Exact and Optimal Conversion of a Hole-free 2d Digital Object into a Union of Balls in Polynomial Time

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Abstract

This paper addresses the problem of converting a 2D digital object, *i.e.* a set S of points in $\mathbb{Z}^2$, into a finite union of balls $\mathcal{B}$ centered on $\mathbb{R}^2$, such that the digitization of $\mathcal{B}$ is exactly S and the cardinality of $\mathcal{B}$ is minimum. We prove that, for the specific case of 2D hole-free digital objects, there exists a greedy polynomial-time algorithm. The algorithm is based on the same principle as the simple greedy optimal algorithm for the interval cover problem. After bringing to light under which conditions the latter algorithm can be extended to tree-like structures, we show that such a structure can be defined for any hole-free 2D digital object, so that the extended algorithm applies.

1 Introduction

Computer representation of shapes is a basic component to digitize, create, visualize or exchange models of physical objects. Different geometric models exist, either to represent the surface (B-rep, point clouds, triangle meshes) or the volume (tetrahedral meshes, digital objects, CSG models) of a solid shape. However, the model used to create or register a shape is not always the one tailored for subsequent processings or applications. Thus, the problem of converting one geometric model into another has been widely studied, for a variety of models. In particular, many provably good conversion algorithms have been designed to output a finite union of balls from other models, including point clouds, polygonal meshes or digital shapes. Indeed, being composed of very simple geometric shapes, finite union of balls are useful in a number of applications, for instance detection of collisions in computer graphics [6], or simulation of physical processes [12]. Various metrics can be used to measure the quality of the conversion such as the number of balls, or the difference in volume between the original model and the union of balls.

In this article, we consider the following problem:

Problem 1. *Given a 2*D *digital object S, compute a finite union of balls $\mathcal{B}$ such that: $\mathcal{B}$ covers exactly the points of S (and no point of $\mathbb{Z}^2 \backslash S$), and the cardinality of $\mathcal{B}$ is minimum.*

This problem is closely related to the more constrained problem where the balls of $\mathcal{B}$ must be centered in $\mathbb{Z}^2$, which is NP-hard [7]. It is also very close to the class of well-studied set cover problems that are also NP-hard [8]. The input of the set cover problem is a pair $(X, \mathcal{R})$, where X is a set of points (generally in $\mathbb{R}^n$) and $\mathcal{R}$ is a family of subsets of X called *ranges*. The problem is to find a minimum subset of $\mathcal{R}$ that covers all the points of X. In our problem, $X = S$ is a subset of $\mathbb{Z}^2$. However, the set of ranges $\mathcal{R}$ is not part of the input, but is constrained to be a set of balls centered on $\mathbb{R}^2$.

We show that, when S is a 4-connected digital object and $\mathbb{Z}^2 \backslash S = S^c$ has exactly one 8-connected component, the problem can be seen as a variant of the interval covering problem (1D set cover problem) for which an optimal greedy algorithm exists. The idea was introduced in [13] in the specific case of (δ, ε)-ball approximation problem: given a shape S, compute a finite union of balls included in the δ-dilation of S while covering its ε-erosion. It was shown that, while the general problem is NP-hard [4], a greedy optimal algorithm exists when the δ-dilation of S has a cycle-free medial axis [14].

In Section 2, we revisit the results of [14, 13] in a more general context. We consider the case where the input is a generic set of ranges and exhibit sufficient conditions on this set to ensure that the greedy algorithm is optimal in this setting. Once the good tools and conditions have been defined, the proofs of termination and optimality unfold as in [14, 13]. In Section 3, we show how to implement this algorithm to compute an exact and optimal conversion of a 2D hole-free digital object into a finite union of balls.

2 General optimal greedy algorithm

2.1 Algorithm specification

For the sake of simplicity, given a subset of ranges R we denote $\bigcup R = \bigcup_{r \in R} r$. We use the same vocabulary as in [14, 13] in the broader context of sets of ranges. A covering of a set of ranges $\mathcal{R}$ is a subset of $\mathcal{R}$ that covers all the points in $\bigcup \mathcal{R}$. More formally,

Definition 1 (Covering)**.** *Let $\mathcal{R}$ be a set of ranges, and R be a subset of $\mathcal{R}$. We say that R is a covering of $\mathcal{R}$ if $\bigcup R = \bigcup \mathcal{R}$.*

A covering R is said to be *minimal* if no range can be removed from R while keeping the covering property, and *minimum* if its cardinality is minimum among all possible coverings. In the following, we assume that $\mathcal{R}$ can be endowed with a partial order $\preceq$ such that the poset $(\mathcal{R}, \preceq)$ is anti-arborescent:

Definition 2 (Anti-arborescence [9])**.** *A poset $(V, \preceq)$ is anti-arborescent if:*

- *for all $v \in V$, the set of its successors $\{v' \in V, v \prec v'\}$ is totally ordered.*
- *for any two incomparable elements $v, v' \in V$, the predecessors of v and the predecessors of v' are pairwise incomparable.*

A range $r \in \mathcal{R}$ is said to be maximal (resp. minimal) in $R \subseteq \mathcal{R}$ if for all $r' \in R$, either $r' \preceq r$ (resp. $r' \succeq r$) or r' and r are incomparable. Given a range $r \in R$, we define the domain covered by ranges smaller than r : $C(\mathcal{R}, \preceq r) = (\bigcup_{r' \in \mathcal{R}, r' \prec r} r') \backslash r$. Similarly, we define the domain covered by ranges larger than or incomparable to r : $C(\mathcal{R}, \not\preceq r) = (\bigcup_{r' \in \mathcal{R}, r' \not\preceq r} r') \backslash r$. Remark that by definition, if $r_1 \preceq r_2$ then $C(\mathcal{R}, \preceq r_1) \cup r_1 \subseteq C(\mathcal{R}, \preceq r_2) \cup r_2$ and $C(\mathcal{R}, \not\preceq r_1) \cup r_1 \supseteq C(\mathcal{R}, \not\preceq r_2) \cup r_2$. It will also be useful later to extend these definitions to a set of ranges $R \subseteq \mathcal{R}$: $C(\mathcal{R}, \preceq R) = \bigcup_{r \in R} C(\mathcal{R}, \preceq r)$ and $C(\mathcal{R}, \not\preceq R) = \bigcap_{r \in R} C(\mathcal{R}, \not\preceq r)$.

Figure 1(c) illustrates these notations in the case of ranges being balls - $\mathcal{R} = \mathscr{B}$: a partial order $\preceq$ on the balls of $\mathscr{B}$ is depicted using arrows on the set of centers of the balls (in red). $(\mathscr{B}, \preceq)$ being an anti-arborescence, it has a root, indicated with a cross. The sets $C(\mathscr{B}, \preceq \mathscr{b})$ and $C(\mathscr{B}, \not\preceq \mathscr{b})$ are depicted respectively in green and orange for a specific ball $\mathscr{b}$ outlined in dashed gray.

Definition 3 (Partial covering)**.** *Let $\mathcal{R}$ be a set of ranges, and R be a subset of $\mathcal{R}$. We say that R is a partial covering of $\mathcal{R}$ if it is a covering of $C(\mathcal{R}, \preceq R)$, i.e. $C(\mathcal{R}, \preceq R) \subseteq \bigcup R$.*

Definition 4 (Candidate range)**.** *Let $R \subset \mathcal{R}$ be a partial covering of $\mathcal{R}$. A range $r \notin R$ is candidate to R if $R' = R \cup \{r\}$ is also a partial covering of $\mathcal{R}$ and $\bigcup R \subsetneq \bigcup R'$.*

A candidate range r with respect to R is said to be *maximal* if it is maximal in the set of candidate ranges. Algorithm 1 describes a greedy algorithm that computes a covering given a finite set of ranges $\mathcal{R}$. It uses the fact that, if $(\mathcal{R}, \preceq)$ is anti-arborescent, a topological ordering of the elements of $\mathcal{R}$ can be defined. The idea is pretty natural: considering ranges in topological order, if a range is critical for the set of uncovered points, then it is added to the covering.

Algorithm 1: GreedyCovering($\mathcal{R}$,$\preceq$)

Preconditions: $\mathcal{R}$ is finite, $(\mathcal{R}, \preceq)$ is an anti-arborescent poset

$R \leftarrow \emptyset$;
$U \leftarrow \bigcup \mathcal{R}$ (points of $\bigcup \mathcal{R}$ not in $\bigcup R$);
for $r \in \mathcal{R}$, *in topological order* **do**
    if r *is a maximal candidate for* U **then**
        $R \leftarrow R \cup \{r\}$;
        $U \leftarrow U \backslash r$
return R

By definition of candidate range, and since Algorithm 1 only inserts candidate ranges to the computed covering, an invariant of Algorithm 1 is that R

is always a partial covering of $\mathcal{R}$. The next section is dedicated to the proof of the fact that, provided that $\mathcal{R}$ fulfills two extra conditions, candidate ranges to non-empty subsets always exist (proving that Algorithm 1 terminates with a covering), and that Algorithm 1 computes a minimum-cardinal covering.

2.2 Correctness, termination and optimality of Algorithm 1

In the following, we prove that if the poset $(\mathcal{R}, \preceq)$ fulfills the two conditions below, Algorithm 1 terminates and computes a minimum covering :

Property (1) for any $r_1, r_2 \in \mathcal{R}$ such that $r_1 \cap r_2 \neq \emptyset$, for all $r_1 \prec r \prec r_2$, $r_1 \cap r_2 \subseteq r$.

Property (2) for $x \in \bigcup \mathcal{R}$, let $Cov(x, \mathcal{R}) = \{r \in \mathcal{R}, x \in r\}$; then $\forall x \in \bigcup \mathcal{R}$, $Cov(x, \mathcal{R})$ admits a greatest element that is called the *critical range* of x and is denoted $\mathrm{Crit}(x, \mathcal{R})$.

The proof of optimality requires several technical lemmas. These lemmas were stated and proven in [14, 13] for a specific family of ranges. We show here that they are still valid when the set of ranges fulfills above properties. The proofs are in general very similar, and simply call properties (1) or (2) when necessary. Space being limited, we only provide the most relevant ones.

The first lemma shows that any range r separates the elements of $\bigcup \mathcal{R}$ into three disjoint subsets of elements: those before, those in, and those after.

Lemma 1 (Proposition 4.10 [13])**.** *Let $r \in \mathcal{R}$. For any $x \in \bigcup \mathcal{R}$, x belongs to one and only one of the three subsets r, $C(\mathcal{R}, \preceq r)$, $C(\mathcal{R}, \npreceq r)$.*

Proof. By definition, r is disjoint from $C(\mathcal{R}, \preceq r)$ and $C(\mathcal{R}, \npreceq r)$. Suppose now that there exists an element $x \in \bigcup \mathcal{R}$ such that $x \in C(\mathcal{R}, \preceq r) \cap C(\mathcal{R}, \npreceq r)$. Let $r^- \prec r$ such that $x \in r^- \backslash r$ and r^+ such that $r \prec r^+$ or r^+ and r are incomparable and $x \in r^+ \backslash r$. By definition, $x \notin r$ but r^- and r^+ are in $Cov(x, \mathcal{R})$. By property (2), $Cov(x, \mathcal{R})$ admits a greatest element $r_M = \mathrm{Crit}(x, \mathcal{R})$, i.e. $r^- \preceq r_M$, $r^+ \preceq r_M$ and $x \in r_M$. If $r_M = r^-$, then $r^+ \preceq r^- \prec r$, a contradiction. Thus r_M is a strict successor of r^-, as r. By Definition 2, they are comparable. If $r \prec r_M$, then by property (1), $r^- \cap r_M \subseteq r$, leading to a contradiction since $x \in r^- \cap r_M$. If $r_M \prec r$, then r is a successor of r_M which is either a successor of r^+ or r^+ itself. Then $r^+ \prec r$ which is a contradiction with the fact that either $r \prec r^+$ or r and r^+ are incomparable. □ □

The following two lemmas were not stated as such in [14, 13], but used in the proofs. Lemma 2 shows that, given a partial covering, there always exists a candidate.

Lemma 2. *Let $R \subseteq \mathcal{R}$ be a minimal covering of $\mathcal{R}$. Let $R_- \subsetneq R$ be a partial covering, and $R_+ = R \backslash R_-$. Then any range r_+ minimal in R_+ is candidate to R_-.*

The proof is similar to part of the proof of Lemma 4.27 [13] and calls Lemma 1 to assert that the points of $C(\mathcal{R}, \preceq r_+)$ are disjoint from $r_+ \cup C(\mathcal{R}, \npreceq r_+)$ and thus cannot be covered by ranges in R_+. Lemma 2 implies in particular that any range $r = min_{x \in (\bigcup \mathcal{R}) \setminus R} Crit(x, \mathcal{R})$ is a candidate to R (there may be several incomparable candidates). By definition of $\mathrm{Crit}(x, \mathcal{R})$, any range $r' \succ r$ does not contain the point $p = \arg\min_{x \in (\bigcup \mathcal{R}) \setminus R} \mathrm{Crit}(x, \mathcal{R})$, $p \in (\bigcup \mathcal{R}) \setminus R$, so that r is actually a maximal candidate to R.

Lemma 3. *Let $R \subseteq \mathcal{R}$ be a minimal covering of $\mathcal{R}$. Let $R_- \subsetneq R$ be a partial covering, and let r be a candidate to R_-. Then any range $r' \in \mathcal{R} \setminus R_-$ such that $r' \prec r$ is also a candidate to R_-.*

Proof. Suppose by contradiction that there exists a range $r' \prec r$ that is not a candidate to R_-. Then there exists a point $x \in C(\mathcal{R}, \preceq (R_- \cup \{r'\}))$ which is not in $R_- \cup \{r'\}$. If x were in $C(\mathcal{R}, \preceq R_-)$, it would be covered by R_- since R_- is a partial covering, a contradiction. So $x \notin C(\mathcal{R}, \preceq R_-)$, which implies $x \in C(\mathcal{R}, \preceq r')$. By definition of C, there exists a range $r'' \prec r'$ that contains x. If $x \in r$, then by Property (1), we get $x \in r'$, a contradiction. Thus $x \notin r$. By transitivity of $\prec$, we have $r'' \prec r$. Using the fact that $x \notin r$, and by definition of C, we have $x \in C(\mathcal{R}, \preceq r)$. Again by definition of C, we have $C(\mathcal{R}, \preceq r) \subseteq C(\mathcal{R}, \preceq (R_- \cup r))$. r being candidate to R_-, $C(\mathcal{R}, \preceq (R_- \cup r)) \subseteq \bigcup (R_- \cup \{r\})$, a contradiction. $\square$ $\square$

Combining the previous lemmas, we can prove that, to complete a partial covering R_-, it is necessary to add a range that is smaller than or equal to a maximal candidate to R_-.

Proposition 1 (Lemma 4.27 [13])**.** *Let $R \subseteq \mathcal{R}$ be a minimal covering of $\mathcal{R}$. Let $R_- \subsetneq R$ be a partial covering, and let r be a maximal candidate to R_-. Then $R \setminus R_-$ contains a candidate range that is smaller than or equal to r.*

Theorem 1 (Theorem 10 [13])**.** *Let $\mathcal{R}$ be a finite set of ranges. Suppose that $\mathcal{R}$ can be endowed with a partial order $\preceq$ such that $(\mathcal{R}, \preceq)$ is an anti-arborescent poset, and fulfills Properties (1) and (2). Then, Algorithm 1 outputs a cardinal minimum covering of $\mathcal{R}$.*

The proofs of the proposition and of the theorem follow exactly the ones of Lemma 4.27 and Theorem 10 in [13]. The proof of Proposition 1 calls Lemmas 2 and 3, and the proof of Theorem 1 appllies Proposition 1 to replace one by one the ranges of any optimal covering by the ranges computed by Algorithm 1.

3 From a digital set to a set of ranges

In this section, we show how Algorithm 1 can be used to solve Problem 1. Here, ranges are balls. Given a digital object S, a set of balls fulfilling Theorem 1 hypothesis is defined. Moreover, this set is such that the result of Algorithm 1 is indeed a collection of balls of minimum cardinality that covers S exactly.

Let $S \subset \mathbb{Z}^2$ be a finite 4-connected digital object such that $S^c = \mathbb{Z}^2 \backslash S$ has one exactly 8-connected component. A digital ball b is a subset of $\mathbb{Z}^2$ for which there exists a ball $\mathscr{b}$ such that $\mathring{\mathscr{b}} \cap \mathbb{Z}^2 = b$, where $\mathring{\mathscr{b}}$ denotes the interior of $\mathscr{b}$. Otherwise said, if Dig denotes the Gauss digitization function, we have which $Dig(\mathscr{b}) = \mathring{\mathscr{b}} \cap \mathbb{Z}^2 = b$. In the following, we assume that balls $\mathscr{b}$ are open, so that $\mathring{\mathscr{b}} = \mathscr{b}$. The preimage of a digital ball b, denoted $Dig^{-1}(b)$ will be useful later on. A digital ball b is said to be valid for a digital object S if $b \subseteq S$. It is said to be maximal if there is no other valid digital ball containing it.

Given a digital object S, we aim at finding a set of ranges $\mathcal{B}$ that are (non empty) valid digital balls and such that $\bigcup \mathcal{B} = S$. Given a set of ranges as input, Algorithm 1 computes a minimum covering for this set of ranges. In order to obtain the minimum covering of a digital object S, the input set of ranges $\mathcal{B}$ must contain all maximal digital balls valid for S. For instance, taking the set of balls ouput by a distance transform of S is not enough to ensure optimality: indeed, all the balls of this set have a center in $\mathbb{Z}^2$, so that it misses all digital balls for which $Dig^{-1}(b)$ contains only balls of center not in $\mathbb{Z}^2$.

The next sections are dedicated to exhibiting a way to grasp the set of all valid maximal digital balls and showing that this set can be endowed with an anti-arborescent poset structure that fulfills sufficient properties (1) and (2).

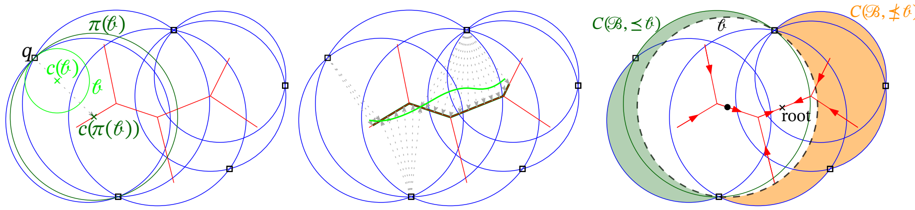
3.1 Getting a grip on valid maximal digital balls

The center of a ball $\mathscr{b}$ is denoted by $c(\mathscr{b})$. For $p \in \mathbb{Z}^2$, let pixel(p) be the unit square centered on p. For any ball $\mathscr{b}$ such that $c(\mathscr{b}) \in pixel(q), q \in S^c$, either $Dig(\mathscr{b}) = \emptyset$ or $Dig(\mathscr{b}) \cap S^c \neq \emptyset$. These balls do not contribute to the set of valid maximal digital balls and can be discarded. Consequently we define $\mathcal{S} = \bigcup_{p \in S} \mathring{\text{pixel}}(p)$ and restrict the study to this set. For $x \in \mathcal{S}$, let $\mathscr{b}^S(x)$ be the maximal ball centered in x such that $Dig(\mathscr{b}^S(x)) \subseteq S$. Note that by maximality, $\partial\mathscr{b}^S(x)$ contains at least one point of S^c. The following Lemma shows that any valid maximal digital ball has a ball in its preimage with at least two points of S^c on its boundary.

Lemma 4. *Let b be a valid maximal digital ball for S. Then there exists $\mathscr{b}$ such that $Dig(\mathscr{b}) = b$ and $|\partial\mathscr{b} \cap S^c| \geq 2$.*

Proof. Let $\mathscr{b}'$ be a ball such that $\mathring{\mathscr{b}}' \cap \mathbb{Z}^2 = b$. If $\partial\mathscr{b}' \cap S^c = \emptyset$, then we increase the radius of $\mathscr{b}'$ until $\mathscr{b}' = \mathscr{b}^S(c(\mathscr{b}'))$. $\partial\mathscr{b}'$ contains at least one point of S^c. Now we use a classical projection from a set of balls to the balls of the medial axis of a shape [11, 10]. The shape considered here is the whole space $\mathbb{R}^2$ punctured by the discrete set S^c. In this simple case, the medial axis is simply the set of edges of the Voronoi diagram of S^c, i.e. $\partial\text{Vor}(S^c)$. The projection is illustrated in Figure 1: it associates to any ball $\mathscr{b}$ a ball $\pi(\mathscr{b})$ centered on $\partial\text{Vor}(S^c)$ and such that $\mathscr{b} \subseteq \pi(\mathscr{b})$. This projection is well defined since S is finite (in particular, no half-space is void of points of S^c). Consider the ball $\pi(\mathscr{b}')$. If $Dig(\pi(\mathscr{b}')) \neq b$, we have a contradiction with the maximality of b, and otherwise, we have found a ball $\mathscr{b}$ such that $Dig(\mathscr{b}) = b$ and $|\partial\mathscr{b} \cap S^c| \geq 2$. $\square$

$\square$



(a) (b) (c)

Figure 1: (a) The projection $\pi(\mathscr{b})$ of $\mathscr{b}$ is defined from the center $c(\mathscr{b})$ and its closest point q in S^c. (b) Projection π is continuous on any continuous path: the continuous path in green is projected on the bolder dark green continuous subpath of the Voronoi diagram. Grey arrows represent the projection. (c) Illustration of a partial order (in red) on $\mathscr{B}$, and of the sets $C(\mathscr{B}, \preceq \mathscr{b})$ and $C(\mathscr{B}, \npreceq \mathscr{b})$.

Consequently, for any valid maximal digital ball b, there exists a ball $\mathscr{b}$ in $Dig^{-1}(b)$ with $c(\mathscr{b}) \in \partial\mathrm{Vor}(S^c) \cap \mathcal{S}$. Note that: (i) all the balls $\mathscr{b}$ with $c(\mathscr{b})$ in this set are such that $Dig(\mathscr{b})$ is valid for S ; (ii) some balls $\mathscr{b}$ with $c(\mathscr{b})$ in this set may however be such that $Dig(\mathscr{b})$ is not maximal. In the following, we denote $\mathrm{Vor}^{\sqcap}(S) = \partial\mathrm{Vor}(S^c) \cap \bigcup \mathring{\mathrm{pixel}}(S)$ (see Figure 2(a)), and we consider the set of balls $\mathscr{B} = \{\mathscr{b}^S(x), x \in \mathrm{Vor}^{\sqcap}(S)\}$. This set contains all the balls which digitization is a valid maximal digital ball for S.

3.2 Ordering balls of $\mathscr{B}$

By construction, $\mathrm{Vor}^{\sqcap}(S)$ is a collection of segments.

Lemma 5. *$Vor^{\sqcap}(S)$ is a geometric embedding of a tree in $\mathbb{R}^2$.*

Proof. Suppose that $\mathrm{Vor}^{\sqcap}(S)$ contains a cycle. This cycle is a Jordan curve, and since it is a subset of $\partial\mathrm{Vor}(S^c)$ it must contain a point of S^c in its interior. Moreover, this cycle is included in $\mathcal{S}$, which is an open polygon containing no point of S^c since S is 4-connected and S^c is 8-connected. A contradiction. □

□

$\mathrm{Vor}^{\sqcap}(S)$ being a tree, it can be endowed with a partial order by picking any point on it as a root: indeed, it is enough to orient each edge/segment from the leaves to the root. This results in an oriented tree, denoted by $\mathcal{T}$, that defines a partial order $\leq_{\mathcal{T}}$ on the set (of centers) of balls $\mathscr{B}$ (see Figure 1(c)). By construction, $(\mathscr{B}, \leq_{\mathcal{T}})$ is an anti-arborescent poset. Moreover, for any $p \in S$, the set of centers of the balls of $Cov(p, \mathscr{B}) = \{\mathscr{b} \in \mathscr{B}, p \in \mathscr{b}\}$ is a connected subset of $\mathrm{Vor}^{\sqcap}(S)$.

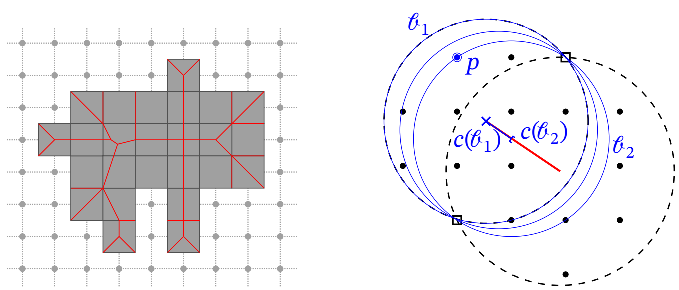
Lemma 6 (Lemma 4.9 [13])**.** *Let $p \in S$. If $p \subseteq \mathring{\mathscr{b}}_1 \cap \mathring{\mathscr{b}}_2$, then $p \subseteq \mathring{\mathscr{b}}$ for all $\mathscr{b}$ such that $c(\mathscr{b})$ is on the unique path $\Gamma(\mathscr{b}_1, \mathscr{b}_2)$ between $c(\mathscr{b}_1)$ and $c(\mathscr{b}_2)$ in $Vor^{\sqcap}(S)$.*

The proof uses projection π defined in the previous section, together with the fact that $\mathrm{Vor}^{\sqcap}(S)$ is the geometric embedding of a tree.

This lemma implies that Property (1) is true for $\mathscr{B}$. It moreover implies that for all p, $Cov(p, \mathscr{B})$ admits a supremum according to the order $\mathcal{T}$. However, since the balls of $\mathscr{B}$ are open, these sets are open too (see illustration in Figure 2(b)), except for points p that belong to the balls that are either the root or leaves of $\mathcal{T}$. A consequence is that, in general, $Cov(p, \mathscr{B})$ does not admit a greatest element, and $p \notin Dig(\sup_{\mathcal{T}} Cov(p, \mathscr{B}))$. This results in the following property:

Lemma 7. *For any $p \in S$ that does not belong to the root of $\mathcal{T}$, $\sup_{\mathcal{T}} Cov(p, \mathscr{B})$ either belongs to an open segment of $\mathrm{Vor}^{\sqcap}(S)$ or, if it is a vertex, the balls of $Cov(p, \mathscr{B})$ are all in the same subtree of predecessors.*

Proof. Suppose that $\sup_{\mathcal{T}} Cov(p, \mathscr{B})$ is a vertex $v \in \mathrm{Vor}^{\sqcap}(S)$, and, by contradiction, pick any ball of $Cov(p, \mathscr{B})$ in a first subtree, and another one in another subtree. Then the unique path between them goes through v, and the ball centered on v must contain p by Lemma 6 and thus be in $Cov(p, \mathscr{B})$. It cannot be the supremum of $Cov(p, \mathscr{B})$. □ □



(a) (b)

Figure 2: (a) Cropped Voronoi diagram $\mathrm{Vor}^{\sqcap}(S)$ for a set of pixels $\mathcal{S}$ depicted in grey. (b) $Cov(p, \mathscr{B})$ is an open set. Part of $\mathrm{Vor}^{\sqcap}(S)$ is depicted in red : the centers of all the balls of $Cov(p, \mathscr{B})$ are on the blue segment, delimited by $c(\mathscr{b}_1)$ and $c(\mathscr{b}_2)$, but $\mathscr{b}_2$ does not belong to $Cov(p, \mathscr{B})$ since it contains p on its boundary.

The set of ranges $\mathscr{B}$ does not fulfill property (2), which is required for Algorithm 1 to be valid. We turn to the set $\mathcal{B} = \{b \subseteq S, \exists \mathscr{b} \in \mathscr{B}\ Dig(\mathscr{b}) = b\}$ instead. Since $\mathcal{B}$ is finite, the sets $Cov(p, \mathcal{B}) = \{b \in \mathcal{B}, p \in b\}$ are also finite and are good candidates to admit a greatest element if equipped with a partial order. We show hereafter how to do this without explicitly computing the set $\mathcal{B}$.

3.3 Ordering digital balls of $\mathcal{B}$

Let the *representative* of b be $\mathrm{Rep}(b) = \sup_{T}\{\mathscr{b} \in \mathscr{B}, \mathscr{b} \in Dig^{-1}(b)\}$. As seen before, usually, $Dig(\mathrm{Rep}(b)) \neq b$. From the partial order $\mathcal{T}$ on $\mathscr{B}$, we define a partial order T on $\mathcal{B}$ as follows:

Definition 5. *Given two digital balls b_1 and b_2 of $\mathcal{B}$, $b_1 \leq_T b_2$ if:*

(1) either $b_1 = b_2$

(2) or $b_1 \neq b_2$ and

(a) either $Rep(b_1) <_{\mathcal{T}} Rep(b_2)$

(b) or $Rep(b_1) = Rep(b_2)$ and $Dig(Rep(b_2)) = b_2$.

Lemma 8. *$(\mathcal{B}, \leq_T)$ is a poset.*

Sketch of proof. Reflexivity follows directly from (1). Antisymmetry is shown by contradiction considering two cases: either $\mathrm{Rep}(b_1) \neq \mathrm{Rep}(b_2)$ and we get a contradiction by Definition 5 and definition of $\mathcal{T}$, or $\mathrm{Rep}(b_1) = \mathrm{Rep}(b_2)$ and we have a contradiction with unicity of $Dig(\mathscr{b})$ using Definition 5 (2)(b). To show transitivity, the case $b_1 = b_2$ or $b_2 = b_3$ is trivial. Otherwise, we distinguish the two cases $\mathrm{Rep}(b_1) \neq \mathrm{Rep}(b_2) \neq Rep(b_3)$ and $\mathrm{Rep}(b_1) = \mathrm{Rep}(b_2)$ and $\mathrm{Rep}(b_2) \neq \mathrm{Rep}(b_3)$ and conclude that $\mathrm{Rep}(b_1) <_{\mathcal{T}} \mathrm{Rep}(b_3)$ using the fact that $\mathcal{T}$ is a partial order. $\square$

In order to prove that the poset $(\mathcal{B}, \leq_T)$ is anti-arborescent, we need two extra lemmas that express properties of the sets $Dig^{-1}(b)$. The following lemma, together with Lemma 6, moreover ensures that Property (1) is fulfilled for the set of ranges $\mathcal{B}$.

Lemma 9. *(i) For any $b \in \mathcal{B}$, $Dig^{-1}(b)$ is connected. (ii) For any $b, b' \in \mathcal{B}$, $b \neq b'$, $Dig^{-1}(b) \cap Dig^{-1}(b') = \emptyset$. As a consequence, we have (iii): let $\mathscr{b} \in Dig^{-1}(b)$ and $\mathscr{b}' \in Dig^{-1}(b')$ with $b \neq b'$: if $\mathscr{b} <_{\mathcal{T}} \mathscr{b}'$, then $\mathscr{b} \leq_{\mathcal{T}} Rep(b) \leq_{\mathcal{T}} \mathscr{b}' \leq_{\mathcal{T}} Rep(b')$.*

Proof. (i) follows from Lemma 6 since $Dig^{-1}(b) = \bigcap_{p \in b} Cov(p, \mathscr{B})$. (ii) is straightforward by unicity of the digitization. To prove (iii), note that $\mathrm{Rep}(b)$ and $\mathscr{b}'$ are comparable since they are both successors of $\mathscr{b}$ and $\mathcal{T}$ is an anti-arborescence. Suppose by contradiction that $\mathscr{b}' \lneq_{\mathcal{T}} \mathrm{Rep}(b)$. Then $\mathscr{b}'$ is on the unique path between $\mathscr{b}$ and $\mathrm{Rep}(b)$, a contradiction with (i) and (ii). $\square$ $\square$

Lemma 10. *Let $b_1, b_2 \in \mathcal{B}$, $b_1 \neq b_2$, and $\mathscr{b} = Rep(b_1) = Rep(b_2)$ such that $\mathscr{b} \notin Dig^{-1}(b_1)$ and $\mathscr{b} \notin Dig^{-1}(b_2)$. Then for all $\mathscr{b}_1 \in Dig^{-1}(b_1)$ and all $\mathscr{b}_2 \in Dig^{-1}(b_2)$, $\mathscr{b}_1$ ans $\mathscr{b}_2$ are incomparable.*

Proof. Suppose by contradiction that $\mathscr{b}_1 <_{\mathcal{T}} \mathscr{b}_2$. Thus $\mathscr{b}_1 \lneq_{\mathcal{T}} \mathrm{Rep}(b_1)$. Since $Dig^{-1}(b_2)$ cannot be empty, $\mathrm{Rep}(b_2) \notin Dig^{-1}(b_2)$ implies that there exists $\mathscr{b}_2 \neq \mathrm{Rep}(b_2)$ such that $\mathscr{b}_2 \in Dig^{-1}(b_2)$. Using Lemma 9, we get $\mathscr{b}_1 \lneq_{\mathcal{T}} \mathrm{Rep}(b_1) \leq_{\mathcal{T}} \mathscr{b}_2 \lneq_{\mathcal{T}} \mathrm{Rep}(b_2)$, which is a contradiction with $\mathrm{Rep}(b_1) = \mathrm{Rep}(b_2)$. $\square$ $\square$

Theorem 2. *The poset $(\mathcal{B}, \leq_T)$ is anti-arborescent.*

Sketch of proof. We first prove by contradiction that the successors of any $b \in \mathcal{B}$ are comparable, by considering three cases: $\text{Rep}(b) = \text{Rep}(b_1) = \text{Rep}(b_2)$, or $\text{Rep}(b) = \text{Rep}(b_1)$ and $\text{Rep}(b) <_{\mathcal{T}} \text{Rep}(b_2)$, or $\text{Rep}(b) <_{\mathcal{T}} \text{Rep}(b_1)$ and $\text{Rep}(b) <_{\mathcal{T}} \text{Rep}(b_2)$. In the first two cases, we have a direct contradiction with Definition 5. The third case is a little bit trickier and uses Lemma 10.

Next, we prove by contradiction that the predecessors of two incomparable balls b_1 and b_2 are also incomparable. Two cases are studied: if $b'_1 = b'_2$, then $\text{Rep}(b'_1) = \text{Rep}(b'_2)$ and we use Lemma 9(iii) and the fact that $(\mathscr{B}, \mathcal{T})$ is an anti-arborescence to get a contradiction; if $b'_1 \leq_T b'_2$ and $\text{Rep}(b'_1) \lneq_{\mathcal{T}} \text{Rep}(b'_2)$, we use again Lemma 9$(iii)$ to get a contradiction. □

It remains to prove that $Cov(p, \mathcal{B})$ admits a greatest element for all $p \in S$. To do so, we remark that the ball $\sup_{\mathcal{T}} Cov(p, \mathscr{B})$ of a point $p \in S$ can be written as the maximum representative ball of $Cov(p, \mathcal{B})$.

$$\begin{aligned}
\sup\nolimits_{\mathcal{T}} Cov(p, \mathscr{B}) &= \sup\nolimits_{\mathcal{T}} \{\mathscr{b} \in \mathscr{B}, p \in \mathscr{b}\} \\
&= \sup\nolimits_{\mathcal{T}} \{\mathscr{b} \in \mathscr{B}, p \in Dig(\mathscr{b})\} \\
&= \sup_{b \in Cov(p,\mathcal{B})}{}_{\mathcal{T}} \; \{\mathscr{b} \in \mathscr{B}, Dig(\mathscr{b}) = b\} \\
&= \sup_{b \in Cov(p,\mathcal{B})}{}_{\mathcal{T}} \; \text{Rep}(b) = \max_{b \in Cov(p,\mathcal{B})}{}_{\mathcal{T}} \; \text{Rep}(b)
\end{aligned} \tag{1}$$

The digital ball $b_{max}(p) \in \mathcal{B}$ that achieves the maximum in Equation (1) is actually the critical ball $\text{Crit}(p, \mathcal{B})$. In the last subsection, we show how to compute it.

Lemma 11. *Let $p \in S$, and $b_{max}(p) \in Cov(p, \mathcal{B})$ be such that $Rep(b_{max}(p)) = \sup_{\mathcal{T}} Cov(p, \mathscr{B})$. Then for any $b \in Cov(p, \mathcal{B})$, $b \leq_T b_{max}(p)$.*

Proof. Let $b \in Cov(p, \mathcal{B})$. If $\text{Rep}(b) <_{\mathcal{T}} \text{Rep}(b_{max}(p))$, by Definition 5, $b <_T b_{max}(p)$. The case $\text{Rep}(b) >_{\mathcal{T}} \text{Rep}(b_{max}(p))$ is not possible by definition of $b_{max}(p)$. The case $\text{Rep}(b) = \text{Rep}(b_{max}(p))$ remains. Since $Dig^{-1}(b)$ are connected and disjoint (Lemma 9), the only way for two balls b_1 and b_2 to have the same representative is when it is a vertex of the anti-arborescence. Then $Dig^{-1}(b_1)$ and $Dig^{-1}(b_2)$ belong to two different subtrees of this vertex, a contradiction with Lemma 7. □ □

3.4 Computing critical balls

The first step is to find the edge of $\text{Vor}^{\sqcap}(S)$ $\text{Rep}(\text{Crit}(p, \mathcal{B}))$ belongs to. It is convenient to note that each edge of $\text{Vor}^{\sqcap}(S)$ corresponds to balls of $\mathscr{B}$ that go through a pair of points of S^c. This edge can then be described as a parabolic pencil of circles [5, 15] defined by two points of S^c and delimited by its two extremities. Each ball $\mathscr{b}_\lambda$ of the pencil can be expressed as a convex combination of the two extremities, according to the following relation: $\forall p$, $\text{pow}(p, \mathscr{b}_\lambda) = (1 - \lambda)\text{pow}(p, \mathscr{b}_1) + \lambda\text{pow}(p, \mathscr{b}_2)$, where pow denotes the power of a point with respect to a ball $\mathscr{b}(c, r)$ and is equal to $\text{pow}(p, \mathscr{b}) = d(c, p) - r^2$.

Given a topological order on the edges of $\mathrm{Vor}^{\sqcap}(S)$, consider the edges $[\mathscr{b}_1, \mathscr{b}_2]$ in increasing order. If p belongs to $Dig(\mathscr{b}_1)$ but not to $Dig(\mathscr{b}_2)$, then $\mathrm{Rep}(\mathrm{Crit}(p, \mathcal{B}))$ belongs to the edge $[\mathscr{b}_1, \mathscr{b}_2[$. Using the fact that $p \in \partial\mathrm{Rep}(\mathrm{Crit}(p, \mathcal{B}))$, and the relation above, we can compute the value λ such that $\mathscr{b}_\lambda = \mathrm{Rep}(\mathrm{Crit}(p, \mathcal{B}))$ on the pencil $[\mathscr{b}_1, \mathscr{b}_2[$, as $\lambda = \frac{\mathrm{pow}(p,\mathscr{b}_1)}{\mathrm{pow}(p,\mathscr{b}_1)-\mathrm{pow}(p,\mathscr{b}_2)}$. For all $0 \leq \lambda' < \lambda$, $Dig(\mathscr{b}_{\lambda'})$ contains p (see Figure 3(a)). We look for a value $\lambda_{crit} < \lambda$ such that $Dig(\mathscr{b}_\lambda) \subset Dig(\mathscr{b}_{\lambda_{crit}})$. Such a value exist thanks to Lemma 7. For all the points $q \in Dig(\mathscr{b}_\lambda) \setminus Dig(\mathscr{b}_1)$, let $\mathscr{b}_{\lambda_q}$ be the ball of $[\mathscr{b}_1, \mathscr{b}_2[$ such that $q \in \partial\mathscr{b}_{\lambda_q}$. For all values $\mu > \lambda_q$, $q \in Dig(\mathscr{b}_\mu)$. By setting $\mu = \max_q\{\lambda_q\}$, we have that $\forall \mu' > \mu$, $Dig(\mathscr{b}_\lambda) \subset Dig(\mathscr{b}'_\mu)$.

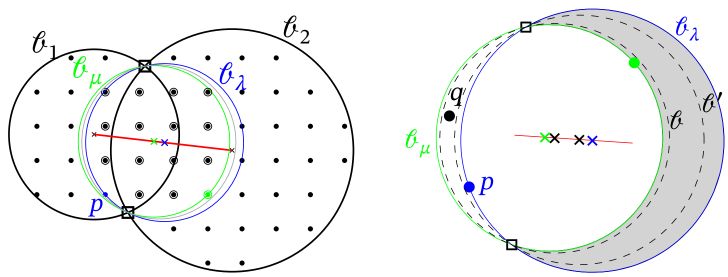


Figure 3: (a) Computation of $\mathrm{Crit}(p, \mathcal{B})$: $\mathscr{b}_\lambda = \mathrm{Rep}(\mathrm{Crit}(p, \mathcal{B}))$; any ball between $\mathscr{b}_\mu$ and $\mathscr{b}_\lambda$ (see for instance the ball in gray) contains p and all the points of $Dig(\mathscr{b}_\lambda)$ (circled). (b) Illustration of the fact that the critical ball is not always maximal.

By setting λ_{crit} to any value strictly between μ and λ, we have $Dig(\mathscr{b}_{\lambda_{crit}}) \supset \{p\} \cup Dig(\mathscr{b}_\lambda)$ as desired. Note that, as mentioned before, $Dig(\mathscr{b}_{\lambda_{crit}})$ may not be maximal. Indeed, as illustrated in Figure 3(b), by definition of $\mathscr{b}_\mu$ there is no point of S in the grey region. However, $Dig(\mathscr{b}_\mu)$ may contain points of S other than p, for instance point q in the figure. Thus, if we consider the two balls $\mathscr{b}$ and $\mathscr{b}'$ both between $\mathscr{b}_\mu$ and $\mathscr{b}_\lambda$, $Dig(\mathscr{b}') \subset Dig(\mathscr{b})$, so that $Dig(\mathscr{b}')$ is not maximal. As proven in the section before, this is not a problem: in the course of the algorithm, either q belongs to the subset not covered yet, and then the critical ball of q, which is equal to $Dig(\mathscr{b})$, is chosen, or q is already covered, and picking $Dig(\mathscr{b}')$ instead of $Dig(\mathscr{b})$ does not change anything.

4 Results

Algorithm 1 was implemented[1] using three open-source libraries: DGtal [2] to handle digital sets, CGAL [1] to compute $\mathrm{Vor}^{\sqcap}(S)$, and Boost Graph [3] to compute topological order on trees. A kernel with exact predicates and constructions was used to avoid rounding errors. As a conclusion, some results are presented in Figure 4.

[1] `https://github.com/isivigno/ConvertDigitalObjectToBalls.git`

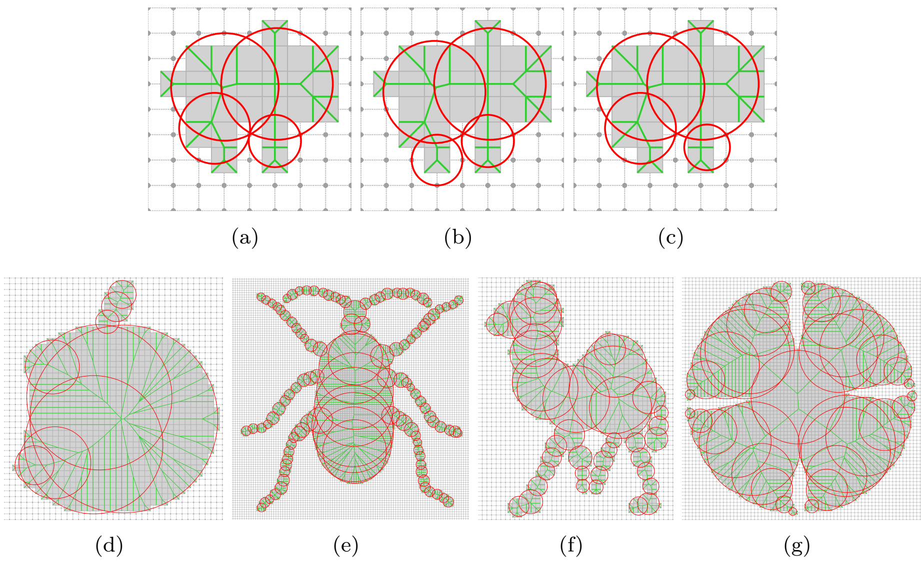


Figure 4: (a-c) Three different optimal coverings of the same toy example with 4 balls, obtained using different roots ; (d-g) Results on images of the database MPEG7 CE Shape-1 Part B :(d) 9 balls (e) 113 balls (f) 40 balls (g) 36 balls.

References

[1] CGAL: Computational Geometry Algorithms Library. `cgal.org`

[2] DGtal: Digital Geometry Tools and Algorithms Library. `dgtal.org`

[3] The Boost Graph Library. `www.boost.org/doc/libs/1_78_0/libs/graph`

[4] Attali, D., Nguyen, T.B., Sivignon, I.: (δ, ε)-ball approximation of a shape: Definition and complexity. Discrete and Computational Geometry 61, 595–625 (2019)

[5] Boissonnat, J.D., Yvinec, M.: Géométrie algorithmique (1995)

[6] Choi, A.R., Sung, M.Y.: Performance improvement of haptic collision detection using subdivision surface and sphere clustering. PLOS ONE 12(9), 1–17 (09 2017)

[7] Coeurjolly, D., Hulin, J., Sivignon, I.: Finding a minimum medial axis of a discrete shape is np-hard. Theoretical Computer Science 406(1), 72–79 (2008)

[8] Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C.: Introduction to Algorithms, Third Edition. The MIT Press, 3rd edn. (2009)

[9] F. Fauvet, F. Menous, D.S.: Explicit linearization of one-dimensional germs through tree-expansions. Bulletin de la SMF 146(2), 241–285 (2018)

[10] Lieutier, A.: Any open bounded subset of $\mathbb{R}^n$ has the same homotopy type as its medial axis. Computer-Aided Design 36(11), 1029–1046 (2004)

[11] Matheron, G., Serra, J. (eds.): Image analysis and mathematical morphology 2: theoretical advances, pp. 216–256. Academic Press (1988)

[12] Mede, T., Chambon, G., Hagenmuller, P., Nicot, F.: A medial axis based method for irregular grain shape representation in dem simulations. Granular Matter 20(16) (2018)

[13] Nguyen, T.B.: Finite unions of balls with inner and outer margins. Ph.D. thesis, Université Grenoble Alpes (2018)

[14] Nguyen, T.B., Sivignon, I.: Epsilon-covering: a greedy optimal algorithm for simple shapes. In: CCCG 2016 - 28th Canadian Conf. on Computational Geometry (2016)

[15] Schwerdtfeger, H.: Geometry of complex numbers: circle geometry, Moebius transformation, non-euclidean geometry. Courier Corporation (1979)