}$$

and

$$v_{\mathcal{M}}(u) := u \cap \Phi. \tag{9.2}$$

The latter definition extends to the important "truth lemma":

Lemma 15. For each formula $\phi \in \mathcal{L}_{\Phi}$ and each $u \in M$ we have:

$$u \Vdash \phi \iff \phi \in u.$$

As an immediate corollary, we note:

Corollary 5. $\forall u, v \in M. u \approx v \implies u = v.$

First, we shall verify, that \mathcal{M} is saturated: Given $u \in M$ and Σ a set of formulas such that for every finite subset $\Sigma_0 \subseteq \Sigma$ there is some v_0 such that $u \to v_0$ and $v_0 \Vdash \bigwedge \Sigma_0$. It follows that every finite subset of the set

$$S := \{ \phi \mid \Box \phi \in u \} \cup \varSigma$$

is satisfied in some v_0 , and hence consistent. Hence the whole set S itself is consistent. Let v be any maximal consistent set containing S, then $v \in M$ and clearly $u \rightarrow v$ as well as $v \Vdash \sigma$ for each $\sigma \in \Sigma$. Therefore:

Lemma 16. \mathcal{M} is saturated.

Let us see that moreover:

Theorem 7. \mathcal{M} is the terminal object in the category of all saturated Kripke structures.

Proof. First note that Corollary 5 yields uniqueness: If for any Kripke structure $\mathcal{X} = (X, R, v)$ we had different homomorphisms $\varphi_1, \varphi_2 : \mathcal{X} \to \mathcal{M}$, then for some $x \in X$ we would have $\varphi_1(x) \neq \varphi_2(x)$. However, $x \approx \varphi_1(x)$ as well as $x \approx \varphi_2(x)$ according to 3.1, whence $\varphi_1(x) \approx \varphi_2(x)$, which contradicts Corollary 5.

For any Kripke structure $\mathcal{X} = (X, R, v)$ we show that the map $\llbracket - \rrbracket : X \to M$ which sends an element $x \in X$ to $\llbracket x \rrbracket := \{ \phi \mid x \Vdash \phi \}$ is a homomorphism, see definition 4:

First, for each $p \in \Phi$ we have: $x \Vdash p$ in \mathcal{X} implies $p \in \llbracket x \rrbracket$, so $\llbracket x \rrbracket \Vdash p$ in \mathcal{M} , by the *Truth Lemma*.

Next, suppose $x, y \in \mathcal{X}$ and $x \to_{\mathcal{X}} y$. Then for each $\phi \in \mathcal{L}_{\Phi}$ with $x \Vdash \Box \phi$ it follows $y \Vdash \phi$, which by the truth lemma says $\Box \phi \in \llbracket x \rrbracket \implies \phi \in \llbracket y \rrbracket$, hence $\llbracket x \rrbracket \to_{\mathcal{M}} \llbracket y \rrbracket$ by 9.1.

Finally, let us assume $\llbracket x \rrbracket \to_{\mathcal{M}} v$ for some maximally consistent set v. We need to find some $y \in \mathcal{X}$ with $x \to_{\mathcal{X}} y$ and $\llbracket y \rrbracket = v$.

For this we invoke a Hennessy–Milner style argument again: Let $(y_i)_{i \in I}$ be the collection of all successors of x. If $[\![y_i]\!] = v$ for some i, then we are done. Otherwise, assume that $[\![y_i]\!] \neq v$ for each $i \in I$, then there are formulae ϕ_i with $\phi_i \in [\![y_i]\!]$ but $\phi_i \notin v$, or, in other words, $y_i \Vdash \phi_i$, but $v \nvDash \phi_i$.

Hence $x \models \Box \bigvee_{i \in I} \phi_i$. By assumption \mathcal{X} is saturated, so $x \in X$ is saturated, which means that we can find a finite subset $I_0 \subseteq I$ with $x \Vdash \Box \bigvee_{i \in I_0} \phi_i$. This is now an honest formula, so from $[\![x]\!] \to v$, and definition 9.1 we conclude $v \models$ $\bigvee_{i \in I_0} \phi_i$. This means that $v \Vdash \phi_i$ for some $i \in I_0$, contradicting our assumption.

Theorem 3 tells us explicitly, how to obtain the terminal Vietoris coalgebra, so we have:

Theorem 8. The category of all Vietoris coalgebras has a terminal object. Its base structure is the canonical model, consisting of all maximally consistent sets of \mathcal{L}_{Φ} -formulas, and its topology is generated by the open sets $\{u \in \mathcal{M} \mid \phi \in u\}$ for all $\phi \in \mathcal{L}_{\Phi}$.

10 Conclusion

Starting from an arbitrary set Φ of atomic proposition, we have characterized modally saturated Kripke structures as *Top*-coalgebras for $\mathbb{V}(-) \times \mathbb{P}(\Phi)$, which is the compact Vietoris functor on the category Top of topological spaces and continuous mappings, augmented with a constant part, representing sets of atomic propositions.

In fact, the categories of saturated Kripke structures and the category of all Vietoris coalgebras over the category Top are isomorphic. We have described the relation of convergence in the Vietoris space $\mathbb{V}(\mathcal{X})$ to convergence in the base space \mathcal{X} , from which it was easy to derive that the Kripke-closure of bisimulations and of subcoalgebras are again bisimulations, resp. subcoalgebras. Finally, we have shown that the final Vietoris coalgebra exists, and is derived from the canonical Kripke model.

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