Stability Analysis of Planar Probabilistic Piecewise Constant Derivative Systems*

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Abstract. In this paper, we study the probabilistic stability analysis of a subclass of stochastic hybrid systems, called the *Planar Probabilistic Piecewise Constant Derivative Systems (Planar PPCD)*, where the continuous dynamics is deterministic, constant rate and planar, the discrete switching between the modes is probabilistic and happens at boundary of the invariant regions, and the continuous states are not reset during switching. These aptly model piecewise linear behaviors of planar robots. Our main result is an exact algorithm for deciding *absolute* and *almost sure stability* of Planar PPCD under some mild assumptions on mutual reachability between the states and the presence of non-zero probability self-loops. Our main idea is to reduce the stability problems on planar PPCD into corresponding problems on Discrete-time Markov Chains with edge weights.

Keywords: Stability · Probabilistic Piecewise Constant Derivative Systems · Discrete-time Markov Chain · Convergence.

1 Introduction

Stability of Stochastic Hybrid Systems (SHS) [28] is a desirable property, as it guarantees eventual convergence of executions to a point of equilibrium, even in the presence of random errors. In this paper, we investigate the stability of a certain kind of SHS where the continuous state space is planar and dynamics has constant rate, where the rates are discrete and chosen probabilistically. More precisely, we study *Probabilistic Piecewise Constant Derivative Systems (PPCD)*, that consist of a finite number of discrete states representing different modes of operation each associated with a constant rate dynamics, and probabilistic mode switches enabled at certain polyhedral boundaries. Such systems can aptly model piecewise linear behaviour of planar robots.

Safety analysis of SHS has been extensively studied in the context of both non-stochastic as well as stochastic hybrid systems [26,17,8,1,18]; stability on the other hand is relatively less explored, especially, from a computational point of view. It is well-known that even for non-stochastic hybrid systems decidability

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(existence of exact algorithms) for safety is achievable only under restrictions on the dynamics and the dimension [14]. More recently, decidability of stability of hybrid systems has been explored in the non-stochastic setting [24]. The main contribution of this paper is the identification of a practically useful subclass of stochastic hybrid systems for which stability is decidable along with an exact stability analysis algorithm.

The classical stability analysis techniques build on the notion of Lyapunov functions that provide a certificate of stability. While the notion of Lyapunov functions have been extended to the hybrid system setting, computing them is a challenge. Typically, they require solving certain complex optimization problems, for instance, to deduce coefficients of polynomial templates, and more importantly, need the exploration of increasingly complex templates. In this paper, we take an alternate route where we present graph theory based reductions to show the decidability of stability analysis.

Our broad approach is to reduce a planar PPCD, that is a potentially infinite state probabilistic system, to that of a Finite State Discrete-time Markov Chain such that the stability of the planar PPCD can be deduced exactly by algorithmically checking certain properties of the reduced system. We study two notions of stability, namely, absolute stability and almost sure stability. In the former, we seek to ensure that every execution converges, while in the latter, we require that the probability of the set of system executions that converge be 1. Absolute convergence ignores the probabilities associated with the transitions, and hence, can be solved using previous results on stability analysis of Piecewise Constant Derivative systems [23], where one checks for certain diverging transitions and cycles. Checking almost sure convergence is much more challenging. We show that almost sure convergence can be characterized by certain constraints based on the stationary distribution of the reduced system. For this result to hold, we need mild conditions on the PPCD that ensure the existence of this stationary distribution. The proof relies on several insights, including the properties of planar dynamics, and convergence results on infinite sequences of random variables.

The rest of the paper is organized as follows. In section 2, we discuss related works. In section 3, we model motion of a planar robot with faulty angle actuator using PPCD. In section 4, we define important definitions and notations related to Markov Chains. In section 5, we develop algorithms for analyzing convergence of Markov Chains. We analyze stability of general and planar PPCDs in section 6. Finally, we conclude in section 7.

2 Related Work

Stability is a well studied problem in classical control theory, where Lyapunov function based methods have been extensively developed. They have been extended to hybrid systems using multiple and common Lyapunov functions [4,9,19,30]. However, constructing Lyapunov functions is computationally challenging, hence, alternate approximate methods have been explored. For example, in one approach the state space is divided into certain regions and shown that the system inevitably ends up in a certain region, thus ensuring stability [12,13,20,21]. Another approach is based on abstraction, where a simplified model (known as the abstract model) is created based on the original model and stability analysis on the simplified model is mapped back to the original one [2,5,25,10,1,8,22,23].

While stability has been extensively studied in non-probabilistic setting, investigations of stability for probabilistic systems are limited. Sufficient conditions for stability of Stochastic Hybrid Systems via Lyapunov functions is discussed in the survey [29]. Almost sure exponential stability [6,7,11,15] and asymptotic stability in distribution [32,31] for Stochastic Hybrid Systems have also been studied. Most of these works on probabilistic stability analysis provide approximate mehtods for analysis. We provide a simple class of Stochastic Hybrid Systems that have practical application in modeling planar robots, and an exact decidable algorithm for probabilistic stability analysis.

3 Case Study: Planar robot with a faulty actuator

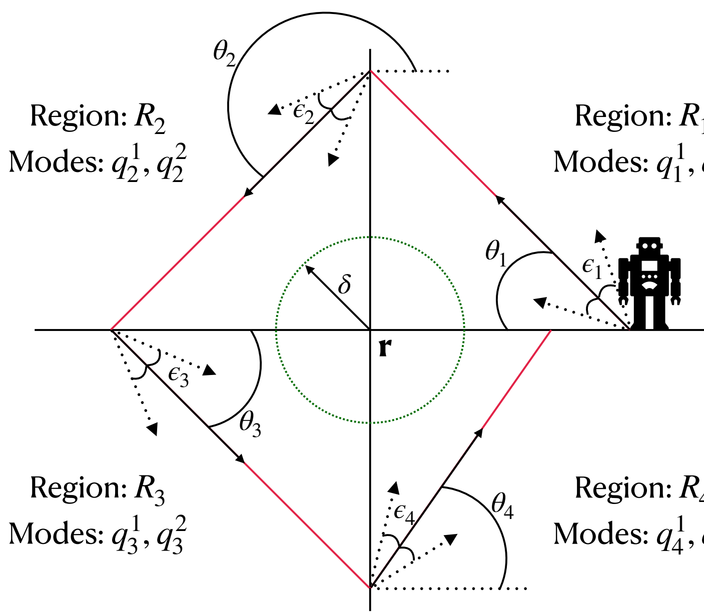

Fig. 1: Motion of planar robot with faulty heading angle actuator

Consider a robot navigating in a 2D plane at some constant speed v as shown in Figure 1. The plane is divided into four regions R_1, R_2, R_3, R_4 corresponding to the four quadrants, and the robot has a unique direction θ_i (mode of operation) in which it moves while in the region R_i, and changes its mode of operation at the boundary of the regions. Due to faulty actuator, the robot heading angle may deviate from θ_i by an amount ϵ_i. We model this as probabilistically choosing one of the k_i uniformly distanced angles $\theta_i^1, \cdots, \theta_i^{k_i}$ in the interval $[\theta_i - \epsilon_i, \theta_i + \epsilon_i]$ with probabilities $p_i^1, \cdots, p_i^{k_i}$, respectively. The whole system can be modelled as a planar PPCD with $\sum_{i=1}^{4} k_i$ modes, where for every i and $1 \leq j \leq k_i$, the mode q_i^j corresponds to the robot traversing with heading angle θ_i^j with speed v in the region R_i. The mode switching is possible between R_i and R_j if they are neighbors, that is, they share a common boundary. For instance, we can switch between quadrants 1 and 2 or 4 and 1 but not 1 and 3. We can move to any mode corresponding to a neighbor q_i^j with probability p_i^j.

The objective of the navigation is to reach a target point r on the 2D plane arbitrarily closely. More precisely, we want to check whether the robot reaches within a $\delta > 0$ ball around r for any arbitrarily small δ. We want to check if all executions of the robot have this property, i.e., if the planar PPCD is absolutely stable, as well as if the probability of convergence is 1, i.e., the planar PPCD is almost surely stable.

4 Preliminaries

In this section, we will discuss important concepts related to Discrete-time Markov Chain (DTMC), Weighted Discrete-time Markov Chain (WDTMC) and convergence of WDTMC.

4.1 Discrete-time Markov Chain

Let $Dist(S)$ denote the set of all probability distributions on the set S. Let us define Discrete-time Markov Chain (DTMC) on the set of states S.

Definition 1 (Discrete-time Markov Chain). *The Discrete-time Markov Chain (DTMC) is defined as the tuple $\mathcal{M} = (S, P)$ where*

- *S is a set of states.*
- *$P : S \mapsto Dist(S)$ is a function from the set of states S to the set of all probability distributions over S, $Dist(S)$.*

We use $\mathrm{P}(s_1, s_2)$ to denote $\mathrm{P}(s_1)(s_2)$ and $\mathrm{P}^n(s_1, s_2)$ to denote the probability of going from s_1 to s_2 in n-steps.

A path of a DTMC $\mathcal{M}$ is a sequence of states $\sigma = s_1, s_2, \ldots$ such that for all $i < |\sigma|$, $\mathrm{P}(s_i, s_{i+1}) > 0$, where $|\sigma|$ is the length of the sequence. A path of length 2 is called an edge and the set of all edges is denoted as $\mathcal{E}$. The i^{th} state of the path σ is denoted by σ_i and the last state of σ is denoted as σ_{end}. $\sigma[i : j]$ denotes the subsequence $\sigma_i, \sigma_{i+1}, \ldots, \sigma_j$. We say s_2 is reachable from s_1 (denoted $s_1 \rightsquigarrow s_2$) if there is a path σ on $\mathcal{M}$ such that $\sigma_1 = s_1$ and $\sigma_{end} = s_2$. The set of all finite paths of a DTMC $\mathcal{M}$ is denoted as $Paths_{fin}(\mathcal{M})$ and the set of all infinite paths is denoted as $Paths(\mathcal{M})$.

The probability of a finite path σ, denoted $\mathrm{P}(\sigma)$, is the product of the probabilities of each of its edges, $\mathrm{P}(\sigma) \coloneqq \prod_{i<|\sigma|} \mathrm{P}(\sigma_i, \sigma_{i+1})$. The probability of σ with respect to a distribution ρ, denoted $P_\rho(\sigma)$ is the product of $\mathrm{P}(\sigma)$ and the probability of σ_1 under ρ, i.e., $P_\rho(\sigma) \coloneqq \rho(\sigma_1) \cdot \mathrm{P}(\sigma)$. We can associate a probability measure Pr to the set of infinite paths $Paths(\mathcal{M})$ of a DTMC $\mathcal{M}$ using probability of the cylinder sets of the finite paths as discussed in [3]. A path property $\mathbb{P}$ is said to be *almost surely* satisfied if the set of all paths having property $\mathbb{P}$ has probability 1, i.e., $Pr\{\sigma \mid \sigma \text{ has } \mathbb{P}\} = 1$.

Next we define some subclasses of DTMC and show that it has some nice convergence properties.

Definition 2 (Irreducibility). *A DTMC $\mathcal{M}$ is called irreducible if for any $s_1, s_2 \in S$, $s_1 \rightsquigarrow s_2$ and $s_2 \rightsquigarrow s_1$.*

Definition 3 (Periodicity). *A state $s \in S$ in a DTMC $\mathcal{M}$ is called periodic if there is a natural number $n > 1$ such that, for any path σ starting and ending at s, $|\sigma|$ is a multiple of n. A DTMC $\mathcal{M}$ is called aperiodic if none of its states is periodic.*

We say a probability distribution is stationary for a DTMC $\mathcal{M}$ if the next step distribution remains unchanged.

Definition 4 (Stationary Distribution). *A distribution $\rho^* \in Dist(S)$ is called the stationary distribution of DTMC $\mathcal{M}$ if,*

$$\rho^*(s) = \sum_{s' \in S} \rho^*(s') P(s', s), \quad \forall s \in S.$$

For finite, irreducible DTMC, the stationary distribution is unique. The following theorem guarantees existence of limiting distribution for finite, irreducible and aperiodic DTMC and associates it with the stationary distribution of the DTMC (see [27]).

Theorem 1. *For a finite, irreducible and aperiodic DTMC $\lim_{n\to\infty} P^n(s_1, s_2)$ exists for all $s_1, s_2 \in S$ and $\lim_{n\to\infty} P^n(s_1, s_2) = \rho^*(s_2)$ where $\rho^* \in Dist(S)$ is the unique stationary distribution of $\mathcal{M}$.*

Note that, $\mathrm{P}^n(s_1, s_2)$ does not depend on s_1 as $n \to \infty$.

4.2 Weighted Discrete-time Markov Chain

Let us now define Weighted Discrete-time Markov Chain (WDTMC) that extend DTMC with weighted edges. Basically, a WDTMC can be observed as a Markov Reward Process where rewards are associated to individual transitions rather than nodes.

Definition 5 (Weighted DTMC). *The weighted DTMC (WDTMC) $\mathcal{M}_W = (S, P, W)$ is a tuple such that (S, P) is a DTMC and $W : \mathcal{E} \mapsto \mathbb{R}$ is a weight function where $\mathcal{E}$ is the set of all possible edges of $\mathcal{M}_W$.*

We also define disjoint union of two WDTMC $\mathcal{M}_W^1$ and $\mathcal{M}_W^2$ as a WDTMC $\mathcal{M}_W^1 \sqcup \mathcal{M}_W^2$ whose states and edges are disjoint unions of states and edges of $\mathcal{M}_W^1$ and $\mathcal{M}_W^2$ respectively. With the weight function W defined, it is possible to associate weights to individual paths of $\mathcal{M}_W$.

Definition 6 (Weight of a path). *The weight of a path σ of WDTMC $\mathcal{M}_W$, denoted $W(\sigma)$, is defined as,*

$$W(\sigma) \coloneqq \sum_{i<|\sigma|} W(\sigma_i, \sigma_{i+1})$$

For $\sigma \in \mathit{Paths}(\mathcal{M}_W)$, the quantity $\lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{W}(\sigma_i, \sigma_{i+1})$ is denoted by $\mathrm{W}(\sigma[1:\infty])$. It is easy to observe that, $\mathrm{W}(\sigma) = \mathrm{W}(\sigma[1:\infty])$. A simple path is a path without state repetition and a simple cycle is a path where only the starting and the ending states are same. We use the notation $\mathcal{SP}(\mathcal{M}_W)$ for the set of all simple paths and the notation $\mathcal{SC}(\mathcal{M}_W)$ for the set of all simple cycles of a WDTMC $\mathcal{M}_W$.

4.3 Convergence of Weighted Discrete-time Markov Chain

Let us define the notions of absolute and probabilistic convergence of WDTMC. A WDTMC is said to be absolutely convergent if the weight of every infinite path diverges to $-\infty$.

Definition 7 (Absolute Convergence of WDTMC). *A WDTMC $\mathcal{M}_W$ is said to be absolutely convergent if for all infinite path $\sigma \in \mathit{Paths}(\mathcal{M}_W)$, $W(\sigma)$ diverges to $-\infty$, i.e.,*

$$W(\sigma[1:\infty]) = -\infty.$$

Further, a WDTMC is said to be almost surely convergent if the weight of an infinite path diverges to $-\infty$ with probability 1.

Definition 8 (Almost Sure Convergence of WDTMC). *We say that a WDTMC $\mathcal{M}_W$ is almost surely convergent if for any path σ of $\mathcal{M}_W$, $W(\sigma)$ diverges to $-\infty$ with probability 1. In other words,*

$$\mathit{Pr}\{\sigma \in \mathit{Paths}(\mathcal{M}_W) : W(\sigma[1:\infty]) = -\infty\} = 1.$$

Remark 1. Let us explain the reason behind defining such a strange notion of convergence. For reasons that will be clarified later, we actually want to check for an infinite path σ of $\mathcal{M}_W$, if the product of weights of the edges converge to 0, i.e., $\lim_{n\to\infty} \prod_{i=1}^{n} \mathrm{W}(\sigma_i, \sigma_{i+1}) = 0$, provided $0 < \mathrm{W}(\sigma_i, \sigma_{i+1}) < \infty$ for all $i \in \mathbb{N}$. This condition is equivalent to $\lim_{n\to\infty} \sum_{i=1}^{n} \log(\mathrm{W}(\sigma_i, \sigma_{i+1})) = -\infty$. Hence for convenience, we consider log of original weights as weights of individual edges, and check if sum of weights of edges of an infinite path diverge to $-\infty$.

4.4 Probabilistic Bisimulation

Probabilistic bisimulation [3] on a WDTMC is an equivalence relation on its set of states such that probabilities of corresponding edges agree for two related states.

Definition 9 (Probabilistic Bisimulation). *A probabilistic bisimulation on a WDTMC $\mathcal{M}_W$ is an equivalence relation $\sim$ on S such that for any $s_1, s_2 \in S$ with $s_1 \sim s_2$, $P(s_1, T) = P(s_2, T)$ for each equivalence class T of $\sim$.*

Note that, $\mathrm{P}(s, T) = \sum_{t \in T} \mathrm{P}(s, t)$ for $s \in S$. Let us now use probabilistic bisimulation to relate infinite paths of a WDTMC.

Definition 10 (Bisimulation-Equivalent Paths). *Given a probabilistic bisimulation $\sim$ on a WDTMC $\mathcal{M}_W$, two infinite paths $\pi = \pi_1, \pi_2, \ldots$ and $\tilde{\pi} = \tilde{\pi}_1, \tilde{\pi}_2, \ldots$ are said to be bisimulation equivalent, denoted $\pi \sim \tilde{\pi}$, if they are statewise related by $\sim$, i.e.,*

$$\pi \sim \tilde{\pi} \text{ iff } \pi_i \sim \tilde{\pi}_i \text{ for all } i \geq 1$$

A set of infinite paths is $\sim$ bisimulation-closed for some probabilistic bisimulation $\sim$, if for any path in the set, all its bisimulation-equivalent paths are also in the set. In other words, $\Pi \subseteq \mathit{Paths}(\mathcal{M}_W)$ is $\sim$ bisimulation-closed if for any $\pi \in \Pi$ and any $\tilde{\pi} \sim \pi$, $\tilde{\pi} \in \Pi$. Let us denote by $Pr_s(\Pi)$ the set of all paths in Π that start from $s \in S$. The following lemma [3] equates the probability of two sets of paths that start from $\sim$ related states and are subset of the same $\sim$ bisimulation-closed set.

Lemma 1. *Let $\sim$ be a probabilistic bisimulation on a WDTMC $\mathcal{M}_W$. For all states s_1, s_2 of $\mathcal{M}_W$, $s_1 \sim s_2$ implies $Pr_{s_1}(\Pi) = Pr_{s_2}(\Pi)$, for all $\sim$ bisimulation-closed events $\Pi \subseteq \mathit{Paths}(\mathcal{M}_W)$.*

4.5 Polyhedral Sets

We denote the set of all polyhedral subsets of $\mathbb{R}^n$ by $Poly(n)$. The facets of a polyhedral subset A are the largest polyhedral subsets of the boundary of A. We denote the boundary of a polyhedral subset A by $\partial(A)$ and the set of all facets of A by $\mathbb{F}(A)$. We say a polyhedral subset P is positive scaling invariant if for all $x \in P$ and $\alpha > 0$, $\alpha x \in P$.

5 Analyzing Convergence of Weighted Discrete-time Markov Chains

In this section, we discuss necessary and sufficient conditions for absolute and almost sure convergence of WDTMC. For our analysis, we will assume all paths of the WDTMC start from a single state called the *initialization point* (denoted s_{init}) of the WDTMC. In other words we restrict our attention to the set of paths $\Sigma' := \{\sigma \in \mathit{Paths}(\mathcal{M}_W) \mid \sigma_1 = s_{init}\}$. Consequently, we consider only those edges $\mathcal{E}' = \Sigma' \cap \mathcal{E}$, which are reachable from s_{init}. We abuse notation and use Σ for Σ' and $\mathcal{E}$ for $\mathcal{E}'$ for the rest of the section.

5.1 Analyzing absolute convergence of Weighted DTMC

Here we provide a necessary and sufficient condition for analyzing absolute convergence of a WDTMC. We begin with the following proposition (proved in the Appendix) which states that for any finite path $\sigma \in Paths_{fin}(\mathcal{M}_W)$, we can get one simple path and a set of simple cycles such that their total weight equals the weight of σ.

Proposition 1. *For any finite path σ of $\mathcal{M}_W$ there exist a simple path $\sigma_s \in \mathcal{SP}(\mathcal{M}_W)$ and a set of simple cycles $\mathcal{SC}_\sigma \subseteq \mathcal{SC}(\mathcal{M}_W)$ such that $W(\sigma) = W(\sigma_s) + \sum_{\mathcal{C} \in \mathcal{SC}_\sigma} W(\mathcal{C})$.*

We use Proposition 1 to prove the following main theorem which states that, a WDTMC is absolutely convergent iff there is no edge of infinite weight and no cycle of weight greater or equal to 0 reachable from the initial point.

Theorem 2. *The WDTMC $\mathcal{M}_W$ is absolutely convergent iff,*

1. *There does not exist an edge $e \in \mathcal{E}$ reachable from s_{init} such that $W(e) = \infty$.*
2. *For any simple cycle $\mathcal{C}$ reachable from s_{init}, $W(\mathcal{C}) < 0$.*

Proof. ($\Rightarrow$) To show that the conditions 1 and 2 are necessary, we have to prove that if either of them is negated then $\mathcal{M}_W$ is not absolutely convergent. If condition 1 is false then there is an edge $e = (s_1, s_2)$ with $\mathrm{W}(s_1, s_2) = \infty$ such that for some finite path σ starting from s_{init}, $\sigma_{|\sigma|-1} = s_1$ and $\sigma_{|\sigma|} = s_2$. But that implies $\mathrm{W}(\sigma) = \sum_{i=1}^{|\sigma|-1} \mathrm{W}(\sigma_i, \sigma_{i+1}) = \infty$. So for any infinite path σ' with prefix σ, $\mathrm{W}(\sigma') = \infty$. Thus $\mathcal{M}_W$ is not absolutely convergent. On the other hand if we suppose condition 2 is false then there is a simple cycle $\mathcal{C} \in \mathcal{SC}(\mathcal{M}_W)$ with $\mathrm{W}(\mathcal{C}) \geq 0$ such that for some finite path σ starting from s_{init}, there exists an index j such that $\mathcal{C} = \sigma[j : |\sigma|]$. Now we can easily construct the following infinite path $\sigma_\infty = \sigma \cdot \mathcal{C} \cdot \mathcal{C} \ldots$ by concatenating $\mathcal{C}$ infinite times to σ. Clearly, σ_∞ starts at s_{init} since σ starts at s_{init} and $\mathrm{W}(\sigma_\infty) = \mathrm{W}(\sigma) + \sum_{n \in \mathbb{N}} \mathrm{W}(\mathcal{C}) \geq \mathrm{W}(\sigma)$. Since for any finite path σ, $\mathrm{W}(\sigma)$ is also finite, $\mathrm{W}(\sigma_\infty)$ is bounded below by some finite quantity and cannot diverge to $-\infty$. Thus, $\mathcal{M}_W$ is not absolutely convergent.

($\Leftarrow$) Conversely, suppose both conditions 1 and 2 hold. Now, let σ be an arbitrary infinite path starting from s_{init} and $\sigma[1 : i]$ be its finite prefix of length $i \in \mathbb{N}$. By Proposition 1, there exist a simple path $\sigma[1 : i]_s$ and a set of simple cycles $\mathcal{SC}_{\sigma[1:i]}$ such that $\mathrm{W}(\sigma[1 : i]) = \mathrm{W}(\sigma[1 : i]_s) + \sum_{\mathcal{C} \in \mathcal{SC}_{\sigma[1:i]}} \mathrm{W}(\mathcal{C})$. Now, for any $i \in \mathbb{N}$, $\mathrm{W}(\sigma[1 : i]_s)$ is at most $\sum_{(s_1,s_2) \in \mathcal{E}} \max\{\mathrm{W}(s_1, s_2) \mid (s_1, s_2) \in \mathcal{E}\} < \infty$. Also, $\mathcal{SC}_{\sigma[1:i]}$ is a set of simple cycles where each cycle has weight at most $\max_{\mathcal{C} \in \mathcal{SC}(\mathcal{M}_W)} \mathrm{W}(\mathcal{C}) < 0$ (here we abuse notation and denote the set of all simple cycles reachable from s_{init} as $\mathcal{SC}(\mathcal{M}_W)$). Thus, for all $K \in \mathbb{R}$, there exists $i \in \mathbb{N}$

such that

$$\begin{aligned}
&\sum_{(s_1,s_2)\in\mathcal{E}} \max\{\mathrm{W}(s_1,s_2) \mid (s_1,s_2)\in\mathcal{E}\} + \sum_{\mathcal{C}\in\mathcal{SC}_{\sigma[1:i]}} \mathrm{W}(\mathcal{C}) < K \\
&\Rightarrow \mathrm{W}(\sigma[1:i]_s) + \sum_{\mathcal{C}\in\mathcal{SC}_{\sigma[1:i]}} \mathrm{W}(\mathcal{C}) < K \\
&\Rightarrow \mathrm{W}(\sigma[1:i]) < K.
\end{aligned}$$

But this implies $\mathrm{W}(\sigma) = \lim_{i\to\infty} \mathrm{W}(\sigma[1:i]) = -\infty$ for any infinite path σ starting from s_{init}, i.e., $\mathcal{M}_W$ is absolutely convergent. □

5.2 Analyzing almost sure convergence of Weighted DTMC

In this subsection, we will provide a necessary and sufficient condition for almost sure convergence of a WDTMC. We assume a WDTMC $\mathcal{M}_W$ is finite, irreducible and aperiodic and thus has the limiting distribution equal to its stationary distribution ρ^* (Theorem 1).

Given a WDTMC $\mathcal{M}_W$, we begin by defining random variables $\{X_j^e \mid e \in \mathcal{E}; j \in \mathbb{N}\}$ on the set of infinite paths $\mathit{Paths}(\mathcal{M}_W)$, that captures the information of whether an edge $e \in \mathcal{E}$ appears on the j^{th} step of an infinite path σ. More precisely,

$$X_j^e(\sigma) = \begin{cases} 1 \text{ if } (\sigma_j, \sigma_{j+1}) = e \\ 0 \text{ else.} \end{cases}$$

Note that for some $e \in \mathcal{E}$ and $\sigma \in \mathit{Paths}(\mathcal{M}_W)$, $\sum_{j=1}^n X_j^e(\sigma)$ gives the number of times e appears on $\sigma[1:n+1]$. Now, the following lemma (proved in Appendix) gives that, for any edge $e \in \mathcal{E}$, the average of $\{X_j^e \mid j \in \mathbb{N}\}$ almost surely converges to $P_{\rho^*}(e)$, which is the probability of e with respect to the stationary distribution ρ^*.

Lemma 2. *For any edge $e \in \mathcal{E}$ of a WDTMC $\mathcal{M}_W$,*

$$Pr\left\{\sigma \in \mathit{Paths}(\mathcal{M}_W) : \lim_{n\to\infty} \frac{\sum_{j=1}^n X_j^e(\sigma)}{n} = P_{\rho^*}(e)\right\} = 1.$$

Next, we define partial average weight upto n for an infinite path σ as

$$\frac{(S_\sigma)_n}{n} := \frac{\sum_{i=1}^n \mathrm{W}(\sigma_i, \sigma_{i+1})}{n},$$

and note that,

$$\begin{aligned}
\frac{(S_\sigma)_n}{n} &= \frac{\sum_{e\in\mathcal{E}} (\#\text{ times } e \text{ appears on } \sigma[1:n+1]) \cdot \mathrm{W}(e)}{n} \\
&= \frac{\sum_{e\in\mathcal{E}} \left(\sum_{j=1}^n X_j^e(\sigma)\right) \cdot \mathrm{W}(e)}{n} \qquad (1)
\end{aligned}$$

We now state the main lemma of this subsection which essentially states that, the average weight of an infinite path almost surely converges to a quantity that depends only on the weights and probabilities of the edges.

Lemma 3. *For a WDTMC $\mathcal{M}_W$,*

$$Pr\left\{\sigma \in Paths(\mathcal{M}_W) : \lim_{n\to\infty} \frac{(S_\sigma)_n}{n} = \sum_{e\in\mathcal{E}} P_{\rho^*}(e)\, W(e)\right\} = 1.$$

Proof. We have already established that,

$$\frac{(S_\sigma)_n}{n} = \frac{\sum_{e\in\mathcal{E}} \left(\sum_{j=1}^{n} X_j^e(\sigma)\right) \cdot \mathrm{W}(e)}{n} \quad \text{[Equation 1]}$$

$$\text{Thus, } \lim_{n\to\infty} \frac{(S_\sigma)_n}{n} = \lim_{n\to\infty} \frac{\sum_{e\in\mathcal{E}} \left(\sum_{j=1}^{n} X_j^e(\sigma)\right) \cdot \mathrm{W}(e)}{n}$$

$$\Rightarrow \lim_{n\to\infty} \frac{(S_\sigma)_n}{n} = \sum_{e\in\mathcal{E}} P_{\rho^*}(e) \cdot \mathrm{W}(e) \text{ almost surely} \quad \text{[by Lemma 2]}$$

□

We say $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)\mathrm{W}(e)$ is the effective weight of the WDTMC $\mathcal{M}_W$ and denote it as $\mathrm{W}_\mathcal{E}$. The main theorem basically states that a WDTMC is almost surely convergent iff its effective weight is strictly less than 0.

Theorem 3. *A WDTMC $\mathcal{M}_W$ is almost surely convergent iff $W_\mathcal{E} < 0$, where $W_\mathcal{E} = \sum_{e\in\mathcal{E}} P_{\rho^*}(e)\, W(e)$ is the effective weight of $\mathcal{M}_W$.*

Proof. Observe that, weight of an infinite path σ, $\mathrm{W}(\sigma)$, can be written as $\lim_{n\to\infty} n \cdot ((S_\sigma)_n/n)$, where $((S_\sigma)_n/n)$ is the partial average weight upto n for the infinite path σ. Since,

$$\lim_{n\to\infty} \left(\frac{(S_\sigma)_n}{n}\right) = \sum_{e\in\mathcal{E}} P_{\rho^*}(e)\mathrm{W}(e) \text{ almost surely} \quad \text{[by Lemma 3]}$$

$$\Rightarrow \mathrm{W}(\sigma) = \lim_{n\to\infty} n \cdot \left(\frac{(S_\sigma)_n}{n}\right) = \lim_{n\to\infty} n \cdot \left(\sum_{e\in\mathcal{E}} P_{\rho^*}(e)\mathrm{W}(e)\right) \text{ almost surely}$$

Thus, $\mathrm{W}(\sigma)$ diverges to $-\infty$ almost surely if and only if $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)\mathrm{W}(e) < 0$. In other words, $\mathcal{M}_W$ is almost surely convergent iff $\mathrm{W}_\mathcal{E} < 0$. □

5.3 Computability

Based on Theorems 2 and 3 we present two algorithms here for checking absolute and almost sure convergence of a WDTMC. For the first algorithm, assuming the WDTMC is finite, we first check for existence of an infinite weight edge by Breadth First Search (BFS) [16] and then for a cycle with non-negative weight using a variant of the Bellman-Ford algorithm [16]. If neither of them is found then the WDTMC is deemed absolutely convergent by Theorem 2. Since BFS takes time linear to the size of its input and Bellman-Ford takes time quadratic to the size of its input, the time complexity of this algorithm is $O(|S|^2)$, where S is the set of states of $\mathcal{M}_W$.

For the second algorithm, assuming the WDTMC is finite, irreducible and aperiodic, existence of an infinite weight edge is checked by Breadth First Search (BFS). If such an edge exists then the WDTMC is deemed not almost surely convergent (by Theorem 3). Otherwise, the stationary distribution ρ^* of the WDTMC is calculated by solving a set of linear equations mentioned in Definition 4. The value $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)\mathrm{W}(e)$ is then calculated (where $\mathcal{E}$ is the set of transitions of the WDTMC) and compared to 0. The WDTMC is deemed almost surely convergent only if $\sum_{e\in\mathcal{E}} P_{\rho^*}(e)\mathrm{W}(e) < 0$. Since BFS takes time linear to its input size and solving a set of linear equations takes time at most cubic in the number of variables, the time complexity of this algorithm is $O(|S|^3)$, where S is the set of states of $\mathcal{M}_W$.

6 Probabilistic Piecewise Constant Derivative Systems

In this section, we present the details of the Probabilistic Piecewise Constant Derivative Systems (PPCD) and provide a characterization of absolute and almost sure stability by a reduction to that of DTMCs.

6.1 Formal Definition of PPCD

We model PPCDs as consisting of a discrete set of modes, each associated with an invariant and probabilistic transitions between modes that are enabled at the boundaries of the invariants.

Definition 11 (PPCD). *The Probabilistic Piecewise Constant Derivative System (PPCD) is defined as the tuple $\mathcal{H} := (Q, \mathcal{X}, \mathit{Inv}, \mathit{Flow}, \mathit{Edges})$ where*

- *Q is the set of discrete locations,*
- *$\mathcal{X} = \mathbb{R}^n$ is the continuous state space for some $n \in \mathbb{N}$,*
- *$\mathit{Inv} : Q \to \mathit{Poly}(n)$ is the invariant function which assigns a positive scaling invariant polyhedral subset of the state space to each location $q \in Q$,*
- *$\mathit{Flow} : Q \to \mathcal{X}$ is the Flow function which assigns a flow vector, say $\mathit{Flow}(q) \in \mathcal{X}$, to each location $q \in Q$,*
- *$\mathit{Edges} \subseteq Q \times (\cup_{q\in Q}\mathbb{F}(\mathit{Inv}(q))) \times \mathit{Dist}(Q)$ is the probabilistic edge relation such that $(q, f, \rho) \in \mathit{Edges}$ where for every (q, f), there is a at most one ρ such that $(q, f, \rho) \in \mathit{Edges}$ and $f \in \mathbb{F}(\mathit{Inv}(q))$. f is called a* Guard *of the location q.*

Next, we discuss the semantics of the PPCD. An execution starts from a location $q_0 \in Q$ and some continuous state $x_0 \in \mathcal{X}$ and evolves continuously for some time T according to the dynamics of q_0 until it reaches a facet f_0 of the invariant of q_0. Then a probabilistic discrete transition is taken if there is an edge (q_0, f_0, ρ_0) and the state q_0 is probabilistically changed to q_1 with probability $\rho_0(q_1)$. The execution (tree) continues with alternating continuous and discrete transitions.

Formally, for any two continuous states $x_1, x_2 \in \mathcal{X}$ and $q \in Q$, we say that there is a *continuous transition* from x_1 to x_2 with respect to q if $x_1, x_2 \in \mathit{Inv}(q)$, there exists $T \geq 0$ such that $x_2 = x_1 + \mathit{Flow}(q) \cdot T$, $x_1 + \mathit{Flow}(q) \cdot t \notin \partial(\mathit{Inv}(q_0))$ for any $0 \leq t < T$ and $x_2 \in \partial(\mathit{Inv}(q_0))$. We note that there is a unique continuous transition from any state (q, x) since it requires the state to evolve until it reaches the boundary for the first time, which corresponds to a unique time of evolution T. Further, if for all $t \geq 0$, $x_1 + \mathit{Flow}(q) \cdot t \in \mathit{Inv}(q)$ then we say x_1 has an infinite edge with respect to q. For two locations $q_1, q_2 \in Q$, we say there is a *discrete transition* from q_1 to q_2 with probability p via $\rho \in \mathit{Dist}(Q)$ and $f \in \mathbb{F}(q_1)$ if $f \subseteq \mathit{Inv}(q_2)$, $(q_1, f, \rho) \in \mathit{Edges}$ and $p = \rho(q_2)$.

We capture the semantics of a PPCD using a WDTMC, wherein we combine a continuous transition and a discrete transition to represent a probabilistic transition of the DTMC. In addition, to reason about convergence, we also need to capture the relative distance of the states from the equilibrium point, which is captured using edge weights. Let us fix 0 as the equilibrium point for the rest of the section. The weight on a transition from (q_1, x_1) to (q_2, x_2) captures the logarithm of the relative distance of x_1 and x_2 from 0, that is, it is $(\|x_2\|/\|x_1\|)$, where $\|x\|$ captures the distance of state x from 0.

Definition 12 (Semantics of PPCD). *Given a PPCD $\mathcal{H}$, we can construct the WDTMC $\mathcal{M}_\mathcal{H} := (S_\mathcal{H}, P_\mathcal{H}, W_\mathcal{H})$ where,*

- $S_\mathcal{H} = Q \times \mathcal{X}$
- *$P_\mathcal{H}$ and $W_\mathcal{H}$ are defined as follows for any (q_1, x_1) and (q_2, x_2):*
  - *If there is a continuous transition from x_1 to x_2 with respect to q_1 and there is a discrete transition from q_1 to q_2 with probability p via some $\rho \in \mathit{Dist}(Q)$ and $f \in \mathbb{F}(q_1)$, and $x_2 \in f$, then $P_\mathcal{H}((q_1, x_1), (q_2, x_2)) = p$ and $W_\mathcal{H}((q_1, x_1), (q_2, x_2)) = \log(\|x_2\|/\|x_1\|)$*
  - *If x_1 has an infinite edge with respect to q_1, then $P_\mathcal{H}((q_1, x_1), (q_2, x_2)) = 1$ if $(q_1, x_1) = (q_2, x_2)$ and 0, otherwise, and $W_\mathcal{H}((q_1, x_1), (q_1, x_1)) = \infty$.*
  - *Otherwise, $P_\mathcal{H}((q_1, x_1), (q_2, x_2)) = W_\mathcal{H}((q_1, x_1), (q_2, x_2)) = 0$.*

Since all executions of the PPCD $\mathcal{H}$ start from location q_0 and state x_0, we consider only those paths of the semantics $\mathcal{M}_\mathcal{H}$ which start from (q_0, x_0) and denote them as $\mathit{Paths}(\mathcal{M}_\mathcal{H})$. We say a path $\sigma = (q_0, x_0), (q_1, x_1), \ldots$ converges to 0 if norm of the corresponding state sequence $\|x_0\|, \|x_1\|, \ldots$ converges to 0. Stability of a PPCD $\mathcal{H}$ is defined in terms of convergence of paths of its semantics $\mathcal{M}_\mathcal{H}$ as follows,

Definition 13 (Stability of PPCD). *A PPCD $\mathcal{H}$ is called absolutely stable if every path of $\mathcal{M}_\mathcal{H}$ converges to 0. Analogously, $\mathcal{H}$ is called almost surely stable if any path of $\mathcal{M}_\mathcal{H}$ converges to 0 with probability 1, i.e.,*

$$Pr\{\sigma \in \mathit{Paths}(\mathcal{M}_\mathcal{H}) : \sigma \text{ converges to } 0\} = 1.$$

We now characterize stability of a PPCD $\mathcal{H}$ in terms of its semantics $\mathcal{M}_{\mathcal{H}}$. Basically we state that, $\mathcal{H}$ is absolutely (almost surely) stable iff $\mathcal{M}_{\mathcal{H}}$ is absolutely (almost surely) convergent.

Theorem 4 (Characterization of Stability). *A PPCD $\mathcal{H}$ is absolutely stable iff its semantics $\mathcal{M}_{\mathcal{H}}$ is absolutely convergent and it is almost surely stable iff $\mathcal{M}_{\mathcal{H}}$ is almost surely convergent.*

Proof. Note that, a path σ of $\mathcal{M}_{\mathcal{H}}$ converges to 0 iff $\mathrm{W}(\sigma)$ diverges to $-\infty$. To observe this, let $\sigma = (q_0, x_0), (q_1, x_1), \ldots$. Then, $\|x_0\|, \|x_1\|, \ldots$ converge to 0 iff,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\|x_n\|}{\|x_0\|} = 0 \quad [\text{since } \|x_0\| \neq 0] \\
\iff &\lim_{n\to\infty} \frac{\|x_1\|}{\|x_0\|} \cdot \frac{\|x_2\|}{\|x_1\|} \cdots \frac{\|x_n\|}{\|x_{n-1}\|} = 0 \\
&\qquad\qquad [\text{since } \|x_i\| \neq 0 \text{ for all } i = 1, \ldots, n-1 \text{ if } \sigma \text{ is infinite}] \\
\iff &\lim_{n\to\infty} \log\left(\frac{\|x_1\|}{\|x_0\|} \cdot \frac{\|x_2\|}{\|x_1\|} \cdots \frac{\|x_n\|}{\|x_{n-1}\|}\right) = -\infty \\
\iff &\lim_{n\to\infty} \log\left(\frac{\|x_1\|}{\|x_0\|}\right) + \log\left(\frac{\|x_2\|}{\|x_1\|}\right) + \cdots \log\left(\frac{\|x_n\|}{\|x_{n-1}\|}\right) = -\infty \\
\iff &\lim_{n\to\infty} \mathrm{W}(\sigma[1:n]) = -\infty
\end{aligned}$$

Thus, every infinite path of $\mathcal{M}_{\mathcal{H}}$ converges to 0 iff weight of every infinite path diverges to $-\infty$ and the set of infinite paths of $\mathcal{M}_{\mathcal{H}}$ converging to 0 has probability 1 iff the set of infinite paths having weight diverging to $-\infty$ has probability 1. In other words, $\mathcal{H}$ is absolutely (almost surely) stable iff $\mathcal{M}_{\mathcal{H}}$ is absolutely (almost surely) convergent. $\square$

6.2 Stability of Planar PPCD

In general, semantics of a PPCD has infinite number of states and thus the algorithms developed in section 5.3 cannot be applied to decide absolute (almost sure) convergence of the semantics. However, if the continuous state space of a PPCD $\mathcal{H}$ is $\mathbb{R}^2$, then we can reduce $\mathcal{M}_{\mathcal{H}}$ to a finite WDTMC that provides an exact characterization of $\mathcal{M}_{\mathcal{H}}$. A PPCD with $\mathcal{X} = \mathbb{R}^2$ is called a planar PPCD. Since for each location q, $Inv(q)$ is positively scaled, the facets of $Inv(q)$ are rays emanating from origin. Given constant flow for each location q, a continuous transition starting at a point of some facet $f_1 \in \cup_{q\in Q}\mathbb{F}(Inv(q))$ ends up at a unique point of a unique facet $f_2 \in \cup_{q\in Q}\mathbb{F}(Inv(q))$. This property is not observed if the continuous state space is of three or higher dimensions (Figure 2). Also, if two continuous transitions start from different points x_1, x_1' of the same facet f_1, they end up in unique points x_2, x_2' (respectively) of a unique facet f_2 such that $\|x_2\|/\|x_1\| = \|x_2'\|/\|x_1'\|$. This gives us the following lemma (proved in Appendix),

Lemma 4. *Let $e = ((q_1, x_1), (q_2, x_2))$, $e' = ((q_1, x_1'), (q_2, x_2'))$ be two edges of $\mathcal{M}_{\mathcal{H}}$ (where $\mathcal{H}$ is a planar PPCD) such that, $P_{\mathcal{H}}(e), P_{\mathcal{H}}(e') > 0$, and $x_1, x_1' \in f_1$ where $f_1 \in \bigcup_{q\in Q}\mathbb{F}(Inv(q))$. Then $P_{\mathcal{H}}(e) = P_{\mathcal{H}}(e')$ and $W_{\mathcal{H}}(e) = W_{\mathcal{H}}(e')$.*

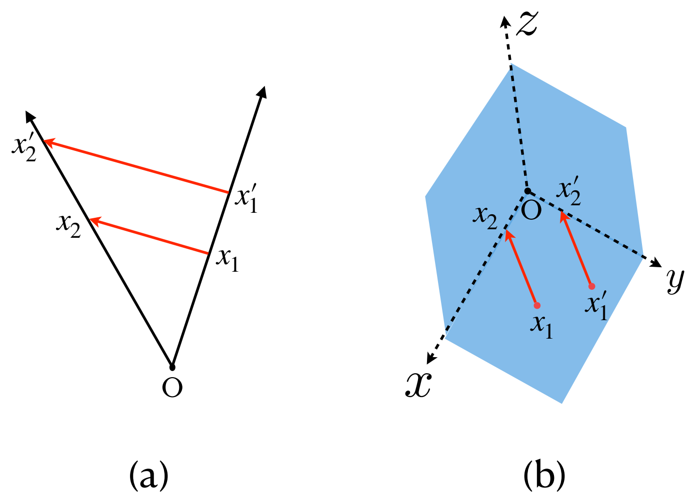

Fig. 2: (a) In $\mathbb{R}^2$, continuous transition with constant rate starting from any point in a facet leads to a unique point in a unique facet. (b) In $\mathbb{R}^3$, even with constant rate, continuous transitions starting from different points in the same facet may end up in different facets.

For the rest of the section, we will assume all paths of the semantics $\mathcal{M}_{\mathcal{H}}$ of a planar PPCD $\mathcal{H}$ start at (q_0, x_0) and $x_0 \in f_0$, where f_0 is a facet in $\cup_{q \in Q} \mathbb{F}(Inv(q))$.

We now define the quotient of a planar PPCD $\mathcal{H}$, which is a finite WDTMC having the same convergence properties as $\mathcal{M}_{\mathcal{H}}$. Here we consider the set of states as $Q \times \cup_{q \in Q} \mathbb{F}(Inv(q))$ instead of $Q \times \mathcal{X}$ and use Lemma 4 to define the probabilistic edges and their weights.

Definition 14 (Quotient of PPCD). *Let $\mathcal{H}$ be a planar PPCD and $\mathcal{M}_{\mathcal{H}}$ be its semantics. We define the WDTMC $\mathcal{H}^{red} = (S^{red}, P^{red}, W^{red})$ as follows,*

- $S^{red} = Q \times \bigcup_{q \in Q} \mathbb{F}(Inv(q))$
- $P^{red}((q_1, f_1), (q_2, f_2)) = P_{\mathcal{H}}((q_1, x_1), (q_2, x_2))$ *for some* $x_1 \in f_1$ *and* $x_2 \in f_2$ *such that* $P_{\mathcal{H}}((q_1, x_1), (q_2, x_2)) > 0$*, and* 0 *otherwise.*
- $W^{red}((q_1, f_1), (q_2, f_2)) = W_{\mathcal{H}}((q_1, x_1), (q_2, x_2))$ *for some* $x_1 \in f_1$ *and* $x_2 \in f_2$ *such that* $P_{\mathcal{H}}((q_1, x_1), (q_2, x_2)) > 0$*, and* 0 *otherwise.*

The above definition is well-defined, that is, the choice of x_1 and x_2 do not matter due to Lemma 4.

We will eventually prove that a planar PPCD $\mathcal{H}$ is absolutely (almost surely) stable if and only if its quotient WDTMC $\mathcal{H}^{red}$ is absolutely (almost surely) convergent. First, let us show that for every infinite path σ of $\mathcal{M}_{\mathcal{H}}$, there is a path π in $\mathcal{H}^{red}$ with same weight and vice versa.

Lemma 5 (Conservation of weight). *For every infinite path σ of $\mathcal{M}_{\mathcal{H}}$, there is a path π in $\mathcal{H}^{red}$ such that $W(\sigma) = W(\pi)$ and vice versa.*

Proof. ($\Rightarrow$) Let $\sigma = \sigma_1, \sigma_2, \ldots$ be an infinite path of $\mathcal{M}_{\mathcal{H}}$. By assumption, $\sigma_i \in f_i$ where $f_i \in \bigcup_{q \in Q} \mathbb{F}(q)$ is a facet, for each $i \in \mathbb{N}$. Suppose for each i, $\sigma_i = (q_i, x_i)$. Since for each i, there is an edge between (q_i, x_i) and (q_{i+1}, x_{i+1}) in $\mathcal{M}_{\mathcal{H}}$, there should be an edge between (q_i, f_i) and (q_{i+1}, f_{i+1}) in $\mathcal{H}^{red}$. Using Lemma 4 we can conclude that for all i, $\mathrm{W}((q_i, x_i), (q_{i+1}, x_{i+1})) = \mathrm{W}((q_i, f_i), (q_{i+1}, f_{i+1}))$. Thus we can construct the infinite path $\pi = ((q_1, f_1), (q_2, f_2), \ldots)$ such that $\mathrm{W}(\pi) = \mathrm{W}(\sigma)$.

($\Leftarrow$) To prove the converse, we show by induction that for any $n \in \mathbb{N}$ if there is a path π of length n in $\mathcal{H}^{red}$ then there is a path σ of length n in $\mathcal{M}_{\mathcal{H}}$ with same weight as π.

Base case: Suppose $((q_1, f_1), (q_2, f_2))$ is an edge of $\mathcal{H}^{red}$. Then there exist $x_1 \in f_1$ and $x_2 \in f_2$ such that $x_2 = x_1 + \mathit{Flow}(q_1) \cdot t$, for some $t \geq 0$, i.e., there is an edge between (q_1, x_1) and (q_2, x_2) in $\mathcal{M}_{\mathcal{H}}$. Also by Lemma 4, $\mathrm{W}((q_1, x_1), (q_2, x_2)) = \mathrm{W}((q_1, f_1), (q_2, f_2))$. Hence base case is proved.

Now suppose $((q_1, f_1), \ldots, (q_n, f_n), (q_{n+1}, f_{n+1}))$ is a path of $\mathcal{H}^{red}$ and by induction hypothesis we have a path $((q_1, x_1), \ldots, (q_n, x_n))$ in $\mathcal{M}_{\mathcal{H}}$ such that $\mathrm{W}((q_1, f_1), \ldots, (q_n, f_n)) = \mathrm{W}((q_1, x_1), \ldots, (q_n, x_n))$. Since there is an edge between (q_n, f_n) and (q_{n+1}, f_{n+1}), there exist $x'_n \in f_n$ and $x'_{n+1} \in f_{n+1}$ such that

$$x'_{n+1} = x'_n + \mathit{Flow}(q_n) \cdot t \qquad (2)$$

for some $t \geq 0$. Since $\mathcal{X} = \mathbb{R}^2$, f_n and f_{n+1} are rays. By Equation 2, there is a straight line of slope $\mathit{Flow}(q_n)$ that intersects both of them. But then any straight line with slope $\mathit{Flow}(q_n)$ intersecting f_n will also intersect f_{n+1}, in fact, if we take the straight line with slope $\mathit{Flow}(q_n)$ passing through x_n, it will intersect f_{n+1}. That means there exists $t \geq 0$ and $x_{n+1} \in f_{n+1}$ such that $x_{n+1} = x_n + \mathit{Flow}(q_n) \cdot t$. This is because for t to be negative, f_n and f_{n+1} must intersect and x_n and x'_n must lie on opposite sides of this intersection point on f_n. But this is impossible since f_n and f_{n+1} intersect only at 0 and both of them get terminated at 0. Thus there exist $x_{n+1} \in f_{n+1}$ such that $((q_n, x_n), (q_{n+1}, x_{n+1}))$ is an edge of $\mathcal{M}_{\mathcal{H}}$. By Lemma 4, $\mathrm{W}((q_n, x_n), (q_{n+1}, x_{n+1})) = \mathrm{W}((q_n, f_n), (q_{n+1}, f_{n+1}))$. Hence our claim is proved for all $n \in \mathbb{N}$, i.e., it holds for infinite paths of $\mathcal{H}^{red}$ as well. $\square$

Using Lemma 5, we now prove the main theorem which states that a PPCD is absolutely (almost surely) stable if and only if its quotient WDTMC is absolutely (almost surely) stable.

Theorem 5. *A planar PPCD $\mathcal{H}$ is absolutely (almost surely) stable iff its quotient WDTMC $\mathcal{H}^{red}$ is absolutely (almost surely) convergent.*

Proof. A PPCD $\mathcal{H}$ is absolutely stable iff $\mathcal{M}_{\mathcal{H}}$ is absolutely convergent (Theorem 4). By Lemma 5, it is easy to observe that every infinite path of $\mathcal{M}_{\mathcal{H}}$ diverge to $-\infty$ if and only if every infinite path of $\mathcal{H}^{red}$ diverge to $-\infty$. Thus, we can conclude that $\mathcal{H}$ is absolutely stable if and only if $\mathcal{H}^{red}$ is absolutely stable.

On the other hand, a PPCD $\mathcal{H}$ is almost surely stable iff $\mathcal{M}_{\mathcal{H}}$ is almost surely convergent (Theorem 4). Let us show that $\mathcal{M}_{\mathcal{H}}$ is almost surely convergent iff $\mathcal{H}^{red}$ is almost surely convergent. Since we have assumed that all paths of $\mathcal{M}_{\mathcal{H}}$ start from (q_0, x_0), all paths of $\mathcal{H}^{red}$ will start from (q_0, f_0), where f_0 is the facet containing x_0. Let us define the equivalence relation $\sim$ on the set of states of the WDTMC $\mathcal{M}_{\mathcal{H}} \sqcup \mathcal{H}^{red}$ as,

$$\begin{aligned}
&(q_i, x_i) \sim (q_j, x_j) \text{ if } q_i = q_j \text{ and } x_i, x_j \text{ belong to the same facet}\\
&(q_i, x_i) \sim (q_j, f_j) \text{ if } q_i = q_j \text{ and } x_i \in f_j\\
&(q_i, f_i) \sim (q_j, f_j) \text{ if } q_i = q_j \text{ and } f_i = f_j,
\end{aligned}$$

where $q_i, q_j \in Q$, $x_i, x_j \in \mathcal{X}$ and $f_i, f_j \in \cup_{q\in Q}\mathbb{F}(\mathit{Inv}(q))$. Note that, the set of equivalence classes of $\sim$ is given by $\{(q, f) \mid q \in Q, f \in \mathbb{F}(\mathit{Inv}(q))\}$. Now by Lemma 4, we can easily deduce that $\sim$ is a probabilistic bisimulation on $\mathcal{M}_{\mathcal{H}} \sqcup \mathcal{H}^{red}$. Observe that, the set

$$\mathit{\Pi} = \{\pi \in \mathit{Paths}(\mathcal{M}_{\mathcal{H}} \sqcup \mathcal{H}^{red}) : \mathrm{W}(\pi[1:\infty]) = -\infty\}$$

is $\sim$ bisimulation-closed. To see this, take any $\pi \in \mathit{\Pi}$ and $\tilde{\pi} \sim \pi$. By Lemma 4, $\mathrm{W}(\pi_i, \pi_{i+1}) = \mathrm{W}(\tilde{\pi}_i, \tilde{\pi}_{i+1})$ for all i. Thus, $\mathrm{W}(\tilde{\pi}[1:\infty]) = -\infty$ as well, i.e., $\tilde{\pi} \in \mathit{\Pi}$. Now, we have $\mathit{Pr}_{(q_0,x_0)}(\mathit{\Pi}) = \mathit{Pr}_{(q_0,f_0)}(\mathit{\Pi})$ as a direct consequence of Lemma 1, i.e.,

$$\begin{aligned}
&\mathit{Pr}\{\sigma \in \mathit{Paths}(\mathcal{M}_{\mathcal{H}}) \mid \sigma_1 = (q_0, x_0) \text{ and } \mathrm{W}(\sigma) \text{ diverges to } -\infty\}\\
=&\mathit{Pr}\{\pi \in \mathit{Paths}(\mathcal{H}^{red}) \mid \pi_1 = (q_0, f_0) \text{ and } \mathrm{W}(\pi) \text{ diverges to } -\infty\}.
\end{aligned}$$

Hence, $\mathit{Pr}\{\sigma \in \mathit{Paths}(\mathcal{M}_{\mathcal{H}}) \mid \sigma_1 = (q_0, x_0) \text{ and } \mathrm{W}(\sigma) \text{ diverges to } -\infty\} = 1$ if and only if $\mathit{Pr}\{\pi \in \mathit{Paths}(\mathcal{H}^{red}) \mid \pi_1 = (q_0, f_0) \text{ and } \mathrm{W}(\pi) \text{ diverges to } -\infty\} = 1$, i.e., $\mathcal{M}_{\mathcal{H}}$ is almost surely convergent iff $\mathcal{H}^{red}$ is almost surely convergent. Thus, $\mathcal{H}$ is almost surely stable iff $\mathcal{H}^{red}$ is almost surely convergent. □

Since $\mathcal{H}^{red}$ is finite, we can use the algorithms developed in section 5.3 to decide its absolute (almost sure) convergence. This in turn decides absolute (almost sure) stability of $\mathcal{H}$ by Theorem 5.

7 Conclusion

In this paper, we showed the decidability of absolute and almost sure convergence of Planar Probabilistic Piecewise Constant Derivative Systems (PPCD), that are a practically useful subclass of stochastic hybrid systems and can model motion of planar robots with faulty actuators. We give a computable characterization of absolute and almost sure convergence through a reduction to a finite state DTMC. In the future, we plan to extend these ideas to analyze higher dimensions PPCD and SHS with more complex dynamics. In particular, the idea of reduction can be applied to higher dimensional PPCD but we will need to extend our analysis to a Markov Decision Process that will appear as the reduced system.

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Appendix

Proof of Proposition 1

Here we provide a detailed proof of Proposition 1 which states that,
For any finite path σ of $\mathcal{M}_W$ there exist a simple path $\sigma_s \in \mathcal{SP}(\mathcal{M}_W)$ and a set of simple cycles $\mathcal{SC}_\sigma \subseteq \mathcal{SC}(\mathcal{M}_W)$ such that $W(\sigma) = W(\sigma_s) + \sum_{\mathcal{C} \in \mathcal{SC}_\sigma} W(\mathcal{C})$.

Proof. We traverse σ and whenever a cycle $\mathcal{C}$ is encountered, remove its edges from σ and add the cycle to the set $\mathcal{SC}_\sigma$. This process is repeated until $\mathcal{SC}_\sigma$ contains only simple cycles and the remaining edges of σ form a simple path $\sigma_s = \sigma - (\cup\{\mathcal{C} \mid \mathcal{C} \in \mathcal{SC}_\sigma\})$. Let $\mathcal{E}^{\sigma_s}$ denote the set of edges of σ_s and for each $\mathcal{C} \in \mathcal{SC}_\sigma$, $\mathcal{E}^{\mathcal{C}}$ denote the set of edges of $\mathcal{C}$. Clearly, $\{\mathcal{E}^{\sigma_s}\} \cup \{\mathcal{E}^{\mathcal{C}} \mid \mathcal{C} \in \mathcal{SC}_\sigma\}$ is a partition of the set of edges of σ. Thus $\mathrm{W}(\sigma) = \mathrm{W}(\sigma_s) + \sum_{\mathcal{C} \in \mathcal{SC}_\sigma} \mathrm{W}(\mathcal{C})$. Hence, our claim is proved. □

Algorithms from Section 5.3

Based on the discussions of section 5.3, we provide pseudocodes for algorithms for checking absolute (almost sure) convergence of a finite (finite, irreducible and aperiodic) WDTMC.

Algorithm 1 Checking absolute convergence of WDTMC

Input: A WDTMC $\mathcal{M}_W \coloneqq (S, \mathrm{P}, \mathrm{W})$
Output: Yes/No

Convert $\mathcal{M}_W$ to a weighted graph $G = (V, E, W')$ where,
$V = S$, $E = \{(s_1, s_2) \in S \times S \mid \mathrm{P}(s_1, s_2) > 0\}$,
and $W' : E \to \mathbb{R}$ defined as $W'(e) \coloneqq -\mathrm{W}(e)$
Run BFS on G to check existence of edge with weight $-\infty$
if (edge with $-\infty$ weight exists) **then**
    Return No
end if
Run Bellman-Ford algorithm on G
if (cycle with negative weight is found) **then**
    Return No
else
    Let $d : V \to \mathbb{R}_{\geq 0}$ define the shortest distance of each $v \in V$ from s_{init}
    Mark in E all edges (u, v) such that $d(v) = d(u) + W'(u, v)$
    Delete from G all unmarked edges
    Run DFS on G (with unmarked edges deleted) to check for a cycle
    if (a cycle is found) **then**
        Return No
    else
        Return Yes
    end if
end if

Algorithm 2 Checking almost sure convergence of WDTMC

Input: A WDTMC $\mathcal{M}_W \coloneqq (S, \mathrm{P}, \mathrm{W})$
Output: Yes/No

```
Convert M_W to a weighted graph G = (V, E, W') where,
  V = S, E = {(s_1, s_2) ∈ S × S | P(s_1, s_2) > 0},
  and W' : E → ℝ defined as W'(e) := W(e)
Run BFS on G to check existence of edge with weight ∞
if (edge with ∞ weight exists) then
    Return No
end if
Calculate stationary distribution ρ* of M_W by solving the set of linear equations,
```

$$\rho^*(s) = \sum_{s' \in S} \rho^*(s')\mathrm{P}(s', s), \quad \forall s \in S$$

$$\sum_{s \in S} \rho^*(s) = 1$$

```
asWeight ← 0
for e ∈ E do
    asWeight = asWeight + P_{ρ*}(e)W'(e)
end for
if asWeight < 0 then
    Return Yes
else
    Return No
end if
```

Proof of Lemma 2

We prove Lemma 2 here which essentially states that,
For any edge $e \in \mathcal{E}$ of a WDTMC $\mathcal{M}_W$,

$$Pr\left\{\sigma \in Paths(\mathcal{M}_W) : \lim_{n\to\infty} \frac{\sum_{j=1}^{n} X_j^e}{n} = P_{\rho^*}(e)\right\} = 1,$$

Proof. Construct the DTMC $\mathcal{M}' = (S', \mathrm{P}')$ from $\mathcal{M}_W$, where $S' = S \cup \mathcal{E}$ and for each $e = (s, s') \in \mathcal{E}$, $(s, e), (e, s') \in \mathcal{E}'$ with $\mathrm{P}'(s, e) = \mathrm{P}(s, s')$ and $\mathrm{P}'(e, s') = 1$ ($\mathcal{E}'$ is the set of edges of $\mathcal{M}'$). Note that, there is a one to one correspondence between $Paths(\mathcal{M}_W)$ and $Paths(\mathcal{M}')$, where each edge $e = (s, s')$ in $\sigma \in Paths(\mathcal{M}_W)$ is replaced by consecutive edges (s, e) and (e, s') in the corresponding path $\sigma' \in Paths(\mathcal{M}')$. Thus, $(\sigma_j, \sigma_{j+1}) = e$ if and only if $\sigma'(2j) = e$, where σ' is the corresponding path of σ. Now, let us define random variables $\{Y_j^x \mid x \in S'; j \in \mathbb{N}\}$ as,

$$\mathrm{Y}_j^x = \begin{cases} 1 \text{ if } \sigma'_j = x \\ 0 \text{ else} \end{cases}$$

for $\sigma' \in Paths(\mathcal{M}')$. Then, it is easy to observe that, $\sum_{j=1}^{n} X_j^e = \sum_{j=1}^{2n} Y_{2j}^e$. Note that, $\mathcal{M}'$ is finite and irreducible. Hence, by strong law of large numbers for any

$x \in S'$ [27],

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{2n} Y_j^x}{2n} = {\rho^*}'(x) \text{ almost surely,}$$

where ${\rho^*}'$ is the stationary distribution of $\mathcal{M}'$. Since for any $x \in S'$,

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{2n} Y_{2j}^x}{2n} = \lim_{n\to\infty} \frac{\sum_{j=1}^{2n} Y_j^x}{2n}$$

$$\text{Thus, } \lim_{n\to\infty} \frac{\sum_{j=1}^{n} X_j^e}{n} = 2\left(\lim_{n\to\infty} \frac{\sum_{j=1}^{2n} Y_{2j}^e}{2n}\right) = 2{\rho^*}'(e) \text{ almost surely.} \tag{3}$$

Consider $\rho : S' \to [0,1]$ as

$$\rho(x) = \begin{cases} \frac{\rho^*(x)}{2} \text{ if } x \in S \\ \frac{\mathrm{P}(x)\rho^*(s)}{2} \text{ if } x = (s,s') \in \mathcal{E}. \end{cases}$$

where ρ^* is the stationary distribution of $\mathcal{M}_W$. Let us observe that, $\sum_{x\in S'} \rho(x) = \sum_{s\in S} \rho^*(s)/2 + \sum_{s\in S}\sum_{(s,s')\in\mathcal{E}} \rho^*(s)\mathrm{P}(s,s')/2 = 1$, i.e., ρ is a probability distribution.

Note that, for any $x \in S$,

$$\begin{aligned}\sum_{x'\in S'} \rho(x')\mathrm{P}'(x',x) &= \sum \{\rho(e)\mathrm{P}'(e,x) : e = (s',x) \in \mathcal{E}\} \\ &= \sum \left\{\frac{\mathrm{P}(e)\rho^*(x)}{2} : e = (s',x) \in \mathcal{E}\right\} \\ &= \frac{\rho^*(x)}{2} = \rho(x).\end{aligned}$$

And for any $x = (s,s') \in \mathcal{E}$,

$$\sum_{x'\in S'} \rho(x')\mathrm{P}'(x',x) = \rho(s)\mathrm{P}'(s,x) = \frac{\rho^*(s)}{2} \cdot \mathrm{P}(x) = \rho(x).$$

Thus, for all $x \in S'$, $\rho(x) = \sum_{x'\in S'} \rho(x')\mathrm{P}'(x',x)$, i.e., ρ is a stationary distribution for $\mathcal{M}'$. Since $\mathcal{M}'$ is finite and irreducible, it has a unique stationary distribution. Thus, $\rho = {\rho^*}'$, which ultimately provides for any $e = (s,s') \in \mathcal{E}$,

$$\begin{aligned}\lim_{n\to\infty} \frac{\sum_{j=1}^{n} X_j^e}{n} &= 2\left(\frac{\mathrm{P}(e)\rho^*(s)}{2}\right) \text{ almost surely} \quad \text{[by Equation 3]} \\ &= \rho^*(s)\mathrm{P}(e) \text{ almost surely} \\ &= P_{\rho^*}(e) \text{ almost surely,}\end{aligned}$$

This proves Lemma 2. $\square$

Proof of Lemma 4

We prove Lemma 4 here which states the following,
Let $e = ((q_1, x_1), (q_2, x_2))$*,* $e' = ((q_1, x'_1), (q_2, x'_2))$ *be two edges of* $\mathcal{M}_{\mathcal{H}}$ *(where* $\mathcal{H}$ *is a planar PPCD) such that,* $P_{\mathcal{H}}(e), P_{\mathcal{H}}(e') > 0$*, and* $x_1, x'_1 \in f_1$ *where* $f_1 \in \bigcup_{q \in Q} \mathbb{F}(Inv(q))$*. Then* $P_{\mathcal{H}}(e) = P_{\mathcal{H}}(e')$ *and* $W_{\mathcal{H}}(e) = W_{\mathcal{H}}(e')$*.*

Proof. Since continuous state space of $\mathcal{H}$ is $\mathbb{R}^2$, there is a unique facet f_2 for f_1 such that $x_2, x'_2 \in f_2$ (assuming $\mathrm{W}_{\mathcal{H}}(e), \mathrm{W}_{\mathcal{H}}(e') \neq \infty$). Now, since $\mathrm{P}_{\mathcal{H}}(e)$ and $\mathrm{P}_{\mathcal{H}}(e')$ depend only on q_1 and f_2, $\mathrm{P}_{\mathcal{H}}(e) = \mathrm{P}_{\mathcal{H}}(e')$. Since any facet is a ray emanating from the origin, it can be depicted by the formula $y = kx$, where $k \in \mathbb{R}$. Let $x_1 = (x_1[1], x_1[2])$ and $x_2 = (x_2[1], x_2[2])$. By property of PPCD, $x_2 = x_1 + Flow(q_1) \cdot T$ for some $T \geq 0$. Thus,

$$(x_2[1], x_2[2]) = (x_1[1], x_1[2]) + (Flow(q_1)[1]), Flow(q_1)[2])T \tag{4}$$

Let $f_1 : y = k_1 x$ and $f_2 : y = k_2 x$. So,

$$x_2[2] = k_2 \cdot x_2[1] \tag{5}$$
$$x_1[2] = k_1 \cdot x_1[1] \tag{6}$$

Using equations 4,5,6 we can write $x_2[1] = c \cdot x_1[1]$ where c depends on k_1, k_2, $Flow(q_1)[1]$ and $Flow(q_1)[2]$. Thus $\frac{\|x_2\|}{\|x_1\|}$ can also be written in terms of k_1, k_2, $Flow(q_1)[1]$ and $Flow(q_1)[2]$ since $\frac{\|x_2\|}{\|x_1\|}$ is equal to either $|x_2[2]|/|x_1[2]|$ or $|x_2[2]|/|x_1[1]|$ or $|x_2[1]|/|x_1[2]|$ or $|x_2[1]|/|x_1[1]|$ and x_1 and x_2 dependent terms on numerator and denomenator always cancel off each other. Same is true for e' as well. Thus, $\mathrm{W}_{\mathcal{H}}(e), \mathrm{W}_{\mathcal{H}}(e')$ depend only on q, f_1 and f_2 and not on the points x_1, x'_1, x_2, x'_2. Hence, they must be equal. $\square$