String Rearrangement Inequalities and a Total Order Between Primitive Words *

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Abstract. We study the following rearrangement problem: Given n words, rearrange and concatenate them so that the obtained string is lexicographically smallest (or largest, respectively). We show that this problem reduces to sorting the given words so that their repeating strings are non-decreasing (or non-increasing, respectively), where the repeating string of a word A refers to the infinite string AAA... Moreover, for fixed size alphabet Σ , we design an O(L) time sorting algorithm of the words (in the mentioned orders), where L denotes the total length of the input words. Hence we obtain an O(L) time algorithm for the rearrangement problem. Finally, we point out that comparing primitive words via comparing their repeating strings leads to a total order, which can further be extended to a total order on the finite words (or all words).

Keywords: String rearrangement inequalities · Primitive words · Combinatorics on words · String ordering · Greedy algorithm.

1 Introduction

Combinatorics on words (MSC: 68R15) have strong connections to many fields of mathematics and have found significant applications to theoretical computer science and molecular biology (DNA sequences) [10,17,21,5,8,14]. Particularly, the primitive words over some alphabet Σ have received special interest, as they have applications in the formal languages and algebraic theory of codes [19,12,13,16]. A word is *primitive* if it is not a proper power of a shorter word.

In this paper, we consider the following rearrangement problem of words: Given n words A_1, \ldots, A_n , rearrange and concatenate these words so that the obtained string S is lexicographically smallest (or largest, respectively). We prove that the lexicographical smallest outcome of S happens when the words are arranged so that their repeating strings are increasing, and the largest outcome of S happens when the words are arranged reversely; see Lemma 6. Throughout, the *repeating string* of a word A refers to the infinite string $R(A) = AAA \ldots$

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Based on the above lemma (we suggest to name its results as "string rearrangement inequalities"), the aforementioned rearrangement problem reduces to sorting the words A_1, \ldots, A_n so that $R(A_1) \leq \ldots \leq R(A_n)$. We show how to sort for the special case where A_1, \ldots, A_n are primitive and distinct in $O(\sum_i |A_i|)$ time. The general case can be easily reduced to the special case and can be solved in the same time bound. Note that we assume bounded alphabet Σ and the size of Σ is fixed. Moreover, |X| always denotes the length of word X.

Our algorithm beats the plain algorithm based on sorting (via comparing several pairs $R(A_i), R(A_j)$) by a factor of log n. The algorithm is simple – it only applies basic data structures such as tries and the failure function [10]. Nevertheless, its correctness and running time analysis is non-straightforward.

We mention that comparing primitive words via comparing their repeating strings leads to a total order \leq_{∞} on primitive words, which can extended to a total order \leq_{∞} on all words (section 5). We show that this order is the same as the lexicographical order over Lyndon words but are different over primitive words and finite words. It is also different from *reflected lexicographic order*, *colexicographic order*, *shortlex order*, *Kleene-Brouwer order*, *V-Order*, *alternative order* [11,9,3,2,1]. It seems that order \leq_{∞} has not been reported in literature.

1.1 Related work

It is shown in [6] that the language of Lyndon words is not context-free. Also, many people conjectured that the language of primitive words is not context-free [12,19,6,13]. But this conjecture is unsettled thus far, to the best of our knowledge. It would be interesting to explore whether the results shown in this paper can be helpful for solving this longstanding open problem in the future. See more introductions about primitive words in [16].

Fredricksen and Maiorana [15] showed that if one concatenates, in lexicographic order, all the Lyndon words that have length dividing a given number n, the result is a de Bruijn sequence. Au [4] further showed that if "dividing n" is replaced by "identical to n", the result is a sequence which contains exactly once every primitive word of length n as a factor. Note that concatenating some Lyndon words by lexicographic order is the same as concatenating by \leq_{∞} order.

The Lyndon words have many interesting properties and have found plentiful applications, both theoretically and practically. Among others, they are used in constructing de Brujin sequence as mentioned above (which have found applications in cryptography), and they are applied in proving the "runs theorem" [20,5,8]. The famous Chen-Fox-Lyndon Theorem states that any word W can be uniquely factorized into $W = W_1 W_2 \dots W_m$, such that each W_i is a Lyndon word, and $W_1 \geq \ldots \geq W_m$ [14,7] (Here \geq refers to the opposite of Lexicographical order, but is the same as the opposite of \leq_{∞}). This factorization is used in the computation of runs in a word [8]. See the Bible of combinatorics on words [17] for more introductions about Lyndon words and primitive words.

2 Preliminaries

Definition 1. The n^{th} **power** of word A is defined as:

$$A^{n} = \begin{cases} AA^{n-1}, & n > 0;\\ \text{empty word, } n = 0. \end{cases}$$

A word A is **non-primitive** if it equals B^k for some word B and integer $k \ge 2$. Otherwise, A is **primitive**. (By this definition the empty word is not primitive.)

The next lemma summarizes three results about the powers proved by Lyndon and Schüzenberger [18]; see their Lemmas 3 and 4, and Corollary 4.1. (More introductions of these results can be found in Section 1.3 "Conjugacy" of [17].)

Lemma 1. [18] Given words A and B, there exist C, k, l such that $A = C^k$ and $B = C^l$ when one of the following conditions holds:

- 1. AB = BA.
- 2. Two powers A^{m_1} and B^{m_2} have a common prefix of length |A| + |B|.

3.
$$A^{m_1} = B^{m_2}$$

Definition 2. The root of a word A, denoted by root(A), is the unique primitive word B such that A is a power of B. The uniqueness of root is obvious, a formal proof can be found in Corollary 4.2 of [18] or in [16].

Lemma 2. Assume A is a non-empty word. Find the largest j < |A| such that the prefix of A with length j equals the suffix of A with length j. Let k = |A| - j > 0. Then,

$$|\operatorname{root}(A)| = \begin{cases} k, & |A| = 0 \pmod{k}; \\ |A|, & |A| \neq 0 \pmod{k}. \end{cases}$$
(1)

Proof. This result should be well-known. A simple proof is as follows.

Fact 1. If S = BB' = B'B and B, B' are non-empty, S is non-primitive.

This is a trivial fact and is implied by Lemma 1 (condition 1); proof omitted. $Claim \ 1. |root(A)| > k.$

Proof: The prefix and suffix of A with length $|A| - |\operatorname{root}(A)|$ are the same, which implies that $j \ge |A| - |\operatorname{root}(A)|$. Consequently, $|\operatorname{root}(A)| \ge |A| - j = k$. Claim 2. If $|\operatorname{root}(A)| < |A|$ (i.e., A is non-primitive), then $k \ge |\operatorname{root}(A)|$.

Proof: Denote $S = \operatorname{root}(A)$ and assume |S| < |A|. Therefore, $A = S^d$ $(d \ge 2)$. Suppose to the opposite that $k < |\operatorname{root}(A)|$. Let B be the prefix of S with length k, and B' be the suffix of S such that S = BB'. As k < |S|, we have $j > |A| - |S| \ge |S|$. Further since the suffix of A with length j (which starts with B'B) equals to the prefix of A with length j (which starts with S = BB'), we get S = B'B. Applying Fact 1, $\operatorname{root}(A) = S$ is non-primitive. Contradictory.

We are ready to prove the lemma. When |A| is a multiple of k, A is a power of its prefix of length k, which means $|\operatorname{root}(A)| \leq k$. Further by Claim 1, $|\operatorname{root}(A)| = k$. Next, assume |A| is not a multiple of k. Since |A| is a multiple of $|\operatorname{root}(A)|$, we see $|\operatorname{root}(A)| \neq k$. Further by Claims 1 and 2, it follows that $|\operatorname{root}(A)| = |A|$. \Box

For a non-empty word A, denote by R(A) the infinite repeating string $AA \dots$

Problem 1. Given non-empty words A_1, \ldots, A_n , sort them so that

$$R(A_1) \le \ldots \le R(A_n)$$

Clearly, R(A) = R(root(A)). To solve Problem 1, we can replace A by root(A) (using a preprocessing algorithm based on Lemma 2), and then it reduces to:

Problem 1'. Given primitive words A_1, \ldots, A_n , sort them so that

$$R(A_1) \leq \ldots \leq R(A_n).$$

Definition 3. For any two non-empty words S and A, denote by $\deg_A(S)$ the largest integer d so that S^d is a prefix of A. Moreover, for non-empty word S and set of non-empty words $\mathcal{A} = \{A_1, \ldots, A_n\}$, denote $\deg_{\mathcal{A}}(S) = \max_j \deg_{A_j}(S)$.

In other words, if we build the trie T of \mathcal{A} , $S^{\deg_{\mathcal{A}}(S)}$ is the longest power of S that equals to some path of the trie T starting from its root.

For any $i \ (1 \le i \le n)$, denote

$$N_i = the \deg_{\mathcal{A}}(A_i) \text{-th power of } A_i \tag{2}$$

$$M_i = N_i A_i^2 = the \; (\deg_{\mathcal{A}}(A_i) + 2) \text{-th power of } A_i \tag{3}$$

The following lemma is fundamental to our algorithm.

Lemma 3. For non-empty words A and B, the relation between R(A) and R(B) is the same as the relation between AB and BA. In other words,

$$R(A) < R(B) \quad \Leftrightarrow \quad AB < BA, \tag{4}$$

$$R(A) > R(B) \quad \Leftrightarrow \quad AB > BA. \tag{5}$$

$$R(A) = R(B) \quad \Leftrightarrow \quad AB = BA, \tag{6}$$

Proof. Assume that A, B are words that consist of the decimal symbols '0',..., '9'. The proof can be easily extended to the more general case.

Let α, β denote the number represented by strings A, B. For example, string '89' represents number 89. Denote a = |A| and b = |B|. Observe that

$$AB < BA \iff \alpha \cdot 10^b + \beta < \beta \cdot 10^a + \alpha \iff \frac{\alpha}{10^a - 1} < \frac{\beta}{10^b - 1}.$$

Moreover,

$$\frac{\alpha}{10^a - 1} = \alpha \frac{\frac{1}{10^a}}{1 - \frac{1}{10^a}} = \alpha [\frac{1}{10^a} + (\frac{1}{10^a})^2 + (\frac{1}{10^a})^3 + \dots] = 0.\alpha\alpha\alpha\dots = 0.\dot{\alpha};$$

$$\frac{\beta}{10^b - 1} = \beta \frac{\frac{1}{10^b}}{1 - \frac{1}{10^b}} = \beta [\frac{1}{10^b} + (\frac{1}{10^b})^2 + (\frac{1}{10^b})^3 + \dots] = 0.\beta\beta\beta\dots = 0.\dot{\beta}.$$

So, $AB < BA \Leftrightarrow 0.\dot{\alpha} < 0.\dot{\beta} \Leftrightarrow R(A) < R(B).$ Similarly, (5) and (6) hold. \Box

A more rigorous but complicated proof of Lemma 3 is given in the appendix. As an interesting corollary of Lemma 3, we obtain that "if $AB \leq BA$ and $BC \leq CB$, then $AC \leq CA$ ". This transitivity is not obvious without Lemma 3.

3 A linear time algorithm for sorting the repeating words

Assume that A_1, \ldots, A_n are **primitive**. Denote $L = \sum_i |A_i|$ for short. This section presents an O(L) time algorithm for solving Problem 1', that is, sorting $R(A_1), \ldots, R(A_n)$. We start with two nontrivial observations.

Lemma 4. The relation between infinitely repeating strings $R(A_i)$ and $R(A_j)$ is the same as the relation between words M_i and M_j , that is,

$$R(A_i) = R(A_j) \quad \Leftrightarrow \quad M_i = M_j,$$

$$R(A_i) < R(A_j) \quad \Leftrightarrow \quad M_i < M_j,$$

$$R(A_i) > R(A_j) \quad \Leftrightarrow \quad M_i > M_j.$$

As a corollary, sorting $R(A_1), \ldots, R(A_n)$ reduces to sorting M_1, \ldots, M_n .

Proof. Consider the comparison of $R(A_i)$ and $R(A_j)$. Assume $|A_i| \le |A_j|$. Otherwise it is symmetric.

First, consider the case $R(A_i) = R(A_j)$. Let $m_1 = |A_j|$ and $m_2 = |A_i|$. We know $A_i^{m_1} = A_j^{m_2}$ because $R(A_i) = R(A_j)$. Applying Lemma 1 (condition 3), $A_i = C^k$ and $A_j = C^l$ for some C, k, l. Further since A_i, A_j are primitive, $A_i = C = A_j$. It follows that $M_i = M_j$. Next, assume that $R(A_i) \neq R(A_j)$.

Let $p = \deg_{A_j}(A_i)$. Thus, $A_j = A_i^p S$, where $p \ge 0$ and A_i is not a prefix of S. Be aware that $p \le \deg_A(A_i)$ by the definition of $\deg_A(A_i)$.

According to Lemma 3, the comparison of $R(A_i)$ and $R(A_j)$ equals to the comparison of A_iA_j and A_jA_i . Further since $A_j = A_i^p S$, it equals to the comparison of $A_i^p A_i S$ and $A_i^p S A_i$. In the following, we discuss two subcases.

Subcase 1. $|S| > |A_i|$, or $|S| \le |A_i|$ and S is not a prefix of A_i .

Recall that A_i is not a prefix of S. In this subcase, we will find an unequal letter if we compare A_i with S (starting from the leftmost letter). Comparing $A_i^p A_i S$ and $A_i^p S A_i$ is thus equivalent to comparing the prefixes $A_i^p A_i$ and $A_i^p S$.

 $A_i^p A_i S$ and $A_i^p S A_i$ is thus equivalent to comparing the prefixes $A_i^p A_i$ and $A_i^p S$. Notice that $A_i^p A_i = A_i^{p+1}$ and $A_i^p S = A_j$ are also prefixes of M_i and M_j , respectively (note that A_i^{p+1} is a prefix of M_i because M_i is the $(\deg_{\mathcal{A}}(A_i)+2)$ -th power of A_i and $\deg_{\mathcal{A}}(A_i) \geq p$ as mentioned above). Therefore, comparing M_i and M_j is also equivalent to comparing the two prefixes $A_i^p A_i$ and $A_i^p S$.

Altogether, comparing $R(A_i)$, $R(A_j)$ is equivalent to comparing M_i , M_j .

Subcase 2. S is a prefix of A_i . (This means S is a proper prefix of A_i as $S \neq A_i$.)

Assume $A_i = ST$. Comparing $A_i^p A_i S$ and $A_i^p SA_i$ is just the same as comparing $A_i^p STS$ and $A_i^p SST$. It reduces to proving that comparing M_i and M_j also reduces to comparing $A_i^p STS$ and $A_i^p SST$.

First, we argue that $ST \neq TS$. Suppose to the opposite that ST = TS. Applying Lemma 1 (condition 1), $S = C^k$ and $T = C^l$ for some C, k, l. This implies that A_i and A_j are both powers of C, and hence $R(A_i) = R(A_j)$, contradictory.

Observe that $A_i^p STS$ is a prefix of $A_i^p STST = A_i^{p+2}$, which is a prefix of M_i (because M_i is the $(\deg_{\mathcal{A}}(A_i) + 2)$ -th power of A_i and $\deg_{\mathcal{A}}(A_i) + 2 \ge p + 2)$.

Observe that p > 0. Otherwise $A_j = A_i^0 S$ is shorter than A_i , which contradicts our assumption $|A_i| \leq |A_j|$. As a corollary, A_i is a prefix of $A_j = A_i^p S$. Therefore, $A_i^p SST = A_i^p SA_i = A_j A_i$ is a prefix of A_j^2 , which is a prefix of M_j .

To sum up, M_i and M_j admit $A_i^p STS$ and $A_i^p SST$ as prefixes, respectively. Further since $TS \neq ST$, comparing M_i, M_j reduces to comparing $A_i^p STS$ and $A_i^p SST$, which is equivalent to comparing $R(A_i), R(A_j)$ as mentioned above. \Box

Assume A_1, \ldots, A_n are **distinct** henceforth in this section. To this end, we can use a trie to reduce those duplicate elements in A_1, \ldots, A_n , which is trivial.

Lemma 5. When A_1, \ldots, A_n are primitive and distinct, $\sum_i |N_i| = O(L)$.

Proof. First, we argue that N_1, \ldots, N_n are distinct. Suppose that $N_i = N_j$ $(i \neq j)$. Recall that $N_i = A_i^m$ (for $m = \deg_{\mathcal{A}}(A_i)$) and $N_j = A_j^n$ (for $n = \deg_{\mathcal{A}}(A_j)$). Applying Lemma 1 (condition 3), $A_i = C^k$ and $A_j = C^l$. Further since A_i, A_j are primitive, $A_i = C = A_j$, which contradicts the assumption that $A_i \neq A_j$.

We say N_i extremal if it is **not** a prefix of any word in $\{N_1, \ldots, N_n\} \setminus \{N_i\}$. Partition N_1, \ldots, N_n into several groups such that (a) for elements in the same group, one of them is the prefix of the other, and (b) the longest element in each group is extremal. (It is obvious that such a partition exists: we can first distribute the extremal ones to different groups, and then distribute the non-extremal ones to suitable group (each non-extremal one is a prefix of some extremal ones).

Now, consider any such group, e.g., N_{i_1}, \ldots, N_{i_x} . It suffices to prove that (X) $|N_{i_1}| + \ldots + |N_{i_x}| = O(|A_{i_1}| + \ldots + |A_{i_x}|)$, and we prove it in the following. Without loss of generality, assume that N_{i_j} is a prefix of $N_{i_{j+1}}$ for j < x.

We state two important formulas: (i) $N_{i_x} = A_{i_x}$. (ii) $|N_{i_j}| < |A_{i_j}| + |A_{i_{j+1}}|$ for j < x. Equation (X) above follows from formulas (i) and (ii) immediately.

Proof of (i). Suppose to the contrary that $N_{i_x} \neq A_{i_x}$. By the definition of N_{i_x} , there exists some A_j such that N_{i_x} is a prefix of A_j . Clearly, $j \neq i_x$ since N_{i_x} is not a prefix of A_{i_x} . Consequently, N_{i_x} is a prefix of some other N_j , which means N_{i_x} is not extremal, contradicting property (b) of the grouping mentioned above.

Proof of (ii). Suppose to the contrary that $|N_{i_j}| \ge |A_{i_j}| + |A_{i_{j+1}}|$. Because N_{i_j} and $N_{i_{j+1}}$ are powers of $A_{i,j}$ and $A_{i_{j+1}}$ and share a common prefix, N_{i_j} , of length at least $|A_{i_j}| + |A_{i_{j+1}}|$. By Lemma 1 (condition 2), $A_{i_j} = C^k$ and $A_{i_{j+1}} = C^l$ for some C, k, l. Hence $A_{i_j} = A_{i_{j+1}}$, as A_{i_j} and $A_{i_{j+1}}$ are primitive. Contradictory.

Our algorithm for sorting $R(A_1), \ldots, R(A_n)$ is simply as follows.

First, we build a trie of A_1, \ldots, A_n and use it to compute N_1, \ldots, N_n . In particularly, for computing N_i , we walk along the trie from the root and search for maximal pieces of A_i , which takes $O(|N_i| + |A_i|) = O(|N_i|)$ time. The total running time for computing N_1, \ldots, N_n is therefore $O(\sum_i |N_i|) = O(L)$.

Second, we compute M_1, \ldots, M_n and build a trie of them. By utilizing this trie, we obtain the lexicographic order of M_1, \ldots, M_n , which equals the order of

 $R(A_1), \ldots, R(A_n)$ according to Lemma 4. The running time of the second step is $\sum_i |M_i| = \sum_i |N_i| + 2\sum_i |A_i| = O(L) + O(L) = O(L)$. To sum up, we obtain

Theorem 1. Problem 1' can be solved in $O(L) = O(\sum_i |A_i|)$ time.

In addition, we can solve Problem 1 within the same time bound.

Theorem 2. Problem 1 can be solved in $O(L) = O(\sum_i |A_i|)$ time.

Proof. It remains to showing that $root(A_i)$ can be computed in $O(|A_i|)$ time.

Applying Lemma 2, computing root(A) reduces to finding the largest j < |A| such that the prefix of A with length j equals the suffix of A with length j. Moreover, the famous KMP algorithm [10] finds this j in O(|A|) time.

As a comparison, there exists a less efficient algorithm for solving Problem 1, which is based on a standard sorting algorithm associated with a naïve gadget for comparing R(A) and R(B) – according to Lemma 3, comparing R(A) and R(B) reduces to comparing AB and BA, which takes O(|A|+|B|) time. The time complexity of this alternative algorithm is higher. For example, when A_1 ="aaaaaa1", A_2 ="aaaaaa2", etc, the running time would be $\Omega(n \log n |A_1|) = \Omega(L \log n)$.

4 The string rearrangement inequalities

We call equation (7) right below the String Rearrangement Inequalities.

Lemma 6. For non-empty words A_1, \ldots, A_n , where $R(A_1) \leq \ldots \leq R(A_n)$, we claim that

$$A_1 A_2 \dots A_n \le A_{\pi_1} A_{\pi_2} \dots A_{\pi_n} \le A_n A_{n-1} \dots A_1, \tag{7}$$

for any permutation π_1, \ldots, π_n of $\{1, \ldots, n\}$.

In other words, if several words are to be rearranged and concatenated into a string S, the lexicographical smallest outcome of S occurs when the words are arranged so that their repeating strings are increasing, and the lexicographical largest outcome of S occurs when the words are arranged so that their repeating strings are decreasing. Here, the repeating string of a word A refers to R(A).

Example 1. Suppose there are four given words: "123", "12", "121", "121". Notice that R(121) < R(12) = R(1212) < R(123). Applying Lemma 6, the lexicographical smallest outcome would be "121121212123", and the lexicographical largest outcome would be "123121212121". The reader can verify this result easily.

Remark 1. If we sort the given words using the lexicographic order instead, the outcome of the concatenation is not optimum. For example, we have "12" < "121" < "1212" < "121", and a concatenation in this order is not the smallest outcome, and a concatenation in its reverse order is neither the largest outcome.

Proof (of Lemma 6). Consider any concatenation $A_{\pi_1} \ldots A_{\pi_n}$. If A_1 is not at the leftmost position, we swap it with its left neighbor A_x . Note that $R(A_1) \leq R(A_x)$ by assumption. According to Lemma 3, $A_1A_x \leq A_xA_1$. This means that the entire string becomes smaller or remains unchanged after the swapping. Applying several such swappings, A_1 will be on the leftmost position. Then, we swap A_2 to the second place. So on and so forth. It follows that $A_1 \ldots A_n \leq A_{\pi_1} \ldots A_{\pi_n}$.

The other inequality in (7) can be proved symmetrically; proof omitted. \Box

Combining Theorem 2 with Lemma 6, we obtain

Corollary 1. Given n words A_1, \ldots, A_n that are to be rearranged and concatenated, the smallest and largest concatenation can be found in $O(\sum_i |A_i|)$ time.

Another corollary of Lemma 6 is the uniqueness of the best concatenation:

Corollary 2. Given primitive and distinct words A_1, \ldots, A_n that are to be rearranged and concatenated, the smallest (largest, resp.) concatenation is unique.

Proof. It follows from Lemma 6 and the fact that $R(A_1), \ldots, R(A_n)$ are distinct (see Proposition 1 below).

Proposition 1. For distinct primitive words A and B, we have $R(A) \neq R(B)$.

Proof. Recall that when A and B are primitive and R(A) = R(B), we can infer that A = B (as proved in the second paragraph of the proof of Lemma 4). Therefore, if A and B are primitive and distinct, $R(A) \neq R(B)$.

5 A total order \leq_{∞} on words

Definition 4. Given primitive words A and B, we state that $A \leq_{\infty} B$ if $R(A) \leq R(B)$. Notice that \leq_{∞} is a total order on primitive words by Proposition 1. Furthermore, we extend \leq_{∞} to the scope of finite nonempty words as follows.

For non-empty words $A = S^k$ and $B = T^l$, where S, T are primitive, we state that $A \leq_{\infty} B$ if

$$(S = T \text{ and } |S| \le |T|), \text{ or } (S \ne T \text{ and } S \le_{\infty} T).$$
(8)

The symbol \leq_{∞} in the equation stands for the relation between primitive words.

For example, $121 \leq_{\infty} 12 \leq_{\infty} 1212 \leq_{\infty} 121212 \leq_{\infty} 122$.

Obviously, the relation \leq_{∞} is a total order on finite nonempty words.

The next lemma shows that within the class of Lyndon words, the order \leq_{∞} is actually the same as the lexicographical order \leq_{lex} (denoted by \leq for short). (Note that Lyndon words are primitive, so the unextended \leq_{∞} is enough here.)

Lemma 7. Given Lyndon words A and B such that $A \leq B$, we have $A \leq_{\infty} B$.

Proof. Assume that $A \neq B$; otherwise we have R(A) = R(B) and so $A \leq_{\infty} B$. By the assumption $A \leq B$, we know A < B. Consider two cases:

1. $|A| \ge |B|$, or |A| < |B| and A is not a prefix of B

Combining the assumption A < B with the condition of this case, we can see that the relation between AB, BA is the same as that between A, B: In comparing AB and BA, the result is settled before the min $\{|A|, |B|\}$ -th character.

2. |A| < |B| and A is a prefix of B, i.e., A is a proper prefix of B

Assume that B = AC where C is nonempty. Because B is a Lyndon word by assumption, AC < CA. Therefore, AB = AAC < ACA = BA.

In both cases, we obtain AB < BA. It further implies that R(A) < R(B) by Lemma 3. This means $A \leq_{\infty} B$.

In fact, it is possible to further extend \leq_{∞} to all (finite and infinite) words. Define the repeating string of an infinite word A, denoted by R(A), to be A itself. We state that $A \leq_{\infty} B$ if R(A) < R(B) or R(A) = R(B) and $|A| \leq |B|$.

6 Conclusions

In this paper, we present a simple proof of the "string rearrangement inequalities" (7). These inequalities have not been reported in literature to the best of our knowledge. We also study the algorithmic aspect of these two inequalities, and present a linear time algorithm for rearranging the strings so that $R(A_1) \leq \ldots R(A_n)$. This algorithm beats the trivial sorting algorithm by a factor of log n.

The algorithm itself is direct (indeed, it looks somewhat brute-force) and easy to implement, yet the analysis of its correctness and complexity is build upon nontrivial observations, namely, Lemma 3, Lemma 4, and Lemma 5.

In the future, it is a problem worth attacking that whether we can improve the running time for sorting $R(A_1), \ldots, R(A_n)$ from O(L) to O(N), where N denotes the number of nodes in the trie of A_1, \ldots, A_n .

The order \leq_{∞} on primitive words has nice connections with repeating decimals as shown in the proof of Lemma 3. It would be interesting to know whether these connections have more applications in the study of primitive words.

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A An alternative proof of Lemma 3

Below we show an alternative proof of Lemma 3. This proof is less clever and much more involved (compared to the other proof in section 2), yet it reflects more insights which helped us in designing our linear time algorithm.

Below we always assume that A, B, X, Y are words.

Definition 5. Word A is **truly less** than word B, if there exists a prefix pair $A_1A_2...A_i$ and $B_1B_2...B_i$, in which $A_1A_2...A_{i-1}$ and $B_1B_2...B_{i-1}$ are equal and A_i is less than B_i . For convenience, let $A <_T B$ denote this case for the rest of this paper. Note that i can be 1 such that A_1 is less than B_1 .

For any pair of nonempty words, A and B, we can generalize 3 following properties with Definition 5. Note that any X or Y in the following properties can be any word, including empty word.

Claim (1). Proposition $A <_T B$ is equivalent to AX < BY, if A is not prefix of B and B is not prefix of A.

Proof. If A is not prefix of B and B is not prefix of A, the proposition AX < BY implies that A and B fits the case described in Definition 5 and thus $A <_T B$ holds. The proposition $A <_T B$, by Definition 5, also indicates that AX < BY.

Claim (2). If $A <_T B$, it holds that $AX <_T BY$.

Proof. If $A <_T B$, by Definition 5, there exists a prefix pair $A_1, A_2...A_i$ and $B_1B_2...B_i$, in which $A_1A_2...A_{i-1}$ and $B_1B_2...B_{i-1}$ are equal and A_i is less than B_i . Since A is the prefix of AX and B is the prefix of BY, AX and BY also have the prefix pair $A_1A_2...A_i$ and $B_1B_2...B_i$ mentioned above, thus it holds that $AX <_T BY$ by Definition 5.

Claim (3). If A < B and |A| = |B|, it holds that $A <_T B$.

Proof. If A < B and |A| = |B|, we can find a substring pair $A_1A_2...A_i$ and $B_1B_2...B_i$, in which $A_1A_2...A_{i-1}$ and $B_1B_2...B_{i-1}$ are equal and A_i is less than B_i . This is exactly the case of Definition 5, so naturally $A <_T B$.

Now, we are ready for proving Lemma 3.

Recall that this lemma states for non-empty words A and B, the relation between R(A) and R(B) is the same as the relation between AB and BA.

We will prove $R(A) = R(B) \Leftrightarrow AB = BA$ and $R(A) < R(B) \Leftrightarrow AB < BA$. Note that $R(A) > R(B) \Leftrightarrow AB > BA$ can be obtained similarly.

Proposition 2. For nonempty words A and B, $AB = BA \Leftrightarrow R(A) = R(B)$.

Proof. From AB = BA or R(A) = R(B), we obtain from Lemma 1 that $A = C^k$ and $B = C^l$ for some C, k, l, which implies that R(A) = R(B) and AB = BA.

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Proposition 3. For nonempty words A and B, $AB < BA \Leftrightarrow R(A) < R(B)$.

We prove the two directions separately in the following.

Proof (of $AB < BA \Rightarrow R(A) < R(B)$). We discuss two subcases.

Subcase 1. $|A| \leq |B|$

Let $B = A^m S$, in which $m = \deg_B(A)$ by Definition 3.

Note that target R(A) < R(B) equals to $R(A) < R(A^m S)$, which then equals to $R(A) < SR(A^m S)$, by eliminating the leading A^m .

With AB < BA, we will get the relation that AB < BA leads to $A^{m+1}S < A^mSA$. And $A^{m+1}S < A^mSA$ leads to AS < SA, which eventually leads to $AS <_T SA$.

We will prove that $AA <_T SA$. And since AA is a prefix of R(A) and SA is a prefix of $SR(A^mS)$, proposition $R(A) < SR(A^mS)$ follows by Claim 2, proving the target proposition.

Now we prove $AA <_T SA$ in two cases.

1. If $|A| \leq |S|$, or |A| > |S| but S is not a prefix of A, note that A is not a prefix of S, since AS < SA, proposition $A <_T S$ follows by Claim 1. Then $AA <_T SA$ follows by Claim 2.

2. If |A| > |S| and S is a prefix of A, let A = ST. Since $AS <_T SA$, we have $STS <_T SST$, then $AA = STST <_T SST = SA$ follows by Claim 2. Thus $AA <_T SA$.

Subcase 2. |A| > |B|

Let $A = B^m S$, in which $m = \deg_A(B)$.

Note that target R(A) < R(B) equals to $R(B^m S) < R(B)$, which then equals to $SR(B^m S) < R(B)$, by eliminating the leading B^m .

With AB<BA, we will get the relation that AB < BA leads to $B^mSB < B^{m+1}S$. And $B^mSB < B^{m+1}S$ leads to SB < BS, which eventually leads to $SB <_T BS$.

We will prove that $SB <_T BB$. And since BB is a prefix of R(B) and SB is a prefix of $SR(B^mS)$, proposition $SR(B^mS) < R(B)$ follows by Claim 2, proving the target proposition.

Now we prove $SB <_T BB$ in 2 cases.

1. If $|B| \leq |S|$, or |B| > |S| but S is not a prefix of B, note that B is not a prefix of S, since SB < BS, proposition $S <_T B$ follows by Claim 1. Then $SB <_T BB$ follows by Claim 2.

2. If |B| > |S| and S is a prefix of B, let B = ST. Since $SB <_T BS$, we have $SST <_T STS$, then $SB = SST <_T STST = BB$ follows by Claim 2. Thus $SB <_T BB$.

With both cases proved, we have $AB < BA \Rightarrow R(A) < R(B)$.

Proof (of $R(A) < R(B) \Rightarrow AB < BA$).

In the following, we discuss two subcases.

Subcase 1. $|A| \leq |B|$.

Let $B = A^m S$, in which $m = \deg_B(A)$.

Note that AB < BA equals to $A^{m+1}S < A^mSA$, which equals to AS < SA by eliminating the leading A^m .

With R(A) < R(B), we will get the relation that R(A) < R(B) equals to $R(A) < R(A^mS)$. And $R(A) < R(A^mS)$ equals to $R(A) < SR(A^mS)$ by eliminating the leading A^m .

Now we will prove AS < SA.

1. If $|A| \leq |S|$, or |A| > |S| and S is not a prefix of A, note that A is not a prefix of S, since $R(A) < SR(A^mS)$, $A <_T S$ follows by Claim 1. Then AS < SA follows by Claim 2.

2. If |A| > |S| and S is a prefix of A, let A = ST. Since $R(A) < SR(A^mS)$, we have $R(ST) < SR((ST)^mS)$, we pay attention to the prefixes with length $2^*|S|+|T|$ of these two infinite words: STS and SST. We argue that $STS \neq SST$ otherwise ST = TS, then S,T,B,A are powers of a common element by Lemma 1, then R(A) = R(B), which is contradictory. Thus, since $R(ST) < SR((ST)^mS)$, we will have STS < SST. It holds that AS = STS < SST = SA. Thus, we end up with AS < SA.

Subcase 2. |A| > |B|.

Let $A = B^m S$, in which $m = \deg_A(B)$.

Note that AB < BA equals to $B^m SB < B^{m+1}S$, which equals to SB < BS by eliminating the leading B^m .

With R(A) < R(B), we will get the relation that R(A) < R(B) equals to $R(B^mS) < R(B)$. And $R(B^mS) < R(B)$ equals to $SR(B^mS) < R(B)$ by eliminating the leading B^m .

Now we will prove SB < BS, in two cases.

1. If $|B| \leq |S|$, or |B| > |S| and S is not a prefix of B, note that B is not a prefix of S, since $SR(B^mS) < R(B)$, $S <_T B$ follows by Claim 1. Then SB < BS follows by Claim 2.

2. If |B| > |S| and S is a prefix of B, let B = ST. Since $SR(B^mS) < R(B)$, we have $SR((ST)^mS) < R(ST)$. we pay attention to the prefixes with length 2 * |S| + |T| of these two infinite words: SST and STS. We can argue that $SST \neq STS$ otherwise ST = TS, then S, T, B, A are powers of a common element by Lemma 1, then R(A) = R(B), which is contradictory. Thus, since $SR((ST)^mS) < R(ST)$, we will have SST < STS. It holds that SB = SST < STS = BS. Thus, we end up with SB < BS.

With both cases proved, we have $R(A) < R(B) \Rightarrow AB < BA$.

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Now, with both subcases proved, we have $R(A) < R(B) \Leftrightarrow AB < BA$.