

Space-efficient data structure for next/previous larger/smaller value queries

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Abstract. Given an array of size n from a total order, we consider the problem of constructing a data structure that supports various queries (range minimum/maximum queries with their variants and next/previous larger/smaller queries) efficiently. In the encoding model (i.e., the queries can be answered without the input array), we propose a $(3.701n + o(n))$ -bit data structure, which supports all these queries in $O(\log^{(\ell)} n)$ time, for any positive integer ℓ (here, $\log^{(1)} n = \log n$, and for $\ell > 1$, $\log^{(\ell)} n = \log(\log^{(\ell-1)} n)$). The space of our data structure matches the current best upper bound of Tsur (Inf. Process. Lett., 2019), which does not support the queries efficiently. Also, we show that at least $3.16n - \Theta(\log n)$ bits are necessary for answering all the queries. Our result is obtained by generalizing Gawrychowski and Nicholson’s $(3n - \Theta(\log n))$ -bit lower bound (ICALP, 15) for answering range minimum and maximum queries on a permutation of size n .

Keywords: Range minimum queries · Encoding model · Balanced parenthesis sequence.

1 Introduction

Given an array $A[1, \dots, n]$ of size n from a total order and an interval $[i, j] \subset [1, n]$, suppose there are k distinct positions $i \leq p_1 \leq p_2 \leq \dots \leq p_k \leq j$ where p_1, p_2, \dots, p_k are the positions of minimum elements in $A[i, \dots, j]$. Then, for $q \geq 1$, *range q -th minimum query* on the interval $[i, j]$ ($\text{RMin}(i, j, q)$) returns the position p_q (returns p_k if $q > k$), and *range minimum query* on the interval $[i, j]$ ($\text{RMin}(i, j)$) returns an arbitrary position among p_1, p_2, \dots, p_k . One can also analogously define *range q -th maximum query* (resp. *range maximum query*) on the interval $[i, j]$, denoted by $\text{RMax}(i, j, q)$ (resp. $\text{RMax}(i, j)$).

In addition to the above queries, one can define next/previous larger/smaller queries as follows. When the position i is given, the *previous smaller value query* on the position i ($\text{PSV}(i)$) returns the rightmost position $j < i$, where $A[j]$ is smaller than $A[i]$ (returns 0 if no such j exists), and the *next smaller value query* on the position i ($\text{NSV}(i)$) returns the leftmost position $j > i$ where $A[j]$ is smaller than $A[i]$ (returns $n + 1$ if no such j exists). The *previous* (resp. *next*)

larger value query on the position i , denoted by $\text{PLV}(i)$ (resp. $\text{NLV}(i)$)) is also defined analogously.

In this paper, we focus on the problem of constructing a data structure that efficiently answers all the above queries. We consider the problem in the *encoding model* [15], which does not allow access to the input A for answering the queries after preprocessing. In the encoding data structure, the lower bound of the space is referred to as the *effective entropy* of the problem. Note that for many problems, their effective entropies have much smaller size compared to the size of the inputs [15]. Also, an encoding data structure is called *succinct* if its space usage matches the optimal up to lower-order additive terms. The rest of the paper only considers encoding data structures and assumes a $\Theta(\log n)$ -bit word RAM model, where n is the input size.

Previous Work The problem of constructing an encoding data structure for answering range minimum queries has been well-studied because of its wide applications. It is well-known that any two arrays have a different set of answers of range minimum queries if and only if their corresponding Cartesian trees [19] are distinct. Thus, the effective entropy of answering range minimum queries on the array A of size n is $2n - \Theta(\log n)$ bits. Sadakane [17] proposed the $(4n + o(n))$ -bit encoding with $O(1)$ query time using the balanced-parenthesis (BP) [11] of the Cartesian tree on A with additional nodes. Fisher and Heun [7] proposed the $(2n + o(n))$ -bit data structure (hence, succinct), which supports $O(1)$ query time using the depth-first unary degree sequence (DFUDS) [2] of the *2d-min heap* on A . Here, a 2d-min heap of A is an alternative representation of the Cartesian tree on A . By maintaining the encodings of both 2d-min and max heaps on A (2d-max heap can be defined analogously to 2d-min heap), the encoding of [7] directly gives a $(4n + o(n))$ -bit encoding for answering both range minimum and maximum queries in $O(1)$ time. Gawrychowski and Nicholson [8] reduced this space to $(3n + o(n))$ -bit while supporting the same query time for both queries. They also showed that the effective entropy for answering the range minimum and maximum queries is at least $3n - \Theta(\log n)$ bits.

Next/previous smaller value queries were motivated from the parallel computing [3], and have application in constructing compressed suffix trees [14]. If all elements in A are distinct, one can answer both the next and previous smaller queries using Fischer and Heun's encoding for answering range minimum queries [7]. For the general case, Ohlebusch et al. [14] proposed the $(3n + o(n))$ -bit encoding for supporting range minimum and next/previous smaller value queries in $O(1)$ time. Fischer [6] improved the space to $2.54n + o(n)$ bits while maintaining the same query time. More precisely, their data structure uses the *colored 2d-min heap* on A , which is a 2d-min heap on A with the coloring on its nodes. Since the effective entropy of the colored 2d-min heap on A is $2.54n - \Theta(\log n)$ bits [10], the encoding of [6] is succinct. For any $q \geq 1$, the encoding of [6] also supports the range q -th minimum queries in $O(1)$ time [9].

From the above, the encoding of Fischer [6] directly gives a $(5.08n + o(n))$ -bit data structure for answering the range q -th minimum/maximum queries and

Table 1. Summary of the upper and lower bounds results of encoding data structures for answering q -th minimum/maximum queries and next larger/smaller value queries on the array $A[1, \dots, n]$, for any $q \geq 1$ (here, we can choose ℓ as any positive integer). Note that all our upper bound results also support the range minimum/maximum and previous larger/smaller value queries in $O(1)$ time (the data structures of [6, 9] with $O(1)$ query time also support these queries in $O(1)$ time).

Array Type	Space (in bits)	Query time	Reference
Upper bounds			
$A[i] \neq A[i+1]$ for all $i \in [1, n-1]$	$4n + o(n)$	$O(1)$	[9]
	$3.585n$	$O(n)$	[18]
	$3.585n + o(n)$	$O(\log^{(\ell)} n)$	This paper
General array	$5.08n + o(n)$	$O(1)$	[6, 9]
	$4.088n + o(n)$	$O(n)$	[9]
	$4.585n + o(n)$	$O(1)$	
	$3.701n$	$O(n)$	[18]
	$3.701n + o(n)$	$O(\log^{(\ell)} n)$	This paper
Lower bounds			
Permutation	$3n - \Theta(\log n)$		[8]
General array	$3.16n - \Theta(\log n)$		This paper

next/previous larger/smaller value queries in $O(1)$ time by maintaining the data structures of both colored 2d-min and max heaps. Jo and Satti [9] improved the space to (i) $4n + o(n)$ bits if there are no consecutive equal elements in A and (ii) $4.585n + o(n)$ bits for the general case while supporting all the queries in $O(1)$ time. They also showed that if the query time is not of concern, the space of (ii) can be improved to $4.088n + o(n)$ bits. Recently, Tsur [18] improved the space to $3.585n$ bits if there are no consecutive equal elements in A and $3.701n$ bits for the general case. However, their encoding does not support the queries efficiently ($O(n)$ time for all queries).

Our Results Given an array $A[1, \dots, n]$ of size n with the interval $[i, j] \subset [1, n]$ and the position $1 \leq p \leq n$, we show the following results:

- (a) If A has no two consecutive equal elements, there exists a $(3.585n + o(n))$ -bit data structure, which can answer (i) $\text{RMin}(i, j)$, $\text{RMax}(i, j)$, $\text{PSV}(p)$, and $\text{PLV}(p)$ queries in $O(1)$ time, and (ii) for any $q \geq 1$, $\text{RMin}(i, j, q)$, $\text{RMax}(i, j, q)$, $\text{NSV}(p)$, and $\text{NLV}(p)$ queries in $O(\log^{(\ell)} n)$ time³, for any positive integer ℓ .
- (b) For the general case, the data structure of (a) uses $3.701n + o(n)$ bits while supporting the same query time.

Our results match the current best upper bounds of Tsur [18] up to lower-order additive terms while supporting the queries efficiently.

The main idea of our encoding data structure is to combine the BP of colored 2d-min and max heap of A . Note that all previous encodings in [8, 9, 18] combine the DFUDS of the (colored) 2d-min and max heap on A . We first consider

³ Throughout the paper, we denote $\log n$ as the logarithm to the base 2

the case when A has no two consecutive elements (Section 3). In this case, we show that by storing the BP of colored 2d-min heap on A along with its color information, there exists a data structure that uses at most $3n + o(n)$ bits while supporting range minimum, range q -th minimum, and next/previous smaller value queries efficiently. The data structure is motivated by the data structure of Jo and Satti [9] which uses DFUDS of colored 2d-min heap on A . Compared to the data structure of [9], our data structure uses less space for the color information. Next, we show how to combine the data structures on colored 2d-min and max heap on A into a single structure. The combined data structure is motivated by the idea of Gawrychowski and Nicholson's encoding [8] to combine the DFUDS of 2d-min and max heap on A .

In Section 4, we consider the case that A has consecutive equal elements. In this case, we show that by using some additional auxiliary structures, the queries on A can be answered efficiently from the data structure on the array A' , which discards all the consecutive equal elements from A .

Finally, in Section 5, we show that the effective entropy of the encoding to support the range q -th minimum and maximum queries on A is at least $3.16n - \Theta(\log n)$ bits. Our result is obtained by extending the $(3n - \Theta(\log n))$ -bit lower bound of Gawrychowski and Nicholson [8] for answering the range minimum and maximum queries on a permutation of size n . We summarize our results in Table 1.

2 Preliminaries

This section introduces some data structures used in our results.

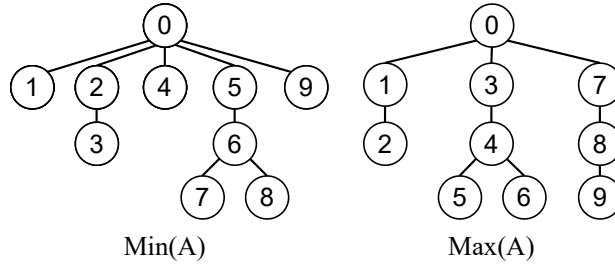


Fig. 1. $\text{Min}(A)$ and $\text{Max}(A)$ on the array $A = 5\ 4\ 5\ 3\ 1\ 2\ 6\ 3\ 1$.

2d min-heap and max-heap Given an array $A[1, \dots, n]$ of size n , the 2d min-heap on A (denoted by $\text{Min}(A)$) [6] is a rooted and ordered tree with $n + 1$ nodes, where each node corresponds to the value in A , and the children are ordered from left to right. More precisely, $\text{Min}(A)$ is defined as follows:

1. The root of $\text{Min}(A)$ corresponds to $A[0]$ ($A[0]$ is defined as $-\infty$).
2. For any $i > 0$, $A[i]$ corresponds to the $(i + 1)$ -th node of $\text{Min}(A)$ according to the preorder traversal.
3. For any non-root node corresponds to $A[j]$, its parent node corresponds to $A[\text{PSV}(j)]$.

In the rest of the paper, we refer to the node i in $\text{Min}(A)$ as the node corresponding to $A[i]$ (i.e., the $(i + 1)$ -th node according to the preorder traversal). One can also define the 2d-max heap on A (denoted as $\text{Max}(A)$) analogously. More specifically, in $\text{Max}(A)$, $A[0]$ is defined as ∞ , and the parent of node $i > 0$ corresponds to the node $\text{PLV}(i)$ (see Figure 1 for an example). In the rest of the paper, we only consider $\text{Min}(A)$ unless $\text{Max}(A)$ is explicitly mentioned. The same definitions, and properties for $\text{Min}(A)$ can be applied to $\text{Max}(A)$.

For any $i > 0$, $\text{Min}(A)$ is the *relevant tree* of the node i if the node i is an internal node in $\text{Min}(A)$. From the definition of $\text{Min}(A)$, Tsur [18] showed the following lemma.

Lemma 1 ([18]). *For any $i \in \{1, 2, \dots, n - 1\}$, the following holds:*

- (a) *If $\text{Min}(A)$ is a relevant tree of the node i , then the node $(i + 1)$ is the leftmost child of the node i in $\text{Min}(A)$.*
- (b) *If A has no two consecutive equal elements, $\text{Min}(A)$ (resp. $\text{Max}(A)$) is a relevant tree of the node i if and only if the node i is a leaf node in $\text{Max}(A)$ (resp. $\text{Min}(A)$).*

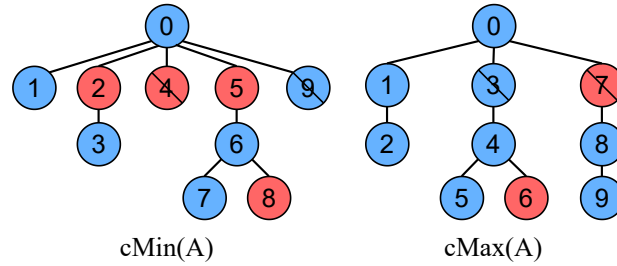


Fig. 2. $\text{cMin}(A)$ and $\text{cMax}(A)$ on the array $A = 5\ 4\ 5\ 3\ 1\ 2\ 6\ 3\ 1$. The nodes with slash line indicate the valid nodes.

Colored 2d min-heap and max-heap The colored 2d-min heap of A (denoted by $\text{cMin}(A)$) [6] is $\text{Min}(A)$ where each node is colored *red* or *blue* as follows. The node i in $\text{cMin}(A)$ is colored red if and only if i is not the leftmost child of its parent node, and $A[i] \neq A[j]$, where the node j is the node i 's immediate left

sibling. Otherwise, the node i is colored blue. One can also define the colored 2d-max heap on A (denoted by $\text{cMax}(A)$) analogously (see Figure 2 for an example). The following lemma says that we can obtain the color of some nodes in $\text{cMin}(A)$ from their tree structures.

Lemma 2. *For any node i in $\text{cMin}(A)$, the following holds:*

- (a) *If the node i is the leftmost child of its parent node, the color of the node i is always blue.*
- (b) *If A has no two consecutive equal elements, the color of the node i is always red if its immediate left sibling is a leaf node.*

Proof. (a) is directly proved from the definition of $\text{cMin}(A)$. Also, if the immediate left sibling j of the node i is a leaf node, j is equal to $i - 1$ (note that the preorder traversal of $\text{cMin}(A)$ visits the node i immediately after visiting the node j). Thus, if A has no consecutive equal elements, the color of the node i is red. \square

We say the node i in $\text{cMin}(A)$ is *valid* if the node i is a non-root node in $\text{cMin}(A)$ which is neither the leftmost child nor the immediate right sibling of any leaf node. Otherwise, the node i is *invalid*. By Lemma 2, if A has no two consecutive equal elements, the color of the invalid nodes of $\text{cMin}(A)$ can be decoded from the tree structure.

Rank and Select queries on bit arrays Given a bit array $B[1, \dots, n]$ of size n , and a pattern $p \in \{0, 1\}^+$, (i) $\text{rank}_p(i, B)$ returns the number of occurrence of the pattern p in $B[1, \dots, i]$, and (ii) $\text{select}_p(j, B)$ returns the first position of the j -th occurrence of the pattern p in B . The following lemma shows that there exists a succinct encoding, which supports both **rank** and **select** queries on B efficiently.

Lemma 3 ([12, 16]). *Given a bit array $B[1, \dots, n]$ of size n containing m 1s, and a pattern $p \in \{0, 1\}^+$ with $|p| \leq \frac{\log n}{2}$, the following holds:*

- *There exists a $(\log \binom{n}{m} + o(n))$ -bit data structure for answering both $\text{rank}_p(i, B)$ and $\text{select}_p(j, B)$ queries in $O(1)$ time. Furthermore, the data structure can access any $\Theta(\log n)$ -sized consecutive bits of B in $O(1)$ time.*
- *If one can access any $\Theta(\log n)$ -sized consecutive bits of B in $O(1)$ time, both $\text{rank}_p(i, B)$ and $\text{select}_p(j, B)$ queries can be answered in $O(1)$ time using $o(n)$ -bit auxiliary structures.*

Balanced-parenthesis of trees Given a rooted and ordered tree T with n nodes, the balanced-parenthesis (BP) of T (denoted by $\text{BP}(T)$) [11] is a bit array defined as follows. We perform a preorder traversal of T . We then add a 0 to $\text{BP}(T)$ when we first visit a node and add a 1 to $\text{BP}(T)$ after visiting all nodes in the subtree of the node. Since we add single 0 and 1 to $\text{BP}(T)$ per each node in T , the size of $\text{BP}(T)$ is $2n$. For any node i in T , we define

$f(i, T)$ and $s(i, T)$ as the positions of the 0 and 1 in $BP(T)$ which are added when the node i is visited, respectively. When T is clear from the context, we write $f(i)$ (resp. $s(i)$) to denote $f(i, T)$ (resp. $s(i, T)$). If T is a 2d-min heap, $f(i, T) = \text{select}_0(i + 1, BP(T))$ by the definition of 2d-min heap.

3 Data structure on arrays with no consecutive equal elements

In this section, for any positive integer ℓ , we present a $(3.585n + o(n))$ -bit data structure on $A[1, \dots, n]$, which supports (i) range minimum/maximum and previous larger/smaller queries on A in $O(1)$ time, and (ii) range q -th minimum/maximum and next larger/smaller value queries on A in $O(\log^{(\ell)} n)$ time for any $q \geq 1$, when there are no two consecutive equal elements in A . We first describe the data structure on $\text{cMin}(A)$ for answering the range minimum, range q -th minimum, and next/previous smaller value queries on A . Next, we show how to combine the data structures on $\text{cMin}(A)$ and $\text{cMax}(A)$ in a single structure.

Encoding data structure on $\text{cMin}(A)$ We store $\text{cMin}(A)$ by storing its tree structure along with the color information of the nodes. To store the tree structure, we use $BP(\text{cMin}(A))$. Also, for storing the color information of the nodes, we use a bit array c_{\min} , which stores the color of all valid nodes in $\text{cMin}(A)$ according to the preorder traversal order. In c_{\min} we use 0 (resp. 1) to indicate the color blue (resp. red). It is clear that $\text{cMin}(A)$ can be reconstructed from $BP(\text{cMin}(A))$ and c_{\min} . Since $BP(\text{cMin}(A))$ and c_{\min} takes $2(n + 1)$ bits and at most n bits, respectively, the total space for storing $\text{cMin}(A)$ takes at most $3n + 2$ bits. Note that a similar idea is used in Jo and Satti's *extended DFUDS* [9], which uses the DFUDS of $\text{cMin}(A)$ for storing the tree structure. However, extended DFUDS stores the color of all nodes other than the leftmost children, whereas c_{\min} does not store the color of all invalid nodes. The following lemma shows that from $BP(\text{cMin}(A))$, we can check whether the node i is valid or not without decoding the entire tree structure.

Lemma 4. *The node i is valid in $\text{cMin}(A)$ if and only if $f(i) > 2$ and both $BP(\text{cMin}(A))[f(i) - 2]$ and $BP(\text{cMin}(A))[f(i) - 1]$ are 1.*

Proof. If both $BP(\text{cMin}(A))[f(i) - 2]$ and $BP(\text{cMin}(A))[f(i) - 1]$ are 1, the preorder traversal of $\text{cMin}(A)$ must complete the traversal of two subtrees consecutively just before visiting the node i for the first time, which implies the node i 's immediate left sibling is not a leaf node (hence the node i is valid).

Conversely, if $BP(\text{cMin}(A))[f(i) - 1] = 0$, $\text{cMin}(A)$ is a relevant tree of the node $i - 1$. Thus, the node i is the leftmost child of the node $(i - 1)$ by Lemma 1. Next, if $BP(\text{cMin}(A))[f(i) - 2] = 0$ and $BP(\text{cMin}(A))[f(i) - 1] = 1$, the node $(i - 1)$ is the immediate left sibling of the node i since $f(i) - 2$ is equal to $f(i - 1)$. Also the node $(i - 1)$ is a leaf node since $f(i) - 1$ is equal to $s(i - 1)$. Thus, the node i is invalid in this case. \square

Now we describe how to support range minimum, range q -th minimum, and next/previous smaller value queries efficiently on A using $\text{BP}(\text{cMin}(A))$ and c_{\min} with $o(n)$ -bit additional auxiliary structures. Note that both the range minimum and previous smaller value query on A can be answered in $O(1)$ time using $\text{BP}(\text{cMin}(A))$ with $o(n)$ -bit auxiliary structures [5, 13]. Thus, it is enough to consider how to support a range q -th minimum and next smaller value queries on A . We introduce the following lemma of Jo and Satti [9], which shows that one can answer both queries with some navigational and color queries on $\text{cMin}(A)$.

Lemma 5 ([9]). *Given $\text{cMin}(A)$, suppose there exists a data structure, which can answer (i) the tree navigational queries (next/previous sibling, subtree size, degree, level ancestor, child rank, child select, and parent⁴) on $\text{cMin}(A)$ in $t(n)$ time, and the following color queries in $s(n)$ time:*

- *color(i): return the color of the node i*
- *PRS(i): return the rightmost red sibling to the left of the node i .*
- *NRS(i): return the leftmost red sibling to the right of the node i .*

Then for any $q \geq 1$, range q -th minimum, and the next smaller value queries on A can be answered in $O(t(n) + s(n))$ time.

Since all tree navigational queries in Lemma 5 can be answered in $O(1)$ time using $\text{BP}(\text{cMin}(A))$ with $o(n)$ -bit auxiliary structures [13], it is sufficient to show how to support $\text{color}(i)$, $\text{PRS}(i)$, and $\text{NRS}(i)$ queries using $\text{BP}(\text{cMin}(A))$ and c_{\min} . By Lemma 3 and 4, we can compute $\text{color}(i)$ in $O(1)$ time using $o(n)$ -bit auxiliary structures by the following procedure: We first check whether the node i is valid using $O(1)$ time by checking the values at the positions $f(i) - 1$ and $f(i) - 2$ in $\text{BP}(\text{cMin}(A))$. If the node i is valid (i.e., both the values are 1), we answer $\text{color}(i)$ in $O(1)$ time by returning $c_{\min}[j]$ where j is $\text{rank}_{110}(f(i), \text{BP}(\text{cMin}(A)))$ (otherwise, by Lemma 4, we answer $\text{color}(i)$ as blue if and only if the node i is the leftmost child of its parent node). Next, for answering $\text{PRS}(i)$ and $\text{NRS}(i)$, we construct the following ℓ' -level structure (ℓ' will be decided later):

- At the first level, we mark every $(\log n \log \log n)$ -th child node and maintain a bit array $M_1[1, \dots, n]$ where $M_1[t] = 1$ if and only if the node t is marked (recall that the node t is the node in $\text{Min}(A)$ whose preorder number is t). Since there are $n/(\log n \log \log n) = o(n)$ marked nodes, we can store M_1 using $o(n)$ bits while supporting rank queries in $O(1)$ time by Lemma 3 (in the rest of the paper, we ignore all floors and ceilings, which do not affect to the results). Also we maintain an array P_1 of size $n/(\log n \log \log n)$ where $P_1[j]$ stores both $\text{PRS}(s)$ and $\text{NRS}(s)$ if s is the j -th marked node according to the preorder traversal order. We can store P_1 using $O(n \log n/(\log n \log \log n)) = o(n)$ bits.
- For the i -th level where $1 < i \leq \ell'$, we mark every $(\log^{(i)} n \log^{(i+1)} n)$ -th child node. We then maintain a bit array M_i which is defined analogously to M_1 . We can store M_i using $o(n)$ bits by Lemma 3.

⁴ refer to Table 1 in [13] for detailed definitions of the queries

Now for any node p , let $cr(p)$ be the child rank of p , i.e., the number of left siblings of p . Also, let $pre_{(i-1)}(p)$ (resp. $next_{(i-1)}(p)$) be the rightmost sibling of p to the left (resp. leftmost sibling of p to the right) which is marked at the $(i-1)$ -th level. Suppose s is the j -th marked node at the current level according to the preorder traversal order. Then we define an array P_i of size $n/(\log^{(i)} n \log^{(i+1)} n)$ as $P_i[j]$ stores both (i) the smaller value between $cr(s) - cr(\text{PRS}(s))$ and $cr(s) - cr(pre_{(i-1)}(s))$, and (ii) the smaller value between $cr(\text{NRS}(s)) - cr(s)$ and $cr(next_{(i-1)}(s)) - cr(s)$. Since both (i) and (ii) are at most $\log^{(i-1)} n \log^{(i)} n$, we can store P_i using $O(n \log^{(i)} n / (\log^{(i)} n \log^{(i+1)} n)) = o(n)$ bits. Therefore, the overall space is $O(n / \log^{(\ell'+1)} n) = o(n)$ bits in total for any positive integer ℓ' .

To answer $\text{PRS}(i)$ (the procedure for answering $\text{NRS}(i)$ is analogous), we first scan the left siblings of i using the previous sibling operation. Whenever the node i_1 is visited during the scan, we check whether (i) $\text{color}(i_1) = \text{red}$, or (ii) $M_{\ell'}[i_1] = 1$ in $O(1)$ time. If i_1 is neither the case (i) nor (ii), we continue the scan. If i_1 is in the case (i), we return i_1 as the answer. If i_1 is in the case (ii), we jump to the i_1 's left sibling i_2 whose child rank is $cr(i_1) - P_{\ell'}[j]$, where $j = \text{rank}_1(i_1, M_{\ell'-1})$. Since the node i_2 always satisfies one of the following: $\text{color}(i_2) = \text{red}$ or $M_{\ell'-1}[i_2] = 1$, we can answer $\text{PRS}(i)$ by iteratively performing child rank and rank operations at most $O(\ell')$ times after finding i_2 . Thus, we can answer $\text{PRS}(i)$ in $O(\ell')$ time in total (we scan at most $O(\ell')$ nodes to find i_2). By choosing ℓ as $\ell' + 2$, we obtain the following theorem.

Theorem 1. *Given an array $A[1, \dots, n]$ of size n and any positive integer ℓ , we can answer (i) range minimum and previous smaller value queries in $O(1)$ time, and (ii) range q -th minimum and next smaller value queries for any $q \geq 1$ in $O(\log^{(\ell)} n)$ time, using $\text{BP}(\text{cMin}(A))$ and c_{\min} with $o(n)$ -bit auxiliary structures.*

Theorem 1 implies that there exists a data structure of $\text{cMax}(A)$ (composed to $\text{BP}(\text{cMax}(A))$ and c_{\max} with $o(n)$ -bit auxiliary structures), which can answer (i) range maximum and previous larger value queries in $O(1)$ time, and (ii) range q -th maximum and next larger value queries for any $q \geq 1$ in $O(\log^{(\ell)} n)$ time.

Combining the encoding data structures on $\text{cMin}(A)$ and $\text{cMax}(A)$

We describe how to combine the data structure of Theorem 1 on $\text{cMin}(A)$ and $\text{cMax}(A)$ using $3.585n + o(n)$ bits in total. We first briefly introduce the idea of Gawrychowski and Nicholson [8] to combine the DFUDS of $\text{Min}(A)$ and $\text{Max}(A)$. In DFUDS, any non-root node i is represented as a bit array $0^{d_i}1$ where d_i is the degree of i [2]. The encoding of [8] is composed of (i) a bit array $U[1, \dots, n]$, where $U[i]$ indicates the relevant tree of the node i , and (ii) a bit array $S = s_1 s_2 \dots s_n$ where s_i is the bit array, which omits the first 0 from the DFUDS of the node i on its relevant tree. To decode the DFUDS of the node i in $\text{Min}(A)$ or $\text{Max}(A)$, first check whether the tree is the relevant tree of the node i by referring to $U[i]$. If so, one can decode it by prepending 0 to s_i . Otherwise, the decoded sequence is simply 1 by Lemma 1(b). Also, Gawrychowski and Nicholson [8] showed that

U and S take at most $3n$ bits in total. The following lemma shows that a similar idea can also be applied to combine $\text{BP}(\text{cMin}(A))$ and $\text{BP}(\text{cMax}(A))$ (the lemma can be proved directly from Lemma 1).

Lemma 6. *For any node $i \in \{1, 2, \dots, n-1\}$, if $\text{cMin}(A)$ is a relevant tree of the node i , $f(i+1, \text{cMin}(A)) = f(i, \text{cMin}(A)) + 1$, and $f(i+1, \text{cMax}(A)) = f(i, \text{cMax}(A)) + k$, for some $k > 1$. Otherwise, $f(i+1, \text{cMax}(A)) = f(i, \text{cMax}(A)) + 1$, and $f(i+1, \text{cMin}(A)) = f(i, \text{cMin}(A)) + k$, for some $k > 1$.*

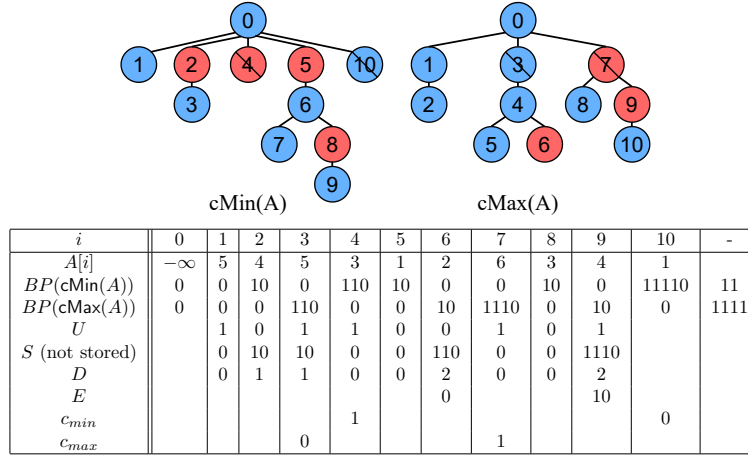


Fig. 3. Combined data structure of $\text{cMin}(A)$ and $\text{cMax}(A)$. i -th column of the table shows (i) the substring of $\text{BP}(\text{cMin}(A))$ and $\text{BP}(\text{cMax}(A))$ begin at position $f(i-1)+1$ and end at position $f(i)$ (shown in the second and the third row, respectively), and (ii) s_i for each i (shown in the fourth row).

We now describe our combined data structure of $\text{cMin}(A)$ and $\text{cMax}(A)$. We first maintain the following structures to store $\text{BP}(\text{cMin}(A))$ and $\text{BP}(\text{cMax}(A))$:

1. The same bit array $U[1, \dots, n-1]$ as in the encoding of [8]. We define $U[i] = 0$ (resp. $U[i] = 1$) if $\text{cMin}(A)$ (resp. $\text{cMax}(A)$) is a relevant tree of the node i . For example, $U[6] = 0$ since $\text{cMin}(A)$ is a relevant tree of the node 6.
2. For each node $i \in \{1, 2, \dots, n-1\}$, suppose the tree $T \in \{\text{cMin}(A), \text{cMax}(A)\}$ is not a relevant tree of i , and let k_i be the number of ones between $f(i, T)$ and $f(i+1, T)$. Now let $S = s_1 s_2 \dots s_{n-1}$ be a bit array, where s_i is defined as $1^{k_i} 0$. For example, since there exist three 1's between $f(6)$ and $f(7)$ in $\text{cMax}(A)$, $s_6 = 110$. Then, S is well-defined by Lemma 6 ($k_i \geq 1$ for all i). Also, since there are at most $n-1$ ones and exactly $n-1$ zeros by lemma 1(b), the size of S is at most $2(n-1)$. We maintain S using the following two arrays:

- (a) An array $D[1, \dots, n-1]$ of size n where $D[i] = 0$ if s_i contains no ones, $D[i] = 1$ if s_i contains a single one, and $D[i] = 2$ otherwise. For example, $D[6] = 2$, since s_6 has two ones. We maintain D using the data structure of Dodis et al. [4], which can decode any $\Theta(\log n)$ consecutive elements of D in $O(1)$ time using $\lceil (n-1) \log 3 \rceil$ bits. Now let k and ℓ be the number of 1's and 2's in D , respectively.
- (b) Let i_2 be the position of the i -th 2 in D . Then, we store a bit array $E = e_1 e_2, \dots, e_\ell$ where e_i is a bit array defined by omitting the first two 1's from s_{i_2} . For example, since the 6 is the first position of D whose value is 2 and $s_6 = 110$, e_1 is defined as 0. The size of E is at most $2(n-1) - (n-1) - (k+\ell) = n - k - \ell$.
- 3. We store both $f(n, \text{cMin}(A))$, and $f(n, \text{cMax}(A))$ using $O(\log n)$ bits.

To store both c_{\min} and c_{\max} , we simply concatenate them into a single array $c_{\min\max}$, and store the length of c_{\min} using $O(\log n)$ bits. Then, by Lemma 4, the size of $c_{\min\max}$ is $k+\ell$. Thus, our encoding of $\text{cMin}(A)$ and $\text{cMax}(A)$ takes at most $(n-1) + (n-1) \log 3 + (n-k-\ell) + (k+\ell) + O(\log n) = (2+\log 3)n + O(\log n) < 3.585n + O(\log n)$ bits in total [18]. An overall example of our encoding is shown in Figure 3. Now we prove the main theorem in this section.

Theorem 2. *Given an array $A[1, \dots, n]$ of size n and any positive integer ℓ , suppose A has no two consecutive equal elements. Then there exists a $(3.585n + o(n))$ -bit encoding data structure which can answer (i) range minimum/maximum and previous larger/smaller value queries in $O(1)$ time, and (ii) range q -th minimum/maximum and next larger/smaller value queries in $O(\log^{(\ell)} n)$ time, for any $q \geq 1$.*

Proof. We show how to decode any $\log n$ consecutive bits of $\text{BP}(\text{cMin}(A))$, which proves the theorem. Note that the auxiliary structures and the procedure for decoding $\text{BP}(\text{cMax}(A))$ are analogous. Let $B[1, \dots, f(n)-1]$ be a subarray of $\text{BP}(\text{cMin}(A))$ of size $f(n)-1$, which is defined as $\text{BP}(\text{cMin}(A))[2, \dots, f(n)]$. Then it is enough to show how to decode $\log n$ consecutive bits of B in $O(1)$ time using $o(n)$ -bit auxiliary structures (note that $\text{BP}(\text{cMin}(A))$ is $0 \cdot B \cdot 1^{2n+2-f(n)}$). We also denote $f(n)-1$ by $f'(n)$ in this proof.

We first define correspondences between the positions of B and D , and between the positions of B and E as follows. For each position $j \in \{1, \dots, f'(n)\}$ of B , let $\alpha(j)$ and $\beta(j)$ be the corresponding positions of j in D and E , respectively. We define both $\alpha(1)$ and $\beta(1)$ as 1, and for each $j \in \{2, \dots, f'(n)\}$, we define $\alpha(j)$ as $\text{rank}_0(j-1, B)$. Next, let k be the number of 2's in $D[1, \dots, \alpha(j)]$ and j' be the number of 1's in B between $B[j]$ and the leftmost 0 in $B[j, \dots, f(\alpha(j+1))]$. Then $\beta(j)$ is defined as (i) 1 if $k=0$, (ii) $\text{select}_0(k, E) + 1$ if $k > 0$ and $D[\alpha(j)] \neq 2$, and (iii) $\text{select}_0(k, E) - \max(j' - 3, 0)$ otherwise. Then any subarray of B starting from the position j can be constructed from the subarrays of U , D and E starting from the positions $\text{rank}_0(j, B)$, $\alpha(j)$ and $\beta(j)$, respectively.

Now, for $i \in \{1, 2, \dots, \lceil (f'(n))/\log n \rceil\}$, let the i -th block of B be $B[\lceil (i-1) \log n + 1 \rceil, \dots, \min(\lceil i \log n \rceil, f'(n))]$. Then, it is enough to decode at

most two consecutive blocks of B to construct any $\log n$ consecutive bits of B . Next, we define the i -th block of U , D , and E as follows:

- i -th block of U is defined as a subarray of U whose starting and ending positions are $\text{rank}_0(\lceil (i-1) \log n \rceil, B)$, and $\text{rank}_0(\min(\lceil i \log n \rceil - 1, f'(n)), B)$, respectively. To decode the blocks of U without B , we mark all the starting positions of the blocks of U using a bit array U_1 of size $f'(n)$ where $U_1[i] = 1$ if and only if the position i is the starting position of the block in U . Then, since U_1 contains at most $O(f'(n)/\log n) = o(n)$ 1's, we can store U_1 using $o(n)$ bits while supporting **rank** and **select** queries in $O(1)$ time by Lemma 3.
- i -th block of D is defined as a subarray of D whose starting and ending positions are $\alpha(\lceil (i-1) \log n \rceil + 1)$ and $\alpha(\min(\lceil i \log n \rceil, f'(n)))$, respectively. Then, the size of each block of D is at most $\log n$, since any position of D has at least one corresponding position in B . We maintain a bit array D_1 analogous to U_1 using $o(n)$ bits. Also, to indicate the case that two distinct blocks of D share the same starting position, we define another bit array D_2 of size $\lceil f'(n)/\log n \rceil$ where $D_2[i] = 1$ if and only if i -th block of D has the same starting position as the $(i-1)$ -th block of D . We store D_2 using the data structure of Lemma 3 using $o(n)$ bits to **rank** and **select** queries in $O(1)$ time. Then, we can decode any block of D in $O(1)$ time using **rank** and **select** operations on D_1 and D_2 .
- i -th block of E is defined as a subarray of E whose starting and ending positions are $\beta(\lceil (i-1) \log n \rceil + 1)$ and $\beta(\min(\lceil i \log n \rceil, f'(n)))$, respectively. To decode the blocks of E , we maintain two bit arrays E_1 and E_2 analogous to D_1 and D_2 , respectively, using $o(n)$ bits.

Note that, unlike D , the size of some blocks in E can be arbitrarily large since some positions in E do not have the corresponding positions in B . To handle this case, we classify each block of E as *bad block* and *good block* where the size of bad block is at least at $c \log n$ for some constant $c \geq 9$, whereas the size of good block is less than $c \log n$. If the i -th block of E is good (resp. bad), we say it as i -th good (resp. bad) block.

For each i -th bad block of E , let F_i be a subsequence of the i -th bad block, which consists of all bits at the position j where $\beta^{-1}(j)$ exists. We store F_i explicitly, which takes $\Theta(n)$ bits in total (the size of F_i is at most $\log n$). However, we can apply the same argument used in [8] to maintain *min-bad block* due to the fact that each position in E corresponds to at least one position in either $\text{BP}(\text{cMin}(A))$ or $\text{BP}(\text{cMax}(A))$. The argument says that for each i -th bad block of E , one can save at least $\log n$ bits by maintaining it in a compressed form. Thus, we can maintain F_i for all i -th bad blocks of E without increasing the total space.

Next, let $g(u, d, e, b)$ be a function, which returns a subarray of B from the subarrays of U , and D , and E as follows (suppose $u = u[1] \cdot u'$ and $d = d[1] \cdot d'$):

$$g(u, d, e, b) = \begin{cases} \epsilon & \text{if } u = \epsilon \text{ or } d = \epsilon \\ 0 \cdot g(u', d', e, b) & \text{if } u[1] \neq b \text{ and } d[1] \neq 2 \\ 0 \cdot g(u', d', e', b) & \text{if } u[1] \neq b, d[1] = 2, \text{ and } e = 1^t 0 \cdot e' \\ 10 \cdot g(u', d', e, b) & \text{if } u[1] = b \text{ and } d[1] = 0 \\ 110 \cdot g(u', d', e, b) & \text{if } u[1] = b \text{ and } d[1] = 1 \\ 1^t \cdot g(u, d, e', b) & \text{if } u[1] = b, d[1] = 2, \text{ and } e = 1^t \cdot e' \\ 1^{t+3} 0 \cdot g(u', d', e', b) & \text{if } u[1] = b, d[1] = 2, \text{ and } e = 1^t 0 \cdot e' \end{cases}$$

We store a precomputed table that stores $g(u, d, e, b)$ for all possible u, d , and e of sizes $\frac{1}{4} \log n$ and $b \in \{0, 1\}$ using $O(2^{\frac{1}{4} \log n + \frac{3}{2} \cdot \frac{1}{4} \log n + \frac{1}{4} \log n} \log n) = O(n^{\frac{7}{8}} \log n) = o(n)$ bits.

To decode the i -th block of B , we first decode the i -block of U and D in $O(1)$ time using **rank** and **select** queries on U_1, D_1 , and D_2 . Let these subarrays be b_u and b_d , respectively. Also, we decode the i -th block of E using **rank** and **select** queries on E_1 and E_2 . We then define b_e as F_i if the i -th block of E is bad. Otherwise, we define b_e as the i -th good block of E . Next, we compute $g(b_u, b_d, b_e, 0)$ in $O(1)$ time by referring to the precomputed table $O(1)$ times, and prepend 0 if we decode the first block of B . Finally, note that there are at most $q \leq 4$ consecutive positions from p to $p + q - 1$ of B whose corresponding positions are the same in both D and E . Because such a case can only occur when $B[p] = B[p+1] = \dots = B[p+q-2] = 1$ and $B[p+q-1] = 0$, we maintain an array R of size $O(n/\log n)$, which stores the four cases of the number of consecutive 1's (0, 1, 2, or at least 3) from the beginning of the i -th block of B . Then, if the number of consecutive 1's from the beginning of $g(b_u, b_d, b_e, 0)$ is at most 3, we delete some 1s from the beginning of $g(b_u, b_d, b_e, 0)$ by referring to R as the final step. \square

4 Data structure on general arrays

In this section, we present a $(3.701n + o(n))$ -bit data structure to support the range q -th minimum/maximum and next/previous larger/smaller value queries on the array $A[1 \dots n]$ without any restriction. Let $C[1, \dots, n]$ be a bit array of size n where $C[1] = 0$, and for any $i > 1$, $C[i] = 1$ if and only if $C[i-1] = C[i]$. If C has k ones, we define an array $A'[1, \dots, n-k]$ of size $n-k$ that discards all consecutive equal elements from A . Then from the definition of colored 2d-min and max heap, we can observe that if $C[i] = 1$, (i) the node i is a blue-colored leaf node, and (ii) i 's immediate left sibling is also a leaf node, both in $\text{cMin}(A)$ and $\text{cMax}(A)$. Furthermore, by deleting all the bits at the positions $f(i, \text{cMin}(A)) - 1$, and $f(i, \text{cMin}(A))$ from $\text{BP}(\text{cMin}(A))$ we can obtain $\text{BP}(\text{cMin}(A'))$. We can also obtain $\text{BP}(\text{cMax}(A'))$ from $\text{BP}(\text{cMax}(A))$ analogously. Now we prove the following theorem.

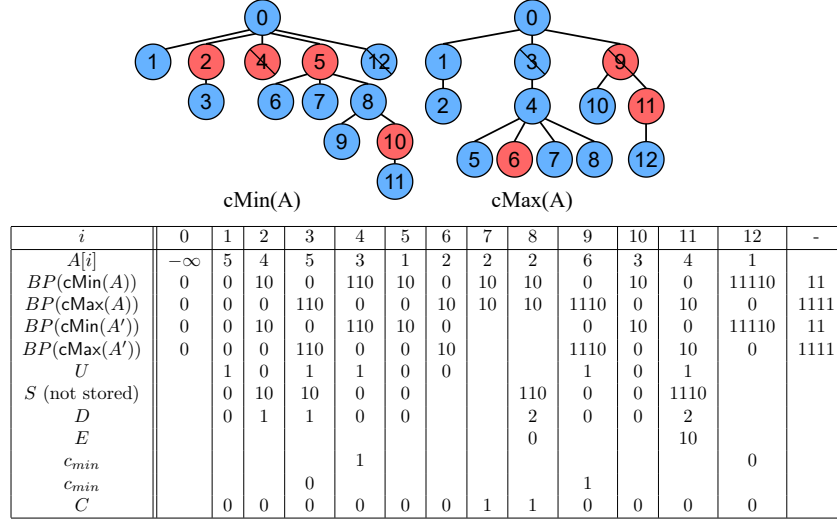


Fig. 4. Combined data structure of $cMin(A)$ and $cMax(A)$. Note that A' is the same array as the array in Figure 3.

Theorem 3. *Given an array $A[1, \dots, n]$ of size n and any positive integer ℓ , there exists a $(3.701n + o(n))$ -bit encoding data structure which can answer (i) range minimum/maximum and previous larger/smaller value queries in $O(1)$ time, and (ii) range q -th minimum/maximum and next larger/smaller value queries in $O(\log^{(\ell)} n)$ time, for any $q \geq 1$.*

Proof. The data structure consists of C and the data structure of Theorem 2 on A' , which can answer all the queries on A' in $O(\log^{(\ell)} n)$ time (see Figure 4 for an example). By maintaining C using the data structure of Lemma 3, the data structure takes at most $(2 + \log 3)(n - k) + \binom{n}{k} + o(n) \leq 3.701n + o(n)$ bits in total [18] while supporting rank and select queries on C in $O(1)$ time. For any node i in $cMin(A)$ and $cMax(A)$, we can compute the color of the node i in $O(1)$ time as follows. If $C[i] = 0$, we return the color of the node $(rank_0(i, C) - 1)$ in $cMin(A')$ and $cMax(A')$, respectively. Otherwise, we return blue. Now we describe how to decode any $\log n$ consecutive bits of $BP(cMin(A))$ in $O(1)$ time using $o(n)$ -bit auxiliary structures, which proves the theorem (the auxiliary structures and the procedure for decoding $BP(cMax(A))$ are analogous). In the proof, we denote $BP(cMin(A))$ and $BP(cMin(A'))$ as B and B' , respectively.

For each position j of B , we say j is *original* if $B[j]$ comes from the bit in B' , and *additional* otherwise. That is, the position j is additional if and only if j is $f(j') - 1$ or $f(j')$ where $C[j'] = 1$. For each original position j , let $b'(j)$ be its corresponding position in B' .

Now we divide B into the blocks of size $\log n$ except the last block, and let s_i be the starting position of the i -th block of B . We then define a bit array $M_{B'}$ of

size $2(n-k)$ as follows. For each $i \in 1, \dots, \lceil (2(n+1)/\log n) \rceil$, we set the $b'(s_i)$ -th position of $M_{B'}$ as one if s_i is original. Otherwise, we set the $b'(s'_i)$ -th position of $M_{B'}$ as one where s'_i is the leftmost original position from s_i to the right in B . All other bits in $M_{B'}$ are 0. Also, let $M'_{B'}$ a bit array of size $\lceil (2(n+1)/\log n) \rceil$ where $M'_{B'}[i]$ is 1 if and only if we mark the same position for s_i and s_{i-1} . Since $M_{B'}$ has at most $\lceil (2(n+1)/\log n) \rceil = o(n)$ ones, we can maintain both $M_{B'}$ and $M'_{B'}$ in $o(n)$ bits while supporting **rank** and **select** queries in $O(1)$ time by Lemma 3. Similarly, we define a bit array M_C of size n as follows. If s_i is original, we set the $(\text{rank}_0(s_i - 1, B))$ -th position of M_C as one. Otherwise, we set the $(\text{rank}_0(s_i - 1, B))$ -th (resp. $(\text{rank}_0(s_i, B))$ -th) position of M_C as one if $B[s_i]$ is 0 (resp. 1). We also maintain a bit array M'_C analogous to $M'_{B'}$. Again, we can maintain both M_C and M'_C using $o(n)$ bits while supporting **rank** and **select** queries on them in $O(1)$ time.

Next, let $h(b, c)$ be a function, which returns a subarray of B from the subarrays of B' and C , defined as follows (suppose $c = c[1] \cdot c'$):

$$h(b, c) = \begin{cases} 1^t & \text{if } b = 1^t \text{ and } c[1] = 0 \\ 1^t 0 \cdot h(b', c') & \text{if } b = 1^t 0 \cdot b' \text{ and } c[1] = 0 \\ 10 \cdot h(b, c') & \text{if } c[1] = 1 \end{cases}$$

We store a precomputed table, which stores $h(b, c)$ for all possible b, c of size $\frac{1}{4} \log n$ using $O(2^{\frac{1}{4} \log n} \log n) = O(\sqrt{n} \log n) = o(n)$ bits.

To decode the i -th block of B , we first decode $\log n$ -sized subarrays of B' and C , $b_{b'}$ and b_c , whose starting positions are $\text{select}_1(\text{rank}_0(i, M'_{B'}), M_{B'})$ and $\text{select}_1(\text{rank}_0(i, M'_C), M_C)$, respectively. We then compute $h(b_{b'}, b_c)$ in $O(1)$ time by referring to the precomputed table $O(1)$ times. Finally, we store a bit array of size $o(n)$, which indicates whether the first bit of the i -th block of B is 0 or not. As the final step, we delete the leftmost bit of $h(b_{b'}, b_c)$ if the i -th block of B starts from 0, and s_i is additional (this can be done by referring to the bit array). \square

5 Lower bounds

This section considers the effective entropy to answer range q -th minimum and maximum queries on an array of size n , for any $q \geq 1$. Note that for any $i \in \{1, \dots, n\}$, both $\text{PSV}(i)$ and $\text{PLV}(i)$ queries can be answered by computing q -th range minimum and maximum queries on the suffixes of the substring $A[1, \dots, i]$, respectively. Similarly, both $\text{NSV}(i)$ and $\text{NLV}(i)$ queries can be answered by computing q -th range minimum and maximum queries on the prefixes of the substring $A[i, \dots, n]$, respectively.

Let \mathcal{A}_n be a set of all arrays of size $n \geq 2$ constructed from the following procedure:

1. For any $0 \leq k \leq n-1$, pick arbitrary k positions in $\{2, \dots, n\}$, and construct a *Baxter Permutation* [1] π_{n-k} of size $n-k$ on the rest of $n-k$ positions.

Here, a Baxter permutation is a permutation that avoids the patterns 2-41-3 and 3-14-2.

2. For k picked positions, assign the rightmost element in π_{n-k} to the left.

Since the number of all possible Baxter permutations of size $n - k$ is at most $2^{3(n-k) - \Theta(\log n)}$ [8], the effective entropy of \mathcal{A}_n is at least $\log|\mathcal{A}_n| \geq \log(\sum_{k=0}^{n-1} 2^{3(n-k) - \Theta(\log n)} \cdot \binom{n-1}{k}) \geq \max_k (3n - 3k + \log \binom{n-1}{k} - \Theta(\log n)) \geq n \log 9 - \Theta(\log n) \geq 3.16n - \Theta(\log n)$ bits [18]. The following theorem shows that the effective entropy of the encoding to support the range q -th minimum and maximum queries on an array of size n is at least $3.16n - \Theta(\log n)$ bits.

Theorem 4. *Any array A in \mathcal{A}_n for $n \geq 2$ can be reconstructed using range q -th minimum and maximum queries on A .*

Proof. We follow the same argument used in the proof of Lemma 3 in [8], which shows that one can reconstruct any Baxter permutation of size n using range minimum and maximum queries.

The proof is induction on n . the case $n = 2$ is trivial since only the possible cases are $\{1, 1\}$ or $\{1, 2\}$, which can be decoded by range first and second minimum queries. Now suppose the theorem statement holds for any size less than $n \geq 3$. Then, both $A_1 = A[1, \dots, n-1]$ and $A_2 = A[2, \dots, n]$ from \mathcal{A}_{n-1} can be reconstructed by the induction hypothesis. Thus, to reconstruct A from A_1 and A_2 , it is enough to compare $A[1]$ and $A[n]$.

If any answer of $\text{RMax}(1, n, q)$ and $\text{RMin}(1, n, q)$ contains the position 1 or n , we are done. Otherwise, let x and y be the rightmost positions of the smallest and largest element in $[2, n-1]$, which can be computed by $\text{RMax}(1, n, q)$ and $\text{RMin}(1, n, q)$, respectively. Without a loss of generality, suppose $x < y$ (other case is symmetric). In this case, [8] showed that (i) there exists a position $i \in [x, y]$, which satisfies $A[1] < A[i] < A[n]$ or $A[1] > A[i] > A[n]$, or (ii) $A[1] < A[n]$, which proves the theorem (note that $A[1]$ cannot be equal to $A[n]$ in this case since the same elements in A always appear consecutively). \square

6 Conclusion

This paper proposes an encoding data structure that efficiently supports range (q -th) minimum/maximum queries and next/previous larger/smaller value queries. Our results match the current best upper bound of Tsur [18] up to lower-order additive terms while supporting the queries efficiently.

Note that the lower bound of Theorem 4 only considers the case that the same elements always appear consecutively, which still gives a gap between the upper and lower bound of the space. Improving the lower bound of the space for answering the queries would be an interesting open problem.

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