# On Higher-Order Reachability Games vs May Reachability 

Kazuyuki Asada ${ }^{1}$, Hiroyuki Katsura ${ }^{2}$, and Naoki Kobayashi ${ }^{2}$<br>${ }^{1}$ Tohoku University<br>${ }^{2}$ The University of Tokyo

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#### Abstract

We consider the reachability problem for higher-order functional programs and study the relationship between reachability games (i.e., the reachability problem for programs with angelic and demonic nondeterminism) and may-reachability (i.e., the reachability problem for programs with only angelic nondeterminism). We show that reachability games for order- $n$ programs can be reduced to may-reachability problems for order$(n+1)$ programs, and vice versa. We formalize the reductions by using higher-order fixpoint logic and prove their correctness. We also discuss applications of the reductions to higher-order program verification.


## 1 Introduction

This paper considers the reachability problem for simply-typed, call-by-name higher-order functional programs with integers, recursion, and two kinds of non-deterministic branches (angelic and demonic ones). The problem of solving reachability games (hereafter, simply called the reachability game problem) asks, given a higher-order functional program and a specific control point succ of the program, whether there exists a sequence of choices on angelic non-determinism that makes the program reach succ no matter what choices are made on demonic non-determinism. Thus, our reachability game problem is just a special case of the notion of two-player reachability games [7], where the game arena is specified as a higher-order functional program. (An important restriction compared to the general notion of reachability games is that each vertex may have only a finite number of outgoing edges, although there can be infinitely many vertices.) Various program verification problems can be reduced to the reachability game problem. For example, the termination problem, which asks whether a given program terminates for any sequence of non-deterministic choices, is a special case of the reachability game problem, where all the non-deterministic branches are demonic, and all the termination points are expressed by succ. The safety verification problem, which asks whether a given program may fall into an error state after some sequence of non-deterministic choices, is also a special case, where all the non-deterministic branches are angelic, and error states are expressed by succ.

We establish relations between the reachability game problem and the may-reachability problem, a special case of the reachability game problem where all the non-deterministic
choices are angelic (hence, may-reachability is a one-player game). We show mutual translations between the reachability game problem for order- $n$ programs and the may-reachability problem for order- $(n+1)$ programs. (Here, the order of a program is defined as the typetheoretic order; the order of a function that takes only integers is 0 , and the order of a function that takes an order-0 function is 1 , etc.) The translations are size-preserving in the sense that for any order- $n$ program $M$, one can effectively construct an order- $(n+1)$ program $M^{\prime}$ such that the answer to the reachability game problem for $M$ is the same as the answer to the may-reachability problem for $M^{\prime}$, and the size of $M^{\prime}$ is polynomial in that of $M$; and vice versa.

The translation from reachability games to may-reachability allows us to use higher-order program verification tools specialized to may-reachability (or, unreachability to error states) such as MoCHi [14] and Liquid types [20] to check a wider class of properties represented as reachability games. Conversely, the translation from may-reachability to reachability games allows us, for example, to use verification tools that can solve reachability games for order-0 programs, such as CHC solvers [17, 8, 4] to check may-reachability of order-1 programs.

We formalize our translations for $\mu \mathrm{HFL}(\mathrm{Z})$, which is a fragment $\operatorname{HFL}(Z)$ [16] without greatest fixpoint operators and modal operators, where $\operatorname{HFL}(Z)$ is an extension of Viswanathan and Viswanathan's higher-order fixpoint logic [24] with integers. The use of higher-order fixpoint formulas rather than higher-order programs in the formalization of the translations is justified by the result of Kobayashi et al. [16, 25], that there is a direct correspondence between the reachability problem for higher-order programs and the validity problem for the corresponding higher-order fixpoint formulas, where angelic and demonic branches in programs correspond to disjunctions and conjunctions respectively.

The rest of this paper is structured as follows. Section 2 introduces $\mu \mathrm{HFL}(\mathrm{Z})$, and clarifies the relationship between the validity checking problem for $\mu \mathrm{HFL}(\mathrm{Z})$ and the reachability problem for higher-order programs. Section 3 formalizes a reduction from the order- $n$ reachability game problem to the order- $(n+1)$ may-reachability problem (as a translation of $\mu \mathrm{HFL}(\mathrm{Z})$ formulas), and proves its correctness. Section 4 formalizes a reduction in the opposite direction, from the order- $(n+1)$ may-reachability problem to the order- $n$ reachability game problem, and proves its correctness. Section 5 discusses applications of the reductions and reports some experimental results. Section 6 discusses related work and Section 7 concludes the paper.

## $2 \mu \mathrm{HFL}(\mathrm{Z})$ and Reachability Problems

In this section, we first introduce $\mu \mathrm{HFL}(\mathrm{Z})$, a fragment of higher-order fixpoint logic HFL(Z) [16] (which is in turn an extension of Viswanathan and Viswanathan's higher-order fixpoint logic [24 with integers) without greatest fixpoint operators. We then review the relationship between $\mu \mathrm{HFL}(\mathrm{Z})$ and reachability problems, and state the main theorem of this paper.

### 2.1 Syntax

The set of (simple) types, ranged over by $\kappa$, is given by:

$$
\begin{aligned}
& \kappa \text { (types) }::=\text { Int } \mid \tau \\
& \tau \text { (predicate types) }::=\star \mid \kappa \rightarrow \tau .
\end{aligned}
$$

$$
\begin{equation*}
\frac{\mathcal{K}, x: \kappa \vdash_{\mathrm{ST}} \varphi: \tau}{\mathcal{K} \vdash_{\mathrm{ST}} \lambda x^{\kappa} \cdot \varphi: \kappa \rightarrow \tau} \quad \quad \quad(\mathrm{T}-\mathrm{ABS}) \tag{T-Mult}
\end{equation*}
$$

Figure 1: Simple Type System for $\mu \mathrm{HFL}(\mathrm{Z})$

For a type $\kappa$, the order and arity of $\kappa$, written $\operatorname{ord}(\kappa)$ and $\operatorname{ar}(\kappa)$ respectively, are defined by:

$$
\begin{aligned}
& \operatorname{ord}(\text { Int })=-1 \quad \operatorname{ord}(\star)=0 \\
& \operatorname{ord}(\kappa \rightarrow \tau)=\max (\operatorname{ord}(\tau), \operatorname{ord}(\kappa)+1) \\
& \operatorname{ar}(\operatorname{Int})=\operatorname{ar}(\star)=0 \quad \operatorname{ar}(\kappa \rightarrow \tau)=\operatorname{ar}(\tau)+1
\end{aligned}
$$

For example, ord $(\operatorname{Int} \rightarrow$ Int $\rightarrow \star)=0$ and $\operatorname{ord}((\operatorname{Int} \rightarrow \star) \rightarrow \star)=1$.
The set of $\mu H F L(Z)$ formulas, ranged over by $\varphi$, is given by:

$$
\begin{aligned}
& \varphi \text { (formulas) }::= \\
& \quad x\left|\varphi_{1} \vee \varphi_{2}\right| \varphi_{1} \wedge \varphi_{2} \\
& \mid \mu x^{\tau} \cdot \varphi \quad \text { (least fixpoint operator) } \\
& \left|\varphi_{1} \varphi_{2}\right| \lambda x^{\kappa} \cdot \varphi \quad \text { ( } \lambda \text {-abstractions and applications) } \\
& |\varphi e| e_{1} \leq e_{2} \quad \text { (extension with integers) } \\
& e \text { (integer expressions) }::=n|x| e_{1}+e_{2} \mid e_{1} \times e_{2} .
\end{aligned}
$$

Intuitively, $\mu x^{\tau} . \varphi$ denotes the least predicate $x$ of type $\tau$ such that $x=\varphi$. We write true and false for $0 \leq 0$ and $1 \leq 0$. For a formula $\varphi$, the order of $\varphi$ is defined as:

$$
\max \left(\{0\} \cup\left\{\operatorname{ord}(\tau) \mid \mu^{\tau} x . \varphi^{\prime} \text { occurs in } \varphi\right\}\right) .
$$

We call a $\mu \mathrm{HFL}(\mathrm{Z})$ formula $\varphi$ disjunctive if the conjunction $\wedge$ occurs in $\varphi$ only in the form of $e_{1} \leq e_{2} \wedge \varphi_{1}$ (i.e., the left-hand side of $\varphi$ is a primitive constraint on integers).

We write $\widetilde{\varphi}_{j, \ldots, k}$ for a sequence of formulas $\varphi_{j}, \ldots, \varphi_{k}$; it denotes an empty sequence if $k<j$. We often omit the subscript and just write $\widetilde{\varphi}$ for $\widetilde{\varphi}_{j, \ldots, k}$ when the subscript is not important. Similarly, we also write $\widetilde{e}$ and $\widetilde{\kappa}$ for sequences of expressions and types respectively. We use the metavariables $\alpha, \beta$, and $\gamma$ to denote either a formula or an integer expression.

The simple type system for $\mu \mathrm{HFL}(\mathrm{Z})$ formulas is defined in Figure 1 Henceforth, we consider only well-typed formulas (i.e., formulas $\varphi$ such that $\mathcal{K} \vdash_{\text {ST }} \varphi: \kappa$ for some $\mathcal{K}$ and $\kappa$ ). A formula $\varphi$ is called a closed formula of type $\kappa$ if $\emptyset \vdash_{\text {ST }} \varphi: \kappa$.

[^0]\[

$$
\begin{align*}
& \overline{\mathcal{K}, x: \kappa \vdash_{\mathrm{ST}} x: \kappa} \quad(\mathrm{T}-\mathrm{VAR}) \quad \frac{\mathcal{K} \vdash_{\mathrm{ST}} \varphi: \operatorname{Int} \rightarrow \tau \quad \mathcal{K} \vdash_{\mathrm{ST}} e: \text { Int }}{\mathcal{K} \vdash_{\mathrm{ST}} \varphi e: \tau} \\
& \frac{\mathcal{K} \vdash_{\mathrm{ST}} \varphi_{1}: \star \quad \mathcal{K} \vdash_{\mathrm{ST}} \varphi_{2}: \star}{\mathcal{K} \vdash_{\text {ST }} \varphi_{1} \vee \varphi_{2}: \star}  \tag{T-OR}\\
& \frac{\mathcal{K} \vdash_{\text {ST }} \varphi_{1}: \star \mathcal{K} \vdash_{\text {ST }} \varphi_{2}: \star}{\mathcal{K} \vdash_{\text {ST }} \varphi_{1} \wedge \varphi_{2}: \star} \\
& \frac{\mathcal{K}, x: \tau \vdash_{\mathrm{ST}} \varphi: \tau}{\mathcal{K} \vdash_{\mathrm{ST}} \mu x^{\tau} \cdot \varphi: \tau} \\
& \frac{\mathcal{K} \vdash_{\mathrm{ST}} \varphi_{1}: \tau_{2} \rightarrow \tau \quad \mathcal{K} \vdash_{\mathrm{ST}} \varphi_{2}: \tau_{2}}{\mathcal{K} \vdash_{\text {ST }} \varphi_{1} \varphi_{2}: \tau}(\mathrm{T}-\mathrm{APP}) \\
& \text { (T-And) } \\
& \begin{array}{cr}
\frac{\mathcal{K} \vdash_{\text {ST }} e_{1}: \text { Int } \quad \mathcal{K} \vdash_{\text {ST }} e_{2}: \text { Int }}{\mathcal{K} \vdash_{\mathrm{ST}} e_{1} \leq e_{2}: \star} & (\mathrm{T}-\mathrm{LE}) \\
\frac{\mathcal{K} \vdash_{\mathrm{ST}} n: \text { Int }}{} & (\mathrm{T}-\mathrm{INT}) \\
\frac{\mathcal{K} \vdash_{\mathrm{ST}} e_{1}: \text { Int } \mathcal{K} \vdash_{\mathrm{ST}} e_{2}: \text { Int }}{\mathcal{K} \vdash_{\mathrm{ST}} e_{1}+e_{2}: \text { Int }} & \text { (T-PLUS) }
\end{array} \\
& \frac{\mathcal{K} \vdash_{\text {ST }} e_{1}: \text { Int } \mathcal{K} \vdash_{\text {ST }} e_{2}: \text { Int }}{\mathcal{K} \vdash_{\text {ST }} e_{1} \times e_{2}: \text { Int }}
\end{align*}
$$
\]

### 2.2 Semantics

For each simple type $\kappa$, we define the partially ordered set $\llbracket \kappa \rrbracket=\left((\Omega), \sqsubseteq_{\kappa}\right)$ where $\sqsubseteq_{\kappa} \subseteq$ $(\kappa) \times(\kappa)$ by:

$$
\begin{aligned}
& (\text { Int })=\mathbf{Z} \quad m \sqsubseteq_{\text {Int }} n \Leftrightarrow m=n \\
& (\star \mid)=\{\perp, \top\} \quad x \sqsubseteq_{\star} y \Leftrightarrow x=\perp \vee y=\top \\
& (\kappa \rightarrow \tau \mid= \\
& \left\{f \in(\kappa \kappa) \rightarrow(\tau \mid) \mid \forall x, y \in(|\kappa|) \cdot x \sqsubseteq_{\kappa} y \Rightarrow f(x) \sqsubseteq_{\tau} f(y)\right\} \\
& f \sqsubseteq_{\kappa \rightarrow \tau} g \Leftrightarrow \forall x \in(\kappa) \cdot f(x) \sqsubseteq_{\tau} g(y) .
\end{aligned}
$$

Here, $\mathbf{Z}$ denotes the set of integers. For each $\tau, \llbracket \tau \rrbracket$ (but not $\llbracket$ Int $\rrbracket)$ forms a complete lattice. We write $\perp_{\tau}\left(T_{\tau}\right)$ for the least (greatest, resp.) element of $\llbracket \tau \rrbracket$, and $\sqcap_{\tau}$ ( $\sqcup_{\tau}$, resp.) for the greatest lower bound (least upper bound, resp.) operation with respect to $\sqsubseteq_{\tau}$. We also define the least fixpoint operator $\mathbf{L F P}_{\tau} \in((\tau \rightarrow \tau) \rightarrow \tau)$ by:

$$
\left.\mathbf{L F P}_{\tau}(f)=\sqcap_{\tau}\{g \in 0 \tau) \mid f(g) \sqsubseteq_{\tau} g\right\}
$$

For a simple type environment $\mathcal{K}$, we write $(\mathcal{K} \$ for the set of maps $\rho$ such that $\operatorname{dom}(\rho)=$ $\operatorname{dom}(\mathcal{K})$ and $\rho(x) \in \ \mathcal{K}(x) \downarrow$ for each $x \in \operatorname{dom}(\rho)$.

For each valid type judgment $\mathcal{K} \vdash_{\text {ST }} \varphi: \kappa$, its semantics $\llbracket \mathcal{K} \vdash_{\text {ST }} \varphi: \kappa \rrbracket \in(\mathcal{K} D \rightarrow(\kappa)$ is defined by:

$$
\begin{aligned}
& \llbracket \Gamma, x: \kappa \vdash_{\text {ST }} x: \kappa \rrbracket(\rho)=\rho(x) \\
& \llbracket \Gamma \vdash_{\text {ST }} \varphi_{1} \vee \varphi_{2}: \star \rrbracket \rho=\llbracket \Gamma \vdash_{\text {ST }} \varphi_{1}: \star \rrbracket \rho \sqcup_{\star} \llbracket \Gamma \vdash_{\text {ST }} \varphi_{2}: \star \rrbracket \rho \\
& \llbracket \Gamma \vdash_{\text {ST }} \varphi_{1} \wedge \varphi_{2}: \star \rrbracket \rho=\llbracket \Gamma \vdash_{\text {ST }} \varphi_{1}: \star \rrbracket \rho \sqcap_{\star} \llbracket \Gamma \vdash_{\text {ST }} \varphi_{2}: \star \rrbracket \rho \\
& \llbracket \Gamma \vdash_{\text {ST }} \mu x^{\tau} \cdot \varphi: \tau \rrbracket \rho=\mathbf{L F P}_{\tau}\left(\lambda v \in\left(\tau \rrbracket \cdot \llbracket \Gamma, x: \tau \vdash_{\text {ST }} \varphi: \tau \rrbracket(\rho\{x \mapsto v\})\right)\right. \\
& \llbracket \Gamma \vdash_{\text {ST }} \lambda x^{\kappa} \cdot \varphi: \tau \rrbracket \rho=\lambda w \in(\kappa) \cdot \llbracket \Gamma, x: \kappa \vdash_{\text {ST }} \varphi: \tau \rrbracket(\rho\{x \mapsto w\}) \\
& \llbracket \Gamma \vdash_{\text {ST }} \varphi_{1} \varphi_{2}: \tau \rrbracket \rho=\llbracket \Gamma \vdash_{\text {ST }} \varphi_{1}: \tau_{2} \rightarrow \tau \rrbracket \rho\left(\llbracket \Gamma \vdash_{\text {ST }} \varphi_{2}: \tau_{2} \rrbracket \rho\right) \\
& \llbracket \Gamma \vdash_{\mathrm{ST}} \varphi e: \tau \rrbracket \rho=\llbracket \Gamma \vdash_{\mathrm{ST}} \varphi: \text { Int } \rightarrow \tau \rrbracket \rho\left(\llbracket \Gamma \vdash_{\mathrm{ST}} e: \text { Int } \rrbracket \rho\right) \\
& \llbracket \Gamma \vdash_{\mathrm{ST}} e_{1} \leq e_{2}: \star \rrbracket \rho= \begin{cases}\top & \text { if } \llbracket \Gamma \vdash_{\mathrm{ST}} e_{1}: \operatorname{Int} \rrbracket \rho \leq \llbracket \Gamma \vdash_{\mathrm{ST}} e_{2}: \operatorname{Int} \rrbracket \rho \\
\perp & \text { otherwise }\end{cases} \\
& \llbracket \Gamma \vdash_{\text {ST }} n: \text { Int } \rrbracket \rho=n \\
& \llbracket \Gamma \vdash_{\text {ST }} e_{1}+e_{2}: \text { Int } \rrbracket \rho=\llbracket \Gamma \vdash_{\text {ST }} e_{1}: \text { Int } \rrbracket \rho+\llbracket \Gamma \vdash_{\text {ST }} e_{2}: \text { Int } \rrbracket \rho \\
& \llbracket \Gamma \vdash_{\text {ST }} e_{1} \times e_{2}: \text { Int } \rrbracket \rho=\llbracket \Gamma \vdash_{\text {ST }} e_{1}: \text { Int } \rrbracket \rho \times \llbracket \Gamma \vdash_{\text {ST }} e_{2}: \text { Int } \rrbracket \rho
\end{aligned}
$$

For a closed formula $\varphi$ of type $\star$, we just write $\llbracket \varphi \rrbracket$ for $\llbracket \emptyset \vdash_{\text {ST }} \varphi: \star \rrbracket$. The validity checking problem for $\mu \mathrm{HFL}(\mathrm{Z})$ is the problem of deciding whether $\llbracket \varphi \rrbracket=T$, given a closed $\mu \mathrm{HFL}(\mathrm{Z})$ formula $\varphi$ of type $\star$.

For closed formulas, the following alternative semantics is sometimes convenient. Let us define the reduction relation $\varphi \longrightarrow \varphi^{\prime}$ by the following rules.

$$
\begin{gathered}
\frac{i \in\{1,2\}}{E\left[\varphi_{1} \vee \varphi_{2}\right] \longrightarrow E\left[\varphi_{i}\right]} \\
\\
E[\text { false } \wedge \varphi] \longrightarrow E[\mathrm{false}]
\end{gathered}
$$

$$
\begin{gathered}
\overline{E[\text { true } \wedge \varphi] \longrightarrow E[\varphi]} \\
\frac{E[\mu x . \varphi] \longrightarrow E[[\mu x . \varphi / x] \varphi]}{} \\
\frac{E[(\lambda x . \varphi) e] \longrightarrow E[[e / x] \varphi]}{} \\
\frac{E[(\lambda x . \varphi) \psi] \longrightarrow E[[\psi / x] \varphi]}{}= \begin{cases}\text { true } & \text { if } \llbracket \vdash_{\text {ST }} e_{1}: \text { Int } \rrbracket \leq \llbracket \vdash_{\text {ST }} e_{2}: \text { Int } \rrbracket \\
\text { false } & \text { otherwise }\end{cases} \\
E\left[e_{1} \leq e_{2}\right] \longrightarrow E[b]
\end{gathered}
$$

Here, $E$ denotes an evaluation context, defined by:

$$
E::=[]|E \wedge \varphi| E \varphi .
$$

We write $\longrightarrow^{*}$ for the reflexive and transitive closure of $\longrightarrow$. We have the following fact (see, e.g., [22]).

Fact 1. Suppose $\vdash_{\mathrm{ST}} \varphi: \star$. Then, $\llbracket \varphi \rrbracket=\top$ if and only if $\varphi \longrightarrow^{*}$ true.
Example 1. Suppose $\vdash_{\text {ST }} \varphi: \operatorname{Int} \rightarrow \star$. Then,

$$
\psi:=\left(\mu x^{\operatorname{Int} \rightarrow \star} \cdot \lambda y \cdot \varphi y \vee \varphi(-y) \vee \varphi(y+1)\right) 0 \longrightarrow^{*} \text { true }
$$

just if $\varphi n \longrightarrow^{*}$ true for some $n$. Thus, the formula $\psi$ represents $\exists z . \varphi z$.
The example above indicates that existential quantifiers on integers can be expressed in $\mu \mathrm{HFL}(\mathrm{Z})$. Henceforth, we treat existential quantifiers as if they were primitives.

### 2.3 Relationship with Reachability Problems

We consider reachability problems for a call-by-name, simply-typed $\lambda$-calculus extended with two kinds of non-determinism ( $\square$ and $\square$ ) and a special term succ, which represents that the designated target has been reached 2 The sets of types and terms, ranged over by $\sigma$ and $M$ respectively, are defined by:

$$
\begin{aligned}
\sigma & ::=\text { Int } \mid \eta \\
\eta & ::=\text { unit } \mid \sigma \rightarrow \eta \\
M & ::=() \mid \text { succ }|x| \lambda x . M\left|M_{1} M_{2}\right| M e \\
& \left|\operatorname{fix}^{\eta}(x, M)\right| M_{1} \square M_{2}\left|M_{1} \square M_{2}\right| \text { assume }\left(e_{1} \leq e_{2}\right) ; M .
\end{aligned}
$$

Here, $\mathrm{fix}^{\eta}(x, M)$ denotes a recursive function $x$ of type $\eta$ such that $x=M$. The term $M_{1} \square M_{2}$ denotes a demonic choice between $M_{1}$ and $M_{2}$, where the choice is up to the environment (or, the opponent O of the reachability game), and $M_{1} \square M_{2}$ denotes an angelic choice between $M_{1}$ and $M_{2}$, where the choice is up to the term (or, the player P of the reachability game). The

[^1]term assume $\left(e_{1} \leq e_{2}\right)$; $M$ first checks whether $e_{1} \leq e_{2}$ holds and if so, proceeds to evaluate $M$; otherwise aborts the evaluation of the whole term. Using assume, we can express a conditional expression if $e_{1} \leq e_{2}$ then $M_{1}$ else $M_{2}$ as (assume $\left.\left(e_{1} \leq e_{2}\right) ; M_{1}\right) \square\left(\right.$ assume $\left(e_{2}+\right.$ $\left.1 \leq e_{1}\right) ; M_{2}$ ). Henceforth, we consider only terms well-typed in the simple type system (which is standard, hence omitted), where () and succ are given type unit.

The order of a type $\sigma$ is defined by:

$$
\begin{aligned}
& \operatorname{ord}(\text { Int })=-1 \quad \operatorname{ord}(\text { unit })=0 \\
& \operatorname{ord}(\sigma \rightarrow \eta)=\max (\operatorname{ord}(\eta), \operatorname{ord}(\sigma)+1)
\end{aligned}
$$

The order of a term $M$ is defined as the largest order of type $\eta$ such that fix ${ }^{\eta}(x, M)$ occurs in $M$. We write Int $^{n} \rightarrow \star$ for $\underbrace{\text { Int } \rightarrow \cdots \text { Int }}_{n} \rightarrow \star$.

For a closed simply-typed term $M$ of type unit, a play is a (possibly infinite) sequence of reductions of $M$. The play is won by the player P if it ends with succ; otherwise the play is won by the opponent 0 . The reachability game for $M$ is the problem of deciding which player ( P or 0 ) has a winning strategy. For the general notion of reachability games and strategies, we refer the reader to [7]. As a special case of the translation of Watanabe et al. 25] from temporal properties of programs to $\operatorname{HFL}(\mathrm{Z})$ formulas, we obtain the following translation $(\cdot)^{\dagger}$ from reachability games to $\mu \mathrm{HFL}(\mathrm{Z})$ formulas.

$$
\begin{aligned}
& ()^{\dagger}=\text { false } \quad \text { succ }^{\dagger}=\text { true } \quad x^{\dagger}=x \\
& (\lambda x \cdot M)^{\dagger}=\lambda x \cdot M^{\dagger} \quad\left(M_{1} M_{2}\right)^{\dagger}=M_{1}^{\dagger} M_{2}^{\dagger} \quad(M e)^{\dagger}=M^{\dagger} e \\
& (\operatorname{fix}(x, M))^{\dagger}=\mu x . M^{\dagger} \quad\left(\operatorname{assume}\left(e_{1} \leq e_{2}\right) ; M\right)^{\dagger}=e_{1} \leq e_{2} \wedge M^{\dagger} \\
& \left(M_{1} \square M_{2}\right)^{\dagger}=M_{1}^{\dagger} \wedge M_{2}^{\dagger} \quad\left(M_{1} \square M_{2}\right)^{\dagger}=M_{1}^{\dagger} \vee M_{2}^{\dagger} .
\end{aligned}
$$

The following is a special case of the result of Watanabe et al. 25.
Theorem 2 ([25). For any closed simply-typed term $M$ of type unit and order $k, \llbracket M \rrbracket$ is a closed $\mu H F L(Z)$ formula of type $\star$ and order $k$. The player P wins the reachability game for $M$, if and only if, $\llbracket M \rrbracket=T$.

Based on the result above, we focus on the validity checking problem for $\mu \mathrm{HFL}(\mathrm{Z})$ formulas, instead of directly discussing the reachability problem. Note that the may-reachability problem (of asking whether, given a closed term $M$ of which all the branches are angelic, there exists a reduction sequence from $M$ to succ) corresponds to the validity checking problem for disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$ formulas.

Example 2. Let us consider the following OCaml program.

```
let rec sum x k =
    assert(x>=0);
    if x=0 then k 0 else sum(x-1)(fun y-> k(x+y))
in sum n (fun r -> assert(r>=n))
```

Suppose we are interested in checking whether the program suffers from an assertion failure. It is modelled as the reachability problem for the term $M_{\text {sum }} n(\lambda r \cdot \operatorname{assume}(r<n)$; succ $)$, where $M_{\text {sum }}$ is:

$$
\begin{gathered}
\text { fix }(\text { sum, } \lambda x \cdot \lambda k .(\text { assume }(x<0) ; \text { succ }) \\
\square(\operatorname{assume}(x=0) ; k 0)
\end{gathered}
$$

$\square(\operatorname{assume}(x>0) ; \operatorname{sum}(x-1)(\lambda y \cdot k(x+y))))$.
Here, note that an assertion failure is modelled as succ in our language. By Theorem 2] the above term is reachable to succ just if the (disjunctive) $\mu \mathrm{HFL}(\mathrm{Z})$ formula $\varphi_{\text {ex1 } 1}:=\varphi_{\text {sum }} n(\lambda r . r<n)$ is valid, where $\varphi_{\text {sum }}$ is:

$$
\begin{aligned}
& \mu \text { sum. } \lambda x \cdot \lambda k \text {. } \\
& x<0 \vee(x=0 \wedge k 0) \vee(x>0 \wedge \operatorname{sum}(x-1)(\lambda y \cdot k(x+y))) .
\end{aligned}
$$

The formula $\varphi_{e x 1}$ is valid only if $n<0$, which implies that the OCaml program suffers from an assertion failure just if $\mathrm{n}<0$.

### 2.4 Main Theorem

The main theorem of this paper is stated as follows.
Theorem 3. There exist polynomial-time translations (.)\# and (.) between order-n $\mu H F L(Z)$ formulas and order- $(n+1)$ disjunctive $\mu H F L(Z)$ formulas that satisfy the following properties.

- For any order-n closed $\mu H F L(Z)$ formula $\varphi, \varphi^{\#}$ is an order- $(n+1)$ closed disjunctive formula such that $\llbracket \varphi \rrbracket=\llbracket \varphi^{\#} \rrbracket$.
- For any order- $(n+1)$ closed disjunctive $\mu H F L(Z)$ formula $\varphi, \varphi^{b}$ is an order-( $n$ ) closed formula such that $\llbracket \varphi \rrbracket=\llbracket \varphi^{b} \rrbracket$.

Due to the connection between reachability problems and $\mu \mathrm{HFL}(\mathrm{Z})$ validity checking problems discussed in Section [2.3, the theorem above implies that any order- $n$ reachability game can be converted in polynomial time to order- $(n+1)$ may-reachability problem, and vice versa. The result allows us to use a tool for checking the may-reachability of higher-order programs (such as MoCHi [14) to solve the reachability game, and conversely, to use a tool for solving the order- $n$ reachability game (such as $\nu \mathrm{HFL}(\mathrm{Z})$ validity checkers [9, 10] and a HoCHC solver [3]) to check the may-reachability of order- $(n+1)$ programs; see Section 5 for more discussion on the applications.

## 3 From Order- $n$ Reachability Games to Order- $(n+1)$ May-Reachability

In this section, we show the translation $(\cdot)^{\#}$ from order- $n \mu \mathrm{HFL}(\mathrm{Z})$ formulas to order- $(n+1)$ disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$ formulas.

The idea is to transform each proposition $\varphi$ (i.e. a formula of type $\star$ ) to a predicate $\varphi^{\#^{\prime}}$ of type $\star \rightarrow \star$, so that true and false are respectively converted to the identity function $\lambda x . x$ and the constant function $\lambda x$.false. We can then encode the conjunction $\varphi_{1} \wedge \varphi_{2}$ as $\lambda x^{\star} \cdot \varphi_{1}^{\#^{\prime}}\left(\varphi_{2}^{\#^{\prime}} x\right)$, which is equivalent to the identity function just if both $\varphi_{1}^{\#^{\prime}}$ and $\varphi_{2}^{\#^{\prime}}$ are.

We first define the translation of types by:

$$
\begin{aligned}
& \star{ }^{\#}=\star \rightarrow \star \quad \text { Int }{ }^{\#}=\text { Int } \\
& (\kappa \rightarrow \tau)^{\#}=\kappa^{\#} \rightarrow \tau^{\#} .
\end{aligned}
$$

We extend it to type environments by:

$$
\left(x_{1}: \kappa_{1}, \ldots, x_{k}: \kappa_{k}\right)^{\#}=x_{1}: \kappa_{1}^{\#}, \ldots, x_{k}: \kappa_{k}^{\#} .
$$

The translation $(\cdot)^{\#}$ of formulas is defined as follows.

$$
\begin{aligned}
& \varphi^{\#}=\varphi^{\#^{\prime}} \text { true } \\
& \left(e_{1} \leq e_{2}\right)^{\#^{\prime}}=\lambda x^{\star} \cdot\left(e_{1} \leq e_{2} \wedge x\right) \\
& \left(\lambda x^{\kappa} \cdot M\right)^{\#^{\prime}}=\lambda x^{\kappa^{\#}} \cdot M^{\#^{\prime}} \quad\left(\varphi_{1} \varphi_{2}\right)^{\#^{\prime}}=\varphi_{1}^{\#^{\prime}} \varphi_{2}^{\#^{\prime}} \quad(\varphi e)^{\#^{\prime}}=\varphi^{\#^{\prime}} e \\
& \left(\mu x^{\tau} \cdot \varphi\right)^{\#^{\prime}}=\mu x^{\tau^{\#}} \cdot \varphi^{\#^{\prime}} \\
& \left(\varphi_{1} \vee \varphi_{2}\right)^{\#^{\prime}}=\lambda x^{\star} \cdot \varphi_{1}^{\#^{\prime}} x \vee \varphi_{2}^{\#^{\prime}} x \quad\left(\varphi_{1} \wedge \varphi_{2}\right)^{\#^{\prime}}=\lambda x^{\star} \cdot \varphi_{1}^{\#^{\prime}}\left(\varphi_{2}^{\#^{\prime}} x\right) .
\end{aligned}
$$

Example 3. Consider the formula $\varphi:=\left(\mu p^{\operatorname{Int} \rightarrow \star} \cdot \lambda y \cdot y=0 \vee(p(y-1) \wedge p(y+1))\right) n$ (where $n$ is an integer constant). We obtain the following formula as $\varphi^{\#}$ :

$$
\begin{aligned}
&\left(\mu p^{\text {Int } \rightarrow \star \rightarrow \star} \cdot \lambda y \cdot \lambda x^{\star} \cdot\left(\lambda x^{\star} \cdot y=0 \wedge x\right) x\right. \\
&\left.\vee\left(\lambda x^{\star} \cdot p(y-1)(p(y+1) x)\right) x\right) n \text { true. }
\end{aligned}
$$

By simplifying the formula with $\beta$-reductions, we obtain:

$$
\begin{aligned}
& \left(\mu p^{\text {Int } \rightarrow \star \rightarrow \star} \cdot \lambda y \cdot \lambda x^{\star}\right. \\
& \quad(y=0 \wedge x) \vee p(y-1)(p(y+1) x)) n \text { true. }
\end{aligned}
$$

The following lemma guarantees that the translation preserves typing.
Lemma 4. If $\mathcal{K} \vdash_{\text {ST }} \varphi: \kappa$, then $\mathcal{K} \# \vdash_{\text {ST }} \varphi^{\#^{\prime}}: \kappa^{\#}$.
Proof. Straightforward induction on the derivation of $\mathcal{K} \vdash_{\text {ST }} \varphi: \kappa$.
Corollary 5. If $\varphi$ is an order-n closed $\mu H F L(Z)$ formula, then $\varphi^{\#}$ is an order- $(n+1)$ closed disjunctive $\mu H F L(Z)$ formula.

Proof. Suppose $\varphi$ is an order- $n$ closed $\mu \mathrm{HFL}(\mathrm{Z})$ formula. By Lemma 4 we have $\emptyset \vdash_{\text {ST }} \varphi^{\#^{\prime}}$ : $\star \rightarrow \star$, which implies $\emptyset \vdash_{\text {ST }} \varphi^{\#}: \star$. Since each $\mu$-formula $\mu x^{\tau} . \varphi^{\prime}$ in $\varphi$ is translated to $\mu x^{\tau^{\#}} . \varphi^{\prime}$ and $\operatorname{ord}\left(\tau^{\#}\right)=\operatorname{ord}(\tau)+1, \varphi^{\#}$ is an order- $(n+1)$ formula. Furthermore, all the conjunctions in $\varphi^{\#}$ are of the form $e_{1} \leq e_{2} \wedge \psi$; hence it is disjunctive.

The following theorem states the correctness of the translation.
Theorem 6. If $\emptyset \vdash_{\text {ST }} \varphi: \star$, then $\llbracket \varphi \rrbracket=\llbracket \varphi^{\#} \rrbracket$.
To prove the theorem, we define the relation $\sim_{\kappa} \subseteq \llbracket \kappa \rrbracket \times \llbracket \kappa^{\#} \rrbracket$ between the values of the source and the target of the translation, by induction on $\kappa$.

$$
\begin{aligned}
& \sim_{\text {Int }}=\{(n, n) \mid n \in \llbracket \mathbf{Z} \rrbracket\} \\
& \sim_{\star}=\{(\perp, \lambda x \in \llbracket \star \rrbracket \cdot \perp)\} \cup\{(\top, \lambda x \in \llbracket \star \rrbracket \cdot x)\} \\
& \sim_{\kappa \rightarrow \tau}= \\
& \quad\left\{(f, g) \mid \forall(v, w) \in \llbracket \kappa \rrbracket \times \llbracket \tau \rrbracket \cdot v \sim_{\kappa} w \Rightarrow f v \sim_{\tau} g w\right\} .
\end{aligned}
$$

We extend $\sim_{\kappa}$ pointwise to the relation $\sim_{\mathcal{K}} \subseteq \llbracket \mathcal{K} \rrbracket \times \llbracket \mathcal{K} \# \rrbracket$ on environments by:

$$
\rho \sim_{\mathcal{K}} \rho^{\prime} \Leftrightarrow \rho(x) \sim_{\mathcal{K}(x)} \rho^{\prime}(x) \text { for every } x \in \operatorname{dom}(\rho) .
$$

We first prepare the following lemma.

Lemma 7. If $f \sim_{\tau \rightarrow \tau} g$, then $\mathbf{L F P}_{\tau}(f) \sim_{\tau} \mathbf{L F P}_{\tau \#}(g)$.
Proof. By Cousot and Cousot's fixpoint theorem [5], there exists an ordinal $\gamma$ such that $\operatorname{LFP}(f)=f^{\gamma}\left(\perp_{\tau}\right)$ and $\operatorname{LFP}(g)=g^{\gamma}\left(\perp_{\tau^{\#}}\right)$. Here, $f^{\gamma}(x)$ is defined by:

$$
f(x)= \begin{cases}x & \text { if } \gamma=0 \\ f\left(f^{\gamma^{\prime}}(x)\right) & \text { if } \gamma=\gamma^{\prime}+1 \\ \sqcup_{\gamma^{\prime}<\gamma} f^{\gamma^{\prime}}(x) & \text { if } \gamma \text { is a limit ordinal. }\end{cases}
$$

Thus, it suffices to show $f^{\gamma}\left(\perp_{\tau}\right) \sim_{\tau} g^{\gamma}\left(\perp_{\tau} \#\right)$ by induction on $\gamma$. The case where $\gamma=0$ or $\gamma=\gamma^{\prime}+1$ is trivial. Suppose $\gamma$ is a limit ordinal. Suppose $\tau=\kappa_{1} \rightarrow \cdots \rightarrow \kappa_{k} \rightarrow \star$, and $v_{i} \sim_{\kappa_{i}} w_{i}$ for $i \in\{1, \ldots, k\}$. It suffices to show

$$
f^{\gamma}\left(\perp_{\tau}\right) v_{1} \cdots v_{k} \sim_{\star} g^{\gamma}\left(\perp_{\tau \#}\right) w_{1} \cdots w_{k} .
$$

By the induction hypothesis, we have $f^{\gamma^{\prime}}\left(\perp_{\tau}\right) \sim_{\tau} g^{\gamma^{\prime}}\left(\perp_{\tau^{\#}}\right)$ for any $\gamma^{\prime}<\gamma$. Thus, we have:

$$
\begin{aligned}
& f^{\gamma}\left(\perp_{\tau}\right) v_{1} \cdots v_{k} \\
& =\left(\sqcup_{\gamma^{\prime}<\gamma} f^{\gamma^{\prime}}\left(\perp_{\tau}\right)\right) v_{1} \cdots v_{k} \\
& =\sqcup_{\gamma^{\prime}<\gamma}\left(f^{\gamma^{\prime}}\left(\perp_{\tau}\right) v_{1} \cdots v_{k}\right) \\
& \sim_{\star} \sqcup_{\gamma^{\prime}<\gamma}\left(g^{\gamma^{\prime}}\left(\perp_{\tau}\right) w_{1} \cdots w_{k}\right) \\
& =\left(\sqcup_{\gamma^{\prime}<\gamma} g^{\gamma^{\prime}}\left(\perp_{\tau}\right)\right) w_{1} \cdots w_{k} \\
& =g^{\gamma}\left(\perp_{\tau}\right) w_{1} \cdots w_{k}
\end{aligned}
$$

Theorem 6 is an immediate corollary of the following lemma.
Lemma 8. Suppose $\mathcal{K} \vdash_{\text {ST }} \varphi: \kappa$. Then $\rho \sim_{\mathcal{K}} \rho^{\prime}$ implies $\llbracket \mathcal{K} \vdash_{\text {ST }} \varphi: \kappa \rrbracket \rho \sim_{\kappa} \llbracket \mathcal{K}^{\#} \vdash_{\text {ST }} \varphi^{\#^{\prime}}$ : $\kappa^{\#} \rrbracket \rho^{\prime}$.
Proof. The proof proceeds by induction on the derivation of $\mathcal{K} \vdash_{\text {ST }} \varphi: \kappa$. Since the other cases are similar or trivial, we show only the main cases.

- Case T-AND: In this case, $\varphi=\varphi_{1} \wedge \varphi_{2}$ and $\varphi^{\#^{\prime}}=\lambda x \cdot \varphi_{1}^{\#^{\prime}}\left(\varphi_{2}^{\#^{\prime}} x\right)$, with $\kappa=\star$ and $\mathcal{K} \vdash_{\mathrm{ST}} \varphi_{i}: \star$. By the induction hypothesis, we have $\llbracket \mathcal{K} \vdash_{\mathrm{ST}} \varphi_{i}: \star \rrbracket \rho \sim_{\star} \llbracket \mathcal{K}^{\#} \vdash_{\mathrm{ST}} \varphi_{i}^{\#^{\prime}}: \star \star^{\#} \rrbracket \rho^{\prime}$ for $i \in\{1,2\}$. If $\llbracket \mathcal{K} \vdash_{\text {ST }} \varphi: \star \rrbracket \rho=\top$, then $\llbracket \mathcal{K} \vdash_{\text {ST }} \varphi_{i}: \star \rrbracket \rho=\top$ for both $i=1$ and 2. Thus, $\llbracket \mathcal{K}^{\#} \vdash_{\text {ST }} \varphi_{i}^{\#^{\prime}}: \star^{\#} \rrbracket \rho^{\prime}=\lambda x \cdot x$ for both $i=1$ and 2 . Therefore, we have $\llbracket \mathcal{K} \# \vdash_{\mathrm{ST}} \varphi^{\#^{\prime}}: \star^{\#} \rrbracket \rho^{\prime}=\lambda x . x$ as required. Otherwise, i.e., if $\llbracket \mathcal{K} \vdash_{\mathrm{ST}} \varphi: \star \rrbracket \rho=\perp$, then $\llbracket \mathcal{K} \vdash_{\mathrm{ST}} \varphi_{i}: \star \rrbracket \rho=\perp$ for $i=1$ or 2 . Thus, $\llbracket \mathcal{K}{ }^{\#} \vdash_{\mathrm{ST}} \varphi_{i}^{\#^{\prime}}: \star{ }^{\#} \rrbracket \rho^{\prime}=\lambda x . \perp$ for such $i$. Therefore, we have $\llbracket \mathcal{K}^{\#} \vdash_{\text {ST }} \varphi^{\#^{\prime}}: \star \# \rrbracket \rho^{\prime}=\lambda x . \perp$ as required.
- Case T-MU: In this case, $\varphi=\mu x^{\tau} \cdot \varphi^{\prime}$ and $\varphi^{\#^{\prime}}=\mu x^{\tau^{\#}} \cdot \varphi^{\prime \#^{\prime}}$ with $\kappa=\tau$ and $\mathcal{K}, x: \tau \vdash_{\text {ST }}$ $\varphi^{\prime}: \tau$. By the induction hypothesis, we have $\llbracket \mathcal{K}, x: \tau \vdash_{\mathrm{ST}} \varphi^{\prime}: \tau \rrbracket(\rho\{x \mapsto v\}) \sim_{\tau} \llbracket \mathcal{K}, x: \tau^{\#} \vdash_{\mathrm{ST}}$ $\varphi^{\prime \#^{\prime}}: \tau^{\#} \rrbracket\left(\rho^{\prime}\{x \mapsto w\}\right)$, which implies $\llbracket \mathcal{K} \vdash_{\text {ST }} \lambda x \cdot \varphi^{\prime}: \tau \rightarrow \tau \rrbracket \rho \sim_{\tau} \llbracket \mathcal{K} \# \vdash_{\text {ST }} \lambda x \cdot \varphi^{\prime \#^{\prime}}:(\tau \rightarrow$ $\tau)^{\#} \rrbracket \rho^{\prime}$. Thus, the required result follows by Lemma 7 .

We are now ready to prove Theorem 6
Proof of Theorem [6. Suppose $\emptyset \vdash_{\text {ST }} \varphi: \star$. By Lemma 8 we have $\llbracket \varphi \rrbracket=\llbracket \emptyset \vdash_{\text {ST }} \varphi: \star \rrbracket \emptyset \sim_{\star}$ $\llbracket \emptyset \vdash_{\text {ST }} \varphi^{\#^{\prime}}: \star \rrbracket \emptyset$. Thus, if $\llbracket \varphi \rrbracket=T$, then $\llbracket \varphi^{\#} \rrbracket=\llbracket \varphi^{\#^{\prime}} \rrbracket \top=(\lambda x \cdot x) \top=\top$. If $\llbracket \varphi \rrbracket=\perp$, then $\llbracket \varphi^{\#} \rrbracket=\llbracket \varphi^{\#^{\prime}} \rrbracket \top=(\lambda x . \perp) \top=\perp$, as required.

## 4 From Order- $(n+1)$ May-Reachability to Order- $n$ Reachability Games

In this section, we show the translation $(\cdot)^{b}$ from order- $(n+1)$ disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$ formulas to order $-n \mu \mathrm{HFL}(\mathrm{Z})$ formulas. The translation $(\cdot)^{b}$ is much more involved than in the translation (.) \#.

To see how such translation can be achieved, let us recall the formula $\varphi_{\text {ex1 }}:=$ $\varphi_{\text {sum }} n(\lambda r . r<n)$ in Example 2 where $\varphi_{\text {sum }}:$ Int $\rightarrow($ Int $\rightarrow \star) \rightarrow \star$ is:

$$
\begin{aligned}
& \text { } \mu \text { sum. } \lambda x \cdot \lambda k \text {. } \\
& x<0 \vee(x=0 \wedge k 0) \vee(x>0 \wedge \operatorname{sum}(x-1)(\lambda y \cdot k(x+y))) .
\end{aligned}
$$

Note that the order of the formula above is 1 . We wish to construct a formula $\psi$ of order 0 , such that $\llbracket \varphi_{e x 1} \rrbracket=\llbracket \psi \rrbracket$. Recall that, by Fact $\mathbb{1} \llbracket \varphi_{e x 1} \rrbracket=\top$ just if $\varphi_{e x 1} \longrightarrow^{*}$ true. There are the following two cases where the formula $\varphi_{\text {ex1 }}$ may be reduced to true.

1. $\phi$ is reduced to true without the order-0 argument $\lambda r . r<n$ being called.
2. $\phi$ is reduced to $(\lambda r . r<n) m$ for some $m$, and then $(\lambda r . r<n) m$ is reduced to true.

Let $\varphi_{\text {sum }_{0}} n$ be the condition for the first case to occur, and let $\varphi_{\text {sum }_{1}} n m$ be the condition that $(\lambda r . r<n) m$ is called. Then, $\varphi_{\text {sum }_{0}}$ and $\varphi_{\text {sum }_{1}}$ can be expressed as follows.

$$
\begin{aligned}
\varphi_{\text {sum }_{0}} & :=\mu \operatorname{sum}_{0} \cdot \lambda x \cdot x<0 \vee\left(x>0 \wedge \operatorname{sum}_{0}(x-1)\right) \\
\varphi_{\text {sum }_{1}} & :=\mu \operatorname{sum}_{1} \cdot \lambda x \cdot \lambda z \cdot(x=0 \wedge z=0) \\
& \vee\left(x>0 \wedge \exists y \cdot \operatorname{sum}_{1}(x-1) y \wedge z=x+y\right)
\end{aligned}
$$

To understand the formula $\varphi_{\text {sum }_{1}}$, notice that $\varphi_{\text {sum }}(x-1)(\lambda y . k(x+y))$ is reduced to $k z$ just if $\operatorname{sum}(x-1)(\lambda y \cdot k(x+y))$ is first reduced to $(\lambda y \cdot k(x+y)) y$ for some $y$ (the condition for which is expressed by $\operatorname{sum}_{1}(x-1) y$ ), and $z=x+y$ holds.

Using $\varphi_{\text {sum }_{0}}$ and $\varphi_{\text {sum }_{1}}$ above, the formula $\varphi_{\text {sum }}$ can be translated to:

$$
\psi:=\varphi_{\text {sum }_{0}} n \vee \exists r . \varphi_{\text {sum }_{1}} n r \wedge r<n .
$$

Note that the order of $\psi$ is 0 .
In general, if $\varphi$ is an order-1 (disjunctive) formula of type

$$
\text { Int }^{k} \rightarrow\left(\text { Int }^{\ell_{1}} \rightarrow \star\right) \rightarrow \cdots \rightarrow\left(\text { Int }^{\ell_{m}} \rightarrow \star\right) \rightarrow \star
$$

and $\psi_{i}(i \in\{1, \ldots, m\})$ is a formula of type $\operatorname{Int}^{\ell_{i}} \rightarrow \star$, then $\varphi \widetilde{e}_{1, \ldots, k} \psi_{1} \cdots \psi_{m}$ can be translated to an order-0 formula of the form:

$$
\varphi_{0} \widetilde{e}_{1, \ldots, k} \vee \bigvee_{i \in\{1, \ldots, m\}} \exists \widetilde{y}_{1, \ldots, \ell_{i}} \cdot\left(\varphi_{i} \widetilde{e}_{1, \ldots, k} \widetilde{y}_{1, \ldots, \ell_{i}} \wedge \psi_{i} \widetilde{y}_{1, \ldots, \ell_{i}}\right),
$$

where the part $\varphi_{0} \widetilde{e}_{1, \ldots, k}$ expresses the condition for $\varphi \widetilde{e}_{1, \ldots, k} \psi_{1} \cdots \psi_{m}$ to be reduced to true without $\psi_{i}$ being called, and the part $\varphi_{i} \widetilde{e}_{1, \ldots, k} \widetilde{y}_{1}, \ldots, \ell_{i}$ expresses the condition for $\varphi \widetilde{e}_{1, \ldots, k} \psi_{1} \cdots \psi_{m}$ to be reduced to $\psi_{i} \widetilde{y}_{1, \ldots, \ell_{i}}$.

For higher-order formulas, the translation is more involved. To simplify the formalization, we assume that a formula as an input or output of our translation is given in the form $\left(\Theta, D, \varphi_{0}\right)$, called an equation system; here $D$ is a set of mutually recursive fixpoint equations of the form $\left\{F_{1} \widetilde{x}_{1}={ }_{\mu} \varphi_{1}, \ldots, F_{n} \widetilde{x}_{n}={ }_{\mu} \varphi_{n}\right\}$ and $\Theta$ is the type environment for $F_{1}, \ldots, F_{n}$.

We sometimes omit $\Theta$ and just write $\left(D, \varphi_{0}\right)$. Here, each $\varphi_{i}(i \in\{0, \ldots, n\})$ should be fixpoint-free, $\varphi_{0}$ is well-typed under $\Theta$, and $\varphi_{i}(i \in\{1, \ldots, n\})$ should have some type $\tau_{i}$ under the type environment:

$$
\Theta, x_{i, 1}: \kappa_{i, 1}, \ldots, x_{i, m_{i}}: \kappa_{i, m_{i}}
$$

where $\Theta\left(F_{i}\right)=\kappa_{i, 1} \rightarrow \cdots \rightarrow \kappa_{i, m_{i}} \rightarrow \tau_{i}$ and $\widetilde{x}_{i}=x_{i, 1} \cdots x_{i, m_{i}}$. The $\mu \mathrm{HFL}(\mathrm{Z})$ formula $\left(D, \varphi_{0}\right)^{\mu}$ represented by $\left(\Theta, D, \varphi_{0}\right)$ is defined by:

$$
\begin{aligned}
& (\emptyset, \varphi)^{\mu}=\varphi \\
& \left(D \cup\left\{F \widetilde{x}={ }_{\mu} \psi\right\}, \varphi\right)^{\mu}=([\mu F \cdot \lambda \widetilde{x} \cdot \psi / F] D,[\mu F \cdot \lambda \widetilde{x} \cdot \psi / F] \varphi)^{\mu} .
\end{aligned}
$$

We write $\llbracket(D, \varphi) \rrbracket$ for $\llbracket(D, \varphi)^{\mu} \rrbracket$.
For an equation system as an input of our translation, we further assume, without loss of generality, the following conditions.
(I) Each $\varphi_{i}(i \in\{1, \ldots, n\})$ on the right-hand side of a definition in $D$ has type $\star$ and is generated by the following grammar (where the metavariable may be a fixpoint variable $F_{j}$ or its parameters):

$$
\begin{equation*}
\varphi::=x\left|\varphi_{1} \vee \varphi_{2}\right| e_{1} \leq e_{2} \wedge \varphi\left|\varphi_{1} \varphi_{2}\right| \varphi e \tag{1}
\end{equation*}
$$

In particular, (i) $\varphi_{i}$ is a disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$ formula, (ii) $\varphi_{i}$ contains neither $\lambda$-abstractions nor fixpoint operators, and (iii) a formula of the form $e_{1} \leq e_{2}$ may occur only in the form $e_{1} \leq e_{2} \wedge \varphi$.
(II) Every integer predicate (i.e., a formula of type of the form Int $^{\ell} \rightarrow \star$ with $\ell \geq 0$ ) that occurs in an argument position has the same arity $M$. In other words, in any function type $\kappa \rightarrow \tau$, either $\kappa=\operatorname{Int}^{M} \rightarrow \star$, or $\operatorname{ord}(\kappa) \neq 0$.
(III) The "main formula" $\varphi_{0}$ is a formula of the form $F \lambda \widetilde{x}_{1, \ldots, M}$.true.

Note that the assumption above does not lose generality. Given an order- $(n+1)$ disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$ formula $\varphi$, it can be first transformed to a formula of the form $\varphi^{\prime}$ true, where true does not occur on the right-hand side of any conjunction in $\varphi^{\prime}$. We then set $M$ to the largest arity of integer predicates that occur in argument positions in $\varphi^{\prime}$ true, and raise the arity of every integer predicate argument to $M$ by adding dummy arguments. For example, given

$$
\left(\lambda f^{\text {Int } \rightarrow \star} \cdot f 1\right)\left(\left(\lambda g^{\text {Int } \rightarrow \operatorname{Int} \rightarrow \star} \cdot g 1\right)\left(\lambda x^{\text {Int }} \cdot \lambda y^{\text {Int }} \cdot x \leq y\right)\right),
$$

we can set $M$ to 2 , and replace the formula with:

$$
\begin{aligned}
& \left(\lambda f^{\prime \operatorname{Int} \rightarrow \operatorname{Int} \rightarrow \star} \cdot f^{\prime} 10\right) \\
& \quad \lambda z_{1} \cdot \lambda z_{2} \cdot\left(\left(\lambda g^{\operatorname{Int} \rightarrow \operatorname{Int} \rightarrow \star} \cdot g 1\right)\left(\lambda x^{\operatorname{Int}} \cdot \lambda y^{\operatorname{Int}} \cdot x \leq y\right)\right) z_{1} .
\end{aligned}
$$

Here, we have inserted dummy (actual and formal) parameters 0 and $z_{2}$ to increase the arities of $f$ and the argument of $\left(\lambda f^{\text {Int } \rightarrow \star} . f 1\right)$. We can then apply $\lambda$-lifting to remove $\lambda$-abstractions and generate a set of top-level definitions $D$.

The formula $\varphi_{\text {sum }}$ given earlier in this section is represented as: $\left(\Theta_{\text {sum }}, D_{\text {sum }}, S \lambda z\right.$. true $)$, where $D_{\text {sum }}$ consists of the following equations. Here, $M$ is set to 1 .

$$
\begin{aligned}
& S t={ }_{\mu} \operatorname{sumtn}(C t) \\
& C t r={ }_{\mu} r<n \wedge t 0 \\
& \operatorname{sum} t x k={ }_{\mu} \\
& (x<0 \wedge t 0) \vee(x=0 \wedge k 0) \vee(x>0 \wedge \operatorname{sum} t(x-1)(K k x)) \\
& K k x y={ }_{\mu} k(x+y),
\end{aligned}
$$

and $\Theta_{\text {sum }}$ is:

$$
\begin{aligned}
& S:(\text { Int } \rightarrow \star) \rightarrow \star, C:(\text { Int } \rightarrow \star) \rightarrow \text { Int } \rightarrow \star, \\
& \text { sum }:(\text { Int } \rightarrow \star) \rightarrow \text { Int } \rightarrow(\text { Int } \rightarrow \star) \rightarrow \star, \\
& K:(\text { Int } \rightarrow \star) \rightarrow \text { Int } \rightarrow \text { Int } \rightarrow \star .
\end{aligned}
$$

We translate each equation $F y_{1} \cdots y_{m}=_{\mu} \varphi$ in $D$ as follows. We first decompose the formal parameters $y_{1}, \ldots, y_{m}$ to two parts: $y_{1}, \ldots, y_{j}$ and $y_{j+1}, \ldots, y_{m}$, where the orders of (the types of) $y_{j+1}, \ldots, y_{m}$ are at most 0 , and the order of $y_{j}$ is at least 1 ; note that the sequences $y_{1}, \ldots, y_{j}$ and $y_{j+1}, \ldots, y_{m}$ are possibly empty. We further decompose $y_{j+1}, \ldots, y_{m}$ into order- 0 variables $x_{1}, \ldots, x_{k}$ and integer variables $z_{1}, \ldots, z_{p}$ (thus, $j+k+p=m$ ). Formally, the decomposition of formal parameters is defined by:

```
\(\operatorname{decomparg}(\epsilon, \star)=(\epsilon, \epsilon, \epsilon)\)
\(\operatorname{decomparg}(u \cdot \widetilde{y}, \kappa \rightarrow \tau)=\)
\(\begin{cases}((u: \kappa) \cdot \mathcal{K}, \widetilde{x}, \widetilde{z}) & \text { if } \operatorname{decomparg}(\widetilde{y}, \tau)=(\mathcal{K}, \widetilde{x}, \widetilde{z}), \mathcal{K} \neq \epsilon \\ (u: \kappa, \widetilde{x}, \widetilde{z}) & \text { if } \operatorname{ord}(\kappa)>0, \\ (\epsilon, u \cdot \widetilde{x}, \widetilde{z}) & \quad \operatorname{decomparg}(\widetilde{y}, \tau)=(\epsilon, \widetilde{x}, \widetilde{z}) \\ & \text { if } \kappa=\operatorname{Int}{ }^{M} \rightarrow \star, \\ (\epsilon, \widetilde{x}, u \cdot \widetilde{z}) & \text { decomparg}(\widetilde{y}, \tau)=(\epsilon, \widetilde{x}, \widetilde{z}) \\ \text { if } \kappa=\operatorname{Int}, \operatorname{decomparg}(\widetilde{y}, \tau)=(\epsilon, \widetilde{x}, \widetilde{z})\end{cases}\)
```

Here, decomparg $\left(\widetilde{y}_{1, \ldots, m}, \Theta(F)\right)$ decomposes the sequence of variables $\widetilde{y}_{1, \ldots, m}$ and returns a triple $(\mathcal{K}, \widetilde{x}, \widetilde{z})$, where $\mathcal{K}$ is the type environment for $y_{1}, \ldots, y_{j}, \widetilde{x}$ is the sequence of integer predicate variables, and $\widetilde{z}$ is the sequence of integer variables.

For example, given an equation

$$
F u_{1} u_{2} u_{3} u_{4} u_{5}={ }_{\mu} \varphi
$$

where $\Theta(F)=$ Int $\rightarrow(($ Int $\rightarrow \star) \rightarrow \star) \rightarrow$ Int $\rightarrow$ (Int $\rightarrow \star) \rightarrow$ Int $\rightarrow \star$, the formal parameters $u_{1} \cdots u_{5}$ are decomposed as follows.

$$
\begin{aligned}
& \operatorname{decomparg}\left(u_{1} \cdots u_{5}, \Theta(F)\right) \\
& =\left(\left\{u_{1}: \operatorname{Int}, u_{2}:(\operatorname{Int} \rightarrow \star) \rightarrow \star\right\}, u_{4}, u_{3} u_{5}\right)
\end{aligned}
$$

Given an equation $F \widetilde{y}={ }_{\mu} \varphi$ where decomparg $(\widetilde{y}, \Theta(F))=\left(\mathcal{K}, \widetilde{x}_{1, \ldots, k}, \widetilde{z}\right)$ with $\mathcal{K}=y_{1}$ : $\kappa_{1}, \ldots, y_{j}: \kappa_{j}$, we generate equations for new fixpoint variables $F_{0}, \ldots, F_{k}$. As in the order-1 case, for $i \in\{1, \ldots, k\}, F_{i} \widetilde{\varphi}_{1, \ldots, j}^{\prime} \widetilde{z} \widetilde{u}_{1, \ldots, M}$ represents the condition for $F \widetilde{\varphi}_{1, \ldots, j}$ to be reduced to $x_{i} \widetilde{u}_{1, \ldots, M}$ (where $\widetilde{\varphi}_{1, \ldots, j}^{\prime}$ is the sequence of formulas obtained by translating $\widetilde{\varphi}_{1, \ldots, j}$ in a recursive manner). $F_{0}$ is a new component required to deal with higher-order formulas; it is used to compute the condition for $F \widetilde{y}$ to be reduced to $x \widetilde{u}_{1, \ldots, \ell_{i}}$ for some order- 0 predicate $x$, which has been passed through higher-order parameters $\widetilde{y}_{1, \ldots, j}$. For example, consider a formula $F(G x) y$ where $F:(($ Int $\rightarrow \star) \rightarrow \star) \rightarrow($ Int $\rightarrow \star) \rightarrow \star, G:($ Int $\rightarrow \star) \rightarrow$ (Int $\rightarrow$ $\star) \rightarrow \star$. Then, the condition for $F(G x) y$ to be reduced to $y n$ is computed by using $F_{1}$, while the condition for $F(G x) y$ to be reduced to $x n$ is computed by using $F_{0}$. A more concrete version of this example is discussed later in Example 4

To compute $F_{0}, \ldots, F_{k}$, we translate each subformula $\varphi$ of the body of $F$ to:

$$
\left(\varphi_{*}, \varphi_{0}, \varphi_{1}, \ldots, \varphi_{k}, \varphi_{k+1}, \ldots, \varphi_{k+\operatorname{gar}(\tau)}\right),
$$

where $\tau$ is the type of $\varphi$, and $\operatorname{gar}(\tau)$ denotes the number of order- 0 arguments passed after the last argument of order greater than 0 . More precisely, we define the decomposition of types as follows.

$$
\begin{aligned}
& \operatorname{decomp}(\star)=(\epsilon, \epsilon, 0) \\
& \operatorname{decomp}(\kappa \rightarrow \tau)= \\
& \begin{cases}(\kappa \cdot \widetilde{\kappa}, m, n) & \text { if } \operatorname{decomp}(\tau)=(\widetilde{\kappa}, m, n), \widetilde{\kappa} \neq \epsilon \\
(\kappa, m, n) & \text { if } \operatorname{ord}(\kappa)>0, \operatorname{decomp}(\tau)=(\epsilon, m, n) \\
(\epsilon, m+1, n) & \text { if } \kappa=\operatorname{Int}^{M} \rightarrow \star, \operatorname{decomp}(\tau)=(\epsilon, m, n) \\
(\epsilon, m, n+1) & \text { if } \kappa=\operatorname{Int}, \operatorname{decomp}(\tau)=(\epsilon, m, n)\end{cases}
\end{aligned}
$$

Then, $\operatorname{gar}(\tau)$ denotes $m$ when $\operatorname{decomp}(\tau)=(\widetilde{\kappa}, m, n)$. For example, for $\tau=($ Int $\rightarrow \star) \rightarrow$ $(($ Int $\rightarrow \star) \rightarrow \star) \rightarrow($ Int $\rightarrow \star) \rightarrow$ Int $\rightarrow($ Int $\rightarrow \star) \rightarrow \star, \operatorname{decomp}(\tau)=(($ Int $\rightarrow \star) \cdot(($ Int $\rightarrow$ $\star) \rightarrow \star), 2,1$ ); hence $\operatorname{gar}(\tau)=2$. Here, $\varphi_{1}, \ldots, \varphi_{k}$ are analogous to $F_{1}, \ldots, F_{k}$ : they are used for computing the condition for $\varphi \widetilde{\psi}$ to be reduced to $x_{i} \widetilde{n}$. Similarly, $\varphi_{k+i}$ (where $i \in\{1, \ldots, \operatorname{gar}(\tau)\})$ is used for computing the condition for $\varphi \widetilde{\psi}$ to be reduced to $\psi_{i} \widetilde{n}$, where $\psi_{i}$ is the $i$-th order- 0 argument of $\varphi$. The component $\varphi_{0}$ is analogous to $F_{0}$, and used to compute the condition $\varphi \widetilde{\psi}$ to be reduced to $x \widetilde{n}$, where $x$ is some order- 0 predicate passed through higher-order arguments of $\varphi$. The other component $\varphi_{*}$ is similar to $\varphi_{0}$, but the target order-0 predicate $x$ may have already been set inside $\varphi_{*}$.

Based on the intuition above, we formalize the translation of a formula as the following relation:

$$
\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k+\operatorname{gar}(\tau)}\right) .
$$

Here, $\Theta$ denotes the type environment for fixpoint variables defined by $D$. If $\varphi$ is a subformula of the body of $F$, and $F$ is defined by $F \widetilde{y}=\varphi_{F}$, then $\mathcal{K}$ and $\widetilde{x}_{F}$ are set to $\mathcal{K}_{F}, \widetilde{z}: \widetilde{\text { Int }}$ and $\widetilde{x}_{F}$ respectively, where decomparg $(\widetilde{y}, \Theta(F))=\left(\mathcal{K}_{F}, \widetilde{x}_{F}, \widetilde{z}\right)$.

The output $\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k+\operatorname{gar}(\tau)}\right)$ of the translation has type $\tau^{b, k+2}$ under the type environment $\Theta^{b^{\prime}}, \mathcal{K}^{b}$, where the translations of types and type environments are defined by:

$$
\begin{aligned}
& \text { Int }{ }^{b, k}=\text { Int } \\
& \tau^{b, k}=\left(\Pi_{i=1, \ldots, k}\left(\widetilde{\kappa}^{b, 2} \rightarrow \text { Int }^{n+M} \rightarrow \star\right)\right) \\
& \quad \times\left(\Pi_{i=1, \ldots, m}\left(\widetilde{\kappa}^{b, 1} \rightarrow \text { Int }^{n+M} \rightarrow \star\right)\right) \\
& \quad(\text { if } \operatorname{decomp}(\tau)=(\widetilde{\kappa}, m, n)) \\
& \emptyset^{b}=\emptyset \\
& (\mathcal{K}, y: \text { Int })^{b}=\mathcal{K}^{b}, y: \text { Int } \\
& (\mathcal{K}, y: \tau)^{b}=\mathcal{K}^{b}, y_{*}: \tau_{*}, y_{0}: \tau_{0}, \ldots, y_{k}: \tau_{k} \\
& \quad \text { where } \tau^{b, 2}=\tau_{*} \times \tau_{0} \times \cdots \times \tau_{k} \\
& \emptyset^{b^{\prime}}=\emptyset \\
& (\Theta, F: \tau)^{b^{\prime}}=\Theta^{b^{\prime}}, F_{0}: \tau_{0}, \ldots, F_{k}: \tau_{k} \\
& \quad \text { where } \tau^{b, 1}=\tau_{0} \times \cdots \times \tau_{k} .
\end{aligned}
$$

Here, we have extended simple types with product types; we extend the definition of the order of a type by: $\operatorname{ord}\left(\tau_{1} \times \cdots \times \tau_{n}\right)=\max \left(\operatorname{ord}\left(\tau_{1}\right), \ldots, \operatorname{ord}\left(\tau_{n}\right)\right)$. In the translation above, $M$ is an integer constant, which denotes an upper-bound of the arities of integer predicates that may occur in the formula to be translated. Note that the translation of a type decreases the order of the type by one, i.e., $\operatorname{ord}\left(\tau^{b, k}\right)=\max (0, \operatorname{ord}(\tau)-1)$.

$$
\begin{aligned}
& \frac{\varphi_{j}= \begin{cases}\lambda \widetilde{z}_{1, \ldots, M} \cdot \lambda \widetilde{w}_{1, \ldots, M} \cdot \wedge_{p=1, \ldots, M}\left(z_{p}=w_{p}\right), & \text { if } j=i \\
\lambda \widetilde{z}_{1, \ldots, M} \cdot \lambda \widetilde{w}_{1, \ldots, M} \cdot f a l \mathrm{se} & \text { otherwise }\end{cases} }{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} x_{i}: \operatorname{Int}^{M} \rightarrow \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k}\right)} \\
& \operatorname{decomp}(\mathcal{K}(y))=(\widetilde{\kappa}, m, p) \\
& \overline{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} y: \mathcal{K}(y) \rightsquigarrow(y_{*}, \underbrace{y_{0}, \ldots, y_{0}}_{k+1}, y_{1}, \ldots, y_{m})} \\
& \operatorname{decomp}(\Theta(F))=(\widetilde{\kappa}, m, p) \\
& \overline{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} F: \Theta(F) \rightsquigarrow(F_{0}, \underbrace{F_{0}, \ldots, F_{0}}_{k+1}, F_{1}, \ldots, F_{m})} \\
& \frac{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k}\right) \quad \psi_{j}=\lambda \widetilde{z}_{1, \ldots, M} . e_{1} \leq e_{2} \wedge \varphi_{j} \widetilde{z}_{1, \ldots, M}}{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} e_{1} \leq e_{2} \wedge \varphi: \star \rightsquigarrow\left(\psi_{*}, \psi_{0}, \ldots, \psi_{k}\right)} \\
& \operatorname{ord}\left(\kappa_{0} \rightarrow \tau\right)>1 \quad \operatorname{gar}\left(\kappa_{0} \rightarrow \tau\right)=m \quad \operatorname{gar}\left(\kappa_{0}\right)=m^{\prime} \\
& \mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \kappa_{0} \rightarrow \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k+m}\right) \\
& \mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \psi: \kappa_{0} \rightsquigarrow\left(\psi_{*}, \psi_{0}, \ldots, \psi_{k+m^{\prime}}\right) \\
& \overline{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi \psi: \tau \rightsquigarrow\left(\varphi_{*}\left(\psi_{*}, \psi_{0}, \psi_{k+1}, \ldots, \psi_{k+m^{\prime}}\right), \varphi_{0}\left(\psi_{0}, \psi_{0}, \psi_{k+1}, \ldots, \psi_{k+m^{\prime}}\right),\right.} \\
& \varphi_{1}\left(\psi_{1}, \psi_{0}, \psi_{k+1}, \ldots, \psi_{k+m^{\prime}}\right), \ldots, \varphi_{k}\left(\psi_{k}, \psi_{0}, \psi_{k+1}, \ldots, \psi_{k+m^{\prime}}\right), \\
& \left.\varphi_{k+1}\left(\psi_{0}, \psi_{k+1}, \ldots, \psi_{k+m^{\prime}}\right), \ldots, \varphi_{k+m}\left(\psi_{0}, \psi_{k+1}, \ldots, \psi_{k+m^{\prime}}\right)\right) \\
& \operatorname{decomp}(\tau)=(\epsilon, m-1, p) \\
& \mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi:\left(\text { Int }^{M} \rightarrow \star\right) \rightarrow \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k+m}\right) \\
& \mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \psi: \operatorname{Int}^{M} \rightarrow \star \rightsquigarrow\left(\psi_{*}, \psi_{0}, \ldots, \psi_{k}\right) \\
& \frac{\xi_{j}=\lambda \widetilde{z}_{1, \ldots, p} \cdot \lambda \widetilde{w}_{1, \ldots, M} \cdot \varphi_{j} \widetilde{z} \widetilde{w} \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{k+1} \widetilde{z} \widetilde{u}_{1, \ldots, M} \wedge \psi_{j} \widetilde{u}_{1, \ldots, M} \widetilde{w}_{1, \ldots, M}\right)}{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi \psi: \tau \rightsquigarrow\left(\xi_{*}, \xi_{0}, \ldots, \xi_{k}, \varphi_{k+2}, \ldots, \varphi_{k+m}\right)} \\
& \frac{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \text { Int } \rightarrow \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{\ell+k}\right)}{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi e: \tau \rightsquigarrow\left(\varphi_{*} e, \varphi_{0} e, \ldots, \varphi_{\ell+k} e\right)} \\
& \mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k}\right) \quad \mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \psi: \star \rightsquigarrow\left(\psi_{*}, \psi_{0}, \ldots, \psi_{k}\right) \\
& \xi_{j}=\lambda \widetilde{z}_{1, \ldots, M} \cdot \varphi_{j} \widetilde{z}_{1, \ldots, M} \vee \psi_{j} \widetilde{z}_{1, \ldots, M} \\
& \mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi \vee \psi: \star \rightsquigarrow\left(\xi_{*}, \xi_{0}, \ldots, \xi_{k}\right) \\
& \text { decomparg }(\widetilde{w}, \Theta(F))=\left(\widetilde{y}: \widetilde{\kappa}, \widetilde{x}_{1, \ldots, k}, \widetilde{z}\right) \\
& y_{1}: \kappa_{1}, \ldots, y_{m}: \kappa_{m}, \widetilde{z}: \widetilde{\text { Int }} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k}\right)
\end{aligned}
$$

$$
\begin{align*}
& \frac{D^{\prime}=\bigcup\left\{D^{\prime \prime} \vdash_{\Theta}\left(F \widetilde{y}={ }_{\mu} \varphi\right) \rightsquigarrow D^{\prime \prime} \mid F \widetilde{y}={ }_{\mu} \varphi \in D\right\}}{(D, S \lambda \widetilde{z} \text {.true }) \rightsquigarrow\left(D^{\prime}, \exists \widetilde{z} \cdot S_{1} \widetilde{z}\right)} \tag{Tr-Def}
\end{align*}
$$

Figure 2: Translation from order- $(n+1)$ disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$ to order- $n \mu \mathrm{HFL}(\mathrm{Z})$.

The translation rules are given in Figure 2. We explain the main rules below. In the rule Tr-VARG for an order-0 variable $x_{i}$ (which should disappear after the translation), $\varphi_{j} \widetilde{z}_{1, \ldots, M} \widetilde{w}_{1, \ldots, M}$ should represent the condition for $x_{i} \widetilde{z}_{1, \ldots, M} \longrightarrow^{*} x_{j} \widetilde{w}_{1, \ldots, M}$; thus $\varphi_{j}$ is defined so that $\widetilde{z}_{1, \ldots, M} \widetilde{w}_{1, \ldots, M}$ is equivalent to true just if $i=j$ and $\widetilde{z}_{1, \ldots, M}=\widetilde{w}_{1, \ldots, M}$. In the rule $\mathrm{Tr}-\mathrm{VAR}$ for a variable $y$ in $\mathcal{K}$, the output of the translation is constructed from $\left(y_{*}, y_{0}, y_{1}, \ldots, y_{m}\right)$, whose values will be provided by the environment. Because the environment does not know order- 0 variables $x_{1}, \ldots, x_{k}$, we use $y_{0}$ to compute the condition for $y \widetilde{\psi}$ to be reduced to $x_{i} \widetilde{m}$. The rule Tr-VARF for fixpoint variables is almost the same as Tr-Var, except that the component $F_{0}$ is reused for $F_{*}$. The rationale for this is as follows: both $\varphi_{*}$ and $\varphi_{0}$ are used for computing the condition for a target order- 0 predicate variable (which is set by the environment) to be reached, and the only difference between them is that the target predicate may have already been set in $\varphi_{*}$, but since $F$ is a closed formula, such distinction does not make any difference; hence $F_{0}$ and $F_{*}$ need not be distinguished from each other.

In the rule Tr-App, the first two components $\left(\varphi_{*}\left(\psi_{*}, \ldots\right)\right.$ and $\left.\varphi_{0}\left(\psi_{0}, \ldots\right)\right)$ are used for computing the condition for some target predicates (set by the environment) to be reached, and the next $k$ components $\left(\varphi_{1}\left(\psi_{1}, \ldots\right), \ldots, \varphi_{k}\left(\psi_{k}, \ldots\right)\right)$ are used for computing the condition for predicate $x_{1}, \ldots, x_{k}$ to be reached. The rule TR-AppG is another rule for applications, where the argument $\psi$ is an order- 0 predicate. The component $\xi_{j}$ of the output is used for computing the condition for the predicate $x_{i}$ to be reached (i.e., the condition for a formula of the form $\varphi \psi \widetilde{\psi^{\prime}}$ (where $\widetilde{\psi^{\prime}}$ consists of order-0 predicates and integer arguments $\widetilde{z}_{1, \ldots, p}$ ) to be reduced to $x_{i} \widetilde{w}_{1, \ldots, \ell_{j}}$.) The formula $\varphi \psi \widetilde{\psi^{\prime}}$ may be reduced to $x_{i} \widetilde{w}_{1, \ldots, \ell_{j}}$ if either (i) $\varphi \psi \widetilde{\psi^{\prime}} \longrightarrow^{*} x_{i} \widetilde{w}_{1, \ldots, \ell_{j}}$ without $\psi$ being called, or (ii) $\varphi \psi \widetilde{\psi}^{\prime}$ is first reduced to $\psi \widetilde{z} \widetilde{u}$ for some $\widetilde{u}$, and then $\psi \widetilde{z} \widetilde{u}$ is reduced to $x_{i} \widetilde{w}_{1, \ldots, \ell_{j}}$. The part $\varphi_{j} \widetilde{z} \widetilde{w}$ represents the former condition, and the part $\exists \widetilde{u} . \cdots$ represents the latter condition. In the rule Tr-Def for definitions, the bodies of the definitions for $F_{0}, \ldots, F_{k}$ are set to the corresponding components of the translation of the body of $F$.

Example 4. Consider $S$ ( $\lambda x$.true), where $S$ is defined by:

$$
\begin{aligned}
& S t={ }_{\mu} F(G t) t \\
& F v w={ }_{\mu} v H \vee w 2 \\
& G p q={ }_{\mu} p 1 \\
& H x={ }_{\mu} H x
\end{aligned}
$$

Notice that there are the following two ways for $S t$ to be reduced to $t n$ for some $n$ :

$$
\begin{aligned}
& S t \longrightarrow F(G t) t \longrightarrow G t H \vee t 2 \longrightarrow G t H \longrightarrow t 1 \\
& S t \longrightarrow F(G t) t \longrightarrow G t H \vee t 2 \longrightarrow t 2 .
\end{aligned}
$$

The output of our transformations (with some simplification) is $\exists z . S_{1} z$ where:

$$
\begin{aligned}
& S_{1}={ }_{\mu} \lambda w_{1} \cdot F_{0}\left(\lambda w_{1} \cdot G_{0} w_{1} \vee G_{1} w_{1}, G_{0}, G_{2}\right) w_{1} \vee F_{1}\left(G_{0}, G_{2}\right) w_{1} \\
& F_{0}\left(v_{*}, v_{0}, v_{1}\right)={ }_{\mu} \lambda z_{1} \cdot v_{*} z_{1} \vee \exists u_{1} \cdot v_{1} u_{1} \wedge H_{0} u_{1} z_{1} \\
& F_{1}\left(v_{0}, v_{1}\right)={ }_{\mu} \lambda z_{1} \cdot v_{0} z_{1} \vee\left(\exists u_{1} \cdot v_{1} u_{1} \wedge H_{0} u_{1} z_{1}\right) \vee 2=z_{1} \\
& G_{0}={ }_{\mu} \lambda w_{1} \cdot \mathrm{false} \\
& G_{1}={ }_{\mu} \lambda w_{1} \cdot 1=w_{1} \\
& G_{2}={ }_{\mu} \lambda w_{1} \cdot \mathrm{false} \\
& H_{0} x={ }_{\mu} H_{0} x .
\end{aligned}
$$

Notice that the formula $S_{1} z$ has the following two reduction sequences that lead to the conditions of the form $z=n$ for some $n$.

$$
\begin{aligned}
S_{1} z & \longrightarrow^{*} F_{0}\left(\lambda w_{1} \cdot G_{0} w_{1} \vee G_{1} w_{1}, G_{0}, G_{2}\right) z \\
& \longrightarrow^{*}\left(\lambda w_{1} \cdot G_{0} w_{1} \vee G_{1} w_{1}\right) z \\
& \longrightarrow^{*} G_{0} z \longrightarrow 1=z \\
S_{1} z & \longrightarrow^{*} F_{1}\left(G_{0}, G_{2}\right) z \\
& \longrightarrow{ }^{*} G_{0} z \vee\left(\exists u_{1} \cdot G_{2} u_{1} \wedge H_{0} u_{1} z\right) \vee 2=z \\
& \longrightarrow{ }^{*} 2=z .
\end{aligned}
$$

The former reduction sequence corresponds to the reduction sequence of the original formula $S t \longrightarrow^{*} t 1$ where $t$ embedded in the first argument of $F$ (in $\left.F(G t) t\right)$ is called, and the latter reduction sequence corresponds to the reduction sequence $S t \longrightarrow{ }^{*} t 2$ where the second argument $t$ of $F$ (in $F(G t) t$ ) is called. Note that the first condition $1=z$ has been computed by using $F_{0}$, and the second condition $2=z$ has been computed by using $F_{1}$.

Example 5. Recall the example of $D_{\text {sum }}$ given earlier in this section. The following is the output of the translation (with some simplification by $\beta$-reductions and simple quantifier eliminations).

$$
\begin{aligned}
& S_{0}={ }_{\mu} \lambda w_{1} . s u m_{0} n w_{1} \vee \exists u_{1} . s u m_{2} n u_{1} \wedge C_{0} u_{1} w_{1} \\
& S_{1}={ }_{\mu} \lambda w_{1} \cdot s u m_{0} n w_{1} \vee \operatorname{sum}_{1} n w_{1} \\
& \vee \exists u_{1} . s u m_{2} n u_{1} \wedge\left(C_{0} u_{1} 0 \vee c_{1} u_{1} w_{1}\right) \\
& C_{0} x={ }_{\mu} \lambda z_{1} \text {.false } \\
& C_{1} x={ }_{\mu} \lambda z_{1} \cdot x<n \wedge 0=z_{1} \\
& \operatorname{sum}_{0} x={ }_{\mu} \lambda z_{1} \cdot\left(x>0 \wedge\left(\operatorname{sum}_{0}(x-1) z_{1}\right.\right. \\
& \left.\left.\vee \exists u_{1} \cdot \operatorname{sum}_{2}(x-1) u_{1} \wedge K_{0} x u_{1} z_{1}\right)\right) \\
& \operatorname{sum}_{1} x={ }_{\mu} x<0 \vee\left(x>0 \wedge\left(\operatorname{sum}_{0}(x-1) 0 \vee \operatorname{sum}_{1}(x-1)\right.\right. \\
& \left.\left.\vee \exists u_{1} \cdot \operatorname{sum}_{2}(x-1) u_{1} \wedge K_{0} x u_{1} 0\right)\right) \\
& \operatorname{sum}_{2} x={ }_{\mu} \lambda z_{1} \cdot x=0 \wedge 0=z_{1} \\
& \vee\left(x>0 \wedge\left(\operatorname{sum}_{0}(x-1) z_{1}\right.\right. \\
& \vee \exists u_{1} \cdot \operatorname{sum}_{2}(x-1) u_{1} \\
& \left.\left.\wedge\left(K_{0} x u_{1} z_{1} \vee \exists u_{2} .\left(K_{1} x u_{1} u_{2} \wedge u_{2}=z_{1}\right)\right)\right)\right) \\
& K_{0} x y={ }_{\mu} \lambda w_{1} . \mathrm{false} \\
& K_{1} x y={ }_{\mu} \lambda w_{1} \cdot x+y=w_{1}
\end{aligned}
$$

Although the output may look complicated, since the order of the resulting formula is 0 , we can directly translate its validity checking problem to a CHC solving problem using the method of [12, for which various automated solvers are available [17, 8, 4].

Example 6. Let us consider the formula $S \lambda z . \operatorname{tru} ⿶^{3}$, where:

$$
\begin{aligned}
& S t={ }_{\mu} \text { sum plus } n(C t) \\
& C t x={ }_{\mu} x<n \wedge t 0 \\
& \text { sum } f x k={ }_{\mu} x \leq 0 \wedge k 0 \vee x>0 \wedge f x(D f x k)
\end{aligned}
$$

[^2]\[

$$
\begin{aligned}
& \text { plus } x k={ }_{\mu} k(x+x) \\
& D f x k y={ }_{\mu} \operatorname{sum} f(x-1)(E y k) \\
& E y k z={ }_{\mu} k(y+z) .
\end{aligned}
$$
\]

It is translated to $\exists z . S_{1} z$, where 4

$$
\begin{aligned}
& S_{1}={ }_{\mu} \lambda w_{1} . \text { sum }_{0}\left(\text { plus }_{0}, \text { plus }_{0}, \text { plus }_{1}\right) n w_{1} \\
& \vee \exists u_{1} \cdot \text { sum }_{1}\left(\text { plus }_{0}, \text { plus }_{1}\right) n u_{1} \wedge\left(C_{0} u_{1} w_{1} \vee C_{1} u_{1} w_{1}\right) \\
& C_{0} x={ }_{\mu} \lambda z_{1} . \mathrm{false} \\
& C_{1} x={ }_{\mu} \lambda w_{1} \cdot x<n \wedge 0=z_{1} \\
& \operatorname{sum}_{0}\left(f_{*}, f_{0}, f_{1}\right) x={ }_{\mu} \lambda z_{1} \cdot x>0 \wedge\left(f_{*} x z_{1}\right. \\
& \left.\vee \exists u_{1} . f_{1} x u_{1} \wedge D_{0}\left(f_{*}, f_{0}, f_{1}\right) x u_{1} z_{1}\right) \\
& \operatorname{sum}_{1}\left(f_{0}, f_{1}\right) x={ }_{\mu} \lambda z_{1} \cdot x \leq 0 \wedge 0=z_{1} \vee x>0 \wedge\left(f_{0} x z_{1} \vee\right. \\
& \left.\exists u_{1} \cdot f_{1} x u_{1} \wedge\left(D_{0}\left(f_{0}, f_{0}, f_{1}\right) x u_{1} z_{1} \vee D_{1}\left(f_{0}, f_{1}\right) x u_{1} z_{1}\right)\right) \\
& \text { plus }_{0} x={ }_{\mu} \lambda w_{1} . \mathrm{fal} \text { se } \\
& \text { plus }_{1} x={ }_{\mu} \lambda w_{1} \cdot x+x=w_{1} \\
& D_{0}\left(f_{*}, f_{0}, f_{1}\right) x y={ }_{\mu} \lambda w_{1} . \operatorname{sum}_{0}\left(f_{*}, f_{0}, f_{1}\right)(x-1) w_{1} \\
& \vee \exists u_{1} \cdot \operatorname{sum}_{1}\left(f_{0}, f_{1}\right)(x-1) u_{1} \wedge E_{0} y u_{1} w_{1} \\
& D_{1}\left(f_{0}, f_{1}\right) x y={ }_{\mu} \lambda w_{1} . \operatorname{sum}_{0}\left(f_{0}, f_{0}, f_{1}\right)(x-1) w_{1} \\
& \vee \exists u_{1} . \operatorname{sum}_{1}\left(f_{0}, f_{1}\right) u_{1} \\
& \wedge\left(E_{0} y u_{1} w_{1} \vee \exists u_{2} . E_{1} y u_{1} u_{2} \wedge u_{2}=w_{1}\right) \\
& E_{0} y z={ }_{\mu} \lambda w_{1} . \mathrm{false} \\
& E_{1} y z={ }_{\mu} \lambda w_{1} \cdot y+z=w_{1} .
\end{aligned}
$$

The order of the original formula is 2 (since sum : (Int $\rightarrow$ (Int $\rightarrow \star$ ) $\rightarrow \star$ ) $\rightarrow$ Int $\rightarrow$ (Int $\rightarrow \star) \rightarrow \star$ ), while the order of the formula obtained by the translation is 1 ; note that $\operatorname{sum}_{0}:\left(\right.$ Int $\left.^{2} \rightarrow \star\right) \times\left(\right.$ Int $\left.^{2} \rightarrow \star\right) \times\left(\right.$ Int $\left.^{2} \rightarrow \star\right) \rightarrow$ Int $^{2} \rightarrow \star$. By further simplifications (note that the 0 -components sum $_{0}, C_{0}, D_{0}, \ldots$ actually return false), we obtain:

$$
\begin{aligned}
& S_{1}={ }_{\mu} \lambda w_{1} \cdot \exists u_{1} \cdot \text { sum }_{1}\left(\text { plus }_{0}, \text { plus }_{1}\right) n u_{1} \wedge C_{1} u_{1} w_{1} \\
& C_{1} x={ }_{\mu} \lambda w_{1} \cdot x<n \wedge 0=z_{1} \\
& \text { sum }_{1}\left(f_{0}, f_{1}\right) x={ }_{\mu} \lambda z_{1} \cdot x \leq 0 \wedge 0=z_{1} \\
& \quad \vee x>0 \wedge\left(f_{0} x z_{1} \vee \exists u_{1} \cdot f_{1} x u_{1} \wedge D_{1}\left(f_{0}, f_{1}\right) x u_{1} z_{1}\right) \\
& \operatorname{plus}_{0} x={ }_{\mu} \lambda w_{1} \cdot \mathrm{false} \\
& \operatorname{plus}_{1} x={ }_{\mu} \lambda w_{1} \cdot x+x=w_{1} \\
& D_{1}\left(f_{0}, f_{1}\right) x y={ }_{\mu} \lambda w_{1} \cdot \exists u_{1} \cdot \text { sum }_{1}\left(f_{0}, f_{1}\right) u_{1} \wedge E_{1} y u_{1} w_{1} \\
& E_{1} y z={ }_{\mu} \lambda w_{1} \cdot y+z=w_{1} .
\end{aligned}
$$

The following lemma states that the output of the translation is well-typed.
Lemma 9. If $\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \tau \rightsquigarrow\left(\varphi_{*}, \widetilde{\varphi}_{0, \ldots, k+\operatorname{gar}(\tau)}\right)$, then $\Theta^{b^{\prime}}, \mathcal{K}^{b} \vdash_{\text {ST }}\left(\varphi_{*}, \widetilde{\varphi}_{0}, \ldots, k+\operatorname{gar}(\tau)\right)$ : $\tau^{b, k+2}$. Also, for $(y: \tau) \in \mathcal{K}, y_{*}$ does not occur free in $\widetilde{\varphi}_{0, \ldots, k+\operatorname{gar}(\tau)}$.

[^3]Proof. Straightforward induction on the derivation of $\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \tau \rightsquigarrow$ $\left(\varphi_{*}, \widetilde{\varphi}_{0, \ldots, k+\operatorname{gar}(\tau)}\right)$.

The following theorem states the correctness of the translation.
Theorem 10. If $\left(D, S \lambda \widetilde{z}_{1, \ldots, M}\right.$.true $) \rightsquigarrow\left(D^{\prime}, \psi\right)$, then $\llbracket\left(D, S \lambda \widetilde{z}_{1, \ldots, M}\right.$.true $) \rrbracket=\llbracket\left(D^{\prime}, \psi\right) \rrbracket$.
Proof. The proof consists of two steps. First we reduce the proof to the case where $D$ is recursion-free. This is achieved by a rather standard technique; we use the finite approximation ( $D^{(m)}, S^{(m)} \lambda \widetilde{z}_{1, \ldots, M}$.true) ( $m \in \mathbb{N}$ ), which behaves like ( $D, S \lambda \widetilde{z}_{1, \ldots, M}$.true) up to $m$-steps, but diverges after $m$-steps.

Then in the recursion-free case, we show a subject reduction property, where we use two substitution lemmas that correspond to Tr-App and Tr-AppG. For the subject reduction, we also uses a reduction relation modified by explicit substitution. The explicit substitution delays the substitution of order-0 arguments, and we extend the translation given in Figure 2 by a new rule for explicit substitution formulas, which "simulates" the rule Tr-AppG. See Appendix A for details.

## 5 Applications

As mentioned already, the translation from order- $n$ reachability games to order- $(n+1)$ mayreachability enables us to use automated (un)reachability checkers for solving the reachability game problem, and the translation in the other direction enables us to use, for example, reachability game solvers for non-higher-order programs as a may-reachability checker for order-1 programs.

As a direct application of the former translation, we have applied it to the $\nu \mathrm{HFL}(\mathrm{Z})$ solver RETHFL 10, which is a refinement-type-based validity checker for formulas of $\nu \mathrm{HFL}(\mathrm{Z})$, the fragment of $\operatorname{HFL}(Z)$ without least fixpoint operators (but with greatest fixpoint operators). The fragment $\nu \mathrm{HFL}(\mathrm{Z})$ is dual to $\mu \mathrm{HFL}(\mathrm{Z})$, in the sense that, for every closed formula $\varphi$ of type $\star$ of $\mu \mathrm{HFL}(\mathrm{Z})$, there exists a $\nu \mathrm{HFL}(\mathrm{Z})$ formula $\bar{\varphi}$ such that $\varphi$ is valid if and only if $\bar{\varphi}$ is invalid, and vice versa; $\bar{\varphi}$ is obtained from $\varphi$ by just replacing each logical operator (including fixpoint operators) with its de Morgan dual, and $e_{1} \leq e_{2}$ with $e_{1}>e_{2}$. Using a refinement type system, RETHFL reduces the validity of a given $\nu \mathrm{HFL}(\mathrm{Z})$ formula in a sound (but incomplete) manner to an extended CHC (constraint Horn clauses) problem, where disjunction is allowed in the head of each clause, and passes the problem to an extended CHC solver called PCSat [21]. For a fragment of $\nu \mathrm{HFL}(\mathrm{Z})$ corresponding to disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$, however, the reduced problem is actually an ordinary CHC problem, for which more efficient tools [17, 8, 4] can be invoked. Thus, we can use the translation in Section 3 to improve the efficiency of RETHFL.

From the benchmark suite of ReTHFL [10] (which originates from [9], https://github.com/Hogeyama/hfl-benchmark/tree/master/inputs/hfl/HO-nontermination), we picked the "non-termination" benchmark set, which consists of formulas obtained from non-termination verification of higher-order programs. All the formulas in that benchmark set do not belong to (the dual of) disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$ (in contrast, the problems in the other benchmark sets belong to disjunctive $\mu \mathrm{HFL}(\mathrm{Z})$, hence our translation is not required). We have implemented the translation in Section 3 applied it to the problems in the "non-termination" benchmark set, and then ran RETHFL with a CHC solver HoIcE as the back-end solver. We have compared the result with plain ReTHFL (without the transformation), which uses the extended CHC solver PCSat.

The results are summarized in Table 1 The column 'RETHFL' shows the result of plain RETHFL with PCSat as the back-end extended CHC solver (since ordinary CHC solvers

Table 1: Experimental results. Times are measured in seconds, with the timeout of 180 seconds.

| input | RETHFL | ReTHFL+i.s. | RETHFL+ tr. |
| :--- | ---: | ---: | ---: |
| fixpoint_nonterm | 11.579 | 0.054 | 0.102 |
| unfoldr_nonterm | timeout | unknown | 4.22 |
| indirect_e | 16.832 | 0.035 | 0.066 |
| alternate | unknown | unknown | unknown |
| fib_CPS_nonterm | timeout | 0.047 | 0.075 |
| foldr_nonterm | 8.447 | unknown | 0.122 |
| passing_cond | 116.423 | unknown | 0.444 |
| indirectHO_e | 11.582 | 0.044 | 0.073 |
| inf_closure | timeout | 20.171 | 9.080 |
| loopHO | timeout | 0.026 | 0.121 |

are inapplicable to this benchmark set, as explained above). The column 'RETHFL+i.s.' show the result of RETHFL where the subtyping relation has been replaced by the imprecise one (equivalent to that of Horus [3], a HoCHC solver that can also be viewed as a $\nu \mathrm{HFL}(\mathrm{Z})$ solver) so that the type checking problem is reduced to ordinary CHC solving. The column 'RETHFL+tr.' shows the result of RETHFL with our translation. In both 'RETHFL+i.s.' and 'ReTHFL+tr.', HoIce was used as the back-end CHC solver. The entry "unknown" indicates that the solver terminated with the answer "ill-typed", in which case, we do not know whether the formula is valid or invalid, due to the incompleteness of the underlying refinement type system 5 The refinement type system used in 'RETHFL+i.s.' is less precise than the one used in RETHFL; hence, it returns more unknowns. As clear from the table, our translation significantly improved the efficiency of RETHFL.

The translation in the other direction presented in Section 4 also sometimes helps ReTHFL, especially for relaxing the limitation caused by the incompleteness of the underlying refinement type system. For example, consider the formula $S$ true, where:

$$
\begin{aligned}
S t & ={ }_{\mu} \operatorname{App}(\lambda x \cdot x \neq 0 \wedge t) 0 \\
A p p p y & ={ }_{\mu} p y \vee \operatorname{App}(\lambda z \cdot p(z-1))(y+1) .
\end{aligned}
$$

The formula is invalid, but RETHFL (nor Horus [3], a higher-order CHC solver based on a refinement type system) cannot prove the validity of the dual formula, due to the incompleteness of the refinement type system (which is related to the incompleteness of a refinement type system addressed by [23] by inserting extra arguments). By applying the transformation in Section 4 we obtain an equivalent order-0 formula, for which the underlying type system of RETHFL is complete and thus automatically proved.

## 6 Related Work

The relationship between order- $n$ reachability games and order- $(n+1)$ may-reachability has some deep connection to the relationship between order- $n$ tree languages and order- $(n+1)$ word languages [6, 1, 2, intuitively because the may-reachability problem is concerned about the set of "paths" of the execution tree of a given program, whereas the reachability game

[^4]problem is also concerned about the branching structures of the execution tree. Indeed, our translations (especially, the use of $\varphi_{*}$ and $\varphi_{0}$ components in the translation in Section (4) have been inspired by Asada and Kobayashi's translations between tree and word languages [2]. Kobayashi et al. [11] have also used a similar idea for a characterization of termination probabilities of higher-order probabilistic programs.

For finite-data programs (programs in Section 2.3 without integers), according to the complexity results on HORS model checking [18, [13, both the order- $n$ reachability game problem and the order- $(n+1)$ may-reachability game problem are $n$-EXPTIME complete, which imply that there are mutual translations between them. Concrete translations have, however, not been given (except unnatural translations through Turing machines). Also, the complexity-theoretic argument for the existence of translations does not apply in the presence of integers.

For HORS model checking, Parys 19 developed an order-decreasing transformation for higher-order grammars, which shares some ideas with our translation in Section 4 The details of the translations are however quite different. His translation makes use of finiteness in a crucial manner, and is not applicable in the presence of integers. Also, his translation is not size-preserving.

For order-1 programs, Kobayashi et al. [12] have shown that linear-time omega regular properties can be translated to order-0 HFL(Z) formulas. Our translation in Section 4 may be viewed as a higher-order extension of their translation, while the properties are restricted to may-reachability.

The fragment $\mu \mathrm{HFL}(\mathrm{Z})$ (or its dual fragment $\nu \mathrm{HFL}(\mathrm{Z})$ ) is essentially (modulo the restriction of data domains to integers) equivalent to HoCHC [3], a higher-order extension of CHC . Therefore, the result of this paper should be useful also for improving HoCHC solvers.

## 7 Conclusion

We have shown translations between order- $n$ reachability games and order- $(n+1)$ mayreachability, and proved their correctness. We have applied the translations to higher-order program verification, and obtained promising results in preliminary experiments. As mentioned in Section 6, our results are closely related to the correspondence between higher-order word and tree languages [2]. A deeper investigation of the relationship and generalization of the translations that subsume the related translations [2, 11] are left for future work.

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## References

[1] K. Asada and N. Kobayashi. On word and frontier languages of unsafe higher-order grammars. In I. Chatzigiannakis, M. Mitzenmacher, Y. Rabani, and D. Sangiorgi, editors, 43 rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy, volume 55 of LIPIcs, pages 111:1-111:13. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.
[2] K. Asada and N. Kobayashi. Size-preserving translations from order-(n+1) word grammars to order-n tree grammars. In Z. M. Ariola, editor, 5th International Conference on Formal Structures for Computation and Deduction, FSCD 2020, June 29-July 6, 2020,

Paris, France (Virtual Conference), volume 167 of LIPIcs, pages 22:1-22:22. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020.
[3] T. C. Burn, C. L. Ong, and S. J. Ramsay. Higher-order constrained horn clauses for verification. Proc. ACM Program. Lang., 2(POPL):11:1-11:28, 2018.
[4] A. Champion, T. Chiba, N. Kobayashi, and R. Sato. ICE-based refinement type discovery for higher-order functional programs. J. Autom. Reason., 64(7):1393-1418, 2020.
[5] P. Cousot and R. Cousot. Constructive versions of Tarski's fixed point theorems. Pacific Journal of Mathematics, 81(1):43-57, 1979.
[6] W. Damm. The IO- and OI-hierarchies. Theor. Comput. Sci., 20:95-207, 1982.
[7] E. Grädel, W. Thomas, and T. Wilke, editors. Automata, Logics, and Infinite Games: A Guide to Current Research, volume 2500 of LNCS. Springer, 2002.
[8] H. Hojjat and P. Rümmer. The eldarica horn solver. In 2018 Formal Methods in Computer Aided Design (FMCAD), pages 1-7, 2018.
[9] N. Iwayama, N. Kobayashi, R. Suzuki, and T. Tsukada. Predicate abstraction and CEGAR for $\nu \mathrm{hfl}_{z}$ validity checking. In D. Pichardie and M. Sighireanu, editors, Static Analysis - 27th International Symposium, SAS 2020, Virtual Event, November 18-20, 2020, Proceedings, volume 12389 of Lecture Notes in Computer Science, pages 134-155. Springer, 2020.
[10] H. Katsura, N. Iwayama, N. Kobayashi, and T. Tsukada. A new refinement type system for automated $\nu \mathrm{hff}_{z}$ validity checking. In B. C. d. S. Oliveira, editor, Programming Languages and Systems - 18th Asian Symposium, APLAS 2020, Fukuoka, Japan, November 30-December 2, 2020, Proceedings, volume 12470 of Lecture Notes in Computer Science, pages 86-104. Springer, 2020.
[11] N. Kobayashi, U. Dal Lago, and C. Grellois. On the termination problem for probabilistic higher-order recursive programs. In Proceedings of LICS 2019. IEEE, 2019.
[12] N. Kobayashi, T. Nishikawa, A. Igarashi, and H. Unno. Temporal verification of programs via first-order fixpoint logic. In B. E. Chang, editor, Static Analysis - 26th International Symposium, SAS 2019, Porto, Portugal, October 8-11, 2019, Proceedings, volume 11822 of Lecture Notes in Computer Science, pages 413-436. Springer, 2019.
[13] N. Kobayashi and C.-H. L. Ong. Complexity of model checking recursion schemes for fragments of the modal mu-calculus. $L M C S, 7(4), 2011$.
[14] N. Kobayashi, R. Sato, and H. Unno. Predicate abstraction and CEGAR for higher-order model checking. In PLDI 2011, pages 222-233. ACM Press, 2011.
[15] N. Kobayashi, T. Tsukada, and K. Watanabe. Higher-order program verification via HFL model checking. CoRR, abs/1710.08614, 2017.
[16] N. Kobayashi, T. Tsukada, and K. Watanabe. Higher-order program verification via HFL model checking. In A. Ahmed, editor, Programming Languages and Systems - 27th European Symposium on Programming, ESOP 2018, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2018, Thessaloniki, Greece, April 14-20, 2018, Proceedings, volume 10801 of Lecture Notes in Computer Science, pages 711-738. Springer, 2018.
[17] A. Komuravelli, A. Gurfinkel, and S. Chaki. Smt-based model checking for recursive programs. Formal Methods Syst. Des., 48(3):175-205, 2016.
[18] C.-H. L. Ong. On model-checking trees generated by higher-order recursion schemes. In LICS 2006, pages 81-90. IEEE Computer Society Press, 2006.
[19] P. Parys. Higher-order model checking step by step. In N. Bansal, E. Merelli, and J. Worrell, editors, 48 th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference), volume 198 of LIPIcs, pages 140:1-140:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.
[20] P. M. Rondon, M. Kawaguchi, and R. Jhala. Liquid types. In PLDI 2008, pages 159-169, 2008.
[21] Y. Satake, H. Unno, and H. Yanagi. Probabilistic inference for predicate constraint satisfaction. In The Thirty-Fourth AAAI Conference on Artificial Intelligence, AAAI 2020, The Thirty-Second Innovative Applications of Artificial Intelligence Conference, IAAI 2020, The Tenth AAAI Symposium on Educational Advances in Artificial Intelligence, EAAI 2020, pages 1644-1651. AAAI Press, 2020.
[22] T. Tsukada. On computability of logical approaches to branching-time property verification of programs. In H. Hermanns, L. Zhang, N. Kobayashi, and D. Miller, editors, LICS '20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, Saarbrücken, Germany, July 8-11, 2020, pages 886-899. ACM, 2020.
[23] H. Unno, Y. Satake, and T. Terauchi. Relatively complete refinement type system for verification of higher-order non-deterministic programs. In Proceeedings of POPL 2018, 2018. to appear.
[24] M. Viswanathan and R. Viswanathan. A higher order modal fixed point logic. In CONCUR, volume 3170 of $L N C S$, pages 512-528. Springer, 2004.
[25] K. Watanabe, T. Tsukada, H. Oshikawa, and N. Kobayashi. Reduction from branchingtime property verification of higher-order programs to HFL validity checking. In M. V. Hermenegildo and A. Igarashi, editors, Proceedings of the 2019 ACM SIGPLAN Workshop on Partial Evaluation and Program Manipulation, PEPM@POPL 2019, Cascais, Portugal, January 14-15, 2019, pages 22-34. ACM, 2019.

## Appendix

## A Proof of Theorem 10

We prove Theorem 10 in the following two steps: (i) we first reduce the proof of Theorem 10 to the case where a given equation system is recursion-free (in Appendix A.1), by using a standard technique of finite approximation, and then (ii) we show the recursion-free case (in Appendix A.3, with some preparation in Appendix A.2). The latter step is the main part of this proof.

For an equation system $(\Theta, D, S$ true $)$, we define $={ }_{D}$ as follows: $\varphi={ }_{D} \psi$ if $\llbracket(D, \varphi) \rrbracket=$ $\llbracket(D, \psi) \rrbracket$. For $\left(F \widetilde{x}={ }_{\mu} \varphi\right) \in D$, we may drop the subscript $\mu$ and write $F \widetilde{x}=\varphi$ if there is no confusion. We write $\left[\psi_{i} / x_{i}\right]_{i=1}^{m} \varphi$ for the substitution $\left[\psi_{1} / x_{1}, \ldots, \psi_{m} / x_{m}\right] \varphi$.

## A. 1 Reduction to the Recursion-free Case

Here we briefly explain how we can reduce Theorem 10 to the recursion-free case.
For an equation system $\left(\Theta, D, \varphi_{0}\right)$ and $m \in \mathbb{N}$, the $m$-th approximation $\left(\Theta^{(m)}, D^{(m)}, \varphi_{0}^{(m)}\right)$ is defined as follows:

$$
\begin{aligned}
\Theta^{(m)} & :=\left\{F^{(i)} \mapsto \Theta(F) \mid F \in \operatorname{dom}(\Theta), 0 \leq i \leq m\right\} \\
\varphi^{(i)} & \left.:=\left[F^{(i)} / F\right]_{F \in \operatorname{dom}(\Theta)} \varphi \quad \text { (for any } \varphi \text { and } i \in\{0, \ldots, m\}\right) \\
D^{(m)} & :=\left\{F^{(i)} \widetilde{x}=\varphi^{(i-1)} \mid(F \widetilde{x}=\varphi) \in D, 1 \leq i \leq m\right\} \\
& \cup\left\{F^{(0)} \widetilde{x}=\text { false } \wedge \varphi^{(0)} \mid(F \widetilde{x}=\varphi) \in D\right\} .
\end{aligned}
$$

For $F^{(0)}$ above, we use false $\wedge \varphi^{(0)}$ rather than false, in order to keep the form of Equation (11). By the technique in [15, Appendix B.1], we can show that

$$
\llbracket\left(D, \varphi_{0}\right) \rrbracket=\sqcup_{\tau}\left\{\llbracket\left(D^{(m)}, \varphi_{0}^{(m)}\right) \rrbracket \mid m \in \mathbb{N}\right\} .
$$

An equation system $\left(\Theta, D, \varphi_{0}\right)$ is called recursion free if there is no cyclic dependency on $D$. More precisely, we define a binary relation $\succ$ on $\operatorname{dom}(\Theta)$ as follows: $F \succ F^{\prime}$ iff $F^{\prime} \in \mathrm{FV}^{\prime}(\varphi)$ where $(F \widetilde{x}=\varphi) \in D$ and $\mathrm{FV}^{\prime}(\varphi)$ is defined by the following:

$$
\begin{aligned}
\mathrm{FV}^{\prime}(x) & =\{x\}, \\
\mathrm{FV}^{\prime}\left(\varphi_{1} \vee \varphi_{2}\right) & =\mathrm{FV}^{\prime}\left(\varphi_{1}\right) \cup \mathrm{FV}^{\prime}\left(\varphi_{2}\right), \\
\mathrm{FV}^{\prime}\left(e_{1} \leq e_{2} \wedge \varphi\right) & = \begin{cases}\emptyset & \left(e_{1} \leq e_{2}=\mathrm{false}\right) \\
\mathrm{FV}^{\prime}(\varphi) & \left(e_{1} \leq e_{2} \neq \mathrm{false}\right)\end{cases} \\
\mathrm{FV}^{\prime}\left(\varphi_{1} \varphi_{2}\right) & =\mathrm{FV}^{\prime}\left(\varphi_{1}\right) \cup \mathrm{FV}^{\prime}\left(\varphi_{2}\right), \\
\mathrm{FV}^{\prime}(\varphi e) & =\mathrm{FV}^{\prime}(\varphi) .
\end{aligned}
$$

Then $D$ is recursion free if the transitive closure $\succ^{*}$ of $\succ$ is irreflexive (i.e., $F \succ^{*} F$ for no $F \in \operatorname{dom}(\Theta)$ ). Clearly $\left(D^{(m)}, \varphi_{0}^{(m)}\right)$ is recursion-free.

Now, since our translation is compositional, we can easily show the following:
Lemma 11. If $\left(D^{(m)},(S \lambda \widetilde{z} \text {.true })^{(m)}\right) \rightsquigarrow\left(D_{m}, \varphi_{m}\right)$, then $\llbracket\left(D_{m}, \varphi_{m}\right) \rrbracket=$ $\llbracket\left(D^{\prime(m)},\left(\exists \widetilde{z} \cdot S_{1} \widetilde{z}\right)^{(m)}\right) \rrbracket$.

Then we can reduce the proof of Theorem 10 to the recursion-free case as follows. Let $(D, S \lambda \widetilde{z}$.true $) \rightsquigarrow\left(D^{\prime}, \exists \widetilde{z} . S_{1} \widetilde{z}\right)$ and $\left(D^{(m)},(S \lambda \widetilde{z} \text {.true })^{(m)}\right) \rightsquigarrow\left(D_{m}, \varphi_{m}\right)$; then

$$
\begin{aligned}
\llbracket(D, S \lambda \widetilde{z} . \text { true }) \rrbracket & =\sqcup_{\star}\left\{\llbracket\left(D^{(m)},(S \lambda \widetilde{z} . \text { true })^{(m)}\right) \rrbracket \mid m \in \mathbb{N}\right\} \\
& =\sqcup_{\star}\left\{\llbracket\left(D_{m}, \varphi_{m}\right) \rrbracket \mid m \in \mathbb{N}\right\} \\
& =\sqcup_{\star}\left\{\llbracket\left(D^{\prime(m)},\left(\exists \widetilde{z} \cdot S_{1} \widetilde{z}\right)^{(m)}\right) \rrbracket \mid m \in \mathbb{N}\right\} \\
& =\llbracket\left(D^{\prime}, \exists \widetilde{z} \cdot S_{1} \widetilde{z}\right) \rrbracket
\end{aligned}
$$

where for the second equation we assume the recursion-free case.

## A. 2 Reduction Relation with Explicit Substitution

In our proof of the recursion-free case, we show a subject reduction property. To this end, we modify the reduction strategy by using explicit substitution, keeping the adequacy for the semantics. For this modification, we first extend the syntax of formulas as follows:

$$
\begin{align*}
\varphi::= & x\left|\varphi_{1} \vee \varphi_{2}\right| e_{1} \leq e_{2} \wedge \varphi\left|\varphi_{1} \varphi_{2}\right| \varphi e  \tag{2}\\
& \mid\left\{\varphi_{1} / x_{1}, \ldots, \varphi_{m} / x_{m}\right\} \varphi
\end{align*}
$$

Here $\left\{\varphi_{1} / x_{1}, \ldots, \varphi_{m} / x_{m}\right\} \varphi$ is called an explicit substitution, and limited to ground types as follows:

$$
\begin{gather*}
\mathcal{K} \vdash_{\mathrm{ST}} \varphi_{i}: \operatorname{Int}^{M} \rightarrow \star \quad(i=1, \ldots, m) \\
\mathcal{K}, x_{1}: \operatorname{Int}^{M} \rightarrow \star, \ldots, x_{m}: \operatorname{Int}^{M} \rightarrow \star \vdash_{\mathrm{ST}} \varphi: \star  \tag{T-ESub}\\
\mathcal{K} \vdash_{\mathrm{ST}}\left\{\varphi_{1} / x_{1}, \ldots, \varphi_{m} / x_{m}\right\} \varphi: \star
\end{gather*}
$$

Its meaning is given through

$$
\left(D,\left\{\varphi_{1} / x_{1}, \ldots, \varphi_{m} / x_{m}\right\} \varphi\right)^{\mu}:=\left(D,\left[\varphi_{1} / x_{1}, \ldots, \varphi_{m} / x_{m}\right] \varphi\right)^{\mu} .
$$

Thus explicit substitution has the same meaning as ordinary substitution, but delays substitution until we need $\varphi_{i}$ for further reduction. As in the definition of $\longrightarrow_{D}$ below, while we use ordinary substitutions for $\beta$-redex to which we can apply Tr-App and Tr-AprI, we use explicit substitution for those corresponding to TR-AppG because, the argument after the translation by Tr-AppG is never substituted.

We extend the translation by adding the following rule for explicit substitutions:

$$
\begin{array}{r}
\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \xi_{i}: \operatorname{Int}^{M} \rightarrow \star \rightsquigarrow\left(\xi_{i, *}, \xi_{i, 0}, \ldots, \xi_{i, k}\right) \\
\quad(i=1, \ldots, m) \\
\mathcal{K} ; \widetilde{x}_{1, \ldots, k}, \widetilde{x}^{\prime}{ }_{1, \ldots, m} \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k+m}\right) \\
\psi_{j}=\lambda \widetilde{w}_{1, \ldots, M} \cdot \varphi_{j} \widetilde{w} \vee \bigvee_{i=1}^{m} \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{k+i} \widetilde{u} \wedge \xi_{i, j} \widetilde{u} \widetilde{w}\right) \\
\frac{(j=*, 0, \ldots, k)}{\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta}\left\{\xi_{1} / x_{1}^{\prime}, \ldots, \xi_{m} / x_{m}^{\prime}\right\} \varphi: \star \rightsquigarrow\left(\psi_{*}, \psi_{0}, \ldots, \psi_{k}\right)} \tag{Tr-ESub}
\end{array}
$$

In the rest of this section, by a formula we mean a formula that may contain extended substitutions, except for formulas in an equation system and except for the case where we explain explicitly. Note that output formulas of the extended translation never contain explicit substitutions.

Let ( $\Theta, D, S \lambda \widetilde{z}$.true) be an equation system. For decomposing actual arguments $\alpha_{1}, \ldots, \alpha_{m}$ of a function $F \in \operatorname{dom}(\Theta)$-recall that $\alpha_{i}$ ranges over formulas and integer expressions-we define decompArg $\left(\alpha_{1}, \ldots, \alpha_{m^{\prime}}, \Theta(F)\right)$ in the same way as decomparg as follows:

$$
\begin{aligned}
& \operatorname{decompArg}(\epsilon, \star)=(\epsilon, \epsilon, \epsilon) \\
& \operatorname{decompArg}(\alpha \cdot \widetilde{\beta}, \kappa \rightarrow \tau)= \\
& \begin{cases}(\alpha \cdot \widetilde{\varphi}, \widetilde{\psi}, \widetilde{e}) & \text { if } \operatorname{decompArg}(\widetilde{\beta}, \tau)=(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{e}), \widetilde{\varphi} \neq \epsilon \\
(\alpha, \widetilde{\psi}, \widetilde{e}) & \text { if } \operatorname{ord}(\kappa)>0, \operatorname{decompArg}(\widetilde{\beta}, \tau)=(\epsilon, \widetilde{\psi}, \widetilde{e}) \\
(\epsilon, \alpha \cdot \widetilde{\psi}, \widetilde{e}) & \text { if } \kappa=\operatorname{Int}^{M} \rightarrow \star, \\
& \text { decompArg }(\widetilde{\beta}, \tau)=(\epsilon, \widetilde{\psi}, \widetilde{e}) \\
(\epsilon, \widetilde{\psi}, \alpha \cdot \widetilde{e}) & \text { if } \kappa=\operatorname{Int}, \operatorname{decompArg}(\widetilde{\beta}, \tau)=(\epsilon, \widetilde{\psi}, \widetilde{e})\end{cases}
\end{aligned}
$$

Now we define the modified reduction relation $\longrightarrow_{D}$ for a formula $\varphi$ such that $\Theta, x_{1}$ : $\operatorname{Int}^{M} \rightarrow \star, \ldots, x_{k}:$ Int $^{M} \rightarrow \star \vdash_{\text {ST }} \varphi: \star$ holds for some $x_{1}, \ldots, x_{k}$. We define the set of evaluation contexts by:

$$
E::=[]|E \vee \varphi| \varphi \vee E \mid\left\{\varphi_{1} / x_{1}, \ldots, \varphi_{m} / x_{m}\right\} E .
$$

Then $\longrightarrow_{D}$ is defined by the following rules:

$$
\begin{gathered}
\llbracket \vdash_{\text {ST }} e_{1}: \text { Int } \rrbracket>\llbracket \vdash_{\text {ST }} e_{2}: \text { Int } \rrbracket \quad\left(e_{1} \leq e_{2}\right) \neq \mathrm{false} \\
E\left[e_{1} \leq e_{2} \wedge \varphi\right] \longrightarrow_{D} E[\text { false } \wedge \varphi] \\
\frac{\llbracket \vdash_{\text {ST }} e_{1}: \text { Int } \rrbracket \leq \llbracket \vdash_{\text {ST }} e_{2}: \text { Int } \rrbracket}{E\left[e_{1} \leq e_{2} \wedge \varphi\right] \longrightarrow_{D} E[\varphi]} \\
\left(F w_{1} \cdots w_{m}=\varphi\right) \in D \\
\operatorname{decompArg}\left(\alpha_{1}, \cdots, \alpha_{m}, \Theta(F)\right)=(\widetilde{\varphi}, \widetilde{\psi}, \widetilde{e}) \\
\operatorname{decomparg}\left(w_{1}, \cdots, w_{m}, \Theta(F)\right)=(\widetilde{y}: \widetilde{\kappa}, \widetilde{x}, \widetilde{z}) \\
\frac{\widetilde{x} \text { do not occur in } E\left[F \alpha_{1} \cdots \alpha_{m}\right]}{E\left[F \alpha_{1} \cdots \alpha_{m}\right] \longrightarrow_{D} E[\{\widetilde{\psi} / \widetilde{x}\}[\widetilde{e} / \widetilde{z}][\widetilde{\varphi} / \widetilde{y}] \varphi]} \\
\frac{E\left[\{\widetilde{\varphi} / \widetilde{x}\}\left(x_{i} \widetilde{e}\right)\right] \longrightarrow_{D} E\left[\varphi_{i} \widetilde{e}\right]}{} \\
\frac{x \notin\left\{x_{1}, \ldots, x_{|\widetilde{x}|}\right\} \cup \operatorname{dom}(\Theta)}{E[\{\widetilde{\varphi} / \widetilde{x}\}(x \widetilde{e})] \longrightarrow_{D} E[x \widetilde{e}]} \\
\frac{E\left[\{\widetilde{\varphi} / \widetilde{x}\}\left(\psi_{1} \vee \psi_{2}\right)\right] \longrightarrow_{D} E\left[\left(\{\widetilde{\varphi} / \widetilde{x}\} \psi_{1}\right) \vee\left(\{\widetilde{\varphi} / \widetilde{x}\} \psi_{2}\right)\right]}{} \\
\frac{E[\{\widetilde{\varphi} / \widetilde{x}\}(\mathrm{false} \wedge \varphi)] \longrightarrow_{D} E[\text { false } \wedge(\{\widetilde{\varphi} / \widetilde{x}\} \varphi)]}{}
\end{gathered}
$$

Note that the above reduction preserves the form of (2) and hence the applicability of the translation $\rightsquigarrow$. For any $\varphi, \varphi$ is a normal form with respect to $\longrightarrow_{D}$ iff $\varphi$ is generated by:

$$
\begin{equation*}
\zeta::=x \widetilde{e}(x \notin \operatorname{dom}(\Theta)) \mid \text { false } \wedge \varphi \mid \zeta \vee \zeta . \tag{3}
\end{equation*}
$$

Clearly we have:
Lemma 12. If $\varphi \longrightarrow_{D} \psi$, then $\llbracket(D, \varphi) \rrbracket=\llbracket(D, \psi) \rrbracket$.

## A. 3 Correctness in the Recursion-free Case

To show the correctness in the recursion-free case, first we prepare two substitution lemmas that correspond to the application rules Tr-App and Tr-AppI. Then we show the subject reduction property and then the correctness.

The following is the substitution lemma corresponding to Tr-AppI:
Lemma 13 (Substitution lemma (integer)). Let $\varphi$ be a formula that does not contain an explicit substitution, e be a closed integer expression, and

$$
\mathcal{K}, z: \text { Int } ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k+m}\right) .
$$

where $m=\operatorname{gar}(\tau)$. Then we have

$$
\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta}[e / z] \varphi: \tau \rightsquigarrow\left([e / z] \varphi_{*},[e / z] \varphi_{0}, \ldots,[e / z] \varphi_{k+m}\right) .
$$

Proof. By straightforward induction on $\varphi$.
Next we show the substitution lemma corresponding to Tr-App. First we prepare some definitions and a lemma.

For a formula $\varphi$, we write $\widetilde{\varphi}_{\backslash k}^{m}$ for $\left(\varphi_{0}, \varphi_{k+1}, \ldots, \varphi_{k+m}\right)$. Note that the translation result of $\varphi \psi$ in TR-APP in Figure 2 can be written as the following:

$$
\begin{array}{r}
\left(\varphi_{*}\left(\psi_{*}, \widetilde{\psi}_{\backslash k}^{m^{\prime}}\right), \varphi_{0}\left(\psi_{0}, \widetilde{\psi}_{\backslash k}^{m^{\prime}}\right), \varphi_{1}\left(\psi_{1}, \widetilde{\psi}_{\backslash k}^{m^{\prime}}\right), \ldots, \varphi_{k}\left(\psi_{k}, \widetilde{\psi}_{\backslash k}^{m^{\prime}}\right)\right. \\
\left.\varphi_{k+1}\left(\widetilde{\psi}_{\backslash k}^{m^{\prime}}\right), \ldots, \varphi_{k+m}\left(\widetilde{\psi}_{\backslash k}^{m^{\prime}}\right)\right)
\end{array}
$$

The following can be shown easily by induction on $\varphi$.
Lemma 14 (weakening). If

$$
\mathcal{K} ; \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \varphi_{1}, \ldots, \varphi_{k}, \varphi_{k+1}, \ldots, \varphi_{k+m}\right)
$$

then

$$
\mathcal{K} ; \widetilde{x}_{1, \ldots, k}, x \vdash_{\Theta} \varphi: \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \varphi_{1}, \ldots, \varphi_{k}, \varphi_{0}, \varphi_{k+1}, \ldots, \varphi_{k+m}\right) .
$$

Now we show the substitution lemma. Here we consider simultaneous substitution, because we cannot apply this lemma repeatedly since $[\widetilde{\psi} / \widetilde{y}] \varphi$ below may contain an explicit substitution.

Lemma 15 (Substitution lemma (higher-order)). Let $\varphi$ be a formula that does not contain an explicit substitution, and

$$
\begin{aligned}
& \widetilde{y}_{1, \ldots, q}: \widetilde{\kappa} ;{\widetilde{x^{\prime}}}_{1, \ldots, m} \vdash_{\Theta} \varphi: \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{m+\operatorname{gar}(\tau)}\right) \\
& m_{i}^{\prime}=\operatorname{gar}\left(\kappa_{i}\right) \quad \operatorname{decomp}\left(\kappa_{i}\right)=\left(\widetilde{\kappa_{i}}, m_{i}^{\prime}, p_{i}\right) \quad(i=1, \ldots, q) \\
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \psi_{i}: \kappa_{i} \rightsquigarrow\left(\psi_{i, *}, \psi_{i, 0}, \ldots, \psi_{i, k+m_{i}^{\prime}}\right) \quad(i=1, \ldots, q) \\
& \widetilde{y}_{i}=\left(y_{i, *}, y_{i, 0}, \ldots, y_{i, \operatorname{gar}\left(\kappa_{i}\right)}\right) \quad \widetilde{y}_{i}^{\circ}=\left(y_{i, 0}, \ldots, y_{i, \operatorname{gar}\left(\kappa_{i}\right)}\right) \\
& \theta^{j}=\left[\left(\psi_{i, j}, \widetilde{\psi}_{i \backslash k}^{m_{i}^{\prime}}\right) / \widetilde{y}_{i}\right]_{i=1}^{q} \quad(j=*, 0, \ldots, k) \\
& \theta^{\circ}=\left[\quad \widetilde{\psi}_{i \backslash k}^{m_{i}^{\prime}} / \widetilde{y}_{i}^{\circ}\right]_{i=1}^{q} \\
& \widetilde{x}_{1, \ldots, k},{\widetilde{x^{\prime}}}_{1, \ldots, m} \vdash_{\Theta}[\widetilde{\psi} / \widetilde{y}] \varphi: \tau \rightsquigarrow\left(\varphi_{*}^{\circ}, \varphi_{0}^{\circ}, \ldots, \varphi_{k+m+\operatorname{gar}(\tau)}^{\circ}\right) .
\end{aligned}
$$

Then we have:

1. $\theta^{0} \varphi_{*}=\theta^{0} \varphi_{0}$.

$$
\begin{aligned}
& \text { 2. } \quad\left(\varphi_{*}^{\circ}, \varphi_{0}^{\circ}, \varphi_{1}^{\circ}, \ldots, \varphi_{k}^{\circ} \quad, \varphi_{k+1}^{\circ}, \ldots, \varphi_{k+m+\operatorname{gar}(\tau)}^{\circ}\right) \\
& =D_{D^{\prime}}\left(\theta^{*} \varphi_{*}, \theta^{0} \varphi_{*}, \theta^{1} \varphi_{*}, \ldots, \theta^{k} \varphi_{*}, \theta^{\circ} \varphi_{1}, \ldots, \theta^{\circ} \varphi_{m+\operatorname{gar}(\tau)}\right) \text {. }
\end{aligned}
$$

Proof. We can show Item 1 easily by induction on $\varphi$ and case analysis on the last rule used for the derivation $\widetilde{y}_{1, \ldots, q}: \widetilde{\kappa} ; \widetilde{x}_{1, \ldots, m}^{\prime} \vdash_{\Theta} \varphi: \tau \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{m+\operatorname{gar}(\tau)}\right)$. We show Item 2 by the same induction and case analysis. Basically the proof is straightforward, where we use the latter part of Lemma 9 Here we show only the cases of Tr-VAR and Tr-App; in the latter case, we use Item 1

- Case of Tr-VAR: Let the last rule be the following, where $i \in\{1, \ldots, q\}$ :

$$
\begin{gathered}
\operatorname{decomp}\left(\kappa_{i}\right)=\left(\widetilde{\kappa_{i}}, m_{i}^{\prime}, p_{i}\right) \\
\widetilde{y}: \widetilde{\kappa} ;{\widetilde{x^{\prime}}}_{1, \ldots, m} \vdash_{\Theta}(\varphi=) y_{i}: \kappa_{i} \\
\rightsquigarrow\left(\left(\varphi_{*}, \varphi_{0}, \varphi_{1}, \ldots, \varphi_{m}, \varphi_{m+1}, \ldots, \varphi_{m+\operatorname{gar}(\tau)}\right)=\right) \\
\left(y_{i, *}, y_{i, 0}, y_{i, 0}, \ldots, y_{i, 0}, y_{i, 1} \quad, \ldots, y_{i, m_{i}^{\prime}}\right)
\end{gathered}
$$

By the weakening lemma (Lemma 14), we have

$$
\begin{aligned}
& \widetilde{x}_{1, \ldots, k},{\widetilde{x^{\prime}}}_{1, \ldots, m} \vdash_{\Theta}([\tilde{\psi} / \widetilde{y}] \varphi=) \psi_{i}: \kappa_{i} \rightsquigarrow \\
& \left(\left(\varphi_{*}^{\circ}, \varphi_{0}^{\circ}, \varphi_{1}^{\circ}, \ldots, \varphi_{k}^{\circ}, \varphi_{k+1}^{\circ}, \ldots, \varphi_{k+m}^{\circ}, \varphi_{k+m+1}^{\circ}, \ldots, \varphi_{k+m+m_{i}^{\prime}}^{\circ}\right)=\right) \\
& \left(\psi_{i, *}, \psi_{i, 0}, \psi_{i, 1}, \ldots, \psi_{i, k}, \psi_{i, 0}, \ldots, \psi_{i, 0} \quad, \psi_{i, k+1} \quad, \ldots, \psi_{i, k+m_{i}^{\prime}}\right)
\end{aligned}
$$

Then we can check the required equation component-wise.

- Case of Tr-App: Let the last rule be the following:

$$
\begin{aligned}
& \operatorname{ord}\left(\kappa^{\prime} \rightarrow \tau\right)>1 \quad \operatorname{gar}\left(\kappa^{\prime} \rightarrow \tau\right)=m^{\prime} \quad \operatorname{gar}\left(\kappa^{\prime}\right)=m^{\prime \prime} \\
& \widetilde{y}: \widetilde{\kappa} ; \widetilde{x}^{\prime}{ }_{1, \ldots, m} \vdash_{\Theta} \varphi^{\prime}: \kappa^{\prime} \rightarrow \tau \rightsquigarrow\left(\varphi_{*}^{\prime}, \varphi_{0}^{\prime}, \ldots, \varphi_{m+m^{\prime}}^{\prime}\right) \\
& \frac{\widetilde{y}: \widetilde{\kappa} ; \widetilde{x}^{\prime}{ }_{1, \ldots, m} \vdash_{\Theta} \psi^{\prime}: \kappa^{\prime} \rightsquigarrow\left(\psi_{*}^{\prime}, \psi_{0}^{\prime}, \ldots, \psi_{m+m^{\prime \prime}}^{\prime}\right)}{\widetilde{y}: \widetilde{\kappa} ; \widetilde{x}^{\prime}{ }_{1, \ldots, m} \vdash_{\Theta}(\varphi=) \varphi^{\prime} \psi^{\prime}: \tau \rightsquigarrow} \\
& \left(\left(\varphi_{*}, \varphi_{0}, \varphi_{1} \ldots, \varphi_{m}, \varphi_{m+1} \ldots, \varphi_{m+\operatorname{gar}(\tau)}\right)=\right) \\
& \left(\varphi_{*}^{\prime}\left(\psi_{*}^{\prime}, \widetilde{\psi}_{\backslash m}^{\prime m^{\prime \prime}}\right), \varphi_{0}^{\prime}\left(\psi_{0}^{\prime}, \widetilde{\psi}_{\backslash m}^{m^{\prime \prime}}\right),\right. \\
& \varphi_{1}^{\prime}\left(\psi_{1}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \ldots, \varphi_{m}^{\prime}\left(\psi_{m}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \\
& \left.\varphi_{m+1}^{\prime}\left({\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \ldots, \varphi_{m+m^{\prime}}^{\prime}\left({\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right)\right)
\end{aligned}
$$

Here note that we have $\operatorname{gar}(\tau)=\operatorname{gar}\left(\kappa^{\prime} \rightarrow \tau\right)=m^{\prime}$ since ord $\left(\kappa^{\prime} \rightarrow \tau\right)>1$.
By induction hypothesis, there exist $\varphi_{*}^{\prime \circ}, \varphi_{0}^{\prime \circ}, \ldots, \varphi_{k+m+m^{\prime}}^{\prime \circ}$ such that

$$
\begin{aligned}
& \widetilde{x}_{1, \ldots, k},{\widetilde{x^{\prime}}}_{1, \ldots, m} \vdash_{\Theta}[\widetilde{\psi} / \widetilde{y}] \varphi^{\prime}: \kappa^{\prime} \rightarrow \tau \rightsquigarrow \\
& \quad\left(\varphi_{*}^{\prime \circ}, \varphi_{0}^{\prime \circ}, \varphi_{1}^{\prime \circ}, \ldots, \varphi_{k}^{\prime \circ}, \varphi_{k+1}^{\prime \circ}, \ldots, \varphi_{k+m+m^{\prime}}^{\prime \circ}\right) \\
& =_{D^{\prime}}\left(\theta^{*} \varphi_{*}^{\prime}, \theta^{0} \varphi_{*}^{\prime}, \theta^{1} \varphi_{*}^{\prime}, \ldots, \theta^{k} \varphi_{*}^{\prime}, \theta^{\circ} \varphi_{1}^{\prime}, \ldots, \theta^{\circ} \varphi_{m+m^{\prime}}^{\prime}\right)
\end{aligned}
$$

and there exist $\psi_{*}^{\prime \circ}, \psi_{0}^{\prime \circ}, \ldots, \psi_{k+m+m^{\prime \prime}}^{\prime \circ}$ such that

$$
\widetilde{x}_{1, \ldots, k}, \widetilde{x}^{\prime}{ }_{1, \ldots, m} \vdash_{\Theta}[\widetilde{\psi} / \widetilde{y}] \psi^{\prime}: \kappa^{\prime} \rightsquigarrow
$$

$$
\begin{aligned}
&\left(\psi_{*}^{\prime \circ}, \psi_{0}^{\prime \circ}, \psi_{1}^{\prime \circ}, \ldots, \psi_{k}^{\prime \circ}, \psi_{k+1}^{\prime \circ}, \ldots, \psi_{k+m+m^{\prime \prime}}^{\prime \circ}\right) \\
&=D^{\prime}\left(\theta^{*} \psi_{*}^{\prime}, \theta^{0} \psi_{*}^{\prime}, \theta^{1} \psi_{*}^{\prime}, \ldots, \theta^{k} \psi_{*}^{\prime}, \theta^{\circ} \psi_{1}^{\prime}, \ldots, \theta^{\circ} \psi_{m+m^{\prime \prime}}^{\prime}\right) .
\end{aligned}
$$

For any $j$ and $j^{\prime} \in\{*, 0,1, \ldots, k, \circ\}$, by the latter part of Lemma $9 \theta^{j} \xi=\theta^{j^{\prime}} \xi$ for any formula $\xi$ that occurs in

$$
\begin{aligned}
& \left(\varphi_{0}^{\prime}\left(\psi_{0}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \varphi_{1}^{\prime}\left(\psi_{1}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \ldots, \varphi_{m}^{\prime}\left(\psi_{m}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right),\right. \\
& \left.\varphi_{m+1}^{\prime}\left({\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \ldots, \varphi_{m+m^{\prime}}^{\prime}\left({\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right)\right) .
\end{aligned}
$$

Especially, for any $j \in\{*, 0,1, \ldots, k, \circ\}$, we have

$$
\begin{aligned}
& \widetilde{\psi^{\prime \circ}}{ }^{m^{\prime \prime}} \\
&=\left(\psi_{0}^{\prime \circ}, \psi_{k+m+1}^{\prime \circ}, \ldots, \psi_{k+m+m^{\prime \prime}}^{\prime \circ},\right) \\
&=D^{\prime}\left(\theta^{0} \psi_{*}^{\prime}, \theta^{\circ} \psi_{m+1}^{\prime}, \ldots, \theta^{\circ} \psi_{m+m^{\prime \prime}}^{\prime}\right) \\
&=\left(\theta^{0} \psi_{0}^{\prime}, \theta^{\circ} \psi_{m+1}^{\prime}, \ldots, \theta^{\circ} \psi_{m+m^{\prime \prime}}^{\prime}\right) \\
&=\left(\theta^{j} \psi_{0}^{\prime}, \theta^{j} \psi_{m+1}^{\prime}, \ldots, \theta^{j} \psi_{m+m^{\prime \prime}}^{\prime}\right)=\theta^{j} \widetilde{\psi^{\prime}}{ }_{\backslash m}^{\prime \prime}
\end{aligned}
$$

Now, with TR-App, we have

$$
\begin{aligned}
& \left(\varphi_{*}^{\circ}, \varphi_{0}^{\circ}, \varphi_{1}^{\circ}, \ldots, \varphi_{k}^{\circ}, \varphi_{k+1}^{\circ}, \ldots, \varphi_{k+m}^{\circ}, \varphi_{k+m+1}^{\circ}, \ldots, \varphi_{k+m+m^{\prime}}^{\circ}\right) \\
& =\left(\varphi_{*}^{\prime \circ}\left(\psi_{*}^{\prime \circ}, \widetilde{\psi^{\prime 0}} \backslash \frac{m^{\prime \prime}}{k+m}\right), \varphi_{0}^{\prime \circ}\left(\psi_{0}^{\prime \circ}, \widetilde{\psi^{\prime \circ}} \widetilde{m}^{m^{\prime \prime}}{ }^{2}\right),\right. \\
& \varphi_{1}^{\prime \circ}\left(\psi_{1}^{\prime \circ},{\widetilde{\psi^{\prime \circ}}}_{\backslash k+m}^{m^{\prime \prime}}\right) \quad, \ldots, \varphi_{k}^{\prime \circ}\left(\psi_{k}^{\prime \circ},{\widetilde{\psi^{\prime \prime}}}^{m^{\prime \prime}}{ }_{k+m}\right) \quad, \\
& \varphi_{k+1}^{\prime \circ}\left(\psi_{k+1}^{\prime \circ}, \widetilde{\psi^{\prime \circ}}{ }_{\backslash k+m}^{m^{\prime \prime}}\right), \ldots, \varphi_{k+m}^{\prime \circ}\left(\psi_{k+m}^{\prime \circ}, \widetilde{\psi^{\prime \prime}}{ }^{m^{\prime \prime}}{ }_{k+m}\right), \\
& \left.\varphi_{k+m+1}^{\prime \circ}\left(\widetilde{\psi^{\prime 0}}{ }_{\backslash k+m}^{m^{\prime \prime}}\right) \quad, \ldots, \varphi_{k+m+m^{\prime}}^{\prime \circ}\left(\widetilde{\psi^{\prime}} \backslash m^{m^{\prime \prime}}{ }^{\prime \prime}{ }^{\prime \prime}\right) \quad\right) \\
& ={ }_{D^{\prime}}\left(\theta^{*} \varphi_{*}^{\prime}\left(\theta^{*} \psi_{*}^{\prime}, \theta^{*}{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \theta^{0} \varphi_{*}^{\prime}\left(\theta^{0} \psi_{*}^{\prime}, \theta^{0} \widetilde{\psi^{\prime}}{ }_{m}^{m^{\prime \prime}}\right)\right. \text {, } \\
& \theta^{1} \varphi_{*}^{\prime}\left(\theta^{1} \psi_{*}^{\prime}, \theta^{1}{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \ldots, \theta^{k} \varphi_{*}^{\prime}\left(\theta^{k} \psi_{*}^{\prime}, \theta^{k}{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \\
& \theta^{\circ} \varphi_{1}^{\prime}\left(\theta^{\circ} \psi_{1}^{\prime}, \theta^{\circ}{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right), \ldots, \theta^{\circ} \varphi_{m}^{\prime}\left(\theta^{\circ} \psi_{m}^{\prime}, \theta^{\circ}{\widetilde{\psi^{\prime}}}_{(m}^{m^{\prime \prime}}\right), \\
& \theta^{\circ} \varphi_{m+1}^{\prime}\left(\theta^{\circ}{\widetilde{\psi^{\prime}} \backslash m}_{m^{\prime \prime}}{ }^{\prime} \quad, \ldots, \theta^{\circ} \varphi_{m+m^{\prime}}^{\prime}\left(\theta^{\circ}{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right) \quad\right) \\
& =\left(\theta^{*}\left(\varphi_{*}^{\prime}\left(\psi_{*}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right)\right), \theta^{0}\left(\varphi_{*}^{\prime}\left(\psi_{*}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right)\right),\right. \\
& \theta^{1}\left(\varphi_{*}^{\prime}\left(\psi_{*}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right)\right), \ldots, \theta^{k}\left(\varphi_{*}^{\prime}\left(\psi_{*}^{\prime},{\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right)\right), \\
& \theta^{\circ}\left(\varphi_{1}^{\prime}\left(\psi_{1}^{\prime}, \widetilde{\psi}^{m^{\prime \prime}}{ }_{m}^{\prime \prime}\right)\right), \ldots, \theta^{\circ}\left(\varphi_{m}^{\prime}\left(\psi_{m}^{\prime}, \widetilde{\psi}_{\backslash m}^{m^{\prime \prime}}\right)\right), \\
& \left.\theta^{\circ}\left(\varphi_{m+1}^{\prime}\left({\widetilde{\psi^{\prime}}}_{\backslash m}^{m^{\prime \prime}}\right)\right), \ldots, \theta^{\circ}\left(\varphi_{m+m^{\prime}}^{\prime}\left({\widetilde{\psi^{\prime}}}_{(m}^{m^{\prime \prime}}\right)\right)\right) \\
& =\left(\theta^{*} \varphi_{*}, \theta^{0} \varphi_{*}, \theta^{1} \varphi_{*}, \ldots, \theta^{k} \varphi_{*}, \theta^{\circ} \varphi_{1}, \ldots, \theta^{\circ} \varphi_{m+m^{\prime}}\right),
\end{aligned}
$$

as required.

Now we show the subject reduction property. For a type $\tau=\kappa_{1} \rightarrow \cdots \rightarrow \kappa_{n} \rightarrow \tau^{\prime}$, we write $\tau^{@ n}$ for $\tau^{\prime}$.

Lemma 16 (subject reduction). Suppose that we have $\left(D, S \lambda \widetilde{z}\right.$.true) $\rightsquigarrow\left(D^{\prime}, \exists \widetilde{z} \cdot S_{1} \widetilde{z}\right)$. If $\varphi \longrightarrow_{D} \psi$ and

$$
\widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{k}\right),
$$

then there exist $\psi_{0}, \ldots, \psi_{k+1}$ such that

$$
\widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \psi: \star \rightsquigarrow\left(\psi_{*}, \psi_{0}, \ldots, \psi_{k}\right)
$$

and $\varphi_{i}={ }_{D^{\prime}} \psi_{i}$ for each $i \in\{*, 0, \ldots, k\}$.
Proof. For the convenience of the proof, we rename the metavariables $\varphi, \psi, \varphi_{i}, \psi_{i}$ with $\varphi^{\prime}, \psi^{\prime}, \varphi_{i}^{\prime}, \psi_{i}^{\prime}$ : so we suppose $\varphi^{\prime} \longrightarrow_{D} \psi^{\prime}$ and $\widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi^{\prime}: \star \rightsquigarrow\left(\varphi_{*}^{\prime}, \varphi_{0}^{\prime}, \ldots, \varphi_{k}^{\prime}\right)$, and prove that we have $\widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \psi^{\prime}: \star \rightsquigarrow\left(\psi_{*}^{\prime}, \psi_{0}^{\prime}, \ldots, \psi_{k}^{\prime}\right)$ and $\varphi_{i}^{\prime}={ }_{D^{\prime}} \psi_{i}^{\prime}$. The proof proceeds by induction on $\varphi^{\prime}$.

Let $\varphi^{\prime}$ be of the form $E\left[\varphi^{\prime \prime}\right]$ where $\varphi^{\prime \prime}$ is a redex of $\longrightarrow_{D}$. The case where $E \neq[]$ can be easily proved by using induction hypothesis. So we consider only the case where $E=[]$. Then we perform case analysis on $\varphi^{\prime} \longrightarrow_{D} \psi^{\prime}$, but we focus only on the non-trivial case where we use the substitution lemmas.

- Case where $\varphi^{\prime} \longrightarrow_{D} \psi^{\prime}$ is of the form

$$
F \alpha_{1} \cdots \alpha_{h^{\prime}} \longrightarrow_{D}\left\{\widetilde{\xi} / \widetilde{x^{\prime \prime}}\right\}\left[\widetilde{e^{\prime \prime}} / \widetilde{z^{\prime \prime}}\right]\left[\widetilde{\varphi^{\prime \prime}} / \widetilde{y^{\prime \prime}}\right] \varphi
$$

with the following conditions:

$$
\begin{aligned}
& \left(F \widetilde{w^{\prime}}=\varphi\right) \in D \\
& \operatorname{decomp}(\Theta(F))=\left(\widetilde{\kappa^{\prime \prime}} 1, \ldots, h^{\prime \prime}, m, p\right) \\
& \left.\operatorname{decomp} A r g(\widetilde{\alpha}, \Theta(F))=\widetilde{\left(\varphi^{\prime \prime}\right.}, \widetilde{\xi}, \widetilde{e^{\prime \prime}}\right) \\
& \left.\operatorname{decomparg}\left(\widetilde{w^{\prime}}, \Theta(F)\right)=\widetilde{\left(y^{\prime \prime}\right.}: \widetilde{\kappa^{\prime \prime}}, \widetilde{x^{\prime \prime}}, \widetilde{z^{\prime \prime}}\right) \\
& \widetilde{x^{\prime \prime}} \text { do not occur in } F \alpha_{1} \cdots \alpha_{m} .
\end{aligned}
$$

By the last condition above, we can assume $\left\{x_{i}\right\}_{i} \cap\left\{x_{i}^{\prime \prime}\right\}_{i}=\emptyset$.
Now there exist $q, r_{1}, \ldots, r_{q+m+1}, \widetilde{\psi}_{1}, \ldots, q, \widetilde{e}_{1, \ldots, r_{q+m+1}}$ that satisfy the following conditions, where we write $\widetilde{e}_{(i)}$ (or simply $\widetilde{e}$ if $i$ is clear) for $\widetilde{e}_{r_{i-1}+1, \ldots, r_{i}}(i=1, \ldots, q+m+1)$ and $r_{0}:=0$ :

$$
\begin{aligned}
& r_{1} \leq \cdots \leq r_{q+m+1} \\
& \left(\alpha_{1}, \ldots, \alpha_{h^{\prime \prime}}\right)=\left(\widetilde{e}_{(1)}, \psi_{1}, \ldots, \widetilde{e}_{(q)}, \psi_{q}\right) \\
& \left(\alpha_{h^{\prime \prime}+1}, \ldots, \alpha_{h^{\prime}}\right)=\left(\widetilde{e}_{(q+1)}, \xi_{1}, \ldots, \widetilde{e}_{(q+m)}, \xi_{m}, \widetilde{e}_{(q+m+1)}\right) \\
& p=r_{q+m+1}-r_{q}
\end{aligned}
$$

Let $\kappa_{i}$ be the type of $\psi_{i}\left(\right.$ i.e., $\left.\kappa_{i}:=\kappa_{r_{i}+i}^{\prime \prime}\right)$. Then we also have

$$
\begin{aligned}
\Theta(F)= & \operatorname{Int}^{r_{1}} \rightarrow \kappa_{1} \rightarrow \cdots \operatorname{Int}^{r_{q}-r_{q-1}} \rightarrow \kappa_{q} \rightarrow \\
& \operatorname{Int}^{r_{q+1}-r_{q}} \rightarrow\left(\operatorname{Int}^{M} \rightarrow \star\right) \rightarrow \\
& \cdots \operatorname{Int}^{r_{q+m}-r_{q+m-1}} \rightarrow\left(\operatorname{Int}^{M} \rightarrow \star\right) \rightarrow \\
& \text { Int }^{r_{q+m+1}-r_{q+m}} \rightarrow \star
\end{aligned}
$$

In the derivation tree of

$$
\widetilde{x}_{1, \ldots, k} \vdash_{\Theta}\left(\varphi^{\prime}=\right) F \alpha_{1} \cdots \alpha_{h^{\prime}}: \star \rightsquigarrow\left(\varphi_{*}^{\prime}, \varphi_{0}^{\prime}, \ldots, \varphi_{k}^{\prime}\right),
$$

the leftmost path from the head position $F$ consists of: (i) Tr-VARF at the leaf, then (ii) repeated applications of either Tr-App or Tr-AppI, and then (iii) repeated applications of either Tr-AppG or Tr-AppI. More specifically, at the leaf of Tr-VarF we have

$$
\widetilde{x}_{1, \ldots, k} \vdash_{\Theta} F: \Theta(F) \rightsquigarrow\left(F_{0}, F_{0},\left(F_{0}\right)^{k}, F_{1}, \ldots, F_{m}\right)
$$

where $\left(F_{0}\right)^{k}$ denotes the sequence of length $k$ whose all components are $F_{0}$. Then by Tr-AppI we have

$$
\begin{aligned}
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} F \widetilde{e}_{(1)}: \Theta(F)^{@ r_{0}} \rightsquigarrow \\
& \quad\left(F_{0} \widetilde{e}_{(1)}, F_{0} \widetilde{e}_{(1)},\left(F_{0} \widetilde{e}_{(1)}\right)^{k}, F_{1} \widetilde{e}_{(1)}, \ldots, F_{m} \widetilde{e}_{(1)}\right) .
\end{aligned}
$$

Then by Tr-App (and Tr-AppI) we have

$$
\begin{aligned}
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \psi_{i}: \kappa_{i} \rightsquigarrow\left(\psi_{i, *}, \psi_{i, 0}, \ldots, \psi_{i, k+m_{i}^{\prime}}\right) \quad(i=1, \ldots, q) \\
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} F \widetilde{e}_{(1)} \psi_{1} \cdots \widetilde{e}_{(q)} \psi_{q}: \Theta(F)^{@ r_{q-1}+q} \rightsquigarrow \\
& \left(F_{0} \widetilde{e}_{(1)}\left(\psi_{1, *}, \widetilde{\psi}_{1 \backslash k}^{m_{1}^{\prime}}\right) \cdots \widetilde{e}_{(q)}\left(\psi_{q, *},{\widetilde{\psi_{q}}}_{m_{q}^{\prime}}^{\prime}\right),\right. \\
& F_{0} \widetilde{e}_{(1)}\left(\psi_{1,0}, \widetilde{\psi}_{1 \backslash k}^{m_{1}^{\prime}}\right) \cdots \widetilde{e}_{(q)}\left(\psi_{q, 0}, \widetilde{\psi}_{q \backslash k}^{m_{q}^{\prime}}\right), \\
& F_{0} \widetilde{e}_{(1)}\left(\psi_{1,1}, \widetilde{\psi}_{1 \backslash k}^{m_{1}^{\prime}}\right) \cdots \widetilde{e}_{(q)}\left(\psi_{q, 1}, \widetilde{\psi}_{q \backslash k}^{m_{q}^{\prime}}\right), \\
& \ldots, F_{0} \widetilde{e}_{(1)}\left(\psi_{1, k},{\widetilde{\psi_{1}}{ }^{m}{ }_{1}^{\prime}}^{\prime}\right) \cdots \widetilde{e}_{(q)}\left(\psi_{q, k},{\left.\widetilde{\psi_{q}}{ }^{m_{q}^{\prime}}\right),}^{\prime}\right. \\
& F_{1} \widetilde{e}_{(1)}\left({\left.\widetilde{\psi_{1}}{ }_{\backslash k}^{m_{1}^{\prime}}\right) \cdots \widetilde{e}_{(q)}\left({\widetilde{\psi_{q}}}_{\backslash k}^{m_{q}^{\prime}}\right), ~}_{\text {, }}\right.
\end{aligned}
$$

$$
\begin{aligned}
& m_{i}^{\prime}:=\operatorname{gar}\left(\kappa_{i}\right) \quad(i=1, \ldots, q) \\
& \operatorname{decomp}\left(\kappa_{i}\right)=\left(\widetilde{\kappa_{i}}, m_{i}^{\prime}, p_{i}\right) \quad(i=1, \ldots, q) .
\end{aligned}
$$

And then by Tr-AppG (and Tr-AppI) we have

$$
\begin{aligned}
& p_{i}^{\circ}:=r_{q+m}-r_{q-1+i} \quad(i=1, \ldots, m) \\
& \varphi_{1}^{\circ}:=F \widetilde{e}_{(1)} \psi_{1} \cdots \widetilde{e}_{(q)} \psi_{q} \widetilde{e}_{(q+1)} \\
& \varphi_{i+1}^{\circ}:=\varphi_{i}^{\circ} \xi_{i} \widetilde{e}_{(q+i+1)} \quad(i=1, \ldots, m) \\
& \tau_{i}:=\Theta(F)^{\text {@ } r_{q+i}+q+i} \quad(i=1, \ldots, m) \\
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi_{i}^{\circ}:\left(\operatorname{Int}^{M} \rightarrow \star\right) \rightarrow \tau_{i} \rightsquigarrow\left(\varphi_{i, *}^{\circ}, \varphi_{i, 0}^{\circ}, \ldots, \varphi_{i, k+m+1-i}^{\circ}\right) \\
& (i=1, \ldots, m+1) \\
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \xi_{i}: \text { Int }^{M} \rightarrow \star \rightsquigarrow\left(\xi_{i, *}, \xi_{i, 0}, \ldots, \xi_{i, k}\right) \\
& (i=1, \ldots, m) \\
& \overline{\xi_{i, j}}:=\lambda \widetilde{z}_{1, \ldots, p_{i}^{\circ}} \cdot \lambda \widetilde{w}_{1, \ldots, M} . \\
& \varphi_{i, j}^{\circ} \widetilde{z} \widetilde{w} \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{i, k+1}^{\circ} \widetilde{z} \widetilde{u} \wedge \xi_{i, j} \widetilde{u} \widetilde{w}\right) \\
& (i=1, \ldots, m, j=*, 0,1, \ldots, k) \\
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta} \varphi_{i+1}^{\circ}\left(=\varphi_{i}^{\circ} \xi_{i} \widetilde{e}_{(q+i+1)}\right):\left(\operatorname{Int}^{M} \rightarrow \star\right) \rightarrow \tau_{i+1} \rightsquigarrow \\
& \left(\overline{\xi_{i, *}} \widetilde{e}, \overline{\xi_{i, 0}} \widetilde{e}, \ldots, \overline{\xi_{i, k}} \widetilde{e}, \varphi_{i, k+2}^{\circ} \widetilde{e}, \ldots, \varphi_{i, k+m+1-i}^{\circ} \widetilde{e}\right) \\
& (i=1, \ldots, m) .
\end{aligned}
$$

Then, for each $i=2, \ldots, m+1$, we have

$$
\left.\begin{array}{rl} 
& \left(\varphi_{i, *}^{\circ}, \varphi_{i, 0}^{\circ}, \ldots, \varphi_{i, k}^{\circ}\right. \\
= & \left(\overline{\xi_{i-1, *}} \widetilde{e}, \varphi_{i, k+1}^{\circ}, \overline{\xi_{i-1,0}} \widetilde{e}, \ldots, \overline{\xi_{i-1, k}} \widetilde{e}, \varphi_{i-1, k+2}^{\circ} \widetilde{e}, \ldots, \varphi_{i-k+m+1-i}^{\circ}\right.
\end{array}\right)
$$

where $\widetilde{e}=\widetilde{e}_{(q+i)}$. Hence, for each $i=1, \ldots, m$,

$$
\begin{aligned}
\varphi_{i, k+1}^{\circ} & =\varphi_{i-1, k+2}^{\circ} \widetilde{e}_{(q+i)}=\varphi_{i-2, k+3}^{\circ} \widetilde{e}_{(q+i-1)} \widetilde{e}_{(q+i)}=\ldots \\
& =\varphi_{1, k+i}^{\circ} \widetilde{e}_{(q+2)} \cdots \widetilde{e}_{(q+i)} \\
& =F_{i} \widetilde{e}_{(1)}\left({\widetilde{\psi_{1}}{ }_{1}^{\prime}{ }_{1 k}^{\prime}}^{\prime} \cdots \widetilde{e}_{(q)}\left(\widetilde{\psi}_{q \backslash k}^{m_{q}^{\prime}}\right) \widetilde{e}_{(q+1)} \cdots \widetilde{e}_{(q+i)}\right.
\end{aligned}
$$

where the last equality follows from the calculation result of Tr-App above. Also, for each $i=2, \ldots, m$ and $j=*, 0, \ldots, k$, we have

$$
\begin{aligned}
& \overline{\xi_{i, j}} \widetilde{z}_{1, \ldots, p_{i}^{\circ}} \widetilde{w}_{1, \ldots, M} \\
={ }_{D^{\prime}} & \overline{\xi_{i-1, j}} \widetilde{e}_{(q+i)} \widetilde{z} \widetilde{w} \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{i, k+1}^{\circ} \widetilde{z} \widetilde{u} \wedge \xi_{i, j} \widetilde{u} \widetilde{w}\right)
\end{aligned}
$$

Now, since $\varphi^{\prime}=\varphi_{m+1}^{\circ}$, for each $j=*, 0, \ldots, k$, we have

$$
\begin{align*}
& ={ }_{D^{\prime}} \overline{\xi_{m-1, j}} \widetilde{e}_{(q+m)} \widetilde{e}_{(q+m+1)} \widetilde{w} \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{m, k+1}^{\circ} \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{m, j} \widetilde{u} \widetilde{w}\right) \\
& ={ }_{D^{\prime}} \overline{\xi_{m-2, j}} \widetilde{e}_{(q+m-1)} \widetilde{e}_{(q+m)} \widetilde{e}_{(q+m+1)} \widetilde{w} \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{m-1, k+1}^{\circ} \widetilde{e}_{(q+m)} \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{m-1, j} \widetilde{u} \widetilde{w}\right) \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{m, k+1}^{\circ} \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{m, j} \widetilde{u} \widetilde{w}\right) \\
& =D^{\prime} \cdots \\
& ={ }_{D^{\prime}} \overline{\xi_{1, j}} \widetilde{e}_{(q+2)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{w} \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{2, k+1}^{\circ} \widetilde{e}_{(q+3)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{2, j} \widetilde{u} \widetilde{w}\right) \\
& \vee \cdots \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{m, k+1}^{\circ} \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{m, j} \widetilde{u} \widetilde{w}\right) \\
& ={ }_{D^{\prime}} \varphi_{1, j}^{\circ} \widetilde{e}_{(q+2)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{w} \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{1, k+1}^{\circ} \widetilde{e}_{(q+2)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{1, j} \widetilde{u} \widetilde{w}\right) \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{2, k+1}^{\circ} \widetilde{e}_{(q+3)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{2, j} \widetilde{u} \widetilde{w}\right) \\
& \vee \ldots \\
& \vee \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{m, k+1}^{\circ} \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{m, j} \widetilde{u} \widetilde{w}\right) \\
& ={ }_{D^{\prime}} \varphi_{1, j}^{\circ} \widetilde{e}_{(q+2)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{w} \\
& \vee \bigvee_{i=1}^{m} \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\varphi_{i, k+1}^{\circ} \widetilde{e}_{(q+i+1)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{u} \wedge \xi_{i, j} \widetilde{u} \widetilde{w}\right) \\
& ={ }_{D^{\prime}} \varphi_{1, j}^{\circ} \widetilde{e}_{(q+2)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{w} \\
& \vee \bigvee_{i=1}^{m} \exists \widetilde{u}_{1, \ldots, M} \cdot\left(F _ { i } \widetilde { e } _ { ( 1 ) } ( \widetilde { \psi } _ { 1 \backslash k } ^ { m _ { 1 } ^ { \prime } } ) \cdots \widetilde { e } _ { ( q ) } \left({\left.\widetilde{\psi_{q}}{ }_{\backslash k}^{m_{q}^{\prime}}\right)}^{\prime}\right.\right.  \tag{4}\\
& \widetilde{e}_{(q+1)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{u} \\
& \left.\wedge \xi_{i, j} \widetilde{u} \widetilde{w}\right) .
\end{align*}
$$

To calculate $\varphi_{1, j}^{\circ}$ and $F_{i}$ above, let us consider the rules of $F_{0}, \ldots, F_{m}$, which are given by Tr-Def as follows. Recall

$$
\left.\operatorname{decomparg}\left(\widetilde{w^{\prime}}, \Theta(F)\right)=\widetilde{\left(y^{\prime \prime}\right.}: \widetilde{\kappa^{\prime \prime}}, \widetilde{x^{\prime \prime}}, \widetilde{z^{\prime \prime}}\right)
$$

and let

$$
\begin{aligned}
& \widetilde{y^{\prime \prime}}: \widetilde{\kappa^{\prime \prime}}, \widetilde{z^{\prime \prime}}: \widetilde{\text { Int }} ; \widetilde{x^{\prime \prime}}{ }_{1, \ldots, m} \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{m}\right) \\
& \widetilde{y^{\prime \prime}}:=\left(y_{i, *}^{\prime \prime}, y_{i, 0}^{\prime \prime}, \ldots, y_{i, \operatorname{gar}\left(\kappa_{i}^{\prime \prime}\right)}^{\prime \prime}\right) \widetilde{y^{\prime \prime}}{ }_{i}^{\circ}:=\left(y_{i, 0}^{\prime \prime}, \ldots, y_{i, \operatorname{gar}\left(\kappa_{i}^{\prime \prime}\right)}^{\prime \prime}\right) \\
& \\
& \left(\underset{y^{\prime \prime}}{i}:=y_{i}^{\prime \prime} \quad{\widetilde{y^{\prime \prime}}}_{i}^{\circ}:=y_{i}^{\prime \prime} \quad\left(i \in\left\{1, \ldots, h^{\prime \prime}\right\} \text { and } \kappa_{i}^{\prime \prime} \neq \text { Int }\right)\right. \\
&
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
& \vdash_{\Theta}\left(F \widetilde{w^{\prime}}=\varphi\right) \rightsquigarrow \\
&\left\{F_{0} \widetilde{y^{\prime \prime}} 1 \cdots \widetilde{y^{\prime \prime}}{ }_{h^{\prime \prime}} \widetilde{z^{\prime \prime}}=\varphi_{*}\right\} \\
& \cup\left\{F_{i} \widetilde{{y^{\prime \prime}}_{1}^{\circ}} \cdots \widetilde{y^{\prime \prime}}{ }_{h^{\prime \prime}} \widetilde{z^{\prime \prime}}=\varphi_{i} \mid i \in\{1, \ldots, m\}\right\} .
\end{aligned}
$$

Recall that $\kappa_{i}:=\kappa_{r_{i}+i}^{\prime \prime}(i=1, \ldots, q)$, and let

$$
\begin{aligned}
& y_{i}:=y_{r_{i}+i}^{\prime \prime} \quad(i=1, \ldots, q) \\
& y_{i, j}:=y_{r_{i}+i, j}^{\prime \prime} \quad\left(i=1, \ldots, q, j=*, 0, \ldots, \operatorname{gar}\left(\kappa_{i}\right)\right) \\
& \widetilde{y}_{i}:={\widetilde{y^{\prime \prime}}}_{r_{i}+i}=\left(y_{i, *}, y_{i, 0}, \ldots, y_{i, \operatorname{gar}\left(\kappa_{i}\right)}\right) \\
& \widetilde{y}_{i}^{\circ}:={\widetilde{y^{\prime \prime}}}_{r_{i}+i}^{\circ}=\left(y_{i, 0}, \ldots, y_{i, \operatorname{gar}\left(\kappa_{i}\right)}\right) .
\end{aligned}
$$

Then let $\widetilde{z}_{1, \ldots, r_{q+m+1}}$ be a sequence of variables of type Int that satisfies the following equations, where we write $\widetilde{z}_{(i)}$ (or simply $\widetilde{z}$ if $i$ is clear) for $\widetilde{z}_{r_{i-1}+1, \ldots, r_{i}}(i=1, \ldots, q+m+1)$ :

$$
\begin{aligned}
& \left(w_{1}^{\prime}, \ldots, w_{h^{\prime \prime}}^{\prime}\right)=\left(y_{1}^{\prime \prime}, \ldots, y_{h^{\prime \prime}}^{\prime \prime}\right)=\left(\widetilde{z}_{(1)}, y_{1}, \ldots, \widetilde{z}_{(q)}, y_{q}\right) \\
& \left(w_{h^{\prime \prime}+1}^{\prime}, \ldots, w_{h^{\prime}}^{\prime}\right)=\left(\widetilde{z}_{(q+1)}, x_{1}^{\prime \prime}, \ldots, \widetilde{z}_{(q+m)}, x_{m}^{\prime \prime}, \widetilde{z}_{(q+m+1)}\right) \\
& \left({\widetilde{y^{\prime \prime}}}_{1}, \ldots,{\widetilde{y^{\prime \prime}}}_{h^{\prime \prime}},{\widetilde{z^{\prime \prime}}}^{\prime \prime}\right)=\left(\widetilde{z}_{(1)}, \widetilde{y}_{1}, \ldots, \widetilde{z}_{(q)}, \widetilde{y}_{q}, \widetilde{z}_{r_{q}+1, \ldots, r_{q+m+1}}\right) \\
& \left({\widetilde{y^{\prime \prime}}}_{1}^{\circ}, \ldots,{\widetilde{y^{\prime \prime}}}_{h^{\prime \prime}}^{\circ}, \widetilde{z^{\prime \prime}}\right)=\left(\widetilde{z}_{(1)}, \widetilde{y}_{1}^{\circ}, \ldots, \widetilde{z}_{(q)}, \widetilde{y}_{q}^{\circ}, \widetilde{z}_{r_{q}+1, \ldots, r_{q+m+1}}\right) .
\end{aligned}
$$

Now, for $j=*, 0, \ldots, k$, we have

$$
\begin{align*}
& \varphi_{1, j}^{\circ} \widetilde{e}_{(q+2)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{w}_{1, \ldots, M} \\
&={ }_{D^{\prime}} F_{0} \widetilde{e}_{(1)}\left(\psi_{1, j}, \widetilde{\psi}_{1 \backslash k}^{m_{1}^{\prime}}\right) \cdots \widetilde{e}_{(q)}\left(\psi_{q, j}, \widetilde{\psi}_{q \backslash k}^{m_{q}^{\prime}}\right) \widetilde{e}_{(q+1)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{w} \\
&={ }_{D^{\prime}}\left(\left[\left(\psi_{i^{\prime}, j}, \widetilde{\psi_{i^{\prime}} \backslash k} \widetilde{m}_{i^{\prime}}^{\prime}\right) / \widetilde{y}_{i^{\prime}}\right]_{i^{\prime}=1}^{q}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi_{*}\right) \widetilde{w} . \tag{5}
\end{align*}
$$

Also for each $i=1, \ldots, m$, we have

$$
\begin{align*}
& F_{i} \widetilde{e}_{(1)}\left({\widetilde{\psi_{1}} \backslash k}_{m_{1}^{\prime}}^{\prime}\right) \cdots \widetilde{e}_{(q)}\left({\left.\widetilde{\psi_{q}}{ }^{m_{q}^{\prime}}\right) \widetilde{e}_{(q+1)} \cdots \widetilde{e}_{(q+m+1)} \widetilde{u}}^{=_{D^{\prime}}\left(\left[{\widetilde{\psi_{i^{\prime}} \backslash k} m_{i^{\prime}}^{\prime}}^{\tilde{y}_{i^{\prime}}^{0}}\right]_{i^{\prime}=1}^{q}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi_{i}\right) \widetilde{u} .}\right.
\end{align*}
$$

Next, let us consider $\psi^{\prime}$. Now we have

$$
\psi^{\prime}=\left\{\widetilde{\xi} / \widetilde{x^{\prime \prime}}\right\}\left[\widetilde{e^{\prime \prime}} / \widetilde{z^{\prime \prime}}\right]\left[\widetilde{\varphi^{\prime \prime}} / \widetilde{y^{\prime \prime}}\right] \varphi
$$

$$
=\left\{\widetilde{\xi} / \widetilde{x^{\prime \prime}}\right\}\left[\psi_{i^{\prime}} / y_{i^{\prime}}\right]_{i^{\prime}=1}^{q}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi .
$$

Recall

$$
\begin{aligned}
& \widetilde{y^{\prime \prime}}: \widetilde{\kappa^{\prime \prime}}, \widetilde{z^{\prime \prime}}: \widetilde{\operatorname{Int}} ; \widetilde{x^{\prime \prime}}{ }_{1, \ldots, m} \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \ldots, \varphi_{m}\right), \\
& \left(\widetilde{y^{\prime \prime}}, \widetilde{z^{\prime \prime}}\right)=\left(\widetilde{z}_{(1)}, y_{1}, \ldots, \widetilde{z_{(q)}}, y_{q}, \widetilde{z}_{(q+1)}, \ldots, \widetilde{z}_{(q+m+1)}\right),
\end{aligned}
$$

and let

$$
\begin{aligned}
& \widetilde{y}: \widetilde{\kappa} ; \widetilde{x^{\prime \prime}}{ }_{1, \ldots, m} \vdash_{\Theta}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi: \star \rightsquigarrow \\
& \quad\left(\psi_{*}^{\prime \prime \prime}, \psi_{0}^{\prime \prime \prime}, \ldots, \psi_{m}^{\prime \prime \prime}\right), \\
& \widetilde{x}_{1, \ldots, k}, \widetilde{x^{\prime \prime}}{ }_{1, \ldots, m} \vdash_{\Theta}\left[\psi_{i^{\prime}} / y_{i^{\prime}}\right]_{i^{\prime}=1}^{q}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi: \star \rightsquigarrow \\
& \quad\left(\psi_{*}^{\prime \prime}, \psi_{0}^{\prime \prime}, \ldots, \psi_{k+m}^{\prime \prime}\right) .
\end{aligned}
$$

Then, by applying Lemma 15 Item 2, and then by applying Lemma 13, we obtain:

$$
\begin{array}{rlr}
\psi_{j}^{\prime \prime} & ={ }_{D^{\prime}}\left[\left(\psi_{i^{\prime}, j}, \widetilde{\psi_{i^{\prime}} \backslash k} m_{i^{\prime}}^{\prime}\right) / \widetilde{y}_{i^{\prime}}\right]_{i^{\prime}=1}^{q} \psi_{*}^{\prime \prime \prime} & (j=*, 0, \ldots, k) \\
& =\left[\left(\psi_{i^{\prime}, j}, \widetilde{\psi_{i^{\prime}} \backslash k} m_{i^{\prime}}^{\prime}\right) / \widetilde{y}_{i^{\prime}}\right]_{i^{\prime}=1}^{q}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi_{*} & \\
\psi_{k+i}^{\prime \prime} & ={ }_{D^{\prime}}^{\prime}\left[\widetilde{\psi_{i^{\prime}} \backslash k} m_{i^{\prime}}^{\prime} / \widetilde{y}_{i^{\prime}}^{o}\right]_{i^{\prime}=1}^{q} \psi_{i}^{\prime \prime \prime} & (i=1, \ldots, m) \\
& =\left[\widetilde{\psi_{i^{\prime}} \backslash k} m_{i^{\prime}}^{\prime} / \widetilde{y}_{i^{\prime}}^{\circ}\right]_{i^{\prime}=1}^{q}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi_{i} & \tag{7}
\end{array}
$$

Now, for $j=*, 0, \ldots, k$, let

$$
\begin{align*}
& \psi_{j}^{\prime}=\lambda \widetilde{w}_{1, \ldots, M} \cdot \psi_{j}^{\prime \prime} \widetilde{w} \\
& \vee \bigvee_{i=1}^{m} \exists \widetilde{u}_{1, \ldots, M} \cdot\left(\psi_{k+i}^{\prime \prime} \widetilde{u}\right.  \tag{8}\\
&\left.\wedge \xi_{i, j} \widetilde{u} \widetilde{w}\right) .
\end{align*}
$$

Then by Tr-ESUB, we have

$$
\begin{aligned}
& \widetilde{x}_{1, \ldots, k} \vdash_{\Theta}\left(\psi^{\prime}=\right)\left\{\widetilde{\xi} / \widetilde{x^{\prime \prime}}\right\}\left[\psi_{i^{\prime}} / y_{i^{\prime}}\right]_{i^{\prime}=1}^{q}\left[e_{j^{\prime}} / z_{j^{\prime}}\right]_{j^{\prime}=1}^{r_{q+m+1}} \varphi: \star \rightsquigarrow \\
& \quad\left(\psi_{*}^{\prime}, \psi_{0}^{\prime}, \ldots, \psi_{k}^{\prime}\right) .
\end{aligned}
$$

Also, by Equations (44) to (8), we have $\varphi_{j}^{\prime}={ }_{D^{\prime}} \psi_{j}^{\prime}$ for $j=*, 0, \ldots, k$, as required.
Now we show the correctness in the recursion-free case. As explained in Appendix A. 1 Theorem 10 follows from the following:
Lemma 17. Suppose that ( $D, S \lambda \widetilde{z}_{1, \ldots, M}$.true) is a recursion-free equation system. If $\left(D, S \lambda \widetilde{z}_{1, \ldots, M}\right.$.true $) \rightsquigarrow\left(D^{\prime}, \exists \widetilde{z} \cdot S_{1} \widetilde{z}\right)$, then $\llbracket\left(D, S \lambda \widetilde{z}_{1, \ldots, M} \cdot\right.$ true $) \rrbracket=\llbracket\left(D^{\prime}, \exists \widetilde{z} \cdot S_{1} \widetilde{z}\right) \rrbracket$.
Proof. Let the rule of $S$ be $S x=\varphi$; then $D^{\prime}$ has the rule $S_{1}=\varphi_{1}$ where $S_{1}$ and $\varphi_{1}$ has type Int $^{M} \rightarrow \star$. Since $\llbracket\left(D, S \lambda \widetilde{z}_{1, \ldots, M}\right.$.true $) \rrbracket=\llbracket\left(D,\left[\lambda \widetilde{z}_{1, \ldots, M}\right.\right.$.true $\left.\left./ x\right] \varphi\right) \rrbracket$, it suffices to show

$$
\llbracket\left(D,\left[\lambda \widetilde{z}_{1, \ldots, M} \cdot \text { true } / x\right] \varphi\right) \rrbracket=\llbracket\left(D^{\prime}, \exists \widetilde{z} \cdot \varphi_{1} \widetilde{z}\right) \rrbracket .
$$

Since ( $D, S \lambda \widetilde{z}_{1, \ldots, M}$.true) is recursion-free, $\varphi$ has a normal form $\zeta$ with respect to $\longrightarrow_{D}$. We have $\varphi \longrightarrow_{D}^{*} \zeta$ and let $x \vdash_{\Theta} \varphi: \star \rightsquigarrow\left(\varphi_{*}, \varphi_{0}, \varphi_{1}\right)$; then by subject reduction
(Lemma 16), we have $x \vdash_{\Theta} \zeta: \star \rightsquigarrow\left(\zeta_{*}, \zeta_{0}, \zeta_{1}\right)$ and $\llbracket\left(D^{\prime}, \varphi_{i}\right) \rrbracket=\llbracket\left(D^{\prime}, \zeta_{i}\right) \rrbracket(i=*, 0,1)$. Also, we have $\llbracket(D, \varphi) \rrbracket=\llbracket(D, \zeta) \rrbracket$ by Lemma 12 and hence $\llbracket\left(D,\left[\lambda \widetilde{z}_{1}, \ldots, M . \operatorname{true} / x\right] \varphi\right) \rrbracket=$ $\llbracket\left(D,\left[\lambda \widetilde{z}_{1}, \ldots, M\right.\right.$.true $\left.\left./ x\right] \zeta\right) \rrbracket$. Therefore we can assume that $\varphi$ is a normal form without loss of generality.

Then we can directly check $\llbracket\left(D,\left[\lambda \widetilde{z}_{1}, \ldots, M \cdot \operatorname{true} / x\right] \zeta\right) \rrbracket=\llbracket\left(D^{\prime}, \exists \widetilde{z} \cdot \zeta_{1} \widetilde{z}\right) \rrbracket$ by induction on the structure of normal forms (3) in Appendix A.2


[^0]:    ${ }^{1}$ Defining the order of Int as -1 is a bit unusual, but convenient for stating our technical result.

[^1]:    ${ }^{2}$ In the context of program verification, we are often interested in (un)reachability to bad states. Thus, in that context, succ in this section is actually interpreted as an error state, and the terms "angelic" and "demonic" below are swapped.

[^2]:    ${ }^{3}$ Taken from 9 .

[^3]:    ${ }^{4}$ This is a mechanically generated output based on the transformation rules, followed by slight manual simplification.

[^4]:    ${ }^{5}$ Although the understanding of the refinement type systems RETHFL is not required below, interested readers may wish to consult 10 .

