The Segment Number: Algorithms and Universal Lower Bounds for Some Classes of Planar Graphs

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Abstract. The *segment number* of a planar graph G is the smallest number of line segments needed for a planar straight-line drawing of G. Dujmović, Eppstein, Suderman, and Wood [CGTA'07] introduced this measure for the *visual complexity* of graphs. There are optimal algorithms for trees and worst-case optimal algorithms for outerplanar graphs, 2-trees, and planar 3-trees. It is known that every *cubic* triconnected planar n-vertex graph (except K_4) has segment number $n/2+3$, which is the only known *universal* lower bound for a meaningful class of planar graphs.

We show that every triconnected planar 4-regular graph can be drawn using at most $n+3$ segments. This bound is tight up to an additive constant, improves a previous upper bound of $7n/4+2$ implied by a more general result of Dujmović et al., and supplements the result for cubic graphs. We also give a simple optimal algorithm for cactus graphs, generalizing the above-mentioned result for trees. We prove the first linear universal lower bounds for outerpaths, maximal outerplanar graphs, 2-trees, and planar 3-trees. This shows that the existing algorithms for these graph classes are constant-factor approximations. For maximal outerpaths, our bound is best possible and can be generalized to circular arcs.

Keywords: Visual complexity · Segment number · Lower/upper bounds

1 Introduction

A drawing of a given graph can be evaluated by various quality measures depending on the concrete purpose of the drawing. Classic examples of such measures include drawing area, number of edge crossings, neighborhood preservation, and stress of the embedding. More recently, Schulz [20] proposed the *visual complexity* of a drawing, determined by the number of geometric objects (such as line segments or circular arcs) that the drawing consists of. It has been experimentally verified that people without mathematical background tend to prefer drawings with low visual complexity [13]. The visual complexity of a graph drawing depends on the drawing style, as well as on the underlying graph properties. A well-studied measure of the visual complexity of a graph is its segment number, introduced by Dujmović, Eppstein, Suderman, and Wood [5]. It is defined as follows. Recall that a *straight-line drawing* of a graph maps (i) the vertices of the graph injectively to points in the plane and (ii) the edges of the graph to straight-line segments that connect the corresponding points. A *segment* in such a drawing is a maximal set of edges that together form a line segment. Given a straight-line drawing Γ of a graph, the set of segments it induces is unique. The cardinality of that set is the *segment number* of Γ. The *segment number*, $\mathrm{seg}(G)$, of a planar graph G is the smallest segment number over all crossing-free straight-line drawings of G.

Previous work. Dujmović et al. [5] pointed out two natural lower bounds for the segment number: (i) $\eta(G)/2$, where $\eta(G)$ is the number of odd-degree vertices of G, and (ii) the *slope number*, $\mathrm{slope}(G)$, of G, which is defined as follows. The slope number $\mathrm{slope}(\Gamma)$ of a straight-line drawing Γ of G is the number of different slopes used by any of the straight-line edges in Γ. Then $\mathrm{slope}(G)$ is the minimum of $\mathrm{slope}(\Gamma)$ over all straight-line drawings Γ of G. Dujmović et al. also showed that any tree T admits a drawing with $\mathrm{seg}(T) = \eta(T)/2$ segments and $\mathrm{slope}(T) = \Delta(T)/2$ slopes, where $\Delta(T)$ is the maximum degree of a vertex in T. These drawings, however, use exponential area. Recall that an *outerplanar graph* is a plane graph that can be drawn such that all vertices lie on the outer face. The *weak dual graph* of an outerplane graph is its dual graph without the vertex corresponding to the outer face; it is known to be a tree. An outerplane graph whose weak dual is a path is called an *outerpath*. A *maximal outerplanar graph* is an outerplanar graph with the maximum number of edges. Dujmović et al. showed that every maximal outerplanar graph G with n vertices admits an *outerplanar* straight-line drawing with at most n segments. They showed that this is worst-case optimal. They also gave (asymptotically) worst-case optimal algorithms for 2-trees and plane (where the combinatorial embedding and outer face is fixed) 3-trees. Finally, they showed that every triconnected planar graph with n vertices can be drawn using at most $5n/2 - 3$ segments. For the special cases of triangulations and 4-connected triangulations, Durocher and Mondal [6] improved the upper bound of Dujmović et al. to $(7n - 10)/3$ and $(9n - 9)/4$, respectively. The former bound implies a bound of $(16n - 3m - 28)/3$ for arbitrary planar graphs with n vertices and m edges. Kindermann et al. [12] observed that this implies that $\mathrm{seg}(G) \leq (8n-14)/3$ for any planar graph G: if $m > (8n-14)/3$ this follows from the bound, otherwise any drawing of G is good enough. Constructive linear-time algorithms that compute the segment number of series-parallel graphs of maximum degree 3 and of maximal outerpaths were given by Samee et al. [19] and by Adnan [1], respectively. Mondal et al. [17] and Igamberdiev et al. [11] showed that every cubic triconnected planar graph (except K_4) has segment number $n/2+3$. Hültenschmidt et al. [10] showed that trees, maximal outerplanar graphs and planar 3-trees admit drawings on a grid of polynomial size, using slightly more segments. Kindermann et al. [12] improved some of these bounds. Concerning the computational complexity, Durocher et al. [7] showed that the segment number of a planar graph is NP-hard to compute, even if one insists that in the resulting planar drawing all faces are convex.

Graph class	Universal lower bound		Existential upper bound		Existential lower bound		Universal upper bound	
planar conn.	1		1		$2n-2$	[5]	$(8n-14)/3$	[6,12]
planar 3-conn.	$\sqrt{2n}$	[5]	$O(\sqrt{n})$	[5]	$2n-6$	[5]	$5n/2-3$	[5]
planar 3-conn. 4-reg.	$\Omega(\sqrt{n})$	R1	$O(\sqrt{n})$	R1	n	P1	$n+3$	T2
planar 3-conn. 3-reg.	$n/2+3$	[5]	—		—		$n/2+3$	[11,17]
triangulation	$\Omega(\sqrt{n})$	[5]	$O(\sqrt{n})$	[5]	$2n-2$	[5]	$(7n-10)/3$	[6]
4-conn. triangulation	$\Omega(\sqrt{n})$	[5]	$O(\sqrt{n})$	[5]	$2n-6$	[5]	$(9n-9)/4$	[6]
planar 3-trees	$n+4$	T6	$n+7$	P6	$3n/2$	P7	$2n-2$	[5]
2-trees	$(n+7)/5$	T8	$(5n+24)/13$	P5	$3n/2-2$	[5]	$3n/2$	[5]
maximal outerplanar	$(n+7)/5$	T8	$(5n+24)/13$	P5	n	[5]	n	[5]
maximal outerpath	$\lfloor n/2 \rfloor + 2$	T3	$\lfloor n/2 \rfloor + 2$	P3	n	[5]	n	[5]
cactus	$\eta/2+\gamma$	L8	—		—		$\eta/2+\gamma$	T7

Table 1: Universal and existential lower and upper bounds on the segment number for subclasses of planar graphs. By *existential upper bound* we mean an upper bound for the universal lower bound. Such a bound is provided by the segment number of a specific graph family within the given graph class.Here, η is the number of odd-degree vertices and $\gamma = 3c_0 + 2c_1 + c_2$, where c_i is the number of simple cycles with exactly i cut vertices. We use "—" to indicate that universal lower bound and the universal upper bound agree for a specific graph class. The corresponding algorithms are thus optimal. Results of this paper are shaded in gray, where we link to remarks (R), lemmas (L), propositions (P), theorems (T).

Other related work. Okamoto et al. [18] investigated variants of the segment number. For planar graphs in 2D, they allowed bends. For arbitrary graphs, they considered crossing-free straight-line drawings in 3D and straight-line drawings with crossings in 2D. They showed that all segment number variants are $\exists\mathbb{R}$-complete to compute, and they gave upper and existential lower bounds for the segment number variants of cubic graphs. The *arc number*, $\mathrm{arc}(G)$, of a graph G is the smallest number of circular arcs in any circular-arc drawings of G. It has been introduced by Schulz [20], who gave algorithms for drawing series-parallel graphs, planar 3-trees, and triconnected planar graphs with few circular arcs. For trees, he reduced the drawing area (from exponential to polynomial). Chaplick et al. [3, 4] considered a different measure of the visual complexity, namely the number of lines (or planes) needed to cover crossing-free straight-line drawings of graphs in 2D (and 3D). Kryven et al. [16] considered spherical covers.

Contribution and outline. In terms of universal upper bounds, we first show that every triconnected planar 4-regular graph with n vertices can be drawn using at most $n+3$ segments (note that there are $2n$ edges); see Sect. 2. This bound is tight up to an additive constant, improves a previous upper bound of $7n/4+2$ implied by a more general result [5, Thm. 15] of Dujmović et al., and supplements the result for cubic graphs due to Mondal et al. [17] and Igamberdiev et al. [11]. Our algorithm works even for *plane* graphs and produces drawings that are *convex*, that is, the boundary of each face corresponds to a convex polygon. We remark that triconnected planar 4-regular graphs are a rich and natural graph class that comes with a simple set of generator rules [2]. It might seem tempting to prove our result inductively by means of these rules, though we have not been able make this idea work. Instead, our algorithm relies on a decomposition of the graph along carefully chosen paths (Lem. 3), which might be of independent interest. We also give a simple optimal (cf. Table 1) algorithm for cactus graphs[3] (see Sect. 4), generalizing the result of Dujmović et al. for trees.

We prove the first linear universal lower bounds for maximal outerpaths ($\lfloor n/2 \rfloor + 2$; see Sect. 3), maximal outerplanar graphs as well as 2-trees ($(n+7)/5$; see Sect. 4), and planar 3-trees ($n+4$; see Sect. 4). This makes the corresponding algorithms of Dujmović et al. constant-factor approximation algorithms. For Adnan's algorithm [1] that computes the segment number of maximal outerpaths, our result provides a lower bound on the size of the solution. For maximal outerpaths, our bound is best possible and can be generalized to circular arcs. For planar 3-trees, the bound is best possible up to the additive constant. Known and new results are listed in Table 1. Claims with "$\star$" are proved in the appendix.

Notation and terminology. All graphs in this paper are simple(i.e., we do not allow parallel edges or self-loops). For any graph G, let $\mathrm{V}(G)$ be the vertex set and $\mathrm{E}(G)$ the edge set of G. Now let Γ be a planar drawing of a planar and connected graph G. The boundary ∂f of each face f of Γ can be uniquely described by a counterclockwise sequence of edges. If G is biconnected, then ∂f is a simple cycle(otherwise, ∂f can visit vertices and edges multiple times). The collection of the boundaries of all faces of Γ is called the *combinatorial embedding* of Γ. The unique unbounded face of Γ is called its *outer* face; the remaining faces are called *internal*. Vertices (edges) belonging to the boundary of the outer face are called *outer* vertices (edges); the remaining vertices (edges) are called *internal*. A *plane* graph is a planar graph equipped with a combinatorial embedding and a distinguished outer face. Note that two drawings of G with the same combinatorial embedding may have different outer faces. A path in a plane graph is *internal* if its edges and interior vertices do not belong to its outer face. We say that an angle is *convex* if it is at most π and *reflex* if it exceeds π. In a *convex* polygon, each internal angle is convex. For any $k \in \mathbb{N}$, we use $[k]$ as shorthand for $\{1, 2, \ldots, k\}$.

2 Triconnected 4-Regular Planar Graphs

This section is concerned with the segment number of 3-connected 4-regular planar graphs. We establish a universal upper bound of $n + 3$ segments, which we complement with an existential lower bound of n segments, where n denotes the number of vertices.

Overview. Towards the upper bound, we will show that each graph of the considered class admits a drawing where all but three of its vertices are placed in the interior of some segment. In such a drawing, each of these vertices is the

[3] A *cactus* is a connected graph where any two simple cycles share at most one vertex.

endpoint of at most two segments. The claimed bound then follows from the fact that each segment has exactly two endpoints.

To construct the desired drawings, we follow a strategy that has already been used in an algorithm by Hong and Nagamochi [9], which was sped up by Klemz [15]. Both algorithms generate convex drawings of so-called hierarchical plane st-graphs, but they can also be applied to "ordinary" plane graphs. In this context, the algorithmic framework is as follows: the input is an internally (defined below, see Definition 1) 3-connected plane graph G and a convex drawing Γ^o of the boundary of its outer face. The task is to extend Γ^o to a convex drawing of G. The main idea of both algorithms is to choose a suitable internal vertex y of the given graph G and compute three disjoint (except for y) paths P_1, P_2, P_3 from y to the outer face. Each of these paths is then embedded as a straight-line segment so that Γ^o is dissected into three convex polygons, for an illustration see Fig. 1a. The graphs corresponding to the interior of these polygons can now be handled recursively. To ensure that a solution exists, the computed paths (as well as the paths corresponding to the segments of Γ^o) need to be *archfree*, meaning that they are not arched by an internal face: a path P is *arched* by a face a between $u, v \in \mathrm{V}(\partial a) \cap \mathrm{V}(P)$ if the subpath P_{uv} of P between u and v is interior-disjoint from ∂a, see Fig. 1a. Indeed, if a is internal, then such a path P cannot be realized as a straight-line segment in a convex drawing since the interior of the segment uv has to be disjoint from the realization of a. We follow the idea of dissecting our graphs along archfree paths. However, to ensure that each internal vertex is placed in the interior of some segment, the way in which we construct our paths is necessarily[4] quite different. Specifically, we will show that a large subfamily of the considered graph class can be dissected along three archfree paths that are arranged in a windmill pattern as depicted in Fig. 2a.

We begin by discussing necessary conditions for the existence of convex drawings and the construction of archfree paths. We then define the desired windmill configuration and give a necessary and sufficient criterion for its existence. Finally, we describe our drawing algorithm, thereby establishing the universal upper bound, and conclude with the existential lower bound.

Existence of convex drawings. It is well-known that a plane graph admits a convex drawing if and only if it is a subdivision of an *internally 3-connected* graph [8, 9, 22, 23]. There are multiple ways to define this property and it will be convenient to refer to all of them. Therefore, we use the following well-known characterization; for a proof, see, e.g., [14].

Definition 1. *Let G be a plane 2-connected graph. Let o denote its outer face. Then G is called* internally 3-connected *if and only if the following equivalent statements are satisfied:*

[4] Note that it is not necessarily possible to embed two of the paths P_1, P_2, P_3 on a common segment since their outer endpoints might already belong to a common segment of Γ^o (in particular, this is the case when $|\mathrm{V}(\Gamma^o)| = 3$). Moreover, the concatenation of the two paths might not be archfree.

(I1) Inserting a new vertex v in o and adding edges between v and all vertices of ∂o results in a 3-connected graph.
(I2) From each internal vertex w of G there exist three paths to o that are pairwise disjoint except for the common vertex w.
(I3) Every separation pair u, v of G is external*, i.e., u and v lie on ∂o and every connected component of the subgraph of G induced by $\mathrm{V}(G) \setminus \{u, v\}$ contains a vertex of ∂o.*

When dissecting a 3-connected plane graph along internal paths, the resulting subgraphs are not necessarily 3-connected anymore. In contrast, *internal* 3-connectivity is preserved:

Observation 1 ($\star$, folklore) *Let G be an internally 3-connected plane graph, and let C be a simple cycle in G. The closed interior C^- of C is an internally 3-connected plane graph.*

In the context of our recursive strategy, we face a special case of the following problem: given an internally 3-connected plane graph G and a convex drawing Γ^o of the boundary of its outer face, extend Γ^o to a convex drawing of G. It is known that such an extension exists if and only if each segment of Γ^o corresponds to an archfree path of G [8, 9, 22, 23]. Hence, we say that Γ^o is *compatible* with G if and only if it satisfies this property.

Construction of archfree paths. The following lemma gives rise to a strategy for transforming a given internal path into an archfree path:

Lemma 1 ([9, Lem. 1]). *Let G be an internally 3-connected plane graph, and let f be an internal face of G. Any subpath P of ∂f with $|\mathrm{E}(P)| \leq |\mathrm{E}(\partial f)| - 2$ is archfree.*

One can simply replace the arched parts by appropriate pieces of the boundaries of the arching faces. More precisely, this strategy works as follows: Let G be an internally 3-connected graph. Consider the edges of the outer face ∂o of G to be directed in counterclockwise direction. Assume that there are two distinct vertices s' and t' on ∂o that are joined by a simple internal path P'. Consider P' to be directed from s' to t' and let $P = (s, \ldots, t)$ be a directed subpath of P'. Suppose that P is arched by an internal face a. Then we say a arches P *from the left* if a is interior to the cycle formed by P' and the directed $t's'$-path on ∂o; otherwise, we say that a arches P *from the right*. The *left-aligned* path $\mathrm{L}_G(P)$ of P is obtained be exhaustively applying the following modification (for an illustration see Fig. 1b): suppose that an internal face a arches P from the left between two vertices u, v such u precedes v along P. Transform P by replacing its uv-subpath with the uv-path obtained by walking along ∂a in counterclockwise direction from u to v. The *right-aligned* path $\mathrm{R}_G(P)$ is defined symmetrically.

Lemma 2 ([8, Lemma 5, Corollary 6]). *Let G be an internally 3-connected plane graph. Let $P = (s, \ldots, t)$ be a subpath of a simple internal directed path P' between two distinct outer vertices of G. Then:*

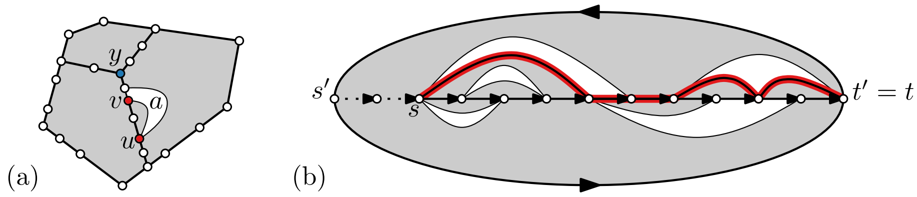


Fig. 1: (a) Splitting Γ^o along three straight-line paths. The subpolygon containing arch a cannot be extended to a convex drawing of its subgraph. (b) Left-aligned path $\mathrm{L}_G(P)$ of $P = (s, \dots, t)$.

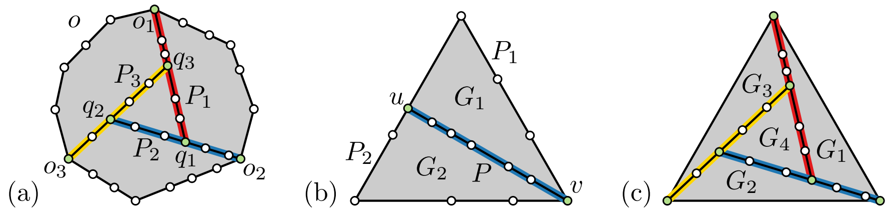


Fig. 2: (a) A windmill (P_1, P_2, P_3). (b,c) The 3-connected case in the proof of Thm. 1.

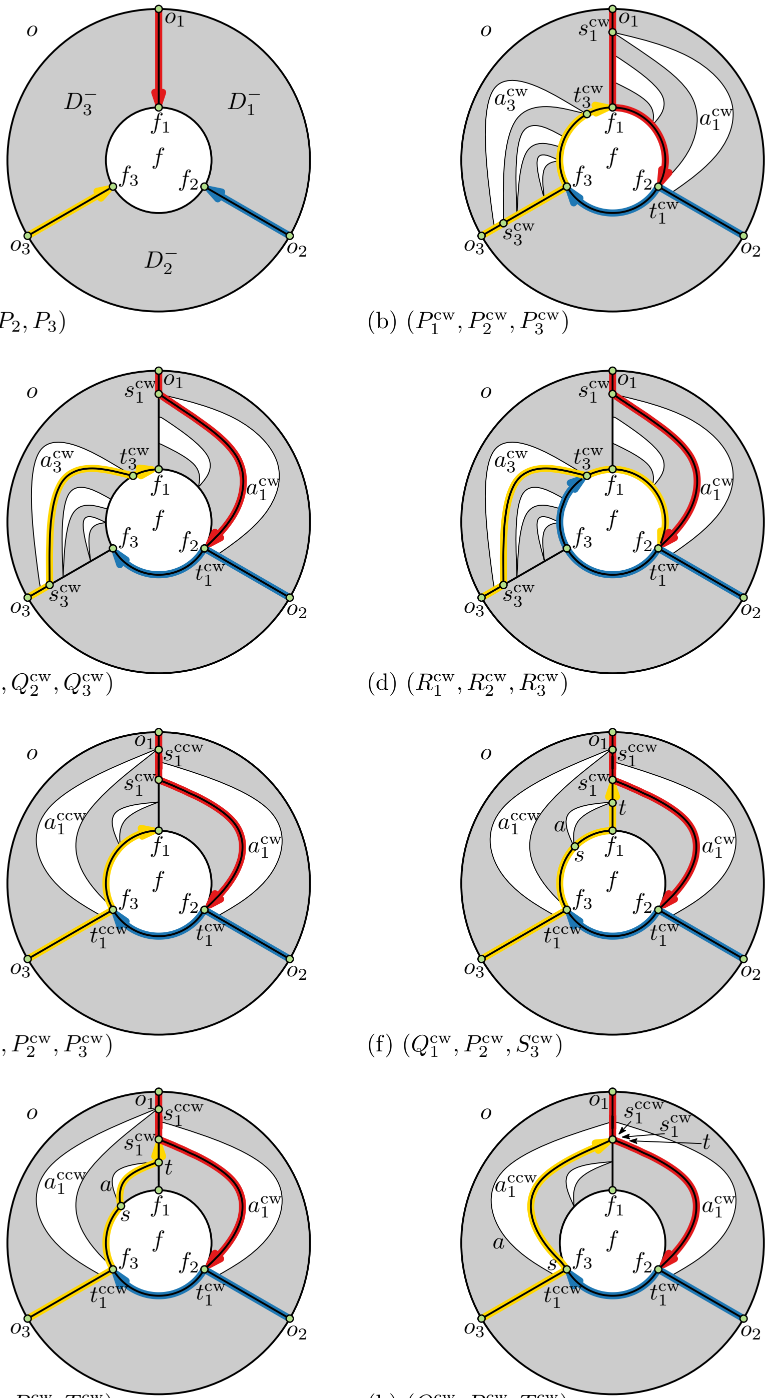
– $\mathrm{L}_G(P)$ *(*$\mathrm{R}_G(P)$*) is a simple internal st-path not arched from the left (right).*
– *If P is not arched from the right (left) by an internal face, then* $\mathrm{L}_G(P)$ *(*$\mathrm{R}_G(P)$*) is archfree.*
– $\mathrm{R}_G(\mathrm{L}_G(P))$ *(*$\mathrm{L}_G(\mathrm{R}_G(P))$*) is archfree.*

Existence of archfree windmills. Recall that our plan is to dissect our given (internally) 3-connected graph along three archfree paths that form a windmill pattern; see Fig. 2a.

Definition 2. *Let G be an internally 3-connected plane graph and let o denote its outer face. For $i \in [3]$, let $P_i = (o_i, \dots, q_i)$ be a simple path in G. We call (P_1, P_2, P_3) a windmill of G if and only if all of the following properties hold (all indices are considered modulo 3):*
(W1) The vertices o_1, o_2, o_3 are pairwise distinct and belong to ∂o.
(W2) For $i \in [3]$, no vertex of $\mathrm{V}(P_i) \setminus \{o_i\}$ belongs to ∂o.
(W3) For $i \in [3]$, no interior vertex of P_i belongs to P_{i+1}.
(W4) For $i \in [3]$, the endpoint q_i is an interior vertex of P_{i+1}.
If (P_1, P_2, P_3) is a windmill of G, we call it archfree if P_1, P_2, P_3 are archfree.

A necessary condition for the existence of an archfree windmill is the existence of a *strictly* internal face (a face without outer vertices). For the considered graph class we show that the condition is sufficient. The following lemma is the main technical contribution of this section:

Lemma 3 ($\star$). *Let G be an internally 3-connected plane graph of maximum degree 4 with a strictly internal face f. Then G contains an archfree windmill.*



(a) (P_1, P_2, P_3) (b) $(P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$

(c) $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$ (d) $(R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}})$

(e) $(Q_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$ (f) $(Q_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, S_3^{\mathrm{cw}})$

(g) $(Q_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, T_3^{\mathrm{cw}})$ (h) $(Q_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, T_3^{\mathrm{cw}})$

Fig. 3: Evolution of the three paths in the first (subfigures (a)–(d)) and second (subfigures (e)–(h)) part of the proof of Lem. 3.

Proof (sketch). Let o be the outer face of G. By means of the internal 3-connectivity of G and Lem. 2, it can be shown that there are three pairwise disjoint archfree paths $P_i = (o_i, \ldots, f_i), i \in [3]$ between ∂o and ∂f as depicted in Fig. 3a. We now walk along ∂f in a clockwise fashion and append appropriate parts of ∂f to the paths P_1, P_2, P_3 to obtain an initial windmill $(P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$ as illustrated in Fig. 3b. Specifically, we extend each P_i by the $f_i f_{i+1}$ subpath of ∂f that does not contain f_{i+2} (indices are considered modulo 3). This windmill is not necessarily archfree, but its paths can only be arched in a controlled way: suppose that P_i^{cw} is arched by an internal face a_i^{cw}. The subpath of P_i^{cw} that belongs to ∂f is archfree by Lem. 1. Combined with the fact that P_i is archfree, it follows that a_i^{cw} arches P_i^{cw} between some vertex $s_i^{\mathrm{cw}} \in \mathrm{V}(P_i) \setminus \{f_i\}$ and a vertex $t_i^{\mathrm{cw}} \in \mathrm{V}(P_i^{\mathrm{cw}}) \setminus \mathrm{V}(P_i)$. Moreover, by planarity, a_i^{cw} has to arch P_i^{cw} from the left, as illustrated in Fig. 3b. We remark that there might be multiple "nested" faces that arch P_i^{cw}. W.l.o.g., we use a_i^{cw} to denote the "outermost" one, that is, the unique arch whose boundary replaces a part of P_i^{cw} in the left-aligned path $Q_i^{\mathrm{cw}} = \mathrm{L}_G(P_i^{\mathrm{cw}})$, see Fig. 3c. The paths of $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$ are now archfree by Lem. 2, though, (W4) from Definition 2 is satisfied only for exactly those $i \in [3]$ where the arch P_{i+1}^{cw} is archfree. For each Q_i^{cw} where (W4) is violated, we append the $f_{i+1} t_{i+1}^{\mathrm{cw}}$-path of ∂f that does not contain f_i, see Fig. 3d. This modification maintains the archfreeness by planarity and Lem. 1. However, the resulting path triple $(R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}})$ might still not be a windmill: suppose that a path P_i^{cw} is not archfree and its arching face a_i^{cw} is *big*, that is, $t_i^{\mathrm{cw}} = f_{i+1}$, while additionally the path P_{i+1}^{cw} is archfree (this is the case for $i = 1$ in Fig. 3b). Then (W3) from Definition 2 is violated for R_{i+1}^{cw} and (W4) is violated for R_{i+2}^{cw}. Suppose that $(R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}})$ is indeed not a windmill. We construct path triples $(P_1^{\mathrm{ccw}}, P_2^{\mathrm{ccw}}, P_3^{\mathrm{ccw}})$, $(Q_1^{\mathrm{ccw}}, Q_2^{\mathrm{ccw}}, Q_3^{\mathrm{ccw}})$, and $(R_1^{\mathrm{ccw}}, R_2^{\mathrm{ccw}}, R_3^{\mathrm{ccw}})$ in a symmetric fashion by walking around ∂f in counterclockwise direction. If $(R_1^{\mathrm{ccw}}, R_2^{\mathrm{ccw}}, R_3^{\mathrm{ccw}})$ is also not a windmill, it follows that both $(R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}})$ and $(R_1^{\mathrm{ccw}}, R_2^{\mathrm{ccw}}, R_3^{\mathrm{ccw}})$ contain a path that is arched by a big face. By planarity and the degree bounds, we can now argue that there is exactly one $i \in [3]$ such that both (P_i^{cw}) and (P_i^{ccw}) are arched by big faces while both (P_{i+1}^{cw}) and (P_{i+2}^{ccw}) are archfree, which is illustrated in Fig. 3e for $i = 1$. Assume w.l.o.g. that $i = 1$ and that s_1^{cw} is not closer to o_1 on P_1 than s_1^{ccw}. In view of the previous observations, it is now easy to argue that the paths of $(Q_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$ are archfree and satisfy all windmill properties with the exception of (W4) for $i = 3$. We restore (W4) by appending the $f_1 s_1^{\mathrm{cw}}$-subpath of P_1 to P_3^{cw}, see Fig. 3f. By means of the degree bounds, it can be argued that (W2) and (W4) are maintained for $i = 3$. The resulting path S_3^{cw} might now be arched (from the left, by planarity), which can be remedied by applying Lem. 2, see Figures 3g and h. By means of the degree bounds and planarity arguments, it can be shown that this modification maintains all windmill properties. □

A plane graph G is *internally 4-regular* if all of its internal vertices have degree 4 and its outer vertices have degree at most 4. In Lem. 3, we established that the existence of an internal face suffices for the existence of an archfree windmill. By means of simple counting arguments, it can be shown that this condition is satisfied if G has a triangular outer face.

Lemma 4 (⋆). *Let G be an internally 3-connected plane graph that is internally 4-regular. Let o denote the outer face of G and assume $|\partial o| = 3$. Then G has a strictly internal face.*

Algorithm. We are now ready to describe our algorithm. As already mentioned in the beginning of Sect. 2, we follow the idea of the recursive combinatorial constructions described by Hong and Nagamochi [9] and Klemz [15], though, the way in which we decompose our graphs is necessarily quite different.

Theorem 1 (⋆). *Let G be an internally 3-connected internally 4-regular plane graph and let Γ^o be a compatible convex drawing of its outer face. There exists a convex drawing Γ of G that uses Γ^o as the realization of the outer face where each internal vertex of G is contained in the interior of some segment of Γ.*

Proof (sketch). Our goal is to (recursively) compute coordinates for the internal vertices to obtain the desired drawing of G. The base case of the recursion is that G contains no internal edges, in which case there is nothing to show. Assume that G is 3-connected – we deal with the case where G is not 3-connected in the appendix. If $|\operatorname{V}(\Gamma^o)| \geq 4$, then there exist two distinct outer vertices u, v that do not belong to a common segment of Γ^o, see Fig. 2b. By 3-connectivity and Lem. 2, they are joined by an archfree internal path P. We split Γ^o into two simple convex polygons along P and handle the two corresponding subgraphs recursively. If $|\operatorname{V}(\Gamma^o)| = 3$, then G contains an archfree windmill (P, S, Q) by Lems. 3 and 4. Since the three outer endpoints of P, S, Q do not belong to a common segment of Γ^o, we can embed them in a straight-line fashion such that Γ^o is dissected into four simple convex polygons, see Fig. 2c. We handle the corresponding four subgraphs recursively. □

Universal upper bound. Recall (from the beginning of Sect. 2) that to establish the claimed upper bound, it suffices to create a drawing where all but three of the vertices of the graph are drawn in the interior of some segment. To achieve this goal, we can now draw the outer face of the graph as a triangle and then apply Thm. 1.

Theorem 2 (⋆). *Every 3-connected internally 4-regular plane graph G admits a convex drawing on at most $n + 3$ segments where n is the number of vertices.*

Existential lower bound. For a graph G, let G^2 denote the *square* of G, that is, G^2 has the same vertex set as G and two vertices in G^2 are adjacent if and only if their distance in G is at most 2. For $n \geq 6$, the square of the n-cycle, C_n^2, is 4-regular and triconnected. By removing three edges from a drawing Γ of C_n^2, we obtain a drawing of a graph whose segment number is n [5, proof of Thm. 7]. Consequently, Γ uses at least $n - 3$ segments, which already shows that Thm. 2 is tight up to an additive constant. In App. B, we examine the situation more closely to prove a slightly stronger bound.

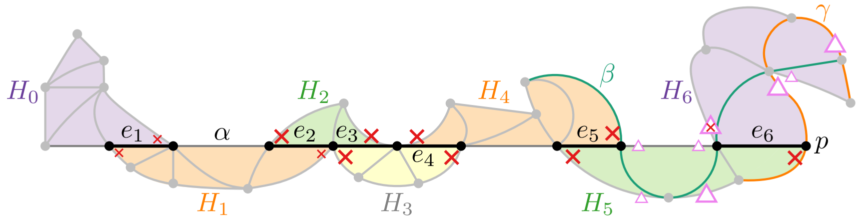


Fig. 4: An outerpath represented by a pseudo-2-arc arrangement. The internal edges $e_1, \ldots, e_6$ of arc α subdivide the outerpath into bays $H_0, \ldots, H_6$. We marked the bay crossings of α and β by red crosses and violet triangles, respectively. For the bay crossings in C that are relevant for our charging scheme we used larger symbols.

Proposition 1 ($\star$). *For even $n \geq 6$, C_n^2 is planar and $\mathrm{seg}(C_n^2) \geq n$.*

It's easy to show a slightly worse bound. Consider the outerpath R_n where every vertex has degree at most 4. By adding three edges to R_n, we obtain C_n^2. Dujmović et al. [5] have shown that $\mathrm{seg}(R_n) = n$. Let Γ be a drawing of R_n with n segments. Each time we insert one of the three missing edges into Γ, we can remove at most two ports, hence $\mathrm{seg}(C_n^2) \geq n - 3$.

Recall that C_6^2 is the octohedron. It is known that $\mathrm{seg}(C_6^2) = 9$ [16]. Hence, for this graph, the bound in Thm. 2 is best possible.

3 Maximal Outerpaths

In this section, we generalize segments and arcs to pseudo-k-arcs (defined below) and give a universal lower bound for the number of pseudo-k-arcs in drawings of maximal outerpaths.

We call a sequence $v_1, v_2, \ldots, v_n$ of the vertices of a maximal outerpath G a *stacking order* of G if for each i, the graph G_i induced by the vertices $v_1, v_2, \ldots, v_i$ is a maximal outerpath. An arrangement of *pseudo-k-arcs* is a set of curves in the plane such that any two of the curves intersect at most k times. (If two curves share a tangent, this counts as two intersections.) We forbid self-intersections, but for $k \geq 2$ we allow a pseudo-k-arc to be closed.

To show the bound, we present a charging scheme that assigns internal edges to pseudo-k-arcs. Any drawing of a maximal outerpath has exactly $n-3$ internal edges. A pseudo-k-arc is *long* if it contains at least $k+1$ internal edges; otherwise it is *short*. Let arc_k denote the number of pseudo-k-arcs, and let arc_k^i denote the number of pseudo-k-arcs with i internal edges. The internal edges of a long arc α subdivide the outerpath into subgraphs $H_0, H_1, \ldots, H_\ell$ called *bays*; see Fig. 4. Given a drawing Γ of a maximal outerpath, we denote the sub-drawings of $G_3, G_4, \ldots, G_n$ within Γ by $\Gamma_3, \Gamma_4, \ldots, \Gamma_n$, respectively. A pseudo-k-arc α is *incident* to a face f if α contains an edge incident to a vertex of f. We say that α is *active* in Γ_i if α is incident to the last face that has been added.

Lemma 5 ($\star$). *For any $i \in \{3, \ldots, n\}$, a partial outerpath drawing Γ_i contains at most one active long pseudo k-arc.*

We do a 2-round assignment to assign each internal edge to a pseudo-k-arc. We start with the *round-1 assignment*. Let I denote the set of internal edges of long pseudo-k-arcs starting at the $(k+1)$-th internal edge (as for the first k internal edges an arc is still short). We assign all $n-3$ internal edges except for the edges in I to their own pseudo-k-arcs:

$$(n-3)-|I| = k\,\mathrm{arc}_k^{\geq k} + (k-1)\,\mathrm{arc}_k^{k-1} + \dots + \mathrm{arc}_k^1 = k\,\mathrm{arc}_k - \textstyle\sum_{i=0}^{k}(k-i)\,\mathrm{arc}_k^i \quad (1)$$

Now we describe the *round-2 assignment*. There, we charge the internal edges of I to specific crossings, which we can charge in turn to pseudo-k-arcs. A *crossing* is a triple (α, β, p) that consists of two pseudo-k-arcs α and β and a point p at which α and β intersect. These specific crossings involve long arcs and we call them *bay crossings*. Next, we define them such that for each long pseudo-k-arc α with ℓ internal edges ($\ell > k$), there are 2ℓ bay crossings $(\alpha, *, *)$ where $*$ is a wildcard. For each bay $H \in \{H_1, \dots, H_{\ell-1}\}$, we have two bay crossings: a crossing of α with another pseudo-k-arc at each of the two vertices of H that have degree 2 within H; see the red crosses in Fig. 4. Clearly, they exist for each H because H is an outerpath. Since these two vertices are distinct for each pair of consecutive bays, their bay crossings are distinct as well. Note that a tangential point may be shared by some H_j and H_{j+2} (for $j \in [\ell-3]$); see, e.g., H_2 and H_4 in Fig. 4. However, we still have distinct bay crossings for H_j and H_{j+2} since a tangential point counts for two crossings. For each of H_0 and H_ℓ, there is one bay crossing defined next. In H_0 and H_ℓ, consider the two crossings of α at the internal edge e_1 and e_ℓ, respectively – one at each of the vertices of the internal edge. One of these vertices is the degree-2 vertex of H_1 ($H_{\ell-1}$) and hence may be identical with a bay crossing of H_1 ($H_{\ell-1}$). E.g., in Fig. 4, the bay crossing (α, γ, p) of H_5 occurs as one of the considered crossings of H_6. The other one of the two considered crossings cannot be a bay crossing in a neighboring bay and this is our bay crossing of H_0 (H_ℓ); see the red crosses at H_0 and H_6 in Fig. 4.

In the round-2 assignment, we charge the surplus internal edges of a long arc α to the other pseudo-k-arcs involved in bay crossings with α. For each internal edge of I, we have two distinct bay crossings of the preceding bay, e.g., in Fig. 4 H_2 provides two bay crossings for e_3. Let C be the set of these bay crossings. The bay crossings of $H_0, \dots, H_{k-1}$, and H_ℓ are not included in C as the internal edges $e_1, \dots, e_k$ are not contained in I and there is no $e_{\ell+1}$. Clearly, $2|I| = |C|$.

Next, we give an upper bound for $|C|$ in terms of arc_k. The main argument we exploit is that, by definition, each pseudo-k-arc can participate in at most k crossings with the (current) long arc and, hence, also in at most k bay crossings with the (current) long arc. However, we need to be a bit careful when one long pseudo-k-arc becomes inactive and a new pseudo-k-arc becomes long, i.e., we consider the transition between one long arc to a new long arc. A (not necessarily long) pseudo-k-arc γ could potentially contribute k crossings in C with each long arc. To compensate for the double counting at transitions, we introduce the *transition loss* t_k, which we define as $t_k = \sum_{\gamma \in \mathcal{A} \setminus \{\alpha_1\}} (|\{c = (*, \gamma, *) \mid c \in C\}| - k)$, where $\mathcal{A}$ is the set of all pseudo-k-arcs and α_1 is the first long arc in Γ. In other words, each pseudo-k-arc, while it is short, contributes to t_k the number of its bay crossings minus k. For example, in Fig. 4, γ contributes 1 to t_k: γ has one bay crossing in C with the long arc α (red cross at e_6) and two bay crossings in C with the long arc β (violet triangles on the top right). The arc β contributes -1 to t_k: β has one bay crossing in C with the long arc α.

Note that, while it is long, an arc does not cross other long arcs. Also, we do not count the crossings of the first k bays and the very last bay. Hence,

$$2|I| = |C| \leq \underbrace{k \cdot (\mathrm{arc}_k \overbrace{-1}^{\substack{\text{The first long pseudo-}k\text{-arc does not provide}\\ \text{crossings with another long pseudo-}k\text{-arc.}}})}_{\substack{\text{Each pseudo-}k\text{-arc intersects the}\\ \text{current long arc at most }k\text{ times.}}} \underbrace{-(2k-1)}_{\substack{\text{Crossings of }H_0, H_1, \ldots, H_{k-1}\text{ of}\\ \text{the first long arc are not in }C.}} \overbrace{-1}^{\substack{\text{The crossing of }H_\ell\text{ of the last}\\ \text{long arc is not in }C.}} \underbrace{+\,t_k}_{\substack{\text{transition}\\ \text{loss}}} \tag{2}$$

Plugging Eq. (2) into Eq. (1), we obtain the following general formula, which gives a lower bound on the number of pseudo-k-arcs for any outerpath.

$$\mathrm{arc}_k \geq \left(2n - 6 + 2 \cdot \textstyle\sum_{i=0}^{k} (k-i)\, \mathrm{arc}_k^i - t_k\right)/(3k) + 1 \tag{3}$$

Since this formula still contains unresolved variables, we now resolve t_k.

Lemma 6 ($\star$). *There is a loss of at most one crossing per transition from one long pseudo-k-arc to another long pseudo-k-arc. Hence,* $t_k \leq \max\{0, \mathrm{arc}_k^{>k} - 1\} \leq \mathrm{arc}_k^{>k} = \mathrm{arc}_k - \sum_{i=0}^{k} \mathrm{arc}_k^i$*, where* $\mathrm{arc}_k^{>k}$ *is the number of long pseudo-k-arcs.*

By Lem. 6 and Eq. (3),

$$\mathrm{arc}_k \geq \left(2n + 3k - 6 + \textstyle\sum_{i=0}^{k} (2k - 2i + 1)\, \mathrm{arc}_k^i\right)/(3k+1)\,. \tag{4}$$

Into this general formula, we plug specific values of k and prove lower bounds on arc_k^i. We start with $k = 1$, i.e., outerpath drawings on pseudo segments.

Lemma 7 ($\star$). *For $k = 1$ and $n \geq 3$, in any outerpath drawing either* $\mathrm{arc}_1^0 \geq 3$ *or (*$\mathrm{arc}_1^0 \geq 2$ *and* $\mathrm{arc}_1^1 \geq 3$*).*

Using Lem. 7, we fill the gaps in Eq. (4) for $k = 1$ and obtain Thm. 3.

Theorem 3 ($\star$). *For any n-vertex maximal outerpath G,* $\mathrm{seg}(G) \geq \lfloor \frac{n}{2} \rfloor + 2$.

For $k = 2$, i.e, for (pseudo) circular arcs, Eq. (4) leads to the following bound.

Theorem 4 ($\star$). *For any n-vertex maximal outerpath G,* $\mathrm{arc}(G) \geq \lceil \frac{2n}{7} \rceil$.

For $k > 2$, it is not obvious how to generalize circular arcs. Still, we can make a similar statement for curve arrangements, which follows directly from Eq. (4).

Proposition 2. *Let G be an n-vertex maximal outerpath drawn on a curve arrangement in the plane s.t. curves intersect pairwise $\leq k$ times, can be closed, but do not self-intersect. Then, the number* $\mathrm{arc}_k(G)$ *of curves required is* $\lceil \frac{2n+3k-6}{3k+1} \rceil$.

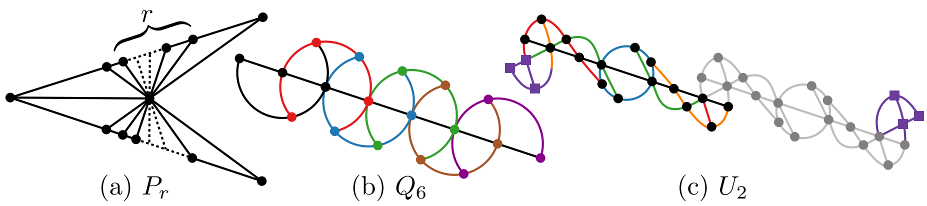


Fig. 5: Families of maximal outerpaths with (a) $n/2+2$ segments (matching the lower bound in Thm. 3), (b) $n/3+1$ circular arcs, and (c) $(5n+18)/16 < n/3$ pseudo 2-arcs.

The infinite families of examples in Prop. 3 and Fig. 5 show that our bounds for segments and arcs are tight. This implies, somewhat surprisingly, that, at least for worst-case instances, using pseudo segments requires as many elements as using straight line segments. Whether this also holds for pseudo circular arcs and circular arcs is an open question. With circular arcs, we could not beat a bound of $n/3$, which we could do for pseudo circular arcs.

Proposition 3 ($\star$). *For every $r \in \mathbb{N}$, maximal outerpaths P_r, Q_r, U_r exist s.t.*
(i) P_r has $n = 2r+6$ vertices and $\text{seg}(P_r) \le r+5 = n/2+2$,
(ii) Q_r has $n = 3r$ vertices and $\text{arc}(Q_r) \le r+1 = n/3+1$,
(iii) U_r has $n = 16r+6$ vertices and $\text{arc}_2(U_r) \le 5r+3 = \frac{5n+18}{16} \approx 0.3125\,n$.

4 Further Results and Open Problems

In App. D, we give an alternative proof for Thm. 3, charging segment ends to vertices. We also give universal lower bounds on the segment numbers of 2-trees and maximal outerpaths. The key idea is to "glue" outerpaths, while adjusting the charging scheme. With a different charging scheme from segment ends to faces, we show an (almost) tight universal lower bound for planar 3-trees.

Theorem 5 ($\star$). *For a 2-tree (or a maximal outerplanar graph) G with n vertices, $\text{seg}(G) \ge (n+7)/5$.*

Theorem 6 ($\star$). *For a planar 3-tree G with $n \ge 6$ vertices, $\text{seg}(G) \ge n+4$.*

For cactus graphs, we can compute the segment number in linear time.

Theorem 7 ($\star$). *Given a cactus graph G, we can compute $\text{seg}(G)$ in linear time. Within this timebound, we can draw G using $\text{seg}(G)$ many segments. If G is given with an outerplanar embedding, the drawing will respect the given embedding.*

Now we turn to open problems. The most prominent one is to close the gaps in Table 1. Since circular-arc drawings are a generalization of straight-line drawings, it is natural to ask about the maximum ratio between the segment number and the arc number of a graph. We make some initial observations regarding this question in App. F. Finally, what is the complexity of deciding whether the arc number of a given graph is strictly smaller than its segment number?

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Appendix

In the following, we provide full proofs and omitted content. First, we introduce some notation that we use throughout the appendix.

Recall that a *cactus* is a connected graph where any two simple cycles share at most one vertex. A graph G is a *k-tree* if it admits a *stacking order* $v_1, v_2, \ldots, v_n$ of the vertices together with a sequence of graphs $G_{k+1}, G_{k+2}, \ldots, G_n = G$ such that (i) G_{k+1} is a clique on $\{v_1, \ldots, v_{k+1}\}$; and (ii) for $k+2 \leq i$, the graph G_i is obtained from G_{i-1} by making v_i adjacent to all vertices of a k-clique in G_{i-1}. A vertex placement in step (ii) is called a *stacking operation*. Similarly, we call the sequence of vertices $v_1, v_2, \ldots, v_n$ of a maximal outerplanar graph G its *stacking order* if for each i the graph G_i induced by the vertices $v_1, v_2, \ldots, v_i$ is a maximal outerplanar graph. If G is an outerpath, each G_i is an outerpath.

In a straight-line drawing Γ of a graph G, each segment terminates at two vertices. Let s be a segment in Γ, and let v be an endpoint of s. Geometrically speaking, we could extend s at v into a face f. We say that s has a *port* at v in f. We call v *open* if v has at least one port and *closed* otherwise. Let $\mathrm{port}(\Gamma)$ be the number of ports in Γ, and let $\mathrm{port}(G)$ be the minimum number of ports over all straight-line drawings of G. Observe that, for any planar graph G, it holds that $\mathrm{seg}(G) = \mathrm{port}(G)/2$. Hence, in a drawing of G, counting segments is equivalent to counting ports.

A An Algorithm for Cactus Graphs

We first state a lower bound for the segment number of cactus graphs. Then, we give a recursive algorithm that produces drawings meeting the bound precisely.

Lemma 8. *Let G be a cactus graph, let η be the number of odd-degree vertices of G, and let $\gamma = 3c_0 + 2c_1 + c_2$, where c_i is the number of simple cycles with exactly i cut vertices in G. Then* $\mathrm{seg}(G) \geq \eta/2 + \gamma$.

Proof. If G is a tree, then $\gamma = 0$ and $\mathrm{seg}(G) = \eta/2$, as shown by Dujmović et al. [5]. If G is a cycle, then all vertices have degree 2 (that is, $\eta = 0$). Moreover, $c_0 = 1$ and $c_1 = c_2 = 0$. A cycle can be drawn as a triangle (but not with less than three segments), that is, $\mathrm{seg}(G) = 3$.

So assume that G is neither a tree nor a cycle. Then G contains at least one cycle and each cycle has at least one cut vertex, that is, $c_0 = 0$. Let Γ be any straight-line drawing of G. Every odd-degree vertex of G has a port in Γ. Hence, Γ has at least η ports.

Additionally, each cycle f of G is a simple polygon in Γ. In other words, f is incident to at least three segments in Γ. If f contains exactly two cut vertices, the drawing of f must contain a bend at some vertex of f that is not a cut vertex, that is, at a degree-2 vertex. This increases the number of ports by 2. Similarly, if f contains exactly one cut vertex, the drawing of f must contain two bends at degree-2 vertices, which increases the number of ports by 4. In total, Γ has at least $\eta + 4c_1 + 2c_2$ ports or $\eta/2 + 2c_1 + c_2$ segments. Since $c_0 = 0$, we have $\mathrm{seg}(G) \geq \eta/2 + 3c_0 + 2c_1 + c_2$ as claimed. □

It is not difficult, but somewhat technical to draw a given cactus such that the lower bound in the above lemma is met exactly. For an idea of how we proceed, refer to Fig. 6.

Theorem 7 (⋆). *Given a cactus graph G, we can compute* $\operatorname{seg}(G)$ *in linear time. Within this timebound, we can draw G using* $\operatorname{seg}(G)$ *many segments. If G is given with an outerplanar embedding, the drawing will respect the given embedding.*

Proof. If G is a tree, we can use the linear-time algorithm of Dujmović et al. [5], which yields a drawing with $\eta/2$ segments, which is optimal. If G is a simple cycle, we can draw G as a triangle, which again is optimal. Otherwise, $c_0 = 0$. In this case, which we treat below, we draw G with $\eta/2 + 2c_1 + c_2$ segments, which is optimal according to Lem. 8.

We draw G recursively, treating its biconnected components as units. Note that, in a cactus graph, the biconnected components (called *blocks*) are exactly its simple cycles and the edges that do not lie on any simple cycle. The *block-cut tree* of a connected graph H has a node for each cut vertex and a node for each block. A block node and a cut-vertex node are connected by an edge in the tree if, in H, the block contains the cut vertex.

We compute the block-cut tree of G, which can be done in linear time [21], and root it at a block node that corresponds to a simple cycle f. We start by drawing this block as a regular p-gon P, where p is the maximum of 3 and the number of cut vertices of f. Let $2r$ be the edge length of P, and let α be the interior angle at each corner of P. Then $\alpha = 180° \cdot (p-2)/p$.

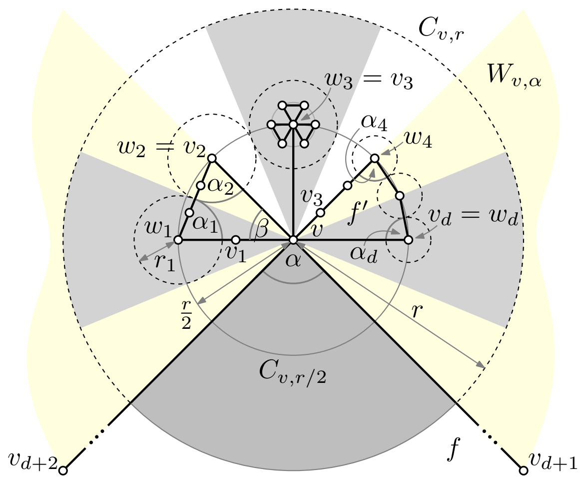


Fig. 6: Recursive approach for drawing cactus graphs. Vertex v is a cut vertex of f (or a degree-2 vertex if f has less than three cut vertices). After f has been drawn, the algorithm recursively draws the subgraph $G(v)$ into $C_{v,r}$ such that v has a port if and only if $\deg(v)$ is odd.

For each cut vertex v of f, we recursively draw the subgraph $G(v)$ of G corresponding to the subtree that hangs off v in the block-cut tree; see Fig. 6. We draw $G(v)$ into the interior of the circle $C_{v,r}$ of radius r centered at v. (Within this circle, we use only the complement of P.) For each pair of cut vertices, the interiors of the corresponding circles are disjoint; hence, the drawing of G has no edge crossings if the drawings of the subgraphs are crossing-free. Our drawing of G will have the following property. Each odd-degree vertex has exactly one port and, in every simple cycle of G with $j < 3$ cut vertices, there are exactly $3 - j$ degree-2 vertices with two ports. This implies that the total number of segments in our drawing meets the bound in Lem. 8 precisely.

Let $d = \deg(v) - 2$, and let $\mathrm{N}(v) = \{v_1, \ldots, v_{d+2}\}$ be the neighborhood of v. Let v_{d+1} and v_{d+2} be the two neighbors of v that lie on f (in clockwise order before and after v on f) and have already been placed. Let $v_1, \ldots, v_{d+2}$ be ordered clockwise around v. We assume that neighbors that belong to the same simple cycle are consecutive in this ordering. (Note that this is the case if G is given with a fixed outerplane embedding.) We now define a set W of vertices in $G(v)$ for which we may call our algorithm recursively. Initially, W is empty. For $i \in \{1, \ldots, d\}$, if v_i and v do not lie on the same simple cycle, then set $w_i = v_i$ and add w_i to W. Now let f' be the simple cycle that contains v, v_i and another neighbor of v, say, v_{i+1}. If f' does not contain a cut vertex other than v, set $w_i = v_i$ and add w_i to W. Otherwise, let w_i be the cut vertex of G closest to v_i in $G(v) - v$. If v_{i+1} has the same closest cut vertex w_j then, if $w_i \neq v_i$, set $w_i = v_i$, otherwise set $w_{i+1} = v_{i+1}$. Add w_i and w_{i+1} and all other cut vertices of f' (if any) to W (except v).

We now place the vertices in W on the circle $C_{v,r/2}$. If d is odd, then we place $w_{(d+1)/2}$ on the line that bisects the angle $\angle v_{d+2} v v_{d+1}$; namely such that $w_{(d+1)/2}$ lies opposite of this angle (as w_3 in Fig. 6). For the remainder of this proof, we assume for simplicity that d is even. Then $d \geq 2$ and we place $w_{d/2}$ on the line vv_{d+1} and $w_{d/2+1}$ on the line vv_{d+2}. We place the remaining neighbors in pairs on opposite sides of lines through v such that these lines equally partition the angle space in the double wedge $W_{v,\alpha}$ (light yellow in Fig. 6) that is bounded by the lines vv_{d+1} and vv_{d+2} and does not contain the angle $\angle v_{d+2} v v_{d+1}$. The angular distance between two consecutive edges incident to v is then $\beta = (360° - 2\alpha)/d$.

We draw each simple cycle f' that contains v and two neighbors v_i and v_{i+1} of v as a simple polygon that connects v to w_i to potential further cut vertices of f' (in their order along f') to w_{i+1} to v.

Now we define, for each newly placed vertex $w \in W$ with $\deg(w) > 2$, values α' and r' so that we can draw the graph $G(w)$ recursively. To this end, if v and w lie on the same simple cycle f', let α' be the interior angle of w in f', and let r' be the distance of w to the closest vertex in $\mathrm{V}(f) \cap W$ divided by 2. Otherwise, let α' be 0 and set r' such that $C_{w,r'}$ fits into a wedge centered at w that has an angle of β at its apex v; see, for example, w_3 in Fig. 6.

Our invariant is that, in each recursive call for $G(w')$, we have $0 \leq \alpha' < 180°$ and $r' > 0$. This ensures that our drawing has no crossings. To finish the proof, note that the segments that we draw end only in odd-degree vertices (one port each) or in degree-2 vertices (two ports each) of simple cycles that have less than three cut vertices.

Concerning the running time, it is easy to see that each recursive call of the algorithm runs in time linear in the size of the subgraph of G that the current call draws without further recursion. Hence, the overall running time is linear in the size of G (including the computation of the block-cut tree). □

Note that the algorithm in the proof of Thm. 7 can draw a cactus with a fixed outerplane embedding such that its embedding is maintained. Unfortunately, the drawing area can be at least exponential, even if the embedding is not fixed.

B Proofs Omitted in Section 2 (4-Regular Planar Graphs)

Observation 1 ($\star$, folklore) *Let G be an internally 3-connected plane graph, and let C be a simple cycle in G. The closed interior C^- of C is an internally 3-connected plane graph.*

Proof. Clearly, Property (I2) of Definition 1 carries over from G to C^-. □

Lemma 3 ($\star$). *Let G be an internally 3-connected plane graph of maximum degree 4 with a strictly internal face f. Then G contains an archfree windmill.*

Proof. Let o be the outer face of G. We begin by constructing three disjoint archfree paths between ∂o and ∂f, as illustrated in Fig. 7a. We plan to use these paths, as well as parts of ∂f to construct a windmill (see Fig. 7b). We then apply Lem. 2 to make its paths archfree (illustrated in Fig. 7c and Fig. 8a)). This may destroy the windmill properties, but it does so in a controlled way, which allows us to successively modify our paths to restore the windmill properties while maintaining the archfreeness.

Claim 1 *G contains three simple paths $P_i = (o_i, \dots, f_i), i \in [3]$ such that*
(P1) P_1, P_2, P_3 are pairwise vertex-disjoint,
(P2) for $i \in [3]$, the endpoint o_i belongs to ∂o,
(P3) for $i \in [3]$, the endpoint f_i belongs to ∂f,
(P4) for $i \in [3]$, the interior vertices of P_i belong to neither ∂f nor ∂o, and
(P5) P_1, P_2, P_3 are archfree in G.

Proof (of Claim 1). For illustrations refer to Fig. 7a. To show that P_1, P_2, P_3 exist, we add a new vertex v_f into f and add edges between v_f every vertex of ∂f. It is easy to see that this modification retains Property (I1) of Definition 1 and, hence, the resulting graph G' is internally 3-connected. By Property (I2) of Definition 1 applied to v_f in G', it follows that G contains three simple paths $P'_i = (o_i, \dots, f_i), i \in \{1, 2, 3\}$, that satisfy Properties (P1)–(P4). For $i = 1, 2, 3$, we consider P'_i to be directed from o_i to f_i and define $P_i = \mathrm{R}_G(\mathrm{L}_G(P'_i))$. By Lem. 2, the paths P_1, P_2, P_3 satisfy Property (P5). To see that the remaining properties are also satisfied, we argue as follows: let C be the simple cycle formed by the paths P_2, P_3, the o_2o_3-path on ∂o that passes through o_1, and the f_2f_3-path on ∂f that passes through f_1. The closed interior C^- of C is internally 3-connected by Obs. 1. Note that $\mathrm{R}_{C^-}(\mathrm{L}_{C^-}(P'_1)) = P_1$ since the only internal face of G that is not an internal face of C^- but shares a vertex with P'_1 is f, which has only a single vertex in common with P'_1 by Properties (P2)–(P4). By Lem. 2 applied to C^- and P'_1, P_1 is an *internal* o_1f_1-path of C^-. Hence, the paths P_1, P'_2, P'_3 satisfy Properties (P1)–(P4) in G. With analogous arguments, we see that P_1, P_2, P'_3 and, finally, P_1, P_2, P_3 also satisfy Properties (P1)–(P4). □

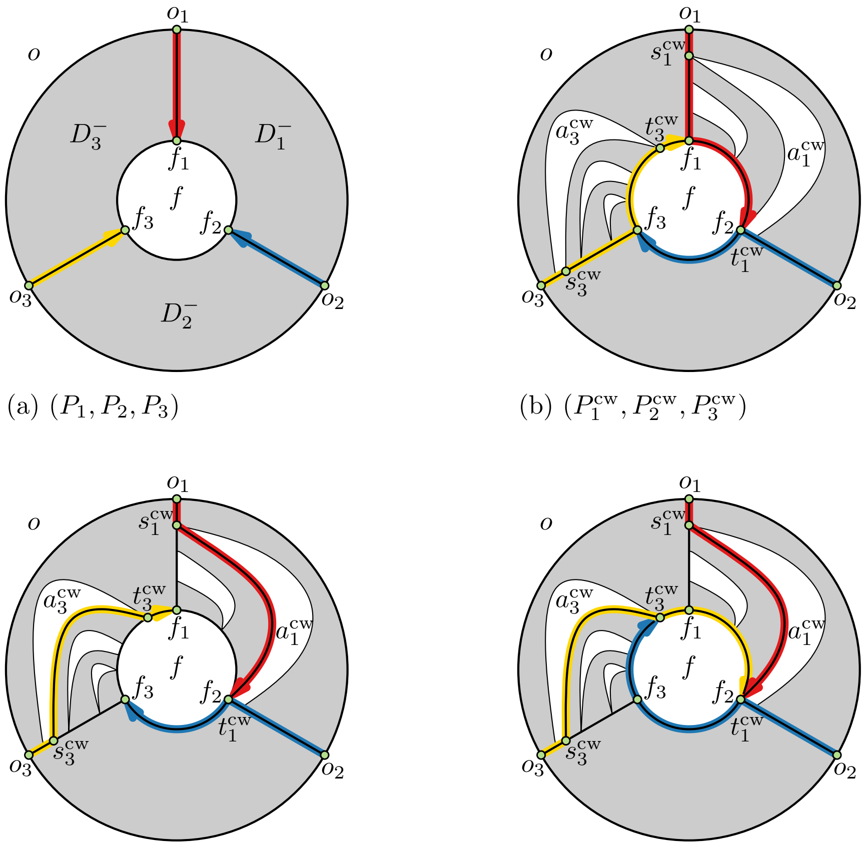
Without loss of generality, assume that o_1, o_2, o_3 appear on ∂f in clockwise order, as depicted in Fig. 7a. For $i \in [3]$, we append the f_if_{i+1}-path on ∂f that does not pass through f_{i-1} to P_i and call the resulting simple path P_i^{cw} (all indices are considered modulo 3), see Fig. 7b. Symmetrically, for $i \in [3]$, we append the f_if_{i-1}-path on ∂f that does not pass through f_{i+1} to P_i and call the resulting simple path P_i^{ccw}. By construction, both $(P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$ and $(P_1^{\mathrm{ccw}}, P_2^{\mathrm{ccw}}, P_3^{\mathrm{ccw}})$ are windmills in G. If one of them is archfree, we are done. So assume otherwise.

We will now deform parts of $(P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$ (or $(P_1^{\mathrm{ccw}}, P_2^{\mathrm{ccw}}, P_3^{\mathrm{ccw}})$) to obtain the desired archfree windmill. To this end, we introduce some notation, for illustrations refer to Fig. 7b: suppose that a path $P_i^{\mathrm{cw}}, i \in [3]$ is arched by a face a_i^{cw}. The subpath P_i of P_i^{cw} is archfree by Claim 1. Moreover, the f_if_{i+1}-subpath of P_i is also archfree by Lem. 1. Consequently, the face a_i^{cw} arches P_i^{cw} between some vertex $s_i^{\mathrm{cw}} \in \mathrm{V}(P_i) \setminus \{f_i\}$ and a vertex $t_i^{\mathrm{cw}} \in \mathrm{V}(P_i^{\mathrm{cw}}) \setminus \mathrm{V}(P_i)$. We consider P_i^{cw} to be directed such that o_i is its source. By planarity, the face a_i^{cw} has to arch P_i^{cw} from the left. We remark that there might be multiple faces that arch P_i^{cw} (from the left), in which case these faces have to be "nested" (as depicted in Fig. 7b). Without loss of generality, we may assume that a_i^{cw} is the "outermost" of these arches. More precisely, we assume that a_i^{cw} is the unique arch such that a $s_i^{\mathrm{cw}}t_i^{\mathrm{cw}}$-path on $\partial a_i^{\mathrm{cw}}$ replaces the $s_i^{\mathrm{cw}}t_i^{\mathrm{cw}}$-path on P_i^{cw} in $\mathrm{L}_G(P_i^{\mathrm{cw}})$. We say that a_i^{cw} is *big* if $t_i^{\mathrm{cw}} = f_{i+1}$ (in Fig. 7b the arch a_1^{cw} is big, while a_3^{cw} is not). Symmetrically, we define the expressions $a_i^{\mathrm{ccw}}, s_i^{\mathrm{ccw}}, t_i^{\mathrm{ccw}}$ for each path $P_i^{\mathrm{ccw}}, i \in [3]$ that is arched (from the right), and we say that a_i^{ccw} is *big* if $t_i^{\mathrm{ccw}} = f_{i-1}$. For $i \in [3]$, let D_i denote the simple cycle that is formed by P_i, P_{i+1}, the f_if_{i+1}-path on ∂f that does not pass through f_{i+2} and the o_io_{i+1}-path on ∂o that does not pass through o_{i+2} (the closed interior D_i^- of D_i is indicated in Fig. 7a).

For $i \in [3]$, we define $Q_i^{\mathrm{cw}} = \mathrm{L}_G(P_i^{\mathrm{cw}})$, for illustrations refer to Fig. 7c.

Claim 2 *The paths $Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}}$ are archfree. Properties (W1)–(W3) from Definition 2 hold for $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$. Property (W4) from Definition 2 is satisfied for exactly those $i \in [3]$ where the arch a_{i+1}^{cw} is undefined.*

Proof (of Claim 2). Lem. 2 implies that the paths are archfree (recall that P_i^{cw} is not arched from the right) and that the Properties (W1) and (W2) from Definition 2 carry over from $(P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$ to $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$.



(a) (P_1, P_2, P_3)

(b) $(P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$

(c) $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$

(d) $(R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}})$

Fig. 7: Evolution of the three paths in the first part of the proof of Lem. 3.

To see that Property (W3) also carries over, we can argue as follows: let C_i be the simple cycle formed by the paths P_{i+1}, P_{i+2}, the $o_{i+1}o_{i+2}$-path on ∂o that passes through o_i, and the $f_{i+1}f_{i+2}$-path on ∂f that does not pass through f_1. The closed interior C_i^- of C_i is internally 3-connected by Obs. 1. Note that $\mathrm{L}_{C_i^-}(P_i^{\mathrm{cw}}) = Q_i^{\mathrm{cw}}$ since the internal faces of G that do not belong to C_i^- intersect P_i^{cw} only in f_{i+1}. Hence, by Lem. 2 applied to C_i^- and P_i^{cw} and by construction, the part of Q_i^{cw} that does not belong to P_i^{cw} is located in the interior of the cycle D_i. By construction, the interior of D_i is disjoint from $P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}$, and P_3^{cw}. Moreover, D_1, D_2, D_3 are pairwise interior disjoint. Consequently, Property (W3) from Definition 2 carries over from $(P_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, P_3^{\mathrm{cw}})$ to $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$, as claimed.

Finally, Property (W4) is violated for those $i \in [3]$ where the arch a_{i+1}^{cw} exists since in this case the endpoint f_{i+1} of Q_i^{cw} is not on Q_{i+1}^{cw}. In contrast, Property (W4) is satisfied for those $i \in [3]$ where the arch a_{i+1}^{cw} undefined since in this case the endpoint f_{i+1} of Q_i^{cw} is on $Q_{i+1}^{\mathrm{cw}}(= P_{i+1}^{\mathrm{cw}})$. □

If $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$ or $(Q_1^{\mathrm{ccw}}, Q_2^{\mathrm{ccw}}, Q_3^{\mathrm{ccw}})$ (which is defined symmetrically and for which a symmetric version of Claim 2 holds) is an archfree windmill, we are done. So assume otherwise. By Claim 2, the only violated property is (W4). To remedy the situation, we will now construct three paths $R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}}$ by appending appropriate parts of ∂f to the corresponding paths in $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$.

For each $i \in [3]$ where the arch a_{i+1}^{cw} is undefined, we set $R_i^{\mathrm{cw}} = Q_i^{\mathrm{cw}}$. For each $i \in [3]$ where the arch a_{i+1}^{cw} exists, we append the $f_{i+1}t_{i+1}^{\mathrm{cw}}$-path on ∂f that does not contain f_i to Q_i^{cw} and denote the resulting path by R_i^{cw}, for illustrations refer to Fig. 7d.

Claim 3 *The paths $R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}}$ are archfree. Properties (W1) and (W2) from Definition 2 hold for $(R_1^{\mathrm{cw}}, R_2^{\mathrm{cw}}, R_3^{\mathrm{cw}})$. Property (W3) from Definition 2 is violated for exactly those $i \in [3]$ where P_{i+2}^{cw} is arched by a big face and a_i^{cw} is undefined. Property (W4) from Definition 2 violated exactly those $i \in [3]$ where P_{i+1}^{cw} is arched by a big face and a_{i+2}^{cw} is undefined.*

Proof (of Claim 3). Let $i \in [3]$. To see that R_i^{cw} is archfree, recall that Q_i^{cw} is archfree by Claim 2. Hence, if $R_i^{\mathrm{cw}} = Q_i^{\mathrm{cw}}$, there is nothing to show, so assume otherwise. The $f_{i+1}t_{i+1}^{\mathrm{cw}}$ subpath of R_i^{cw} is archfree by Lem. 1. Therefore, if R_i^{cw} is arched by some internal face $a \neq f$, then a has to arch R_i^{cw} between some vertex in $\mathrm{V}(R_i^{\mathrm{cw}}) \setminus \mathrm{V}(Q_i^{\mathrm{cw}})$ and some vertex in $\mathrm{V}(Q_i^{\mathrm{cw}}) \setminus \{f_{i+1}\}$, which is impossible by planarity (specifically, the boundary of a would have to cross the cyle D_{i+1}). Moreover, by construction, R_i^{cw} is not arched by f. Hence, R_i^{cw} is archfree, as claimed.

Clearly, Properties (W1) and (W2) of Definition 2 carry over from $(Q_1^{\mathrm{cw}}, Q_2^{\mathrm{cw}}, Q_3^{\mathrm{cw}})$.

Regarding Property (W4), let q_j denote the (internal) endpoint of R_j^{cw} that is not o_j for $j \in [3]$. By construction, q_i belongs to R_{i+1}^{cw}. Hence, Property (W4) is violated if and only if q_i is not an *interior* vertex of R_{i+1}^{cw}, which, by construction, is the case if and only if P_{i+1}^{cw} is arched by a big face and a_{i+2}^{cw} is undefined (in which case $q_i = t_{i+1}^{\mathrm{cw}} = f_{i+2} = q_{i+1}$), for an illustration refer to Fig. 7d with $i = 3$.

Regarding Property (W3), we argue in two steps: let I_i^{old} be the set of interior vertices of R_i^{cw} that are also interior vertices of Q_i^{cw}, i.e, $I_i^{\mathrm{old}} = \mathrm{V}(Q_i^{\mathrm{cw}}) \setminus \{o_i, f_{i+1}\}$. Further, let I_i^{new} be the set of interior vertices of R_i^{cw} that are not interior vertices of Q_i^{cw}. Property (W3) holds if and only if none of these sets intersects $\mathrm{V}(R_{i+1}^{\mathrm{cw}})$.

We first consider the set I_i^{new}. If $R_i^{\mathrm{cw}} = Q_i^{\mathrm{cw}}$, then $I_i^{\mathrm{new}} = \emptyset$ and, hence, $I_i^{\mathrm{new}} \cap \mathrm{V}(R_{i+1}^{\mathrm{cw}}) = \emptyset$. Otherwise, $I_i^{\mathrm{new}} = \{f_{i+1}\} \cup (\mathrm{V}(R_i^{\mathrm{cw}}) \setminus (\mathrm{V}(Q_i^{\mathrm{cw}} \cup t_{i+1})))$. By construction, this set is disjoint from both $\mathrm{V}(Q_{i+1}^{\mathrm{cw}})$ and $\mathrm{V}(R_{i+1}^{\mathrm{cw}}) \setminus \mathrm{V}(Q_{i+1}^{\mathrm{cw}})$. Hence, $I_i^{\mathrm{new}} \cap \mathrm{V}(R_{i+1}^{\mathrm{cw}}) = \emptyset$.

It remains to consider the set I_i^{old}. By Property (W3) for Q_i^{cw}, we have $I_i^{\mathrm{old}} \cap \mathrm{V}(Q_{i+1}^{\mathrm{cw}}) = \emptyset$. So if $I_i^{\mathrm{old}} \cap \mathrm{V}(R_{i+1}^{\mathrm{cw}}) \neq \emptyset$, then $I_i^{\mathrm{old}} \cap (\mathrm{V}(R_{i+1}^{\mathrm{cw}}) \setminus \mathrm{V}(Q_{i+1}^{\mathrm{cw}})) \neq \emptyset$. All vertices in I_i^{old} belong to the closed interior D_i^- of the cycle D_i. The $f_{i+2}t_{i+2}^{\mathrm{cw}}$-path on ∂f intersects D_i^- if and only if P_{i+2} is arched by a big face (i.e., $t_{i+2}^{\mathrm{cw}} = f_i$), namely in f_i. Hence, Property (W3) is violated for i if and only if P^{cw}_{i+2} is arched by a big face and a^{cw}_i is undefined (in which case $Q^{\text{cw}}_i = P^{\text{cw}}_i$ and, hence, $f_i \in \mathrm{V}(R^{\text{cw}}_i)$). □

If $(R^{\text{cw}}_1, R^{\text{cw}}_2, R^{\text{cw}}_3)$ or $(R^{\text{ccw}}_1, R^{\text{ccw}}_2, R^{\text{ccw}}_3)$ (which is defined symmetrically and for which a symmetric version of Claim 3 holds) is an archfree windmill, we are done. So assume otherwise. By Claim 3, it follows that both $\{P^{\text{cw}}_1, P^{\text{cw}}_2, P^{\text{cw}}_3\}$ and $\{P^{\text{ccw}}_1, P^{\text{ccw}}_2, P^{\text{ccw}}_3\}$ contain a path that is arched by a big face. Specifically, we may assume without loss of generality that the arch a^{cw}_1 exists and is big (i.e., $t^{\text{cw}}_1 = f_2$) and a^{cw}_2 is undefined; for an illustration refer to Fig. 8a. By planarity, the path P^{ccw}_2 cannot be arched (the boundary of a^{ccw}_2 would have to cross the boundary of a^{cw}_1). Moreover, the path P^{ccw}_3 cannot be arched by a big face since this would imply $\deg(f_2) \geq 5$, contradicting the degree bounds. Hence, the arch a^{ccw}_1 of P^{ccw}_1 exists and is big (by assumption and Claim 3). Note that, by planarity, the path P^{cw}_3 cannot be arched (the boundary of a^{cw}_3 would have to cross the boundary of a^{ccw}_1). Without loss of generality, we may assume that s^{cw}_1 is not closer to o_1 on P_1 than s^{ccw}_1 (otherwise, we can argue symmetrically), for illustrations refer to Figures 8a and d.

Claim 4 *The paths $Q^{\text{cw}}_1, P^{\text{cw}}_2, P^{\text{cw}}_3$ are archfree. Properties (W1)–(W3) from Definition 2 hold for $(Q^{\text{cw}}_1, P^{\text{cw}}_2, P^{\text{cw}}_3)$. Property (W4) from Definition 2 is satisfied for $i = 1, 2$, but violated for $i = 3$.*

Proof (of Claim 4). Since neither P^{cw}_2 nor P^{cw}_3 are arched, Claim 2 implies the claimed properties hold for $(Q^{\text{cw}}_1, Q^{\text{cw}}_2, Q^{\text{cw}}_3)$. Moreover, $P^{\text{cw}}_2 = Q^{\text{cw}}_2$ and $P^{\text{cw}}_3 = Q^{\text{cw}}_3$, which proves the claim. □

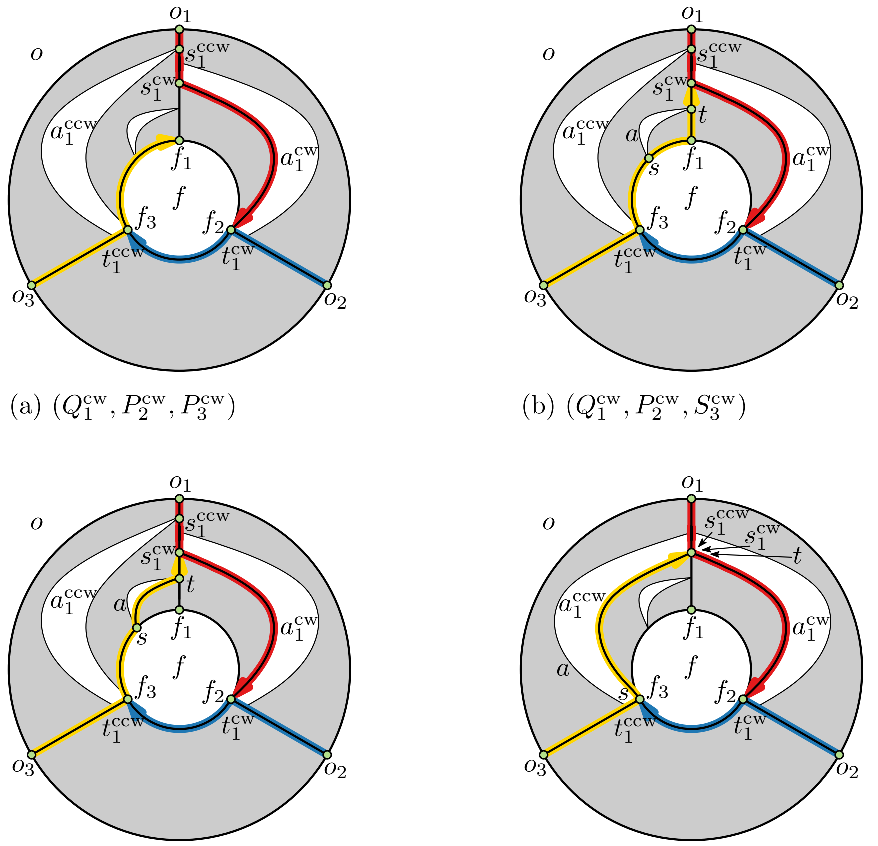
We now append the $f_1 s^{\text{cw}}_1$-subpath of P_1 to P^{cw}_3 and denote the resulting path by S^{cw}_3, for an illustration see Fig. 8b.

Claim 5 *The paths Q^{cw}_1 and P^{cw}_2 are archfree. Properties (W1)–(W4) from Definition 2 hold for $(Q^{\text{cw}}_1, P^{\text{cw}}_2, S^{\text{cw}}_3)$.*

Proof (of Claim 5). To see that Properties (W2) and (W4) hold for $i = 3$, we argue as follows: towards a contradiction, assume that $s^{\text{cw}}_1 \in \partial o$, i.e., $s^{\text{cw}}_1 = o_1$. By our assumption about the positions of s^{cw}_1 and s^{ccw}_1 on P_1, it follows that $s^{\text{cw}}_1 = s^{\text{ccw}}_1 = o_1$. However, this implies that $\deg o_1 \geq 5$, contradicting the degree bounds.

Clearly, by construction, the remaining properties of $(Q^{\text{cw}}_1, P^{\text{cw}}_2, P^{\text{cw}}_3)$ guaranteed by Claim 4 carry over to $(Q^{\text{cw}}_1, P^{\text{cw}}_2, S^{\text{cw}}_3)$. □

If $(Q^{\text{cw}}_1, P^{\text{cw}}_2, S^{\text{cw}}_3)$ is an archfree windmill, we are done, so assume otherwise. By Claim 5, it follows that S^{cw}_3 is arched by an internal face. The paths P^{cw}_3 and P_1 are archfree, so an internal face a that is arching S^{cw}_3 has to do so between some vertex in $t \in \mathrm{V}(S^{\text{cw}}_3) \setminus \mathrm{V}(P^{\text{cw}}_3)$ and some vertex in $s \in \mathrm{V}(P^{\text{cw}}_3) \setminus \{f_1\}$. By planarity, it is not possible that a arches S^{cw}_3 from the right, so a arches S^{cw}_3 from the left. As in the definition of the arches a^{cw}_i, there may be multiple nested arches that arch S^{cw}_3 from the left, and we assume a to be the"outermost" one.



(a) $(Q_1^{\text{cw}}, P_2^{\text{cw}}, P_3^{\text{cw}})$ (b) $(Q_1^{\text{cw}}, P_2^{\text{cw}}, S_3^{\text{cw}})$

(c) $(Q_1^{\text{cw}}, P_2^{\text{cw}}, T_3^{\text{cw}})$ (d) $(Q_1^{\text{cw}}, P_2^{\text{cw}}, T_3^{\text{cw}})$

Fig. 8: Evolution of the three paths in the second part of the proof of Lem. 3.

More precisely, we assume that a is this unique arch such that a st-path on ∂a replaces the st-path on S_3^{cw} in $\mathrm{L}_G(S_3^{\text{cw}})$.

As the final modification, we replace S_3^{cw} by $T_3^{\text{cw}} = \mathrm{L}_G(S_3^{\text{cw}})$, for an illustration see Figures 8c and d.

Claim 6 $(Q_1^{\text{cw}}, P_2^{\text{cw}}, T_3^{\text{cw}})$ *is an archfree windmill.*

Proof (of Claim 6). By Claim 5, the paths Q_1^{cw} and P_2^{cw} are archfree and Properties (W1)–(W4) from Definition 2 hold for $(Q_1^{\text{cw}}, P_2^{\text{cw}}, S_3^{\text{cw}})$. By Lem. 2, T_3^{cw} is archfree. To conclude the proof, it remains to show that Properties (W1)–(W4) hold for the set $(Q_1^{\text{cw}}, P_2^{\text{cw}}, T_3^{\text{cw}})$.

Properties (W1) and (W2) are preserved from $(Q_1^{\text{cw}}, P_2^{\text{cw}}, S_3^{\text{cw}})$ (by Lem. 2 for $i = 3$). To show that the remaining two properties hold, we first show that $s \notin \mathrm{V}(P_3) \setminus \{f_3\}$. If $s_1^{\text{cw}} \neq s_1^{\text{ccw}}$ (see Fig. 8c), this is clear by planarity (the boundary of a would have to cross the boundary of a_1^{ccw}). Towards a contradiction, assume that $s_1^{\mathrm{cw}} = s_1^{\mathrm{ccw}}$ (see Fig. 8c) and $s \in \mathrm{V}(P_3) \setminus \{f_3\}$. By planarity, it follows that $t = s_1^{\mathrm{cw}} = s_1^{\mathrm{ccw}}$ (otherwise, the boundary of a would have to cross the boundary of a_1^{ccw}). However, this implies that $\deg(t) \geq 5$, contradicting the degree bounds. So indeed, $s \notin \mathrm{V}(P_3) \setminus \{f_3\}$ as claimed.

Clearly, Property (W4) for $i = 1, 3$ is preserved from $(Q_1^{\mathrm{cw}}, P_2^{\mathrm{cw}}, S_3^{\mathrm{cw}})$ (by Lem. 2 for $i = 3$). Since $s \notin \mathrm{V}(P_3) \setminus \{f_3\}$, it follows that the endpoint f_3 of P_2^{cw} is not an interior vertex of the st-path on S_3^{cw}. Consequently, Property (W4) is also preserved for $i = 2$.

It remains to establish Property (W3). It is clearly preserved for $i = 1$. Consider the simple cycle C' formed by Q_1^{cw}, the $f_2 f_3$-path on ∂f that does not pass through f_1, P_3, and the $o_1 o_3$-path on o that does not pass through o_2. By Obs. 1, the closed interior C'^{-} of C' is internally 3-connected. By construction, all vertices of Q_1^{cw} and P_2^{cw} belong to the closed exterior of C'. By planarity, there is no internal face of G that is not an internal face of C'^{-} and incident to more than one vertex of the st-subpath of S_3^{cw}. Hence, by Lem. 2 applied to C'^{-} and the st-subpath of S_3^{cw}, it follows that Property (W3) is preserved for $i = 2$ and $i = 3$. □

By Claim 6, there is an archfree windmill, which concludes the proof. □

Lemma 4 (⋆). *Let G be an internally 3-connected plane graph that is internally 4-regular. Let o denote the outer face of G and assume $|\partial o| = 3$. Then G has a strictly internal face.*

Proof. Let n, m, f denote the number of vertices, edges, and faces of G, respectively. The existence of a separation pair would clearly violate Property (I3) of Definition 1. Hence, the graph is 3-connected and it follows that each vertex of ∂o has degree 3 or 4. The handshaking lemma implies that the number η of odd degree vertices is even. Towards a contradiction, assume that G has no strictly internal face, i.e., each face is incident to one of the vertices of ∂o. By 3-connectivity, this implies that $f = 7 - \eta$. By Euler's polyhedron formula, the handshaking lemma, and internal 4-regularity, we obtain:

$$\begin{aligned} & n - m + f = 2 \\ \Leftrightarrow\ & n - (2n - \frac{\eta}{2}) + (7 - \eta) = 2 \\ \Leftrightarrow\ & n = 5 - \frac{\eta}{2} \end{aligned}$$

If $\eta = 2$, it follows that $n = 4$, contradicting the internal 4-regularity. If $\eta = 0$, then G is a 4-regular graph on 5 vertices, i.e., it is isomorphic to K_5; a contradiction to the fact that G is planar. □

Theorem 1 (⋆). *Let G be an internally 3-connected internally 4-regular plane graph and let Γ^o be a compatible convex drawing of its outer face. There exists a convex drawing Γ of G that uses Γ^o as the realization of the outer face where each internal vertex of G is contained in the interior of some segment of Γ.*

Proof. The coordinates of the outer vertices of G are already fixed. Our goal is to (recursively) compute coordinates for the internal vertices to obtain the desired drawing of G. The base case of the recursion is that G contains no internal edges, in which case there is nothing to show. So assume that G has at least one internal edge. Without loss of generality, we may assume that G contains no (outer) degree-2 vertices whose outer angle in Γ^o is π (we can just iteratively merge the two incident edges of such vertices, compute the drawing, and then reinsert the removed vertices at their prescribed coordinates). We distinguish two main cases:

Case 1: G is not 3-connected. We distinguish two subcases:

Case 1.1: G contains a degree-2 vertex v. For illustrations refer to Fig. 9a. By internal 4-regularity, v belongs to $\mathrm{V}(\Gamma^o)$. By our assumption about degree-2 vertices, its outer angle in Γ^o is reflex. Let u and w denote the two neighbors of v. Note that if $uw \in E$, then it belongs to the boundary of the (triangular) internal face incident to v since otherwise u, w would form a separation pair that separates the interior of the cycle uvw from the outer face; contradicting Property (I3) of Definition 1. If $uw \in E$ we set $G_1' = G$. Otherwise, we add the edge uw in the internal face incident to v and call the resulting graph G_1'. Adding an internal edge to an internally 3-connected graph clearly preserves Property (I2) of Definition 1, so, in both cases, G' is internally 3-connected. We delete v from G_1' and call the resulting graph G_1. This modification preserves Property (I2) of Definition 1, so G_1 is internally 3-connected.

Since Γ^o is compatible, the vertices u and w cannot belong to a common segment s of Γ^o (otherwise the internal face of G incident to v would arch s). Consequently, we can replace the edges uv and vw of Γ^o with the edge uw to obtain a *simple* convex polygon Γ_1^o, which is a convex drawing of the outer face of G_1. By Lem. 1, the edge uw is archfree in G_1. Combined with the fact that Γ^o is compatible with G, it follows that Γ_1^o is compatible with G_1.

We recursively compute the coordinates of the internal vertices in a convex drawing of G_1 with Γ_1^o as the realization of the outer face such that all internal vertices of G_1 are placed in the interior of some segment. Since each internal vertex of G is also an internal vertex of G_1, these coordinates combined with the coordinates of v correspond to the desired drawing of G.

Case 1.2: G contains no degree-2 vertex. Let u be a vertex that belongs to a separation pair in G. By Property (I3) of Definition 1, all vertices that belong to separation pairs are on the boundary of the outer face o of G. Let v be the first vertex encountered when walking from u along ∂o in clockwise direction such that u, v is a separation pair, for an illustration see Fig. 9b. By 2-connectivity, there is an internal face f such that $u, v \in \mathrm{V}(\partial f)$. The boundary ∂f contains two simple interior disjoint uv-paths. By the case assumption and the definition of v, at least one of these paths, say P', is internal. In fact, if $|\,\mathrm{E}(P')| = |\,\mathrm{E}(\partial f)| - 1$, then the other path just consists of a single edge and is therefore also internal. So in any case, the boundary ∂f contains an internal uv-path P with $|\,\mathrm{E}(P)| \leq |\,\mathrm{E}(\partial f)| - 2$, which, by Lem. 1, is archfree.

The boundary ∂o contains two interior disjoint paths P_1, P_2 between u and v. Each of these two paths forms a simple cycle together with P. The closed interior

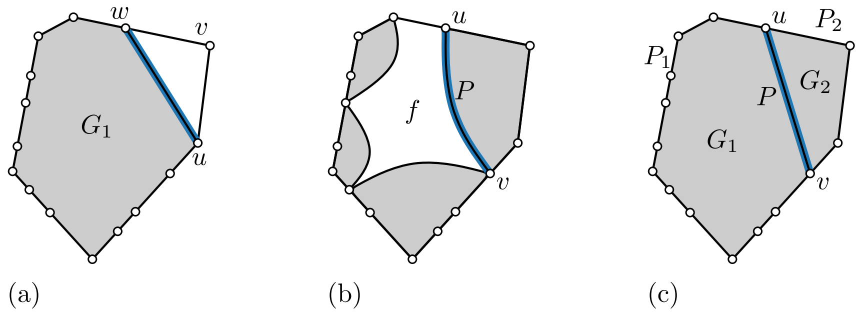


Fig. 9: (a) Case 1.1 and (b,c) Case 1.2 in the proof of Thm. 1. Note that in (b) the boundary of f contains only one internal uv-path.

of each of these two cycles describes an internally 3-connected plane graph by Obs. 1. We denote these two graphs by G_1 and G_2 such that G_i, $i \in [2]$ has P_i on its outer face. We define Γ_1^o to be the polygon resulting from replacing the part of Γ^o that corresponds to P_2 with P drawn as a straight-line segment, see Fig. 9c. The drawing Γ_2^o is defined analogously. Since Γ^o is compatible with G, the vertices u, v cannot belong to a common segment s of Γ^o (otherwise, since there are no degree-2 vertices (with outer angle π), s would be arched by the internal face f). Hence, Γ_1^o and Γ_2^o correspond to *simple* (convex) polygons. Moreover, since Γ^o is compatible and P is archfree, Γ_1^o and Γ_2^o are compatible with G_1 and G_2 respectively. We recursively compute the coordinates of the internal vertices in convex drawings of G_1 and G_2 with outer face Γ_1^o and Γ_2^o, respectively, where each internal vertex is placed in the interior of some segment. Since, additionally, the interior vertices of P are contained in the interior of the segment corresponding to P, the combination of these drawings corresponds to the desired drawing of G.

Case 2: G is 3-connected. We distinguish two subcases:

Case 2.1: $|\mathrm{V}(\Gamma^o)| \geq 4$. For illustrations refer to Fig. 10a. Then there exist two distinct outer vertices u, v that do not belong to a common segment of Γ^o. By 3-connectivity, G contains an internal uv-path P'. Consequently, by Lem. 2, G contains an archfree internal uv-path P. The boundary of the outer face of G contains two interior disjoint paths P_1, P_2 between u and v. Each of these two paths forms a simple cycle together with P. The closed interior of each of these two cycles describes an internally 3-connected plane graph by Obs. 1. We denote these two graphs by G_1 and G_2 such that G_i, $i \in [2]$ has P_i on its outer face. We define Γ_1^o to be the polygon resulting from replacing the part of Γ^o that corresponds to P_2 with P drawn as a straight-line segment. The drawing Γ_2^o is defined analogously. By definition of u and v, Γ_1^o and Γ_2^o correspond to *simple* (convex) polygons. More, since Γ^o is compatible and P is archfree, Γ_1^o and Γ_2^o are compatible with G_1 and G_2 respectively. We recursively compute the coordinates of the internal vertices in convex drawings of G_1 and G_2 with outer face Γ_1^o and Γ_2^o, respectively, where each internal vertex is placed in the interior of

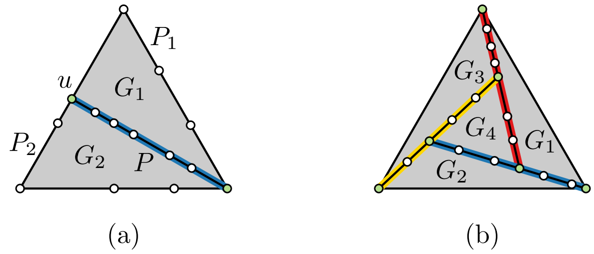


Fig. 10: (a) Case 2.1 and (b) Case 2.2 in the proof of Thm. 1.

some segment. Since, additionally, the interior vertices of P are contained in the interior of the segment corresponding to P, the combination of these drawings corresponds to the desired drawing of G.

Case 2.2: $|\mathrm{V}(\Gamma^o)| = 3$. By Lem. 4, G contains a strictly internal face. Thus, by Lem. 3, it contains an archfree windmill (P, Q, S). The paths P, Q, S dissect G into four plane graphs G_1, G_2, G_3, G_4, which are internally 3-connected by Obs. 1. The outer endpoints of P, Q, S correspond to the exactly three vertices of Γ^o. Consequently, they do not belong to a *common* segment of Γ^o. Hence, it is possible to draw each of P, Q, S as a straight-line segment such that the polygon Γ^o is dissected into four simple convex polygons, as depicted in Fig. 10b. Each of these polygons corresponds to a convex drawing of the outer face of one of G_1, G_2, G_3, G_4. Moreover, these drawings are compatible with their respective subgraphs since Γ^o is compatible and P, S, Q are archfree. We recursively draw G_1, G_2, G_3, G_4 into their respective compatible convex polygons in a convex fashion such that each of their internal vertices is placed in interior of some segment. Since, additionally, all internal vertices of G that belong to P, Q, or S have been drawn in the interior of one of the segments corresponding to P, Q, and S, the combination of the four drawings corresponds to the desired drawing of G. □

It is easy to see that the proof of Thm. 1 corresponds to a polynomial-time algorithm. In fact, it seems very plausible that it can be implemented in quadratic time, though, we have not worked out the details yet.

Theorem 2 ($\star$). *Every 3-connected internally 4-regular plane graph G admits a convex drawing on at most $n + 3$ segments where n is the number of vertices.*

Proof. We create a convex drawing Γ^o of the outer face of G on exactly 3 segments. Let u, v, w be the three vertices of Γ^o whose outer angles are reflex. By 3-connectivity, none of the segments of Γ^o can correspond to a path that is arched by an internal face. Consequently, Γ^o is compatible with G and, by Thm. 1, we can create a convex drawing Γ of G that uses Γ^o as the outer face such that each vertex in $\mathrm{V}(G) \setminus \{u, v, w\}$ is drawn in the interior of some segment of Γ. Hence, for each vertex $x \in \mathrm{V}(G) \setminus \{u, v, w\}$ at most two segments of Γ have x as an endpoint. For each vertex $y \in \{u, v, w\}$ at most four segments of Γ

have y as an endpoint. Since each segment has exactly two endpoints, it follows that the number of segments is at most $\frac{2(n-3)+4\cdot 3}{2} = n+3$, which concludes the proof. □

Proposition 1 (⋆). *For even $n \geq 6$, C_n^2 is planar and $\operatorname{seg}(C_n^2) \geq n$.*

Proof. Suppose that, for $n \geq 6$, the graph C_n^2 has a drawing Γ with at most $n-1$ segments. For $i \in \{0,2,4\}$, let n_i be the number of vertices in C_n^2 with i ports. Clearly, $n = n_0 + n_2 + n_4$. The drawing Γ has $2n_2 + 4n_4$ ports and hence $n_2 + 2n_4$ segments. If $n_4 \geq n_0$, then Γ has $n_2 + 2n_4 \geq n_2 + (n_0 + n_4) = n$ segments, which would contradict our assumption. Hence, $n_0 > n_4$. Each vertex on the convex hull of C_n^2 has four ports, which implies that $n_4 \geq 3$. This in turn yields that $n_2 = n - n_0 - n_4 \leq n - 7$.

We label the vertices of C_n^2 such that $\langle v_1, v_2, \ldots, v_n \rangle$ forms the simple cycle C_n. Since $n_0 > n_4$, there must be two indices $1 \leq j < \ell \leq n$ such that vertices v_j and v_ℓ have zero ports and every vertex v_k with $j < k < \ell$ has two ports. Let $V' = N(v_j) \cup \{v_j, \ldots, v_\ell\} \cup N(v_\ell)$ and $n' := |V'|$. We have $n' \leq n-1$ since V' contains at most $n-7$ vertices (with two ports) strictly between v_j and v_ℓ, plus six further vertices. Hence, w.l.o.g., we can choose our labeling of C_n^2 such that $V' = \{v_{j-2}, \ldots, v_{\ell+2}\}$. Let $G' = G[V']$, but drop any edge that connects one of the first two with one of the last two vertices. Then G' is isomorphic to the outerpath $R_{n'}$ where every vertex has degree at most 4. Dujmović et al. have shown that $\operatorname{seg}(R_{n'}) = n'$ [5, Proof of Theorem 7]. The graph G', however, has only $2n' - 2$ ports: all vertices have two ports, except for v_j and v_ℓ with zero ports and v_{j-1} and $v_{\ell+1}$ (both of degree 3) with at most three ports. This contradicts the fact that $\operatorname{seg}(R_{n'}) = n'$. □

Remark 1. As a universal lower bound for the class of 3-connected 4-regular planar graphs, note that in each vertex either at least two segments end or two segments cross. In order to generate n vertices, we need at least $\Omega(\sqrt{n})$ segments as Dujmović et al. [5] observed. It is not hard to see that (grid-like) 3-connected 4-regular planar graphs with segment number $O(\sqrt{n})$ exist.

C Proofs Omitted in Section 3 (Maximal Outerpaths)

Lemma 5 (⋆). *For any $i \in \{3, \ldots, n\}$, a partial outerpath drawing Γ_i contains at most one active long pseudo k-arc.*

Proof. Suppose that Γ_i contains two pseudo-k-arcs α and β that are both active and long. Let α have its first internal edge before β. For β to become long, β must have $k+1$ internal edges, while α remains active. Let $H_0, H_1, \ldots, H_\ell$ be the subgraphs into which the internal edges of β subdivide the complete outerpath drawing Γ; see Fig. 4. Now for α to leave H_0, α needs either to enter H_1 (which requires an intersection between α and β) or to enter H_2 (which requires a tangential point of α at β and is counted as two intersections). For α to be active when β is long, α needs to reach H_{k+1} (or some H_j with $j > k+1$). This however, requires at least $k+1$ intersection points between α and β, a contradiction to the definition of pseudo-k-arcs. $\square$

Lemma 6 ($\star$). *There is a loss of at most one crossing per transition from one long pseudo-k-arc to another long pseudo-k-arc. Hence, $t_k \leq \max\{0, \mathrm{arc}_k^{>k} - 1\} \leq \mathrm{arc}_k^{>k} = \mathrm{arc}_k - \sum_{i=0}^{k} \mathrm{arc}_k^i$, where $\mathrm{arc}_k^{>k}$ is the number of long pseudo-k-arcs.*

Proof. Of course, the loss cannot be negative and the number of transitions from one long arc to the other is $\mathrm{arc}_k^{>k} - 1$.

Summing up the losses over all pseudo-k-arcs of the drawing, we obtain t_k. Being counted in a crossing with a long arc more than k times is no contradiction to the definition of pseudo-k-arcs because the long arc may change. We distinguish two cases for the transition of a long arc α (with internal edge $e_1, \dots, e_\ell$ and subgraphs $H_1, \dots, H_\ell$) to a long arc β (with internal edge $e'_1, e'_2, \dots$ and subgraphs $H'_1, H'_2, \dots$).

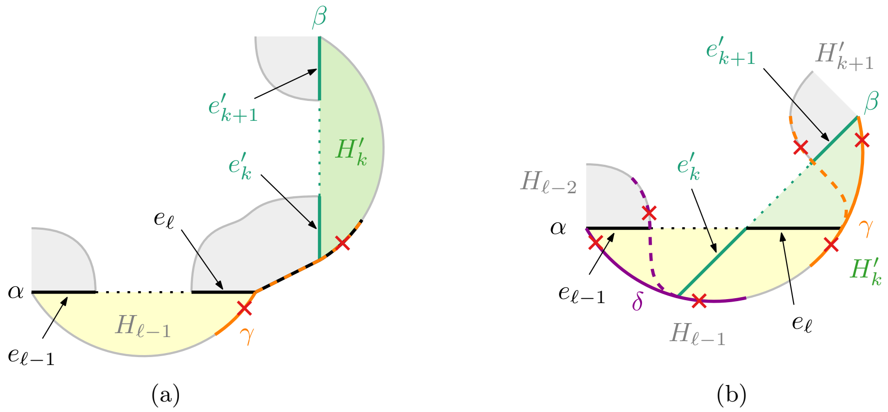


Fig. 11: Cases for the transition of one long pseudo segment α to another long pseudo segment β.

In the **first case**, e_ℓ precedes e'_k; see Fig. 11a. Say an arc γ has been counted in q crossings with a long arc before reaching e'_k. (If there has been a transition of a long arc before, we have already subtracted its loss and hence we assume $q \leq k$.) Next we show that γ is part of at most $k-q+1$ counted crossings with β. When γ reaches e'_k, it must have intersected β already at least $q-1$ times. This is due to the fact that $k-1 \geq q-1$ internal edges of β precede e'_k and while an arc is active, it intersects all arcs of the internal edges. We know that γ has intersected α (or a previous long arc) q times, so it has been in at least the last $(q-1)$ bays of α (or a previous long arc). By then, β has also already been active and also has been in at least the last $(q-1)$ bays of α (or a previous long arc).

In any bay H, all arcs that leave H intersect all other arcs that leave H at least once. Hence, β and γ have intersected pairwise at least $q-1$ times. This means that γ can be part of at most $q+(k-q+1)=k+1$ counted crossings with a long arc – regarding all long arcs up to and including β. It remains to argue that there is at most one arc γ with $k+1$ counted crossings with a long arc per transition. Suppose there was another arc γ' with the same property, which has been in q' crossings with long arcs before e'_k. Then, without loss of generality, γ has been in the crossing with α at e_ℓ. However, γ' has also intersected α at e_ℓ but without being counted in a crossing. So, γ' has been in the last q' Hs of α together with β and contributes at most $k-q'$ crossings with β.

In the **second case**, e_ℓ succeeds e'_k; see Fig. 11b. If we started counting crossings with β in bay H'_{k+1} instead of H'_k, we would have the same situation as in the first case. Now consider the counted crossing of H'_k at e'_{k+1}. Similar to the first case, if an arc δ reaches this crossing and was part of counted crossings before, it has intersected α at e_ℓ. Again, only one of γ and some other pseudo-k-arc γ' can contribute the crossing with α at e_ℓ and then be part of $k+1$ crossings with long edges. For the counted crossing of bay H'_k at e'_k, we cannot rule out the possibility that the involved arc δ is part of more than k crossings. So, we consider this crossing as being lost, but then there exists a crossing of α and β that has not been counted – namely at the common vertex of e_ℓ and e'_k. Therefore, also in the second case we have a loss of at most one counted crossing. □

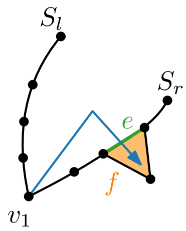
Lemma 7 ($\star$). *For $k=1$ and $n\geq 3$, in any outerpath drawing either $\mathrm{arc}_1^0\geq 3$ or ($\mathrm{arc}_1^0\geq 2$ and $\mathrm{arc}_1^1\geq 3$).*

Proof. Consider v_1 and v_n, i.e., the first and the last vertex in the stacking order of G. Each of them is incident to two pseudo segments. If they would lie on only one pseudo segment S, S would intersect the pseudo segment connecting the two neighbors of v_1 (or v_n) twice.

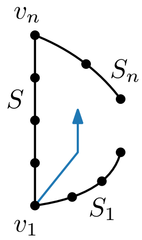
First, we show that v_1 and v_n have at least one incident pseudo segment with zero internal edges each (**Case 0**). Without loss of generality, assume that v_1 is incident to the pseudo segments S_l and S_r, both have at least one internal edge, and in the stacking order of the outerpath, the first internal edge e of S_r precedes the first internal edge of S_l; see Fig. 12a. The path of faces reaches the face f when passing over e. However, S_l is not incident to f and becomes inactive. (S_l cannot be incident to f because then S_l and S_r would intersect twice or v_1 would have a degree >2.) Therefore, S_l has zero internal edges. The same holds when traversing the outerpath backwards starting at v_n.

Using this property, we now can make the following case distinction.

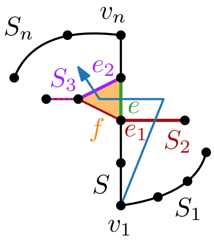
Case 1: v_1 and v_n are incident to the same pseudo segment S having zero internal edges. Let the other pseudo segments being incident to v_1 and v_n be S_1 and S_n, respectively (clearly, they are distinct); see Fig. 12(b). This means that S is incident to all faces in the outerpath. So, if S_1 or S_n had an internal edge, they would intersect S a second time. Hence, S_1 and S_n have also zero internal edges and we have at least three pseudo segments with zero internal edges in total.



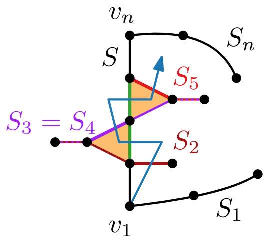
(a) Case 0



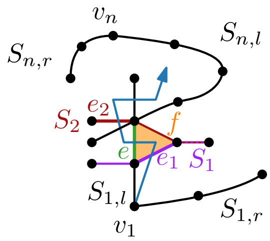
(b) Case 1



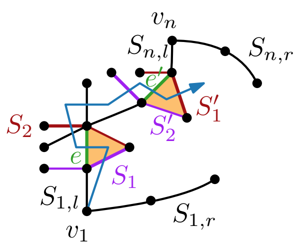
(c) Case 2



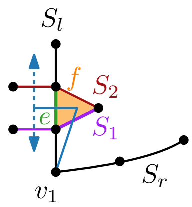
(d) Case 3



(e) Case 4



(f) Case 5



(g) Case 6

Fig. 12: Cases to show Lem. 7.

Case 2: v_1 and v_n are incident to the same pseudo segment S having one internal edge. Let the other pseudo segments being incident to v_1 and v_n be S_1 and S_n, respectively (clearly, they are distinct); see Fig. 12c. Since S has an internal edge, S_1 and S_n have zero internal edges. Consider the face f following the internal edge e of S. Beside S, the two other distinct bounding pseudo segments of f are S_2 and S_3. Let S_3 have an internal edge e_2 following e along the sequence of internal faces. All faces of the outerpath are incident to S, hence S_3 cannot have a second internal edge as it intersects S incident to f. Similarly, S_2 can have at most one internal edge e_1 when it intersects S incident to f. Thus, S_1 and S_n have zero internal edges, while S, S_2, and S_3 have at most one internal edge each.

Case 3: v_1 and v_n are incident to the same pseudo segment S having at least two internal edges. As in Case 2, when the sequence of faces of the outerpath passes over S, there are two pseudo segments S_2 and S_3 each having at most one internal edge. We have this situation at least twice – we denote the next corresponding pair of segments that has at most one internal edge per pseudo segment by S_4 and S_5. Observe that maybe $S_3 = S_4$; see Fig. 12d. Then, however, $S_2 \neq S_5$ as otherwise S_2 and S_3 would intersect twice. Therefore, we have two pseudo segments with zero internal edges (S_1 and S_n) and we have at least three pseudo segments with at most one internal edge (S_2, S_3, and S_5).

Case 4: v_1 and v_n are incident to four distinct pseudo segment and exactly one of these pseudo segments has at least two internal edges. This case is similar to Case 2 and Case 3. Without loss of generality, let the segments of v_1 and v_n be $S_{1,l}$, $S_{1,r}$ and $S_{n,l}$, $S_{n,r}$, respectively, and let $S_{1,l}$ have at least two internal edges; see Fig. 12e. Consider the first internal edge e of $S_{1,l}$ and the face f preceding e along the sequence of faces. Let the other pseudo segments bounding f be S_1 and S_2 and let the internal edge e_1 for entering f be contained in S_1. Because until the sequence of faces passes over $S_{1,l}$ a second time, all faces are neighboring $S_{1,l}$. Hence, S_1 and S_2 have at most one internal edge each. Moreover, observe that neither $S_{n,l}$ nor $S_{n,r}$ can be equal to S_1 or S_2 as otherwise they would intersect $S_{1,l}$ twice. This gives us our bound – the three pseudo segments with at most one internal edge are S_1, S_2, and one of $S_{n,l}$ and $S_{n,r}$.

Case 5: v_1 and v_n are incident to four distinct pseudo segment and two of these pseudo segments have at least two internal edges. We have a very similar situation as in Case 4, but now we have S_1 and S_2 in the forward direction and S'_1 and S'_2 symmetrically in the backward direction; see Fig. 12f. Let $S_{1,l}$ and $S_{n,l}$ be the segments originating at v_1 and v_n, respectively, that have at least two internal edges each. We have to be a bit more careful about the case that $S_{1,l}$ and $S_{n,l}$ intersect. However, even in this case S_1, S_2, S'_1, and S'_2 are four distinct pseudo segments since the first internal edge e of $S_{1,l}$ precedes all internal edges of $S_{n,l}$ and the last internal edge e' of $S_{n,l}$ succeeds all internal edges of $S_{1,l}$ (otherwise the drawing would not be an outerpath).

Case 6: v_1 and v_n are incident to four distinct pseudo segment and each of them has at most one internal edge. If three of them have zero internal edges, we are done. So assume that the pseudo segment S_l of v_1 (and one pseudo segment of v_n) has one internal edge e; see Fig. 12g. Consider the face f preceding e in the sequence of faces in the outerpath. Beside S_l, let f be bounded by S_1 and S_2. The key insight is that S_1 and S_2 pass over S_l at e, but on the other side of S_l, they cannot intersect a second time and so the path of faces in the outerpath can yield another internal edge at most for one of S_1 and S_2. Hence, either S_1 has at most one internal edge (when entering f) or S_2 has zero internal edges, which provides our bound. We have to be careful about the case that S_1 or S_2 are pseudo segments of v_n. Note that not both of them can reach v_n because then they would intersect a second time. If S_2 reaches v_n, then S_1 is our third pseudo segment with at most one internal edge. If S_1 reaches v_n, then S_2 is our third pseudo segment without any internal edges. $\square$

Theorem 3 (⋆). *For any n-vertex maximal outerpath G,* $\mathrm{seg}(G) \geq \lfloor \frac{n}{2} \rfloor + 2$.

Proof. Clearly, $\mathrm{seg}(G) \geq \mathrm{arc}_1(G)$. Hence, it suffices to show $\mathrm{arc}_1(G) \geq \lfloor \frac{n}{2} \rfloor + 2$.

We plug in the result from Lem. 6, into Eq. (4) for $k = 1$ and use Lem. 7 to observe $3\,\mathrm{arc}_1^0 + \mathrm{arc}_1^1 \geq 9$:

$$\mathrm{arc}_1 \geq \frac{2n - 3 + 3\,\mathrm{arc}_1^0 + \mathrm{arc}_1^1}{4} = \frac{n}{2} + \frac{3\,\mathrm{arc}_1^0 + \mathrm{arc}_1^1 - 3}{4} \geq \frac{n+3}{2}$$

As we cannot have partial (pseudo) segments, we can round up to $\lceil \frac{n+3}{2} \rceil = \lfloor \frac{n}{2} \rfloor + 2$.
□

Theorem 4 (⋆). *For any n-vertex maximal outerpath G,* $\mathrm{arc}(G) \geq \lceil \frac{2n}{7} \rceil$.

Proof. Clearly, $\mathrm{arc}(G) \geq \mathrm{arc}_2(G)$. Hence, it suffices to show $\mathrm{arc}_2(G) \geq \lceil \frac{2n}{7} \rceil$.

For $k = 2$ and Eq. (4), we plug in the result from Lem. 6 and we get

$$\mathrm{arc}_2 \geq \frac{2n + 5\,\mathrm{arc}_2^0 + 3\,\mathrm{arc}_2^1 + 1\,\mathrm{arc}_2^2}{7} \geq \frac{2n}{7}\,.$$

Since we can only have an integral number of arcs, we can round up this value.
□

Proposition 3 (⋆). *For every $r \in \mathbb{N}$, maximal outerpaths P_r, Q_r, U_r exist s.t.*

(i) P_r has $n = 2r + 6$ vertices and $\mathrm{seg}(P_r) \leq r + 5 = n/2 + 2$,
(ii) Q_r has $n = 3r$ vertices and $\mathrm{arc}(Q_r) \leq r + 1 = n/3 + 1$,
(iii) U_r has $n = 16r + 6$ vertices and $\mathrm{arc}_2(U_r) \leq 5r + 3 = \frac{5n+18}{16} \approx 0.3125\,n$.

Proof. (i) Consider Fig. 5a. In the base case ($m = 0$), there obviously is a drawing with six vertices on five line segments. When we increase m by one, we add a line segment going through the central vertex and increasing the number of vertices by two.

(ii) Consider Fig. 5b, where $m = 6$. The main structure is a long horizontal line segment (this is a circular arc with radius ∞). In the base case ($m = 2$), we have two more circular arcs that look like the first and the last arc in Fig. 5b – two of their vertices are shared with each other, which gives us 6 vertices in total. When we increase m by one, we add a circular arc as in Fig. 5b. It has six vertices, where three of them are new.

(iii) Consider Fig. 5c. In the base case ($k = 0$), we have only the first three and the last three vertices (in purple) using three pseudo-2-arcs. When we increase k by one, we add the colored part k times (to show the repeating pattern, there is another copy in gray). This colored part has 16 vertices, it extends three pseudo-2-arcs and introduces five new pseudo 2-arcs. Observe that each pair of pseudo-2-arcs intersects at most twice. □

D Maximal Outerplanar Graphs and 2-Trees

Consider a straight-line drawing $\varGamma_G$ of a 2-tree G. The main idea for a universal lower bound for 2-trees (and for its subclass of maximal outerplanar graphs) is that G either has many degree-2 vertices and thus requires many segments (recall that, in a 2-tree, all faces are triangles, hence degree-2 vertices cannot be closed) or G can be obtained by gluing few outerpaths for which we know (tight) universal lower bounds on the segment number. By gluing we mean the following. Let G be a 2-tree and P a maximal outerpath. Let f_G be a triangle of G that is not incident to a degree-2 vertex and let f_P be a triangle of P that is incident to a degree-2 vertex (i.e., f_P is the first or last triangle of P). Let $\varGamma_P$ be a straight-line drawing of P. Then we define the *gluing* of $\varGamma_P$ to $\varGamma_G$ as the straight-line drawing $\varGamma_{G\oplus P}$ of the 2-tree $G \oplus P$ obtained by identifying f_P and f_G; see Fig. 13. Note that $|V(G \oplus P)| = |V(G)| + |V(P)| - 3$. In $\varGamma_{G\oplus P}$, we call f_G and f_P the *gluing faces* of G and P, respectively.

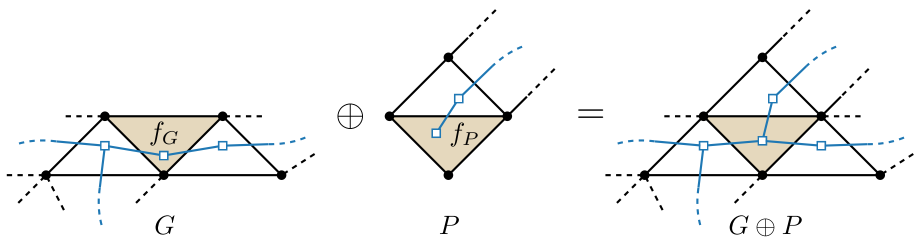


Fig. 13: Gluing drawings of an outerplanar graph G and an outerpath P.

Unfortunately, for gluing outerpaths, we cannot directly employ Thm. 3 because it does not tell us how many ports we lose when gluing. Therefore, we first investigate the distribution of ports within a straight-line drawing of a maximal outerpath. We will see that, by some careful counting arguments, we lose only few (counted) ports when gluing outerpaths. We start by formally proving some auxiliary properties; see Fig. 14.

Lemma 9. *Let P be a maximal outerpath given with a stacking order, and let v be a vertex of P. Then, in any outerplanar straight-line drawing of P, all of the following holds.*

(P1) *If $\deg(v) = 2$ or $\deg(v)$ is odd, then v is open.*

(P2) *If $\deg(v) \geq 5$, then v is succeeded by $\deg(v) - 4$ many neighbors of degree 3, which we call* companions.

(P3) *If $\deg(v) \geq 6$ and v is closed, then v has a companion with three ports, which we call* bend companion.

(P4) *If subsequent vertices u and v both have degree 4, then at least one of u and v is open.*

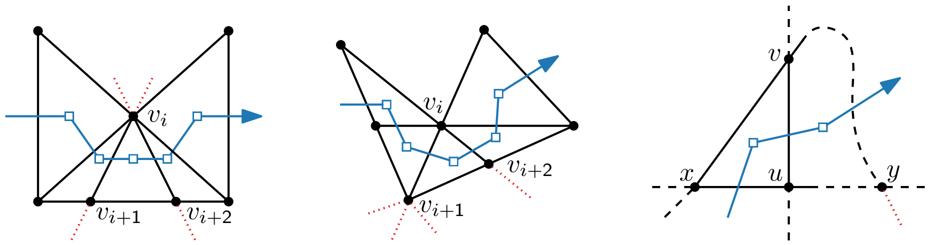


Fig. 14: For property (P3) in Lem. 9, observe that vertex v_i (which has degree 6) is (a) either open or (b) has a bend companion (here v_{i+1}) with three ports; (c) for property (P4), note that two subsequent degree-4 vertices u and v cannot both be closed because of the two triangles they form with their common neighbors x and y.

(P5) Let v be stacked upon the edge uw and u, v be subsequent vertices. If v is closed, $\deg(v) = 4$, $\deg(u) = 3$, *and* $\deg(w) = 5$, *then either u or w has at least three ports.*

Proof. We consider each of the statements individually.

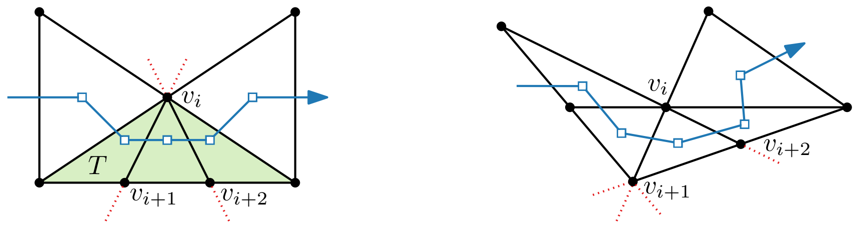
(P1) If $\deg(v)$ is odd, the claim is trivial. Otherwise, v and its two neighbors form a triangle in any 2-tree and cannot be collinear.

(P2) Since v has degree at least five, constructing P with a sequence of stacking operations involves $\deg(v) - 2$ consecutive stacking operation on edges incident to v. Consequently, all succeeding neighbors of v, except for the last two, must have degree three.

(P3) Let $v = v_i$. Consider the companions $v_{i+1}, \dots, v_{i+\deg(v_i)-4}$ of v. Suppose neither of them has three ports (two is not possible since they have degree 3), then (at least) $\deg(v_i) - 2$ neighbors of v_i are collinear and thus result in a triangle T with v at one corner and these neighbors on the opposing side of T; see Fig. 15a. Then, however, v_i cannot be closed since $\deg(v_i)-2 > \deg(v_i)/2$ and at most two segments can pass through v_i, which is a contradiction. Hence, one of the companion vertices of v has three ports; see Figure 15b.

(P4) Let x, u, v, y be a stacking subsequence in P where both u and v have degree four. Assume, for the sake of contradiction, that there exists a planar straight-line drawing of P where both u and v are closed. Let s be the segment that contains the edge uv. Then s intersects at u the segment s' that contains xu, and s intersects at v the segment s'' that contains xv; see Fig. 16a. Observe that s' and s'' need to intersect again in y since P is a maximal outerpath and both u and v are closed. However, this would only be possible if x, u, v, and y are collinear; which is a contradiction to the drawing being a planar straight-line drawing.

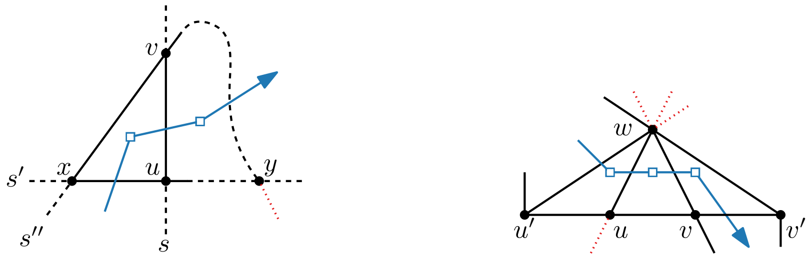
(P5) If u has three ports, we are done. Otherwise, u has only one port; see Fig. 16b. Then note that u and v need to be collinear with a successor v' of v and predecessor u' of u. Observe that these four vertices are adjacent to w. However, since w has degree 5, only one of the edges $\{u', w\}, \{u, w\}, \{v, w\}, \{v', w\}$



(a) For (P3), if all companions of v are collinear, then v cannot be closed.

(b) For (P3), if v is closed, then one of its companion neighbors has 3 ports.

Fig. 15: Configurations in the proof of Lem. 9, where v has degree 6 (or a higher even degree).



(a) For (P4), two subsequent degree-4 vertices u and v cannot both be closed because of the two triangles with their common neighbors x and y.

(b) For (P5), in case u has only one port, w has at least three ports because four of its neighbors are collinear.

Fig. 16: Configurations in the proof of Lem. 9 where v is closed and has degree 4.

can be extended at w. (In Fig. 16b, the edge $\{v', w\}$ lies on a segment passing through w.) Therefore, w has at least three ports.

This finishes the proof. □

Proposition 4. *Let P be a maximal outerpath with $n \geq 4$ vertices. Then* $\mathrm{port}(P) \geq n + 1$. *Moreover, for any planar straight-line drawing of P, we can find an injective assignment of ports to vertices such that every port is assigned to its own vertex or to a neighboring vertex.*

Proof. Given any straight-line drawing Γ_P and any stacking order $\langle v_1, \dots, v_n \rangle$ of P, we describe an assignment of ports to vertices in their vicinities such that no two ports are assigned to the same vertex. This immediately proves that $\mathrm{port}(P) \geq n$. For the one remaining port, observe that v_1 has an additional unassigned port.

Let $i \in [n]$. We consider different situations for vertex v_i. Each situation is illustrated by a vertex in the example shown in Fig. 17. If v_i is open (such as v_1, v_2, or v_4 in Fig. 17), we assign one of the ports to itself. If v_i is closed, then

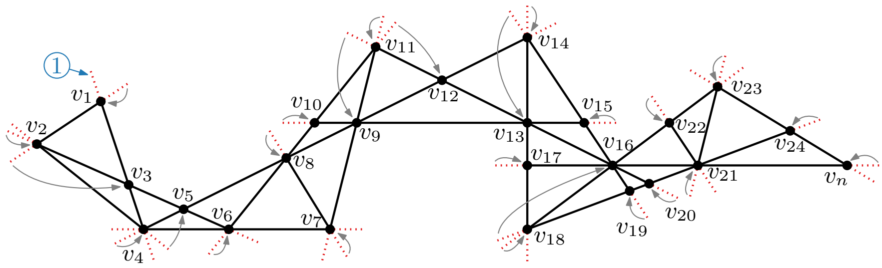


Fig. 17: A straight-line drawing of a maximal outerpath where each vertex is assigned a port (indicated by grey arrows). Several ports remain unassigned (e.g., ①).

$\deg(v_i)$ is even and at least 4 by (P1). First assume $\deg(v_i) \geq 6$ (such as v_9, v_{13}, v_{16} in Fig. 17). Then, by (P3), we know that v_i has a bend companion v_j with three ports. Only one of the three ports of v_j is assigned to v_j itself, so we assign one of the remaining ports of v_j to v_i. (In Fig. 17 such a port would be supplied by v_{11}, v_{14} and v_{18}, respectively.)

If $\deg(v_i) = 4$, then either $\deg(v_{i-1}) = 4$ (such as v_4 preceding v_5) or $\deg(v_{i-1}) = 3$ (such as v_{11} preceding v_{12}) since, by (P2), v_{i-1} has degree at most 4. In the former case, v_{i-1} has at least two ports by (P4) and we can assign one of the ports to v_i (such as v_4 to v_5). In the latter case, we distinguish three subcases. If $v_i = v_3$ in the stacking order of P, then v_2 has degree 3 and cannot be closed (as v_2 and v_3 in Fig. 17). If $v_i = v_4$ in the stacking order of P, then v_2 or v_3 has three ports. Otherwise, observe that the common neighboring predecessor of v_{i-1} and v_i has degree at least 5; hence one of (P3) or (P5) applies (see v_9, v_{11}, and v_{12} in Fig. 17; this is the only case where a vertex provides ports for itself and two other vertices). ◻

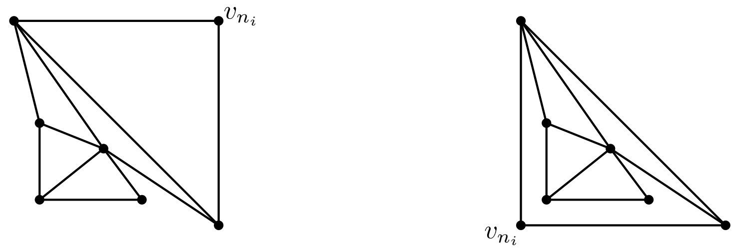
Proposition 4 implies a universal lower bound of $(n+1)/2$ for the segment number of an n-vertex outerpath. In Thm. 3, we improve this by a constant.

Theorem 8 ($\star$). *For a 2-tree (or a maximal outerplanar graph) G with n vertices,* $\operatorname{seg}(G) \geq (n+7)/5$.

Proof. For now, assume that G is a maximal outerplanar graph. We consider the case that G is a 2-tree at the end of this proof. If the weak dual T of G has at least $(n+7)/5$ leaves, we are done since G has at least as many segments as T has leaves.

Otherwise, let $\mathcal{P} = \{P_1, \dots, P_p\}$ be a minimum-size set of maximal outerpaths such that when we define $G_1 = P_1$ and $G_i = G_{i-1} \oplus P_i$, for $i \in \{2, \dots p\}$, we get that $G = G_p$. In other words, we can obtain G by $p-1$ consecutive gluing operations of the paths in $\mathcal{P}$. Note that p is at most $(n+7)/5 - 1 = (n+2)/5$ because P_1 contains two leaves and, for $i \in \{2, \dots p\}$, P_i contains one leaf of T.

Next, we show a lower bound on the number of ports on any straight-line drawing of G. To this end, we use the assignment of ports to vertices that we established in Prop. 4 and apply it to each outerpath P_i in $\mathcal{P}$. Further, we use



(a) Initial outerplanar drawing. (b) Drawing after flipping v_{n_i}.

Fig. 18: Straight-line drawing of a maximal outerpath where we "flip" v_{n_i} over the rest of the drawing such that the resulting drawing remains planar. This way, we can append maximal outerpaths to inner faces of 2-trees.

the stacking order of P_i that starts at the degree-2 vertex of P_i that is not incident to the gluing face of P_i. For $i \in \{1, \dots, p\}$, let $n_i = |V(P_i)|$. Note that $|V(G)| = \sum_{i=1}^{p} n_i - 3(p-1)$.

First, we compute $\mathrm{port}(\mathcal{P})$, the sum of ports counted for $P_1, \dots, P_p$:

$$\mathrm{port}(\mathcal{P}) = \sum_{i=1}^{p} \mathrm{port}(P_i) \geq \sum_{i=1}^{p} (n_i + 1) = n + 3(p-1) + p = n + 4p - 3$$

Second, we analyze the number of *counted* ports that we lose by the $p-1$ gluing operations. Consider the gluing operation $G_i = G_{i-1} \oplus P_i$ and let $f_{G_{i-1}}$ and f_P, respectively, be the gluing faces identified to face f of G_i.

Observe that we counted three ports at f_P since neither v_{n_i} nor one of its neighbors needs to assign a port to another vertex (we assign only ports to vertices coming later in the stacking order except for bend companions, but the last three vertices cannot be bend companions). We assume to lose all of these three ports when gluing. This means that every vertex has at most as many counted ports in G_i as it had in G_{i-1}. For the ports lost at $f_{G_{i-1}}$, observe that the vertex that is identified with v_{n_i} at P_i cannot lose any ports. The other two vertices are neighbors in G_{i-1}. In the assignment that we established in Prop. 4, any two such vertices provide ports for at most four vertices in total. We assume also to lose all of these ports, which results in a total loss of at most seven ports per gluing operation. Hence, with $p \leq (n+2)/5$, we get

$$\mathrm{seg}(G) = \frac{\mathrm{port}(G)}{2} \geq \frac{\mathrm{port}(\mathcal{P}) - \mathrm{loss}}{2} \geq \frac{n + 4p - 3 - (7p - 7)}{2} \geq \frac{n+7}{5}.$$

It remains to consider the case that G is a 2-tree. As for maximal outerplanar graphs, we can also construct a 2-tree by gluing multiple outerpaths. Similar to leaves in the dual drawing, each attached outerpath provides at its ending a vertex of degree 2 with two ports. The only exception is that we are not restricted on gluing to the outside – we may also draw a outerpath within an inner face of the current 2-tree drawing.

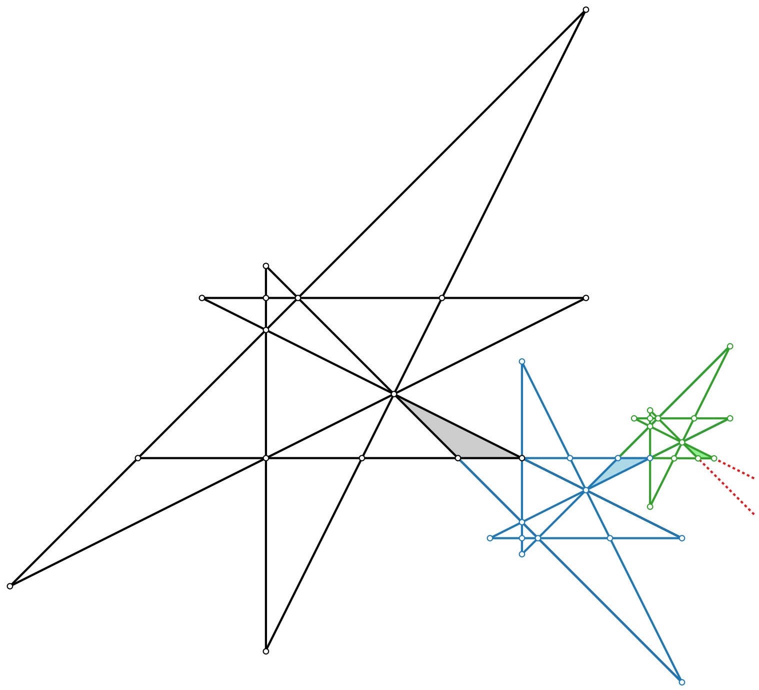


Fig. 19: The maximal outerplanar graph G_3 with 42 vertices drawn on 18 segments. (G_1 in black)

A difficulty is how to identify the faces $f_{G_{i-1}}$ and f_P if we want to draw the rest of P within this unified face. However, consider an outerplanar straight-line drawing of P where we "flip" the last vertex v_{n_i} over the rest of the drawing such that the drawing remains planar; see Fig. 18. Clearly, the number of ports in the drawing of the maximal outerpath P did not change and the assignment scheme from Prop. 4 is still applicable. We may use such flips also along inner edges of a outerpath drawing to obtain a "folded" outerpath drawing with the same properties. Hence, we can apply gluing operations to inner faces with at most the same loss as analyzed before. □

We remark that, though we get the same lower bound for maximal outerplanar graphs and 2-trees, the actual (tight) numbers might be different. In other words, maybe there are 2-trees requiring less segments than any maximal outerplanar graph with the same number of vertices. This is because our current analysis is most likely not tight as we see by comparison with our existential upper bound.

For an existential upper bound of maximal outerplanar graphs, consider the construction in Fig. 19. It defines a family of graphs $G_1, G_2, \ldots$ where the base graph G_1 has 16 vertices and admits a drawing Γ_{G_1} with eight segments. From G_{i-1} to G_i, we glue a scaled and rotated copy of Γ_{G_1} to the drawing of G_{i-1}

(gluing faces are shaded). With each step, we get 13 more vertices with only 5 more segments and hence the following result.

Proposition 5. *For every* $k \in \mathbb{N}$*,* G_k *has* $n_k = 13k + 3$ *vertices and* $\mathrm{seg}(G_k) \leq 5k + 3 = (5n_k + 24)/13$*.*

E Planar 3-Trees

In this section we study the segment number of planar 3-trees. For a 3-tree G with $n \geq 6$ and an arbitrary planar straight-line drawing Γ of G, we observe that we can assign at least (i) one port to each internal face of Γ and (ii) twelve ports to the outer face of Γ; see Fig. 20. By Euler, any n-vertex triangulation has $2n - 5$ internal faces. Hence, Γ has $2n + 7$ ports. This yields the following bound, which is tight up to a constant.

Theorem 6 (⋆). *For a planar 3-tree* G *with* $n \geq 6$ *vertices,* $\mathrm{seg}(G) \geq n + 4$*.*

Proof. For claim (i), consider a sequence of stacking operations that starts with a drawing of K_4 and yields Γ. Let v be the current vertex in this process, and let f be the face into which v is stacked. Let $V(f) = \{x, y, z\}$ be the set of vertices incident to f, and let f_x, f_y, and f_z be the three newly created faces such that $V(f_x) = \{v, y, z\}$ etc.; see Fig. 20a. Since f is a triangle, no two of the edges xv, yv, zv can share a segment. Thus, v has three ports. In particular, the segment xv points into f_x, yv points into f_y, and zv points into f_z. We assign the ports of v accordingly to f_x, f_y, and f_z. When the stacking process ends with Γ, each internal face of Γ has a port assigned to it.

For claim (ii), note that the number of ports on the outer face equals the degree sum of the three vertices on the outer face. Thus, K_4 has nine ports. The next (fifth) vertex in the stacking sequence is incident to two vertices on the outer face and hence contributes two more ports. Similarly, the sixth vertex contributes at least one more port; see Fig. 20b. Hence, in total, the outer face

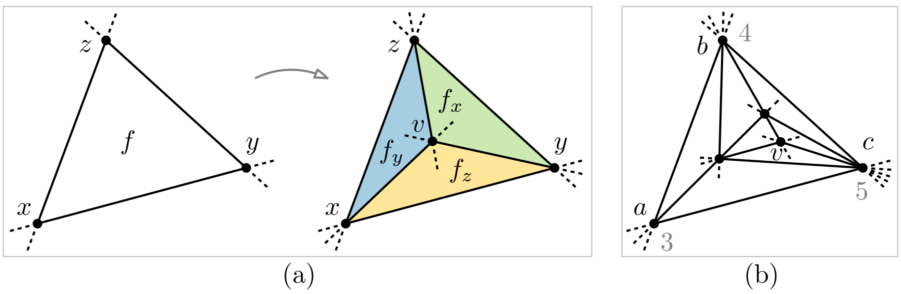


Fig. 20: (a) Stacking a vertex v into an internal face $f = xyz$ creates a port in each new face (f_x, f_y, and f_z); (b) a planar 3-tree with $n \geq 6$ vertices has at least twelve ports on the outer face abc.

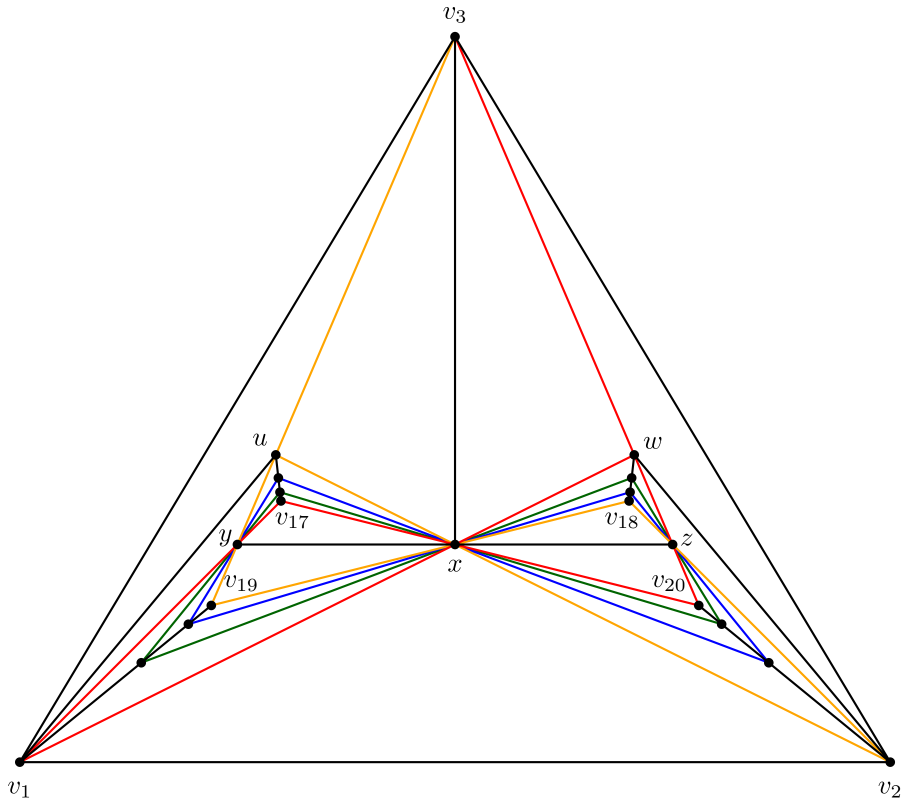


Fig. 21: A straight-line drawing of T_3 from Prop. 6 with 20 vertices and 27 segments.

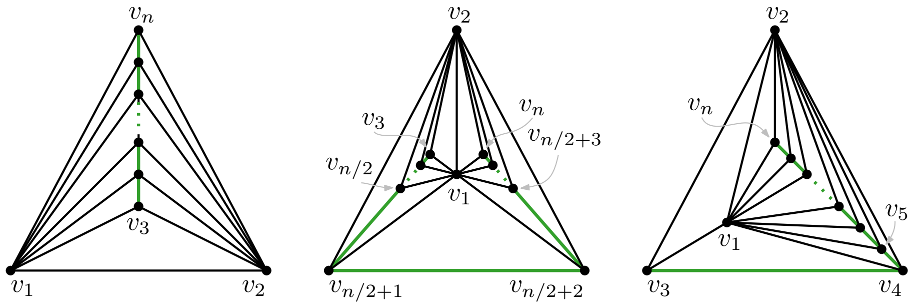
has at least twelve ports. (Note that this bound is tight since any further vertex can be stacked into an internal face that is not adjacent to the outer face.)

To finish the proof, we treat the remaining small graphs. For $n = 5$, we have one port less on the outer face, and there exists a drawing of this unique graph using eight segments (see Fig. 20b without vertex v). It is easy to verify the claim for $n = 4$. $\square$

In Fig. 21, we draw an n-vertex planar 3-tree using $n + 7$ segments. This yields an existential upper bound as formalized in Prop. 6. Hence, the universal lower bound in Thm. 6 is tight up to an additive constant of 3.

Proposition 6. *For every $k \geq 1$ there exists a 3-tree T_k, whose construction is illustrated in Fig. 21, with $n = 4k + 8$ vertices and* $\operatorname{seg}(T_k) \leq 4k + 15 = n + 7$.

Proof. Consider Fig. 21. We start by drawing the outer triangle $\triangle v_1 v_2 v_3$ using three segments. As fourth vertex, we add the central vertex x introducing three more segments. For the fifth and sixth vertex, u and w, we re-use the line segments xv_1 and xv_2 and, consequently, add only four new segments. For the seventh



(a) $2n-2$ segments (b) $3n/2+1$ segments (c) $2n-2$ segments

Fig. 22: Straight-line drawings of the 3-tree B_n (with $n \geq 6$ even) for two different embeddings that were analyzed by Dujmović et al. [5]

and eighth vertex, y and z, we re-use line segments uv_3 and wv_3, respectively. Moreover, they share a segment for the edges yx and zx, which results in three new segments. This gives us 13 vertices for the base construction.

Now in k rounds, we iteratively stack four vertices into the faces $\triangle uyx$, $\triangle wxz$, $\triangle v_1xy$, and $\triangle v_2zx$. We stack along four new (black) line segments (see e.g. $\overline{v_1v_{19}}$ in Fig. 21) such that the final drawing uses four more segments once as well as four more per iteration (colored line segments through y, z and x in Fig. 21). We re-use the segments uy, wz, v_1y, and v_2z for one edge each, which saves us two more segments. Together with the 13 segments of the base construction, we get $\text{seg}(T_k) \leq 13 + 4 + 4k - 2 = 4k + 15 = n + 7$. $\square$

Consider the universal upper bound of $2n-2$ on the segment number of planar 3-trees due to Dujmović et al. [5, Lemma 18]. They show the tightness of their result in a fixed-embedding setting, that is, they prove that there is a family $(B_n)_{n\geq 4}$ of *plane* 3-trees (see Fig. 22a) such that B_n has n vertices and requires $2n-2$ segments in any straight-line drawing that adheres to the given embedding. They remark that, given a different embedding, B_n can be drawn using roughly $3n/2$ segments; see Fig. 22b. We formalize this to compute the exact segment number of B_n, which will be useful in App. F.

Proposition 7. *For every $n \geq 6$ there exists a 3-tree B_n (see Fig. 22) with n vertices and* $\text{seg}(B_n) = \lceil 3n/2 \rceil + 1$.

Proof. We first show the lower bound $\text{seg}(B_n) \geq \lceil 3n/2 \rceil + 1$.

Let B_n be the graph depicted in Fig. 22 with vertex set $\{v_1, v_2, \dots, v_n\}$ and edge set $\{v_1v_i, v_2v_i : 3 \leq i \leq n\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-1\}$. If v_1 and v_2 are on the outer face (see Fig. 22a), we have at least $2(n-2)+2$ segments. For $n \geq 6$, $2n-2 \geq \lceil 3n/2 \rceil + 1$.

So w.l.o.g. let v_1 not be on the outer face. Consequently, v_2 lies on the outer face because any triangle of the graph contains at least one vertex of $\{v_1, v_2\}$ and, hence, also the triangle of the outer face. This implies that there are $n-1$ distinct segments incident to v_2.

For every $i \in \{3, \dots, n\}$, the path $\langle v_1, v_i, v_2 \rangle$ is drawn with a bend at v_i because otherwise it would coincide with the edge $v_1 v_2$. Therefore, the $n-2$ edges $v_1 v_3, \dots, v_1 v_n$ form at least $(n-2)/2$ new segments.

Consider the two other vertices on the outer face – we call them u and w. The edge uw yields another segment. Moreover, v_3 and v_n cannot both be on the outer face as they are not adjacent. Therefore, w.l.o.g., u has degree 4. So far, we have counted the segments of the edges uv_1, uv_2 and uw. This means that there is another segment for the fourth edge incident to u.

If w, too, has degree 4, we count another segment by the same argument. Overall, this sums up to at least $(n-1) + (n-2)/2 + 1 + 2 = 3n/2 + 1$ segments.

Otherwise w has degree 3. Assume w.l.o.g. that $w = v_3$. Consequently, the outer face is the triangle $\triangle v_2 v_3 v_4$; see Fig. 22c. Observe now that $\triangle v_1 v_2 v_4$ separates v_3 on the outside from all other vertices in the inside. Thus, the $n-3$ edges $v_1 v_4, \dots, v_1 v_n$ reach v_1 in an angle smaller than 180° and, hence, require $n-3$ distinct segments. This results in at least $(n-1) + (n-3) + 1 + 1 = 2n - 2 \geq 3n/2 + 1$ segments.

Finally, we show that this lower bound is tight. Consider the drawing of B_n in Fig. 22b. It uses exactly the $\lceil 3n/2 \rceil + 1$ segments that we counted above for the lower bound. In particular, $u = v_{\lfloor n/2 \rfloor + 1}$ and $w = v_{\lfloor n/2 \rfloor + 2}$. □

F The Ratio of Segment Number and Arc Number

Since circular-arc drawings are a natural generalization of straight-line drawings, it is natural to also ask about the maximum ratio between the segment number and the arc number of a graph. In this section, we make some initial observations regarding this question. Clearly, $\operatorname{seg}(G)/\operatorname{arc}(G) \geq 1$ for any graph G. Note that $\operatorname{seg}(K_3)/\operatorname{arc}(K_3) = 3$. We investigate the ratio for two classes of planar graphs. We construct families of graphs showing that, for maximal outerpaths, (and, hence, for maximal outerplanar graphs and 2-trees) the minimum ratio is 1 (Prop. 8/Fig. 5a) and the maximum ratio is at least 2 (Prop. 9/Fig. 5b). For planar 3-trees, the minimum ratio is at most 4/3 (Prop. 10/Fig. 21) and the maximum ratio is at least 3 (Prop. 11/Fig. 23).

It would be interesting to find out how much of an improvement in terms of visual complexity circular-arc drawings offer over straight-line drawings for arbitrary planar graphs. Can the ratio between segment and arc number be bounded by 3 for every planar graph?

Proposition 8. *For $r \in \mathbb{N}$, let P_r be the maximal outerpath from Prop. 3; see Fig. 5a. Then,* $\lim_{r\to\infty} \operatorname{seg}(P_r)/\operatorname{arc}(P_r) = 1$.

Proof. Consider Fig. 5a for a drawing of P_r on $n/2 + 2$ segments where n is the number of vertices of P_r. Observe that the central vertex has degree $(n-1)$ and, thus, is contained in at least $n/2$ different arcs in any arc-drawing. Hence the segment number and the arc number of P_n differ by at most a constant of 2. □

Proposition 9. *For every positive integer k, let Q_k be the maximal outerpath with $n_k = 3k + 3$ vertices shown in Fig. 5b. Then* $\lim_{k\to\infty} \operatorname{seg}(Q_k)/\operatorname{arc}(Q_k) \geq 2$.

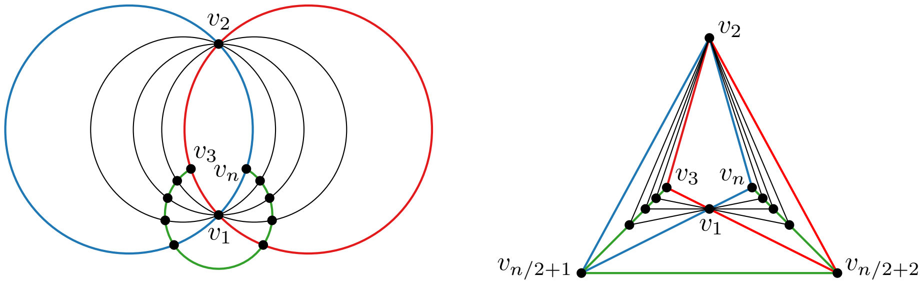


Fig. 23: The planar graph B'_{12} from Prop. 11 drawn with $n/2 = 6$ arcs and with $3n/2 = 18$ line segments.

Proof. The outerpath Q_k contains $n_k/3-2$ degree-6 vertices and for each of them two degree-3 neighbors, with at least one port each. The degree-6 vertices either have two ports themselves or their bend companions have three ports. In either case, we find four ports for each degree-6 vertex. The remaining six vertices around the first and the last face have at least ten ports. Therefore, $\mathrm{seg}(Q_k) \geq 2n_k/3+1$. Fig. 5b yields that $\mathrm{arc}(Q_k) \leq n_k/3+1$. Hence, $\mathrm{seg}(Q_k)/\mathrm{arc}(Q_k) \geq 2-2/(k+2)$. □

Proposition 10. *For $k \geq 2$, let T_k be the planar 3-tree shown in Fig. 21. Then* $\lim_{k\to\infty} \mathrm{seg}(T_k)/\mathrm{arc}(T_k) \leq 4/3$.

Proof. See Fig. 21 for a drawing of T_k on $4k+11$ segments. Let v be the unique vertex of degree $n-1$ and u, w be the two degree-$(n/2-1)$ vertices.

There is a set of $4k+2$ unique paths, one half from u to v and the other from v to w. Each of these paths needs to be covered by at least one arc. Obviously no arc may cover more than two paths. Now observe that any arc covering one path on each side connects all three vertices, such that only one such arc may exist. Hence, of the remaining $4k$ paths we may cover only two with the same arc if both lie on the same side of v. However, every such arc must have the same tangent in v in order not to cross the other paths. Therefore, we may only do this on one side of v. Thus, k arcs may suffice for one side, but the other needs $2k$ arcs, which yields a total of $3k+1$ necessary arcs. Hence, $\mathrm{seg}(T_k)/\mathrm{arc}(T_k) \leq 4/3+29/(9k+3)$. □

Proposition 11. *For every even $n \geq 8$, let $B'_n = B_n - v_1v_2$ be the planar graph shown in Fig. 23. Then* $\mathrm{seg}(B'_n)/\mathrm{arc}(B'_n) = 3$ *and* $\lim_{n\to\infty} \mathrm{seg}(B_n)/\mathrm{arc}(B_n) = 3$.

Proof. Figure 23 shows drawings of B'_n with $n/2$ arcs and with $3n/2$ segments. Clearly, $\mathrm{arc}(B'_n) = n/2$ since $\deg(v_2) = n-2$ and there are two vertices of odd degree (v_3 and v_n), where some arc(s) must start and end. For the same reason $\mathrm{arc}(B_n) = n/2+1$. By Prop. 7, $\mathrm{seg}(B_n) = 3n/2+1$. Recall that removing the edge v_1v_2 from B_n yields B'_n. Observe that B'_n is still triconnected. Therefore, the set of embeddings is the same as for B_n (except that we have the face

$\langle v_1, v_n, v_2, v_3 \rangle$ instead of the triangular faces $\triangle v_1 v_2 v_3$ and $\triangle v_1 v_n v_2$) and depends only on the choice of the outer face. Analyzing the different embeddings of B'_n as those of B_n in the proof of Prop. 7, shows that $\operatorname{seg}(B'_n) = 3n/2$. In particular, while we could straighten the path $\langle v_2, v_3, v_1 \rangle$ in Fig. 23, this would introduce a new bend in the path $\langle v_3, v_1, v_{n/2+2} \rangle$, and the number of segments remains $3n/2$. Hence $\operatorname{seg}(B'_n)/\operatorname{arc}(B'_n) = 3$ and $\lim_{n\to\infty} \operatorname{seg}(B_n)/\operatorname{arc}(B_n) = 3$. $\square$