On Minimally Non-Firm Binary Matrices

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Abstract. For a binary matrix \mathbf{X} , the Boolean rank $br(\mathbf{X})$ is the smallest integer k for which \mathbf{X} equals the Boolean sum of k rank-1 binary matrices, and the isolation number $i(\mathbf{X})$ is the maximum number of 1s no two of which are in a same row, column and a 2×2 submatrix of all 1s. In this paper, we continue Lubiw's study of firm matrices. \mathbf{X} is said to be firm if $i(\mathbf{X}) = br(\mathbf{X})$ and this equality holds for all its submatrices. We show that the stronger concept of superfirmness of \mathbf{X} is equivalent to having no odd holes in the rectangle cover graph of \mathbf{X} , the graph in which $br(\mathbf{X})$ and $i(\mathbf{X})$ translate to the clique cover and the independence number, respectively. A binary matrix is minimally non-firm if it is not firm but all of its proper submatrices are. We introduce two matrix operations that lead to generalised binary matrices and use these operations to derive four infinite classes of minimally non-firm matrices. We hope that our work may pave the way towards a complete characterisation of firm matrices via forbidden submatrices.

Keywords: Boolean rank · Rectangle covering number · Firm matrices

1 Introduction

The Boolean rank of a binary matrix \mathbf{X} , $br(\mathbf{X})$, is the smallest integer k for which \mathbf{X} equals the sum of k rank-1 binary matrices, using Boolean arithmetic in which 1+1=1 holds [10]. A rectangle of \mathbf{X} is a submatrix of all 1s. Note that the support of a rank-1 binary matrix is precisely a rectangle, hence $br(\mathbf{X})$ is the minimum number of rectangles needed to cover $\sup(\mathbf{X}) := \{(i,j) : x_{i,j} = 1\}$.

An isolated set of **X** is a set $S \subseteq \text{supp}(\mathbf{X})$ such that for any distinct (i_1, j_1) , (i_2, j_2) in S, it holds $i_1 \neq i_2$, $j_1 \neq j_2$ and $x_{i_1, j_2} = 0$ or $x_{i_2, j_1} = 0$. The isolation number of **X**, $i(\mathbf{X})$, is the maximum cardinality of an isolated set [8]. In the field of communication complexity, quantities $br(\mathbf{X})$ and $i(\mathbf{X})$ are often referred to as the rectangle covering number and the fooling set bound [11].

In the bipartite graph whose biadjacency matrix is \mathbf{X} , $br(\mathbf{X})$ is the minimum number of bicliques (complete bipartite subgraphs) needed to cover the edge set, while $i(\mathbf{X})$ is the maximum cardinality of a matching in which no two edges are in a 4-cycle. Both $br(\mathbf{X})$ and $i(\mathbf{X})$ are NP-hard to compute for general binary [16,17] and totally balanced matrices as well [14,15].

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For any binary matrix X, it can be readily checked that $i(X) \leq br(X)$. This inequality may however be strict for many matrices. In fact, the complement of the identity matrix shows that the gap between $i(\mathbf{X})$ and $br(\mathbf{X})$ may be arbitrarily large [3]. We say **X** is firm if $i(\mathbf{X}) = br(\mathbf{X})$ and this equality also holds for all its submatrices. The concept of firmness, along with many results that form the basis of this paper were introduced by Lubiw in [13]. A key tool in Lubiw's work is to define the rectangle cover graph of X (the 1's graph in her words) in which $i(\mathbf{X})$ and $br(\mathbf{X})$ translate to the independence and clique cover number, respectively. Lubiw defines X to be superfirm if X's rectangle cover graph is perfect and demonstrates that superfirm matrices are a strict subset of firm matrices. In addition, she shows that covering rectilinear polygons by a minimum number of continuous rectangles is a special case of the rectangle cover problem on binary matrices [13]. In the bipartite setting, firmness is later redefined under the name 'edge-perfection' [15], while superfirmness is investigated under the name 'crossperfection' from a polyhedral perspective [6]. The following important classes of matrices have been shown to be firm. Interval matrices, matrices whose columns can be permuted so the 1s appear consecutively in each row, are proved to be firm by a deep result of Győri [9]. Linear matrices, matrices that have no 2×2 submatrix of 1s, and matrices that can be decomposed into linear matrices via the matrix equivalent of split decomposition on bipartite graphs are shown to be superfirm by Lubiw [13]. The firmness of biadjacency matrices of domino-free bipartite graphs is implied by a result of Amilhastre et al. [1].

In this paper, we start the investigation of minimally non-firm matrices. A binary matrix X is minimally non-firm if i(X) < br(X) and i(X') = br(X') for all proper submatrices X' of X. Our main tool is looking at the problem through the rectangle cover graph. First, we extend a theorem of Lubiw and show that interestingly odd antiholes cannot appear without odd holes in rectangle cover graphs. Then we characterise the necessary and sufficient submatrices for 5-holes to appear. We define simplicial 1s and a procedure for their removal which leads to generalised binary matrices. We introduce the stretching matrix operation which then along with the simplicial 1 removal procedure are used to give a general recipe for the construction of minimally non-firm matrices. We then prove by using this general recipe that four infinite classes of matrices are minimally non-firm. To the best of our knowledge, minimally non-firm matrices have not been studied before. We believe that studying them is a natural approach to better understand firmness, akin to the study of perfect graphs via minimally imperfect graphs. We hope that our results may pave the way towards a complete characterisation of firm and superfirm matrices via forbidden submatrices.

This paper is organised as follows. Section 2 gives a brief recap on the work of Lubiw introducing the concept of rectangle cover graphs, superfirmness and generalised binary matrices. In Section 3, simplicial 1s and the stretching operation are introduced. In Section 4, we show that a matrix is superfirm if and only if it has no odd holes in its rectangle cover graph. In Section 5 we prove our main theorem, which we then use to derive four infinite classes of minimally non-firm binary matrices. We conclude in Section 6 and mention two open problems.

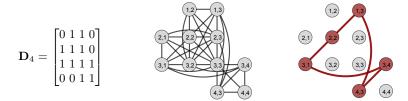


Fig. 1: \mathbf{D}_4 , its rectangle cover graph $\mathcal{G}(\mathbf{D}_4)$ and the 5-hole in $\mathcal{G}(\mathbf{D}_4)$ highlighted

2 Preliminaries

Let $\mathbf{X} \in \{0,1\}^{m \times n}$. For $I \subseteq [n] := \{1,\ldots,n\}$ and $J \subseteq [m]$, a submatrix of \mathbf{X} identified by $I \times J$ is obtained by deleting the rows not in I and the columns not in J. If $I \subseteq [n]$ or $J \subseteq [m]$ then $I \times J$ is a proper submatrix of \mathbf{X} . A submatrix is a rectangle if $I \times J \subseteq \operatorname{supp}(\mathbf{X}) = \{(i,j) : x_{i,j} = 1\}$. As the 1s in a row or column form a rectangle, we have $br(\mathbf{X}) \subseteq \min\{m,n\}$. In addition, note that $br(\mathbf{X})$ is invariant under transposition and under duplicating rows and columns.

For an isolated set S and rectangle $I \times J$, we have $|S \cap (I \times J)| \leq 1$, hence $i(\mathbf{X}) \leq br(\mathbf{X})$. Recall that **X** is firm if $i(\mathbf{X}') = br(\mathbf{X}')$ holds for all submatrices \mathbf{X}' of \mathbf{X} , including \mathbf{X} . The rectangle cover graph $\mathcal{G}(\mathbf{X})$ of \mathbf{X} is the graph on vertex set supp(X), where two vertices are adjacent if they can be covered by a common rectangle of X. We adopt the convention that vertices of $\mathcal{G}(X)$ are drawn in the positions of the corresponding 1s' of X. See Figure 1 for an example of $\mathcal{G}(X)$ for matrix \mathbf{D}_4 . Clearly, the independent sets of $\mathcal{G}(\mathbf{X})$ are just the isolated sets of X. Lubiw shows that maximal cliques of $\mathcal{G}(\mathbf{X})$ are in direct correspondence with maximal rectangles of X [13]. Therefore we have $i(X) = \alpha(\mathcal{G}(X))$ and $br(\mathbf{X}) = \theta(\mathcal{G}(\mathbf{X}))$, where $\alpha(G)$ and $\theta(G)$ denote the independence and clique cover number of a graph G, respectively. A graph G is perfect if $\alpha(H) = \theta(H)$ holds for every induced subgraph H of G. A hole is an induced chordless cycle of length at least four. An *odd hole* is a hole of odd length and an *odd antihole* is the complement of an odd hole. Perfect graphs are exactly those that have no odd holes and no odd antiholes by the Strong Perfect Graph Theorem [4]. X is said to be superfirm if $\mathcal{G}(\mathbf{X})$ is perfect [13]. Superfirm matrices are a strict subset of firm matrices [13], as for instance \mathbf{D}_4 is an interval matrix hence firm by Győri's Theorem [9] but not superfirm as $\mathcal{G}(\mathbf{D}_4)$ contains a 5-hole as shown in Figure 1. Note that this is because not every induced subgraph of $\mathcal{G}(\mathbf{X})$ corresponds to a submatrix of **X** and firmness requires $\alpha(H) = \theta(H)$ to hold for only those subgraphs H of $\mathcal{G}(\mathbf{X})$ where $H = \mathcal{G}(\mathbf{X}')$ for a submatrix \mathbf{X}' of \mathbf{X} .

Replacing a 1 of \mathbf{X} at (i,j) with a 0 does not necessarily correspond to the deletion of vertex (i,j) from $\mathcal{G}(\mathbf{X})$ as edges not incident to (i,j) may get deleted. To represent all induced subgraphs of $\mathcal{G}(\mathbf{X})$ in matrix form, Lubiw introduces a new entry type? which may be part of a rectangle but need not be covered in a feasible covering. A matrix over $\{0,1,?\}$ is called a *generalised binary matrix* [13]. A rectangle of a generalised binary matrix \mathbf{Y} is a submatrix containing no 0s, while an isolated set of \mathbf{Y} is a subset of supp $(\mathbf{Y}) := \{(i,j) : y_{i,j} = 1\}$ in

which no two elements are contained in a common rectangle of \mathbf{Y} . Then $i(\mathbf{Y})$, $br(\mathbf{Y})$ and firmness are analogously defined as for standard binary matrices. For $\mathbf{X} \in \{0,1\}^{m \times n}$ and $P \subseteq \operatorname{supp}(\mathbf{X})$, let \mathbf{X}^P be the generalised binary matrix obtained from \mathbf{X} by replacing all 1s in P by ?s, i.e. $x_{i,j}^P = ?$ for $(i,j) \in P$ and $x_{i,j}^P = x_{i,j}$ otherwise. For \mathbf{X}^P define its rectangle cover graph $\mathcal{G}(\mathbf{X}^P)$ to be the subgraph of $\mathcal{G}(\mathbf{X})$ induced by $\operatorname{supp}(\mathbf{X}) \setminus P$. Superfirmness of \mathbf{X} is then equivalent to the requirement that $i(\mathbf{X}^P) = br(\mathbf{X}^P)$ for all $P \subseteq \operatorname{supp}(\mathbf{X})$ [13].

3 Simplicial 1s and Stretching

Let **Y** be a generalised binary matrix. We say $(\ell, k) \in \text{supp}(\mathbf{Y})$ is a *simplicial* 1 of **Y** if $I \times J$ with $I = \{i : y_{i,k} \in \{1,?\}\}$ and $J = \{j : y_{\ell,j} \in \{1,?\}\}$ satisfies $I \times J \subseteq \{(i,j) : y_{i,j} \in \{1,?\}\}$, that is $I \times J$ is a rectangle of **Y**. Note that $I \times J$ is a maximal rectangle and the only maximal rectangle of **Y** that covers the simplicial 1 at (ℓ, k) . To remove the simplicial 1 at (ℓ, k) of **Y** we delete row ℓ and column k and set all remaining entries that are in $I \times J$ to ?s.

Lemma 1. If \mathbf{Y}' is obtained by removing a simplicial 1 of a generalised binary matrix \mathbf{Y} , then $i(\mathbf{Y}) = i(\mathbf{Y}') + 1$ and $br(\mathbf{Y}) = br(\mathbf{Y}') + 1$.

Proof. Let (ℓ, k) be the simplicial 1 and $I \times J$ its unique maximal rectangle. For a maximum isolated set S' and a minimum rectangle cover \mathcal{R}' of $\mathbf{Y}', S' \cup \{(\ell, k)\}$ and $\mathcal{R}' \cup (I \times J)$ are clearly feasible for \mathbf{Y} . Conversely, if S is a maximum isolated set of \mathbf{Y} , then $S \cap (I \times J) = \{(i, j)\}$ for some $(i, j) \in I \times J$, as otherwise $S \cup \{(\ell, k)\}$ would be a larger isolated set of \mathbf{Y} . So $S \setminus \{(i, j)\}$ is a feasible isolated set of \mathbf{Y}' . As (ℓ, k) is a simplicial 1, we may assume that $I \times J$ is used in a minimum cover \mathcal{R} of \mathbf{Y} . Then $\mathcal{R} \setminus \{I \times J\}$ is a feasible cover of \mathbf{Y}' .

Our definition of simplicial 1s for a standard binary matrix \mathbf{X} is identical to the definition of $bisimplicial\ edges$ [7] in the bipartite graph whose biadjacency matrix is \mathbf{X} . The key difference is how we remove a simplicial 1 and transition into generalised binary matrices.

We have seen that not every induced subgraph of $\mathcal{G}(\mathbf{X})$ corresponds to a submatrix of \mathbf{X} , but by turning 1s to ?s we can consider arbitrary induced subgraphs of $\mathcal{G}(\mathbf{X})$ in matrix form. The idea behind the next matrix operation is to expose induced subgraphs of rectangle cover graphs without explicitly setting matrix entries to ?s. Let $\mathbf{X} \in \{0,1\}^{m \times n}$. By stretching a 1 at $(\ell,k) \in \text{supp}(\mathbf{X})$ we get the $(m+1) \times (n+1)$ binary matrix $\mathcal{S}^{(\ell,k)}(\mathbf{X})$ which satisfies

$$\mathcal{S}^{(\ell,k)}(\mathbf{X})_{i,j} = x_{i,j} \qquad i \in [m], j \in [n], \tag{1}$$

$$S^{(\ell,k)}(\mathbf{X})_{i,j} = 1 \qquad (i,j) \in \{(\ell,n+1), (m+1,k), (m+1,n+1)\},$$
 (2)

and $S^{(\ell,k)}(\mathbf{X})_{i,j} = 0$ otherwise. For instance, if $(m,n) \in \text{supp}(\mathbf{X})$ then by stretching (m,n) we obtain

$$S^{(m,n)}(\mathbf{X}) = \begin{bmatrix} x_{1,1} & \dots & x_{1,n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m-1,n} & 0 & \vdots & \vdots \\ x_{m-1,n} & 1 & 1 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}.$$
 (3)

Stretching (ℓ, k) adds in a simplicial 1 at position (m+1, n+1) whose unique maximal rectangle covers only (ℓ, k) from supp(**X**). By Lemma 1, removing the simplicial 1 at (m+1, n+1), we get

$$i(\mathcal{S}^{(\ell,k)}(\mathbf{X})) = i(\mathbf{X}^{(\ell,k)}) + 1, \quad br(\mathcal{S}^{(\ell,k)}(\mathbf{X})) = br(\mathbf{X}^{(\ell,k)}) + 1, \tag{4}$$

where $\mathbf{X}^{(\ell,k)}$ is a shorter notation for \mathbf{X}^P with $P = \{(\ell,k)\}.$

For a non-empty set $Q \subseteq \operatorname{supp}(\mathbf{X})$, the matrix obtained by stretching each 1 in Q is denoted by $\mathcal{S}^Q(\mathbf{X})$. We adopt the convention to stretch 1s in Q in non-decreasing order of row and then column index, so $\mathcal{S}^Q(\mathbf{X})$ may be written in block form as

$$S^{Q}(\mathbf{X}) = \begin{bmatrix} \mathbf{X} & \mathbf{U} \\ \mathbf{L} & \mathbf{I}_{|Q|} \end{bmatrix}$$
 (5)

where **U** is an $m \times |Q|$ matrix with |Q| 1s exactly one in each column that have non-decreasing row index from left to right, **L** is an $|Q| \times n$ matrix with |Q| 1s exactly one in each row and **I**_t is the $t \times t$ identity matrix.

If (ℓ, k) is a simplicial 1 of **X** then we say that $\mathcal{S}^{(\ell, k)}(\mathbf{X})$ is obtained by *simplicial stretching*. Looking at $\mathcal{G}(\mathbf{X})$ and using Lemma 1 and the Clique Cutset Lemma [2], the following can be proved.

Lemma 2. Let X be superfirm. Then $S^{(\ell,k)}(X)$ is firm. Furthermore, if (ℓ,k) is a simplicial 1 of X, then $S^{(\ell,k)}(X)$ is superfirm.

This lemma is tight in two ways. First, non-simplicial stretching may destroy superfirmness. Second, both simplicial and non-simplicial stretching do not preserve firmness. In Section 5, we will exploit the superfirmness and firmness destroying properties of stretching to create minimally non-firm matrices.

For $n \geq 3$, let $\mathbf{C}_n \in \{0,1\}^{n \times n}$ be the *n*-th *cycle matrix* with exactly two 1s in each row and column such that no proper submatrix has this property. A binary matrix is *totally balanced* if it has no \mathbf{C}_n submatrices for any $n \geq 3$. Totally balanced matrices are exactly those that have a Γ -free ordering [12], where $\Gamma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The following result can be verified by a Γ -free ordering.

Lemma 3. If **X** is totally balanced then so is $S^Q(X)$ for any $Q \subseteq \text{supp}(X)$.

4 Superfirm Matrices and Odd Holes

The Strong Perfect Graph Theorem [4] tells us that a binary matrix \mathbf{X} is superfirm if and only if $\mathcal{G}(\mathbf{X})$ has no odd holes and no odd antiholes. But which are the necessary submatrices so that odd holes or odd antiholes appear in $\mathcal{G}(\mathbf{X})$? In this section, we show that forbidding odd antiholes in $\mathcal{G}(\mathbf{X})$ is unnecessary. Then we study when a 5-hole in $\mathcal{G}(\mathbf{X})$ exists.

A theorem of Lubiw in [13] states that for $\mathcal{G}(\mathbf{X})$ to have an odd antihole of size 7 or more, \mathbf{X} needs to have the 3×3 cycle matrix \mathbf{C}_3 as a submatrix. Note that \mathbf{C}_3 is superfirm. Let $\mathbf{1}$ be the all 1s column vector of appropriate size and define $\mathbf{W} := \begin{bmatrix} \mathbf{C}_3 & \mathbf{1} \\ \mathbf{1}^\top & 1 \end{bmatrix}$ and $\bar{\mathbf{I}}_4 := \begin{bmatrix} \mathbf{C}_3 & \mathbf{1} \\ \mathbf{1}^\top & 0 \end{bmatrix}$ in $\{0,1\}^{4\times 4}$. Considering a slight extension of Lubiw's proof, we show that these two larger matrices are necessary for the appearance of odd antiholes.







Fig. 2: The three 5-holes in $\mathcal{G}(\mathbf{H}_3)$

Lemma 4. If G(X) contains an odd antihole of size 7 or more then X has W or $\overline{\mathbf{I}}_4$ as a submatrix.

Proof. Following the proof structure of [13, Theorem 6.3], suppose that \mathbf{X} has no such submatrices but $\mathcal{G}(\mathbf{X})$ contains an antihole $A \subseteq \operatorname{supp}(\mathbf{X})$ of odd size $k = |A| \geq 7$. By duplicating rows and columns of \mathbf{X} , we may assume that no two 1s in A are in the same row or column. Note that row and column duplication cannot introduce \mathbf{W} or $\bar{\mathbf{I}}_4$ submatrices into \mathbf{X} . Then the submatrix \mathbf{X}' of \mathbf{X} that consists of the rows and columns of the 1s in A is of dimension $k \times k$ and may be permuted so that the vertices of A appear on the main diagonal and are non-adjacent to the two vertices that are directly above and below them. Then \mathbf{X}' has the form as below where each undecided entry pair (i,j),(j,i) denoted by *s satisfies $|\operatorname{supp}(\mathbf{X}') \cap \{(i,j),(j,i)\}| \leq 1$ so that A is indeed an antihole in $\mathcal{G}(\mathbf{X})$.

Assume without loss of generality that $x'_{1,2} = 0$. Suppose that $x'_{2,3} = 0$. If $x'_{5,6} = x'_{6,5} = 0$ then the submatrix $I \times J$ of \mathbf{X}' with $I = \{1, 2, 5, 6\}$, $J = \{2, 3, 5, 6\}$ is $\overline{\mathbf{I}}_4$. Moreover, if $x'_{5,6} + x'_{6,5} = 1$ then $I \times J$ is \mathbf{W} . Hence, $x'_{2,3} \neq 0$. In general, exactly one of (i, j) and (i+1, j+1) can be a 0 for all *s, so the zeros of \mathbf{X}' must zigzag as shown in the right of Equation (6). But as k is odd, this is impossible.

The importance of Lemma 4 over Lubiw's theorem, is that both \mathbf{W} and $\overline{\mathbf{I}}_4$ contain the submatrix $\mathbf{H}_3 := [\mathbf{1}, \mathbf{C}_3]$ and $\mathcal{G}(\mathbf{H}_3)$ contains three 5-holes as shown in Figure 2, whereas \mathbf{C}_3 is superfirm. This shows that a rectangle cover graph cannot contain an odd antihole of size 7 or larger if it does not contain an odd hole. Recalling that a 5-antihole is just a 5-hole, we obtain the following result.

Theorem 1. X is superfirm if and only if G(X) has no odd holes.

Theorem 1 motivates us to study when $\mathcal{G}(\mathbf{X})$ has odd holes. We initialise this by characterising when a 5-hole exists in $\mathcal{G}(\mathbf{X})$. The proof is skipped but it is of similar nature to that of Lemma 4. Let $\mathbf{K}_5 \in \{0,1\}^{5\times 5}$ be the circulant matrix with exactly three 1s per row and column and recall \mathbf{D}_4 from Figure 1.

Theorem 2. $\mathcal{G}(X)$ contains a 5-hole if and only if X has at least one of D_4 , H_3 , H_3^{\top} or K_5 as a submatrix.

5 Four Infinite Classes of Minimally Non-Firm Matrices

In this section we prove a theorem which shows how minimally non-firm matrices may arise by using the stretching operation. Then using this theorem we show that four infinite classes of matrices are minimally non-firm.

Recall that a standard binary matrix \mathbf{X} is minimally non-firm (mnf) if it is not firm but all proper submatrices of it are. This definition naturally extends to generalised binary matrices \mathbf{Y} , \mathbf{Y} is mnf if $i(\mathbf{Y}) < br(\mathbf{Y})$ and $i(\mathbf{Y}') = br(\mathbf{Y}')$ for all proper submatrices \mathbf{Y}' of \mathbf{Y} . Note that as $br(\mathbf{Y})$ and $i(\mathbf{Y})$ are invariant under transposition, the transpose of any mnf matrix is mnf as well. The following two simple results apply to both standard and generalised mnf matrices.

Lemma 5. Each row and column of an mnf matrix has at least two non-zeros.

Proof. Suppose **Y** is mnf and its *i*-th row only has a single nonzero at entry (i, j). If $y_{i,j} = ?$ then row *i* can clearly be dropped without changing $i(\mathbf{Y})$ or $br(\mathbf{Y})$. If $y_{i,j} = 1$ then (i, j) is a simplicial 1. By Lemma 1, removing it we obtain a firm submatrix \mathbf{Y}' with $i(\mathbf{Y}) - 1 = i(\mathbf{Y}') = br(\mathbf{Y}') = br(\mathbf{Y}) - 1$, a contradiction. \square

Lemma 6. If **Y** is mnf then $i(\mathbf{Y}) = br(\mathbf{Y}) - 1$.

Proof. Let **Y** be mnf. By deleting a single row or column of **Y** we get a submatrix \mathbf{Y}' which by definition is firm and satisfies $i(\mathbf{Y}) - 1 \le i(\mathbf{Y}') \le i(\mathbf{Y})$ and $br(\mathbf{Y}) - 1 \le br(\mathbf{Y}') \le br(\mathbf{Y})$ as a row or column forms a rectangle and may contain at most one element of an isolated set. So, we must have $br(\mathbf{Y}') = br(\mathbf{Y}) - 1$ as otherwise **Y** is firm. But then $i(\mathbf{Y}') = br(\mathbf{Y}') = br(\mathbf{Y}) - 1 \le i(\mathbf{Y})$ which together with $i(\mathbf{Y}) < br(\mathbf{Y})$ implies $br(\mathbf{Y}) - 1 = i(\mathbf{Y})$.

By Theorem 1, \mathbf{X} is superfirm if $\mathcal{G}(\mathbf{X})$ has no odd holes, so for \mathbf{X} to be mnf $\mathcal{G}(\mathbf{X})$ must contain odd holes. Using Theorem 2 one can show that the smallest mnf standard binary matrices are of dimension 4×4 and there are exactly two of them: $\overline{\mathbf{I}}_4$ and $\overline{\mathbf{I}}_4'$, where $\overline{\mathbf{I}}_4'$ is obtained from $\overline{\mathbf{I}}_4$ by turning a single 1 to a 0 (for instance at (1,4), but due to symmetry any other 1 would work).

Let **X** be a standard binary matrix with an odd hole C in $\mathcal{G}(\mathbf{X})$ of size |C| = 2k + 1. Stretching all 1s at $Q = \text{supp}(\mathbf{X}) \setminus C$ of **X**, by Lemma 1 we get

$$i(\mathcal{S}^{Q}(\mathbf{X})) - |Q| = i(\mathbf{X}^{Q}) = k < k + 1 = br(\mathbf{X}^{Q}) = br(\mathcal{S}^{Q}(\mathbf{X})) - |Q|,$$
 (7)

so $\mathcal{S}^Q(\mathbf{X})$ is non-firm. This recipe however, does not guarantee that $\mathcal{S}^Q(\mathbf{X})$ is *minimally* non-firm. By adding extra conditions on Q, minimality can be enforced.

Theorem 3. Let $X \in \{0,1\}^{m \times n}$. If X^Q is a minimally non-firm generalised binary matrix for some non-empty $Q \subset \operatorname{supp}(X)$ and X^P is firm for all $P \subsetneq Q$, then $S^Q(X) \in \{0,1\}^{(m+|Q|) \times (n+|Q|)}$ is minimally non-firm.

Proof. $S^Q(\mathbf{X})$ may be written as a block matrix with four blocks $\mathbf{X}, \mathbf{L}, \mathbf{U}$ and $\mathbf{I}_{|Q|}$ as in Equation (5). By construction all 1s in block $\mathbf{I}_{|Q|}$ are simplicial, hence

removing them we obtain the mnf generalised binary matrix \mathbf{X}^Q . By Lemma 1 then $i(\mathcal{S}^Q(\mathbf{X})) = i(\mathbf{X}^Q) + |Q| < br(\mathbf{X}^Q) + |Q| = br(\mathcal{S}^Q(\mathbf{X}))$.

Suppose that not all proper submatrices of $S^Q(\mathbf{X})$ are firm and let \mathbf{Y} be the smallest non-firm proper submatrix indexed by $I \times J$. Then Y is mnf. Note that the four block matrices of $S^Q(\mathbf{X})$ are all firm: (1) \mathbf{X} is firm as it is just \mathbf{X}^{\emptyset} . (2) $\mathbf{I}_{|O|}$ is clearly firm. (3) U has exactly one 1 per column, so it can be obtained from an identity matrix by duplicating columns and adding zero rows, and thus firm. (4) Similarly, as L has exactly one 1 per row, it is firm. Hence Y cannot be fully contained in any of the four blocks. As Y is a mnf standard binary matrix it has at least two 1s in each row and column by Lemma 5. Since block $[L I_{|Q|}]$ has exactly two 1s in each row, if Y has a row from this block, then Y must also contain the columns of both 1s in this row. Similarly, if Y contains a column from block $\begin{bmatrix} \mathbf{I}_{[O]} \end{bmatrix}$, it must contain the rows of both 1s in this column. Therefore, the rows in I from block $\begin{bmatrix} \mathbf{L} \ \mathbf{I}_{|Q|} \end{bmatrix}$ and the columns in J from block $\begin{bmatrix} \mathbf{U} \\ \mathbf{I}_{|Q|} \end{bmatrix}$ come in pairs and may be identified with their 1 in block $I_{|Q|}$. Let P be the subset of Q whose stretching created the 1s in block $I_{|Q|}$ which are in Y. Removing all |P| simplicial 1s present in Y from block $\mathbf{I}_{|Q|}$ we obtain a generalised binary matrix which is fully contained in block X and is just a submatrix Z of X^P . By Lemma 1, **Z** satisfies $i(\mathbf{Z}) + |P| = i(\mathbf{Y})$ and $br(\mathbf{Z}) + |P| = br(\mathbf{Y})$. If P = Q, then I contains all the rows and columns from block $\mathbf{I}_{|Q|}$ so \mathbf{Z} must be a proper submatrix of \mathbf{X}^Q , hence firm. If $P \neq Q$, then **Z** is a submatrix of the firm matrix \mathbf{X}^P . In both cases $i(\mathbf{Z}) = br(\mathbf{Z})$ which implies $i(\mathbf{Y}) = br(\mathbf{Y})$, a contradiction. \square

One can see that a partial converse of the above theorem also holds, i.e. if a standard binary mnf matrix has some simplicial 1s then by removing those we obtain a generalised binary mnf matrix for which the theorem's conditions hold. Note however, that not all mnf matrices have simplicial 1s, e.g. $\overline{\mathbf{I}}_4$, hence certainly not all mnf matrices arise via Theorem 3.

Recall \mathbf{C}_n is the $n \times n$ cycle matrix. For $n \geq 3$, let $\mathbf{M}_{n+1} := \mathcal{S}^{(n,n)}(\mathbf{C}_n)$ be the $(n+1) \times (n+1)$ matrix and $\mathbf{H}_n := [\mathbf{1}, \mathbf{C}_n]$ be the $n \times (n+1)$ matrix,

$$\mathbf{M}_{n} = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & \ddots & \ddots & & \\ 1 & & & 1 & 1 \\ 1 & & & & 1 & 1 \end{bmatrix}, \qquad \mathbf{H}_{n} = \begin{bmatrix} 1 & 1 & 1 & & & \\ 1 & 1 & 1 & & & \\ \vdots & \ddots & \ddots & \ddots & \\ \frac{1}{1} & 1 & & & & 1 \end{bmatrix}. \tag{8}$$

Matrices \mathbf{M}_n appear in the work of Lubiw [13] as forbidden submatrices for a subset of superfirm matrices that can be decomposed into linear matrices by applying the matrix equivalent of split decomposition [5] on bipartite graphs.

Recall matrix \mathbf{D}_4 from Figure 1 and for $n \geq 5$, let $\mathbf{D}_n := \mathcal{S}^{(3,n-1)}(\mathbf{D}_{n-1})$. In addition, let $\mathbf{T}_5 \in \{0,1\}^{5 \times 5}$ as below and for $n \geq 6$ define $\mathbf{T}_n := \mathcal{S}^{(4,n-1)}(\mathbf{T}_{n-1})$,

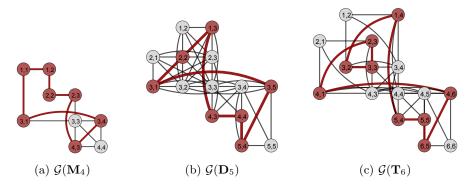


Fig. 3: Odd holes highlighted in the rectangle cover graphs of M_4 , D_5 and T_6

All these matrices contain odd holes in their rectangle cover graph as shown in Figure 2 for \mathbf{H}_3 and Figure 3 for $\mathbf{M}_4, \mathbf{D}_5$ and \mathbf{T}_6 . In the remaining parts of this section, we will prove that by choosing the set Q in Theorem 3 to be $\{(n,n)\}$ for \mathbf{M}_n , $G_n = \{(n,2), (n,n+1)\}$ for \mathbf{H}_n , $Q_n = \{(1,2), (2,1), (n,n)\}$ for \mathbf{D}_n and \mathbf{T}_n , the conditions of Theorem 3 are satisfied and thus we get our main theorem.

Theorem 4. For $n \geq 4$, $S^{(n,n)}(M_n)$, $S^{G_{n-1}}(H_{n-1})$, $S^{Q_n}(D_n)$ and $S^{Q_{n+1}}(T_{n+1})$ are mnf standard binary matrices. In addition, $S^{Q_n}(D_n)$ and $S^{Q_{n+1}}(T_{n+1})$ are totally balanced.

The claim of total balancedness is immediate by Lemma 3 as D_4 is an interval matrix and T_5 is Γ -free [12]. Lubiw observed that $\mathcal{S}^{Q_4}(\mathbf{D}_4)$ and $\mathcal{S}^{Q_5}(\mathbf{D}_5)$ are non-firm [13]. Her observation served as a motivation to us to define the stretching operation and matrices \mathbf{D}_n .

For the first two classes, \mathbf{M}_n and \mathbf{H}_n , the proofs that Theorem 3's conditions hold are similar because both are minimally non-superfirm. A standard binary matrix is minimally non-superfirm (mnsf) if it is not superfirm but all proper submatrices of it are. Next, we show that the conditions hold for the class \mathbf{H}_n .

Lemma 7. For $n \geq 3$, \mathbf{H}_n^P is firm for all $P \subsetneq G_n = \{(n,2), (n,n+1)\}$ and $\mathbf{H}_n^{G_n}$ is a mnf generalised binary matrix. In addition, \mathbf{H}_n is mnsf.

Proof. For $P \subseteq G_n$, at least n rectangles are needed to cover \mathbf{H}_n^P as

$$C_n := \operatorname{supp}(\mathbf{H}_n) \setminus (\{(i,1) : i \in [n-1]\} \cup G_n), \tag{10}$$

is a 2n-1-hole in $\mathcal{G}(\mathbf{H}_n^P)$. As \mathbf{H}_n^P only has n rows, $br(\mathbf{H}_n^P)=n$. Note that submatrix $[1,n-1]\times[2,n+1]$ (where $[\ell,k]:=\{\ell,\ell+1,\ldots,k\}$) has two isolated sets of size n-1. For $P \subsetneq G_n$, $(i,j) \in G_n \setminus P$ may be added to one of these two isolated sets to get an isolated set of size n for \mathbf{H}_n^P . For $\mathbf{H}_n^{G_n}$ however, none of the 1s can be added to these two isolated sets, so we only have $i(\mathbf{H}_n^{G_n}) \geq n-1$. Suppose $\mathbf{H}_n^{G_n}$ has an isolated set T_n of size n. Then T_n needs to contain a 1 from each row, so $(n,1) \in T_n$. But then T_n cannot contain (1,2) and (n-1, n+1), the only 1s in columns 2 and n+1, as they are in a rectangle with (n, 1). Hence T_n has n elements from n-1 distinct columns, which is impossible.

 $\mathcal{G}(\mathbf{H}_n)$ has no odd antiholes of size 7 or more by Lemma 4 but it contains n 2n-1-holes, one of which is \mathcal{C}_n . Any other hole in $\mathcal{G}(\mathbf{H}_n)$ is either contained in the submatrix \mathbf{C}_n and hence it is the 2n-hole, or contains at most two vertices from column 1. Note that if $(\ell, 1)$ is a vertex of a hole then the hole cannot have another vertex from row ℓ . If a hole contains a single vertex from column 1 then it is easy to see that it must be one of the n 2n-1-holes. If the hole has two vertices from column 1, then it must contain an even number of vertices from submatrix \mathbf{C}_n , so it is an even hole. Therefore, the n 2n-1-holes are the only odd holes in $\mathcal{G}(\mathbf{H}_n)$ which all have a vertex from every row and column. For $P \subseteq G_n$, $\mathcal{G}(\mathbf{Y})$ for any proper submatrix \mathbf{Y} of \mathbf{H}_n^P then has no odd holes and no odd antiholes, so $\mathcal{G}(\mathbf{Y})$ is perfect by the Strong Perfect Graph Theorem [4]. \square

Observe that $\mathcal{G}(\mathbf{M}_n)$ contains a single odd-hole of size 2n-1 as shown in Figure 3a for m=4. To prove that conditions of Theorem 3 are satisfied by class \mathbf{M}_n , the same structure of proof as for \mathbf{H}_n may be applied to get the following.

Lemma 8. For $n \geq 4$, M_n is firm and mnsf, and $M_n^{(n,n)}$ is a mnf generalised binary matrix.

Although \mathbf{D}_4 and \mathbf{T}_5 are mnsf, for larger n as both \mathbf{D}_n and \mathbf{T}_n are defined recursively, they have proper submatrices which are not superfirm. Hence the argument used in the proof of the previous two classes does not work for \mathbf{D}_n and \mathbf{T}_n . Next we prove that class \mathbf{D}_n satisfies the conditions of Theorem 3.

Lemma 9. For $n \geq 4$, \mathbf{D}_n^P is firm for all $P \subsetneq Q_n = \{(1,2),(2,1),(n,n)\}$ and $\mathbf{D}_n^{Q_n}$ is a mnf generalised binary matrix. In addition, \mathbf{D}_4 is mnsf.

Proof. I. For all $P \subseteq Q_n$, $\mathcal{G}(\mathbf{D}_n^P)$ contains the 2n-3-hole

$$C_n = \{(3,1), (2,2), (1,3), (4,3), \dots, (3,n)\},\tag{11}$$

and thus $br(\mathbf{D}_n^P) \ge n-1$. On the other hand, \mathbf{D}_n has a feasible cover using n-1 rectangles in which each row $i \ne 3$ is covered by a distinct rectangle.

II. In $\mathcal{G}(\mathbf{D}_n)$, each $(i,j) \in Q_n$ is adjacent to two consecutive vertices of \mathcal{C}_n , and not adjacent to the others. For $P \subsetneq Q_n$, let $(\ell,k) \in Q_n \setminus P$ and S_n be an independent set of \mathcal{C}_n of size n-2 which does not use the two vertices of \mathcal{C}_n that are adjacent to (ℓ,k) . Then $S_n \cup \{(\ell,k)\}$ is a feasible isolated set of \mathbf{D}_n^P .

For $\mathbf{D}_n^{Q_n}$, S_n is a feasible isolated set. Suppose that $\mathbf{D}_n^{Q_n}$ has an isolated set T_n of size n-1. Then as $\mathbf{D}_n^{Q_n}$ is of size $n\times n$, there is exactly one row and one column that does not have a 1 in T_n . Since columns 1 and n each have a single 1 which are both in row 3, exactly one of these 1s must be in T_n . (a) Suppose that $(3,1)\in T_n$. Then $(3,j)\not\in T_n$ for any $j\neq 1$. Observe that (2,2) can also not be in T_n as it is adjacent to (3,1). But column 2 only has the 1s at (2,2) and (3,2), so T_n contains no 1s from column 2 and n and it has n-1 isolated 1s from n-2 columns, which is a contradiction. (b) Suppose that $(3,n)\in T_n$. Then $(3,j)\not\in T_n$ for any $j\neq n$. As $(3,2)\not\in T_n$, we must have the only available

1 at (2,2) from column 2 in T_n . But then as (2,2) is in a rectangle with (1,3), we cannot have (1,3) in T_n . As (1,3) is the only 1 in row 1, T_n has no 1s from row 1. But T_n can also not have any 1s from row n, as row n only has a 1 at (n,n-1) which is adjacent to $(3,n) \in T_n$. Hence T_n has n-1 isolated 1s from n-2 rows, which is impossible. Therefore, $i(\mathbf{D}_n^{Q_n}) = n-2$.

III. We use induction on n. For the base case take n=4 and observe that $\mathcal{G}(\mathbf{D}_4)$ has the 5-hole \mathcal{C}_4 as an only odd hole and \mathcal{C}_4 contains a vertex from each row and column of \mathbf{D}_4 . Therefore, any proper submatrix of \mathbf{D}_4^P is superfirm for any $P\subseteq Q_4$. Assume that for k< n, all proper submatrices of $\mathbf{D}_k^{P'}$ are firm for any $P'\subseteq Q_k$. Let $P\subseteq Q_n$, and suppose that not every proper submatrix of \mathbf{D}_n^P is firm and let \mathbf{Y} be a smallest non-firm proper submatrix indexed by $I\times J$. Note that we have $n\in I$ or $n\in J$, as otherwise \mathbf{Y} is a submatrix of $\mathbf{D}_k^{P'}$ for some k< n and $P'\subseteq \{(1,2),(2,1)\}$ and firm by either the induction hypothesis or by parts \mathbf{I} . and \mathbf{II} . of this proof as $P'\subseteq Q_k$. By the minimality of \mathbf{Y} it must be mnf. So \mathbf{Y} has at least two non-zero entries in each row and column by Lemma 5. Hence $n\in I$ implies $n-1, n\in J$ and $n\in J$ implies n=1. Thus we must have $n\in I$ and $n\in I$ and $n\in I$ implies n=1, $n\in I$ and $n\in I$ in the $n\in I$ and $n\in I$ and $n\in I$ in the $n\in I$ in the $n\in I$ in the $n\in I$ and $n\in I$ and $n\in I$ and $n\in I$ in the $n\in I$ and $n\in I$ and $n\in I$ and $n\in I$ in the $n\in I$ in the $n\in I$ and $n\in I$ and if $n\in I$ then $n\in I$ then $n\in I$ and if $n\in I$ then $n\in I$ and if $n\in I$ then $n\in I$ and if $n\in I$ then $n\in I$ then $n\in I$ and if $n\in I$ then $n\in I$ then $n\in I$ and if $n\in I$ then $n\in I$ then $n\in I$ and if $n\in I$ then $n\in I$ and if $n\in I$ then $n\in I$ then $n\in I$ and $n\in I$ then $n\in I$ t

If I = [n], then by the above we must have $J = [n] \setminus \{1\}$. Then $(3,2) \cup S_n$ and $\{\{1,2,3\} \times \{2,3\}\} \cup \mathcal{R}_n$ with $S_n := \{(i,i-1) : i \in [4,n] := \{4,\ldots,n\}\}$ and $\mathcal{R}_n := \{\{3,i\} \times \{i-1,i\} : i \in [4,n]\}$ give a feasible isolated set and rectangle cover of size n-2 of \mathbf{Y} , hence we cannot have I = [n].

So let ℓ be the largest row index of \mathbf{D}_n^P for which $\ell \notin I$. (a) If $\ell = 1$, then $I = [n] \setminus \{1\}$. Then $[4, n] \subset I$ implies $[3, n] \subseteq J$, and $2 \in I$ implies that column 1 or 2 are in J, so let $k \in J \cap \{1, 2\}$. Then $(3, k) \cup S_n$ and $\{\{2, 3\} \times (J \cap \{1, 2, 3\})\} \cup \mathcal{R}_n$ give a feasible isolated set and rectangle cover of size n - 2 of \mathbf{Y} , so $\ell \neq 1$.

- (b) If $\ell=2$, then we have $1 \not\in J$. If $1 \in I$, then $2,3 \in J$ must hold, so we have $I=[n]\setminus\{2\}$ and J=[2,n]. Then $(3,2)\cup S_n$ and $\{\{1,3\}\times\{2,3\}\}\cup \mathcal{R}_n$ give a feasible isolated set and rectangle cover of size n-2 of \mathbf{Y} . If $1 \not\in I$, then $2 \not\in J$, so we have I=[3,n] and J=[3,n]. Then S_n and \mathcal{R} give a feasible isolated set and rectangle cover of size n-3 of \mathbf{Y} .
- (c) If $\ell > 3$, then $(\ell + 1, \ell)$ is a simplicial 1 of **Y** and its unique maximal rectangle is $\{3, \ell + 1\} \times \{\ell, \ell + 1\}$. Remove this simplicial 1 at $(\ell + 1, \ell)$. But then $(\ell + 2, \ell + 1)$ becomes a simplicial 1, so it can also be removed. We may repeat this process until at last (n, n 1) becomes a simplicial 1 and can be removed. Once (n, n 1) is removed, column n only consist of 0s and a single?, hence can be dropped. Let the resulting matrix be **Y**'. As dropping a column which does not have any 1s does not impact the isolation number and Boolean rank, by Lemma 1 **Y**' satisfies $i(\mathbf{Y}') + n \ell = i(\mathbf{Y})$ and $br(\mathbf{Y}') + n \ell = br(\mathbf{Y})$. But then **Y**' is just a proper submatrix of **Y** formed by rows $(I \cap [\ell 1]) \times (J \cap [\ell 1])$, so firm. Hence $i(\mathbf{Y}) = br(\mathbf{Y})$ which contradicts **Y** being mnf.

A proof which is very similar to the above may be applied to class \mathbf{T}_n to get our final lemma below and by this completing the proof of Theorem 4.

Lemma 10. For $n \geq 5$, T_n^P is firm for all $P \subsetneq Q_n = \{(1,2),(2,1),(n,n)\}$ and $T_n^{Q_n}$ is a mnf generalised binary matrix. In addition, T_5 is mnsf.

6 Conclusion

In this paper, we studied firm and superfirm binary matrices. We showed that superfirmness is equivalent to having no odd holes in the rectangle cover graph. Then we presented four infinite classes of minimally non-firm binary matrices.

We close with two future research directions. We suspect that every minimally non-superfirm matrix is firm and any minimally non-firm matrix $\mathbf{X} \in \{0,1\}^{m \times n}$ satisfies $|m-n| \leq 1$.

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