

The Hamiltonian Path Graph is Connected for Simple s, t Paths in Rectangular Grid Graphs

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Abstract. A *simple s, t path* P in a rectangular grid graph \mathbb{G} is a Hamiltonian path from the top-left corner s to the bottom-right corner t such that each *internal* subpath of P with both endpoints a and b on the boundary of \mathbb{G} has the minimum number of bends needed to travel from a to b (i.e., 0, 1, or 2 bends, depending on whether a and b are on opposite, adjacent, or the same side of the bounding rectangle). Here, we show that P can be reconfigured to any other simple s, t path of \mathbb{G} by *switching 2×2 squares*, where at most $5|\mathbb{G}|/4$ such operations are required. Furthermore, each *square-switch* is done in $O(1)$ time and keeps the resulting path in the same family of simple s, t paths. Our reconfiguration result proves that the *Hamiltonian path graph* \mathcal{G} for simple s, t paths is connected and has diameter at most $5|\mathbb{G}|/4$ which is asymptotically tight.

1 Introduction

An $m \times n$ *rectangular grid graph* \mathbb{G} is an induced subgraph of the infinite integer grid embedded on m rows and n columns. The outer boundary of \mathbb{G} is a rectangle $R_{\mathbb{G}}$ composed of four *boundaries*: \mathcal{E} , \mathcal{W} , \mathcal{N} and \mathcal{S} ; the inner faces of \mathbb{G} are 1×1 grid cells. An s, t *Hamiltonian path* P of \mathbb{G} is a Hamiltonian path with endpoints at the top left and bottom right vertices s and t of $R_{\mathbb{G}}$. Path P is called *simple* if each *internal* subpath (i.e., a subpath of P that starts and ends on the outer boundary of \mathbb{G} and contains only vertices internal to \mathbb{G} otherwise) contains the minimum possible number of bends. In other words, P is simple if an internal subpath $P_{u,v}$ contains no bends when u and v are on opposite boundaries (\mathcal{E} and \mathcal{W} , or \mathcal{N} and \mathcal{S}), exactly one bend when they are on adjacent boundaries (e.g., \mathcal{E} and \mathcal{N} etc.), and two bends when they are on the same boundary.

The *reconfiguration of simple paths in \mathbb{G}* asks the following question: given any two simple s, t paths P and P' of \mathbb{G} , is there an *operation*, preferably local to a small subgrid, and a sequence of simple paths $P = P_0, P_1, \dots, P'$ of \mathbb{G} such that each path in the sequence can be obtained from the previous path by applying the operation? Alternately, suppose that we define the *simple s, t Hamiltonian path graph* of \mathbb{G} with respect to an *operation* as the graph \mathcal{G} , where each simple s, t Hamiltonian path of \mathbb{G} is represented by a vertex, and two vertices u, v of \mathcal{G}

are connected by an edge if the defined *operation* reconfigures the Hamiltonian path represented by the one to the other. Then, the reconfiguration problem for simple paths in \mathbb{G} stated above asks whether \mathcal{G} is connected with respect to the *operation*. See Figure 1.

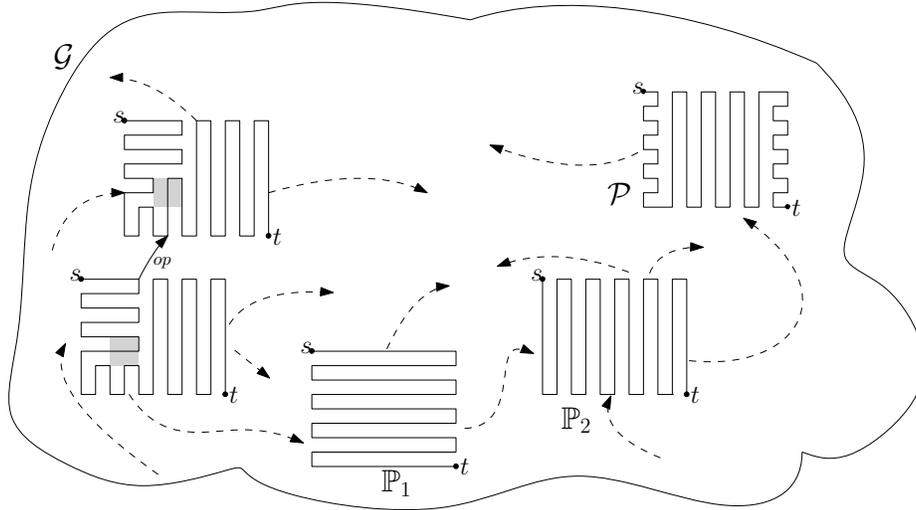


Fig. 1: The simple s, t Hamiltonian path graph \mathcal{G} with respect to the *square-switch* operation. Nodes \mathbb{P}_1 , \mathbb{P}_2 , and \mathcal{P} represent canonical and almost canonical paths (defined in Section 2).

In previous work [13], we provided a partial answer to this question. We introduced simple s, t paths in rectangular grid graphs in [13] and gave a *structure theorem* that we used to design an $O(|\mathbb{G}|)$ time algorithm to find a sequence of s, t Hamiltonian paths between two given simple s, t paths of \mathbb{G} . We used *pairs of cell-switch* operations. See Figure 2 for an example of a *cell-switch*, which exchanges two parallel edges of P on a cell for two non-edges of P on that cell. However, our approach had two limitations: the intermediate paths in that sequence obtained by pairs of cell-switches were not necessarily simple, and the pair of cells that were switched at each step were not always close to each other in \mathbb{G} . In other words, the pair of switch operations was not *local* in \mathbb{G} .

In this paper, we overcome these limitations and solve the reconfiguration problem for simple paths of \mathbb{G} completely. We introduce a new local operation we call *square-switch* or *switching a square*. Briefly, in a square subgrid sq consisting of 4 cells, the operation exchanges four edges of P for four non-edges of P , and leaves the other grid edges of sq unchanged. The four edges and non-edges of P occur in two diagonally opposite cells of sq , and the square-switch can be viewed as switching these two cells as illustrated in Figure 3(a) (other conditions apply; see Section 2 for details). We give an $O(|\mathbb{G}|)$ time algorithm to reconfigure any

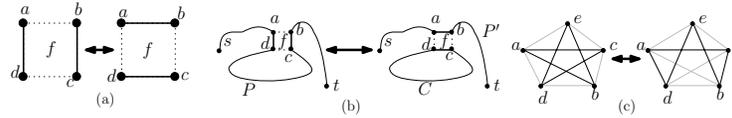


Fig. 2: (a) A cell-switch, (b) the cell-switch breaks an s, t Hamiltonian path for \mathbb{G} into a *path-cycle cover* for \mathbb{G} consisting of one cycle and one s, t path.

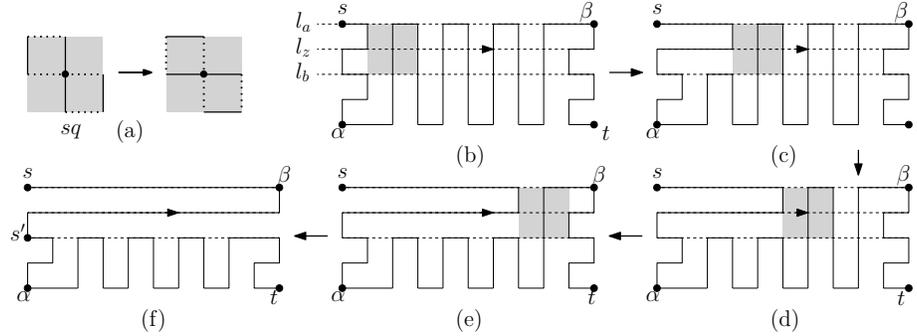


Fig. 3: (a) a square-switch on sq (b)–(e) (read clockwise) using square-switches to make lines.

simple s, t path of \mathbb{G} to another such path using square-switches. The core ideas in our algorithm are shown in Figure 3(b)–(f), where the simple path in (b) is transformed into the simple path in (f) using a sequence of square-switch operations. Moreover, we show that our reconfiguration algorithm uses at most $5|\mathbb{G}|/4$ such operations. This implies that the *diameter* of \mathcal{G} with respect to the square-switch operation (see Figure 1) is at most $5|\mathbb{G}|/4$.

Our Contributions. (1) We introduce a new operation called *square-switch*. A square-switch is a local operation on a small subgrid, only changing edges in the square. Our square-switch operation maintains Hamiltonicity after each square-switch in the reconfiguration. (2) We give a $O(|\mathbb{G}|)$ time algorithm that reconfigures a simple s, t Hamiltonian path in a rectangular grid graph to another using $5|\mathbb{G}|/4$ square-switches in such a way that the intermediate paths remain simple after each square-switch. (3) We give an affirmative answer to the connectivity question for the simple s, t Hamiltonian path graph, \mathcal{G} , of an $m \times n$ grid graph \mathbb{G} . Our algorithm provides a constructive proof that \mathcal{G} has diameter at most $5|\mathbb{G}|/4$ which is asymptotically tight.

Related Work and Applications. Reconfiguration problems have attracted attention for some time [7,17]. Takaoka [21] and Lignos [10] studied reconfiguration of Hamiltonian cycles in *unembedded* graphs using *switches*. However, for *embedded* graphs, a single switch operation increases the number of components in the path-cycle cover of the graph and hence needs to be paired with another switch operation right after the first one to restore the number of the compo-

nents. This observation led us to use pairs of cell-switches in reconfiguration of simple s, t paths [13] and 1-complex s, t paths [14].

Previously, Nishat and Whitesides studied reconfiguration of 1-complex Hamiltonian cycles in grid graphs without holes [15,16,12], where each internal vertex is connected to a boundary of the grid graph with a single turn-free segment on the cycle. They used two operations *flip* and *transpose*, and showed that the *Hamiltonian cycle graph* with respect to those two operations is connected for 1-complex cycles in rectangular grids and L -shaped grid graphs.

Apart from reconfiguration, the complexity of finding Hamiltonian cycles and paths in grid graphs has been extensively studied [6,3,9,22,20], as well as various combinatorial aspects of the problem [8,19,2], which has many possible application areas (e.g., in robot navigation [5], 3D printing [11], and polymer science [18]). The reconfiguration of paths and cycles has the potential to reduce turn costs and travel time and to increase navigation accuracy (e.g., [1], [4], [23]).

2 Preliminaries

In this section, we define the *square-switch* operation and show that it preserves Hamiltonicity. In Sections 3 and 4, we show how a carefully chosen sequence of square-switch operations, called the *zip*, can be applied repeatedly to design a reconfiguration algorithm for simple s, t paths. We start with basic terminology, some of which has been defined in [13], and is repeated here for completeness.

A *simple path* always means a simple s, t Hamiltonian path of \mathbb{G} ; it visits each node of \mathbb{G} exactly once and uses only edges in \mathbb{G} .

A *cell* of \mathbb{G} is an internal face of \mathbb{G} . A vertex of \mathbb{G} with coordinates (x, y) is denoted by $v_{x,y}$, where $0 \leq x \leq n-1$ and $0 \leq y \leq m-1$. The top left corner vertex s of \mathbb{G} has coordinates $(0, 0)$, and the positive y -direction is downward. We use the two terms *node* and *vertex* interchangeably.

Column x of \mathbb{G} is the shortest path of \mathbb{G} between $v_{x,0}$ and $v_{x,m-1}$, and *Row y* is the shortest path between $v_{0,y}$ and $v_{n-1,y}$. We call Columns 0 and $n-1$ the *west* (\mathcal{W}) and *east* (\mathcal{E}) boundaries of \mathbb{G} , respectively, and Rows 0 and $m-1$ the *north* (\mathcal{N}) and *south* (\mathcal{S}) boundaries.

Let P be a simple path of \mathbb{G} . The *directed subpath* of P from vertex u to w is denoted by $P_{u,w}$. Straight subpaths are called segments, denoted $seg[u, v]$, where u and v are the segment endpoints. An *internal subpath* $P_{u,v}$ of P , defined in Section 1, is called a *cookie* if both u, v are on the same boundary (i.e., \mathcal{N} , \mathcal{S} , \mathcal{E} , and \mathcal{W}); otherwise, $P_{u,v}$ is called a *separator*.

Cookies and Separators. A cookie can be an $\mathcal{E}, \mathcal{W}, \mathcal{N}, \mathcal{S}$ cookie, according to the boundary where the cookie has its end points. A cookie c is formed by three segments of P . The common length of the two parallel segments measures the *size* of c . The boundary edge between the endpoints of c is the *base* of c , and it does not belong to P .

Assumption [13]. Let α and β denote the bottom left and top right corner vertices of \mathbb{G} . Without loss of generality, we assume the input simple path

$P_{s,t}$ visits α before β . The target simple path for the reconfiguration as well as intermediate configurations may visit β before α .

Since separators of P have endpoints on distinct boundaries, there are two kinds: a *corner separator* μ_i or ν_i has one bend, and a *straight separator* η_i has no bends. Traveling along $P_{s,t}$, we denote the i -th straight separator we meet by η_i , where $1 \leq i \leq k$. The endpoints of η_i are denoted $s(\eta_i)$ and $t(\eta_i)$, where $s(\eta_i)$ is the first endpoint met. We say a corner separator *cuts off* a corner (s or t). Traveling along $P_{s,t}$, we denote the i -th corner separator cutting off s by μ_i , where $1 \leq i \leq j$. We denote its internal bend by $b(\mu_i)$, and its endpoints by $s(\mu_i)$ and $t(\mu_i)$, where $s(\mu_i)$ is the first endpoint met. Similarly, we denote the i -th corner separator cutting off t by ν_i ; endpoint $s(\nu_i)$ is met before $t(\nu_i)$. Corner separator ν_i has an internal bend at $b(\nu_i)$, where $1 \leq i \leq \ell$. A corner separator that has one of its endpoints connected to s or t by a segment of P is called a *corner cookie*. It is regarded as a cookie and not counted as a corner separator. We have j corner separators μ_i cutting off s , and k straight separators η_i where k must be odd (see [13]), and ℓ corner separators ν_i cutting off t .

Square-Switch Operation. A zipline $l_z^{q_1, q_2}$ is an *internal* gridline (i.e., a row or a column that is not a boundary) directed from endpoint q_1 to the other endpoint q_2 . Since l_z is not a boundary, it has two adjacent and parallel grid lines, which we denote by $l_a = [a_1, a_2]$ and $l_b = [b_1, b_2]$, where a_1 and b_1 are adjacent to q_1 on a boundary of \mathbb{G} and a_2 and b_2 are adjacent to q_2 on the opposite boundary. We call the rectangular region enclosed by l_a and l_z the *main track* tr , and the rectangular region enclosed by l_z and l_b the *side track* tr' .

Let P be a simple s, t path of \mathbb{G} . A cell c of \mathbb{G} is *switchable with respect to* P if two parallel sides of c lie on P and the other two parallel sides of c are *non-edges* of P . A cell-switch of such a cell is illustrated in Figure 2. Let l_z be a zipline of \mathbb{G} . A *square* $sq_{x,y}$ is a set of 4 cells of \mathbb{G} that share a common vertex $v_{x,y}$ at the center of sq . Coordinates may be dropped when the meaning is clear. We say square $sq = sq_{x,y}$ is *on the zipline* l_z if $v_{x,y}$ is on l_z . For sq on l_z we assign local names to the nodes of sq , denoting the nodes on l_a as p_1, p_2, p_3 , with index increasing along directed line l_a . We name the nodes of sq on l_z similarly, labelling the center of sq as p_5 , and then continue for the nodes on l_b as in Figure 4 (later used to illustrate Definition 1 below). Walking along directed l_z from q_1 to q_2 , we have the *left* or *right* side of the zipline; *near* side is closer to q_1 than q_2 , and *far* side is closer to q_2 . Based on this terminology, we denote by $c_{nl}, c_{nr}, c_{fl},$ and c_{fr} the near-left, near-right, far-left, and the far-right cell of sq , respectively.

Definition 1 (Switchable square for P and its switch operation).

Let sq be a square on zipline l_z . Square sq is switchable for P if: (i) the far cell in tr' and the near cell in tr are switchable, and (ii) switching the far cell in tr' creates a path-cycle cover comprising a cycle in tr and an s, t path $p_{s,t}$ through the nodes of \mathbb{G} not on the cycle. A square-switch of such a square sq is the operation that exchanges the four edges of P in the far cell of tr' and the near cell of tr for the four non-edges of P in those cells.

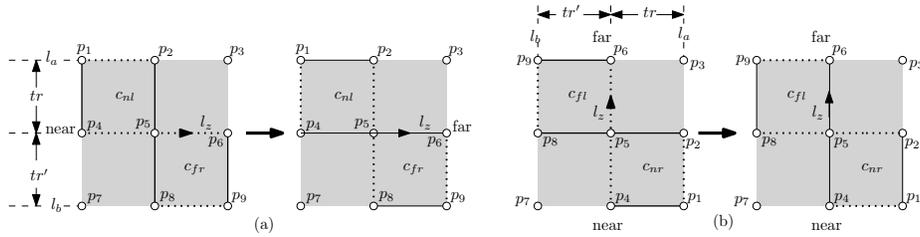


Fig. 4: Square-switch for sq on (a) horizontal l_z , where c_{fr} is the far cell of tr' and (b) vertical l_z , where c_{fl} is the far cell of tr' .

Figure 4(a),(b) shows a square sq on l_z before and after a square-switch for two different orientations of the zipline. We only perform a square-switch on switchable squares. By Definition 1(ii), we must have the edge (p_2, p_3) not shown in the figure to form a part of the cycle in tr both for horizontal (part (a)) and vertical zipline (part (b)).

The following observation shows that square-switch, when applied to switchable squares meeting the criteria of Definition 1, preserves Hamiltonicity. However, to achieve our reconfiguration goal, we must later take care to use square-switch in a way that keeps the path simple after each switch.

Observation 1 (Square-switch Hamiltonicity). Let sq be a switchable square for P on zipline l_z . Then performing square-switch on sq yields a new s, t Hamiltonian path P' .

Proof. By condition (ii) of Definition 1, P contains a subpath in tr that joins p_5 and p_6 , where grid edge (p_5, p_6) is a non-edge of P on l_z . The square-switch of sq can be thought of as carried out in two steps: first exchange the two edges of P in the far cell of tr' for the two non-edges in that cell, and then exchange the edges and non-edges of the near cell of tr . The first step creates a cycle in tr by turning grid edge (p_5, p_6) on l_z into an edge, and it also replaces the subpath of P connecting the endpoints p_8 and p_9 with an edge, resulting in a path $p_{s,t}$ containing the remaining nodes of \mathbb{G} . The second step, the switch of the near cell in tr , breaks the cycle and replaces the edge (p_1, p_4) of $p_{s,t}$ with a subpath joining its endpoints p_1 and p_4 . \square

We conclude this section with definitions of two special types of simple paths that will be used by our reconfiguration algorithm.

Canonical Paths. A *canonical path* is a simple path P with no bends at internal vertices. If m is odd, P can be $\mathcal{E}\text{-}\mathcal{W}$ and fill rows of \mathbb{G} one by one; if n is odd, P can be $\mathcal{N}\text{-}\mathcal{S}$ and fill columns. See the nodes \mathbb{P}_1 and \mathbb{P}_2 in Figure 1. There are no other types.

Almost Canonical Paths. A simple s, t Hamiltonian path is said to be *almost canonical* if it is not canonical, and contains straight separators in Columns 2 to $n - 3$, or in Rows 2 to $m - 3$. By definition, an almost canonical path must have

at least one of the following: unit size \mathcal{W} cookies covering the \mathcal{W} boundary, or unit size \mathcal{E} cookies covering \mathcal{E} . See the node \mathcal{P} in Figure 1 for an example that contains both.

3 Square-switches and the zip operation

In this section, we show how we use the square-switch operation to reconfigure simple s, t paths. We define a *zip* operation, which is a sequence of square-switches for squares on a directed zipline l_z , where the centers of the squares occur at every other position on l_z . We prove that switching these squares in order of occurrence along an interval of l_z produces a new simple path after each square switch. We first describe zip for a special case, where the input path is almost canonical and we want to reconfigure it to a canonical path (Section 3.1). We then discuss the more general case, where the input path is neither canonical nor almost canonical and we want to reconfigure it to a canonical or almost canonical form (Section 3.2).

3.1 P almost canonical

Let P be an almost canonical path of \mathbb{G} that visits α before β . Then P must have either unit \mathcal{W} cookies, or unit \mathcal{E} cookies, or unit size cookies on both \mathcal{E} and \mathcal{W} boundaries. In this section, we show how we can reconfigure P to an \mathcal{E} - \mathcal{W} canonical path by switching squares such that each such operation gives a simple s, t path of \mathbb{G} . See Figure 3(b)–(f).

We take Row 1 as the zipline l_z directed from \mathcal{W} to \mathcal{E} , l_a is the \mathcal{N} boundary and l_b is Row 2. Walking on l_z from q_1 on the \mathcal{W} boundary to q_2 on the \mathcal{E} , we define the first switchable square with respect to P to have center on η_1 , and denote the square by $sq(\eta_1)$. The next switchable square $sq(\eta_3)$ on l_z has center on η_3 , and so on. We show that each of the squares $sq(\eta_i)$, $1 \leq i \leq k$ and i odd, is switchable with respect to P .

Lemma 1. *Let P be an almost canonical path of \mathbb{G} visiting α before β , and let l_z and l_a be the Rows 1 and 0, respectively. Then each of the squares $sq(\eta_i)$, $1 \leq i \leq k$ and i odd, is switchable with respect to P .*

Proof. We prove the claim by showing that for each $sq(\eta_i)$, the cells c_{nl} in tr and c_{fr} in tr' are the switchable cells, and switching c_{fr} creates a path-cycle cover with a cycle in tr .

Case 1: If $k = 1$, then there is just one square $sq(\eta_1)$. The two vertical edges of c_{nl} are contributed by η_1 and the unit \mathcal{W} cookie in tr , or by η_1 and $seg[s, \alpha]$ on the \mathcal{W} boundary. The two vertical edges in c_{fr} are contributed by η_1 and the unit \mathcal{E} cookie in tr' , or by η_1 and the segment $seg[\beta, t]$ on the \mathcal{E} boundary. If there is no \mathcal{E} cookie in P , then the cell c_{fl} contains three edges of P , and switching c_{fr} creates a 1×1 cycle in tr containing only the cell c_{fl} . Therefore, $sq(\eta_1)$ is switchable. Otherwise, switching c_{fr} creates a cycle of two cells in tr that contains c_{fl} and goes through β , making $sq(\eta_1)$ switchable.

Case 2: If $k > 1$, the cell c_{nl} of $sq(\eta_1)$ and cell c_{fr} of $sq(\eta_k)$ will be the same as Case 1. For $i < k$, cell c_{fr} of $sq(\eta_i)$, i odd, will be between cross separators η_i and η_{i+1} in tr' and thus will be switchable; similarly cell c_{fl} will be in track tr between the same two cross separators that are connected by an edge on the \mathcal{N} . Therefore, switching c_{fr} will create a 1×1 cycle in tr . Therefore, square $sq(\eta_i)$ is switchable for $i < k$. The last square $sq(\eta_k)$ can be proved to be switchable with respect to P in a similar way as in Case 1. \square

We now define a *zip* operation that applies switches to the above squares.

Definition 2 (Zip operation \mathcal{W} to \mathcal{E}). *Let P be an almost canonical path of \mathbb{G} visiting α before β , and let l_z and l_a be Rows 1 and 0, respectively, directed \mathcal{W} to \mathcal{E} . Then the zip \mathcal{W} to \mathcal{E} operation $Z = zip(\mathbb{G}, P, l_z, l_a)$ applies switches to the squares $sq(\eta_i)$, $1 \leq i \leq k$ and i odd, in order from $i = 1$ to k .*

We now show that after every square switching of this zip operation we get a simple s, t path.

Lemma 2. *Let P be an almost canonical path of \mathbb{G} visiting α before β , let l_z be Row 1 directed eastward, and l_a be Row 0. The path after switching each square $sq(\eta_i)$, i odd, in the zip operation $Z = zip(\mathbb{G}, P, l_z, l_a)$ is a simple s, t path of \mathbb{G} .*

Proof. By Lemma 1, each of $sq(\eta_i)$, $1 \leq i \leq k$ and i odd, is switchable with respect to P . By Observation 1, switching the square $sq(\eta_i)$, i odd, yields a Hamiltonian path P_i of \mathbb{G} . We now prove that P_i is simple.

For $i < k$, switching c_{fr} in $sq(\eta_i)$ creates an \mathcal{S} cookie by shortening the cross separators η_i and η_{i+1} ; Switching c_{nl} increases the size of the corner \mathcal{W} cookie in track tr by 2. The cross separators η_{i+2} to η_k and the \mathcal{E} cookies, if there is any, are the same in P and P_i . Therefore, P_i is a simple path. See Figure 3 (c)-(e).

In the path P_k obtained after switching the square $sq(\eta_k)$, l_a and l_z are two line segments $seg[a_1, a_2 = \beta]$ and $seg[q_1, q_2]$ connected by edge $(a_2 = \beta, q_2)$. Then, P_k visits β before α , has one horizontal straight separator; and the final subpath of P_k contains \mathcal{S} cookies, and probably unit \mathcal{E} cookies preceded by the only corner separator ν_1 created from η_k . In fact, the subpath from $s' = v_{0,2}$ to t is almost canonical. Therefore, P_k is simple. See Figure 3 (f). \square

3.2 P neither canonical, nor almost canonical

Let P be a simple path of \mathbb{G} that visits α before β . We abbreviate $x(\eta_1) - 1$ to -1 . Thus $Col(-1)$ lies one unit west of η_1 and node $v_{-1,0}$ is the grid node on \mathcal{N} one unit west of $t(\eta_1)$. Lines l_a , l_z (the zipline), and l_b lie in $Col(-1)$, $Col(-2)$, and $Col(-3)$, respectively. Since P is neither almost canonical with unit-size west cookies nor in canonical form, $Col(-1)$, $Col(-2)$, and $Col(-3)$ are well defined. Zipline l_z is directed from q_1 on \mathcal{S} to q_2 on \mathcal{N} . See Figure 5(a).

The following observation shows that $Col(-1)$ contains at least one node that is joined by a horizontal segment of P to \mathcal{W} . We denote the row index of the highest and lowest such nodes by hi and lo . It may occur that $lo = hi$. See Figure 5(b)-(c).

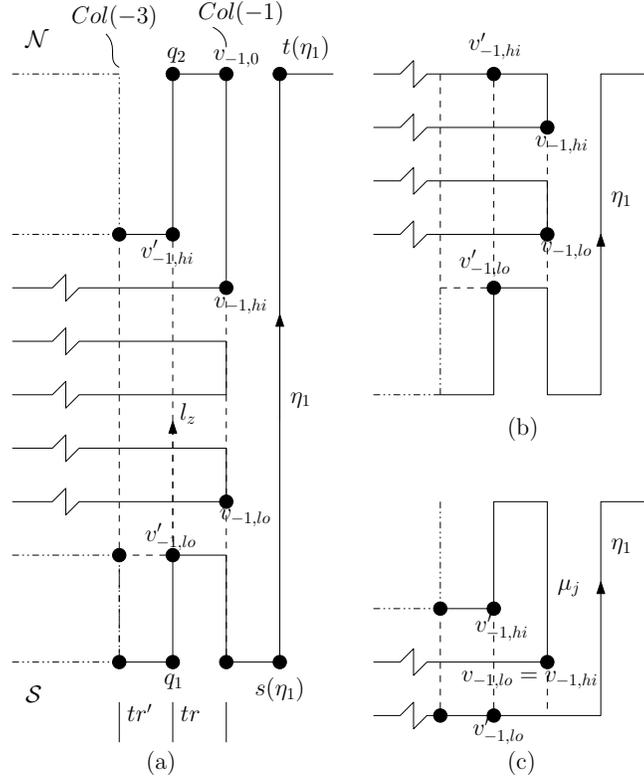


Fig. 5: Simple path P in tr and tr' with l_z in $Col(-2)$ (a) corner separator, \mathcal{W} cookies that reach $Col(-1)$, and an \mathcal{S} cookie in tr ; (b) a \mathcal{W} corner cookie that reaches $Col(-1)$; (c) bend b_j of μ_j in $Row(m-2)$ and $v_{-1,y_{hi}} = v_{-1,y_{lo}}$.

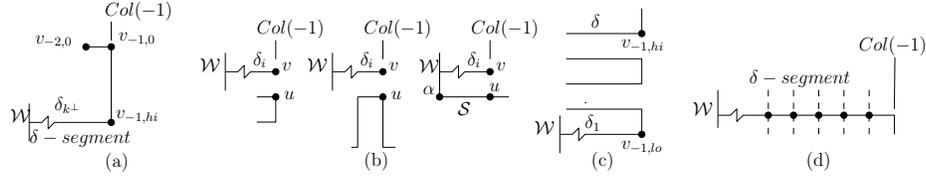


Fig. 6: Illustrations for: (a) Observation 2 (b) Observation 3 (c) Observation 4 and (d) Observation 5

Observation 2. Path P contains the following: i) edge $(v_{-2,0}, v_{-1,0})$ on \mathcal{N} ; ii) a vertical segment with endpoints $v_{-1,0}$ and $v_{-1,hi}$, where $v_{-1,hi}$ is an internal node of $Col(-1)$; and iii) a horizontal segment in $Row(hi)$ that extends from $v_{-1,hi}$ to \mathcal{W} . Furthermore, either $v_{-1,hi} = b_j$, where b_j is the bend in μ_j , or $v_{-1,hi}$ lies on a \mathcal{W} corner cookie. See Figure 6(a).

Proof. Node $v_{-1,0}$ has degree 3 in \mathbb{G} , and in P it has degree 2 but is not adjacent to $t(\eta_1)$. Hence $v_{-1,0}$ is incident in P to the boundary edge between $v_{-2,0}$ and $v_{-1,0}$ and is also incident to a vertical edge belonging to a maximal segment of P between nodes $v_{-1,0}$ and $v_{-1,hi}$, where $v_{-1,hi}$ cannot lie on \mathcal{S} as η_1 is the first straight separator. The internal path of $v_{-1,hi}$ cannot be an \mathcal{N} cookie due to the edge $(v_{-2,0}, v_{-1,0})$ on \mathcal{N} . The only other internal paths that could bend at $v_{-1,hi}$ are a \mathcal{W} corner cookie and a \mathcal{W} corner separator. In the latter case, the corner separator must be μ_j . \square

The next two observations give some properties of the coverage of nodes on l_a in $Col(-1)$ by P .

Observation 3. Let v be a node of $Col(-1)$ that is joined to its neighbor above and to \mathcal{W} by a horizontal segment of P , and let u be the grid node one unit below v . Then u lies in one of the following positions: i) at the top corner of a \mathcal{W} cookie; ii) at the top right corner of an \mathcal{S} cookie in tr ; or iii) on segment $seg[\alpha, s(\eta_1)]$ of P on \mathcal{S} . See Figure 6(b).

Proof. If u is an internal node $v_{x,y}$ of \mathbb{G} , then P joins u to its west and south neighbors $v_{x-1,y}$ and $v_{x,y+1}$. The only possibilities for the internal path of u are a \mathcal{W} and an \mathcal{S} cookie in tr . If u is not internal to \mathbb{G} , then $u = v_{-1,m-1}$ on \mathcal{S} , where no nodes strictly between α and $s(\eta_1)$ can lie on a vertical edge of P . \square

Observation 4. From $v_{-1,hi}$ to $v_{-1,lo}$ inclusive, there are an odd number of nodes in $Col(-1)$ on horizontal segments of P that extend to \mathcal{W} , and these nodes appear consecutively in $Col(-1)$. Any such segments below the topmost one occur as horizontal sides of \mathcal{W} cookies. See Figure 6(c).

Proof. By Observation 3, either $v_{-1,lo} = v_{-1,hi}$, or else the nodes in $Col(-1)$ with y -index in the range $[hi, lo]$ occur in pairs on \mathcal{W} cookies. \square

We denote by k^\perp the number of segments of P that extend from $Col(-1)$ to \mathcal{W} . We call them δ -segments. By the observation above, k^\perp is odd. Analogous to the η_i , we denote them $\delta_1, \dots, \delta_{k^\perp}$, with indices increasing along l_z . Similarly, we index the squares on l_z with the indices of the δ -segments through their centers: $sq_1, \dots, sq_{k^\perp}$. Note that $Row(lo)$ is the row of δ_1 , and $Row(hi)$ is the row of δ_{k^\perp} . Thus $v_{-1,lo}$ and $v_{-1,hi}$ occur in sq_1 and sq_{k^\perp} in position p_2 of each respective square. See Figure 5. The next easy observation is very useful.

Observation 5. The nodes internal to a δ -segment of P cannot be adjacent in P to grid nodes one unit above or below them, as their two incident horizontal edges give them degree 2 on P . See Figure 6(d).

Lemma 3. *The square sq_1 on l_z is switchable for P .*

Proof. There are two cases: i) $k^\perp > 1$ ($lo \neq hi$) and ii) $k^\perp = 1$ ($lo = hi$).

Case 1, $k^\perp > 1$: Cells $cell_{fl}$ and $cell_{fr}$ have lower sides in δ_1 (the segment of P extending from $v_{-2,lo}$ to \mathcal{W}). By Observation 4, these cells lie inside a \mathcal{W} cookie

that ends in $Col(-1)$. Hence $cell_{fl}$ is switchable for P . The diagonally opposite cell $cell_{nr}$ is switchable for P , as $cell_{nr}$ has its upper side in δ_1 (the lower side of the \mathcal{W} cookie) and its lower side at the end of an \mathcal{S} cookie or on \mathcal{S} . Thus condition i) of Definition 1 holds. Condition ii) is satisfied by the subpath P_{p_5, p_6} of P in tr . This subpath consists of the edge at the end of the \mathcal{W} cookie and its two adjacent edges on δ_1 and δ_2 (the lower and upper sides of the cookie). Therefore, switching c_{fl} will create a 1×1 cycle in tr . Thus sq_1 on l_z is switchable for P in case 1.

Case 2, $k^\perp = 1$: By Observation 4, there are no \mathcal{W} cookies. By Observation 2, segment δ_1 forms part of a \mathcal{W} corner cookie or part of μ_j . By Observation 3, node p_1 of sq_1 either sits on top of an \mathcal{S} cookie in tr , or sits on \mathcal{S} . Thus c_{nr} of sq_1 is the cell with a side at the top of an \mathcal{S} cookie in tr , or the cell with a side in \mathcal{S} and a side in either μ_j or a \mathcal{W} corner cookie. In the former case, P contains $seg[\alpha, s(\eta_1)]$. It follows that c_{nr} is switchable for P .

Next we show c_{fl} is switchable for P . By Observation 2, c_{fl} of sq_1 is either a cell in a \mathcal{W} corner cookie or a cell with a lower horizontal side on μ_j . Cell c_{fr} shares a non-edge of P (i.e., (p_5, p_6)) on l_z with c_{fl} and either lies at the end of a \mathcal{W} corner cookie or has b_j as its lower right vertex (i.e., p_2). Either way, c_{fr} has edges of P on its right and lower sides and a non-edge of P on its left side in l_z . Cell c_{fl} contains the node p_6 above the center p_5 of sq_1 . If p_6 is an interior node of \mathbb{G} , the only possibilities for its internal path are a \mathcal{N} cookie and μ_{j-1} ; either way, $cell_{fl}$ has both horizontal sides in P . If p_6 lies on \mathcal{N} , then c_{fl} lies inside a \mathcal{W} corner cookie that ends in $Col(-1)$, and thus $cell_{fl}$ is switchable. This completes the proof that both c_{fl} and c_{nr} are switchable for P .

To complete the proof that sq_1 on l_z is switchable, we show that switching c_{fl} creates a path-cycle cover whose cycle lies in tr . By Observation 2, switching c_{fl} creates a cycle in tr consisting of the following: the segment of P on l_z with an endpoint $v_{-2,0}$ on \mathcal{N} (this is of length 0 if P has a \mathcal{W} corner cookie); the boundary edge $(v_{-2,0}, v_{-1,0})$; the segment of P of positive length from $v_{-1,0}$ to v_{-1,l_0} ; and the horizontal edge of P incident to the center p_5 of sq_1 , where the edge lies on δ_1 (belongs to a segment of P extending to \mathcal{W}).

This completes the proof that sq_1 on l_z is switchable for P in case ii), and concludes the proof of the statement of the lemma. \square

The next two lemmas will help to show that switching the odd-indexed squares in order from sq_1 to sq_{k^\perp} yields a simple s, t Hamiltonian path after each square switch, and the switch of square sq_{k^\perp} results in a simple s, t Hamiltonian path with $k' = k + 2$ cross separators, joined by an edge of P on \mathcal{N} .

Lemma 4. *If P has \mathcal{W} cookies, then $k^\perp > 1$, and switching square sq_1 on l_z yields a new simple s, t Hamiltonian path P' . Paths P and P' are the same outside sq_1 , and the horizontal segments of P' are the remaining segments of P , namely δ_i , for $3 \leq i \leq k^\perp$.*

Proof. Using Lemma 3 and Observation 1, sq_1 is switchable and switching it yields a s, t Hamiltonian path P' . The new path is simple, as the internal paths

of nodes are the same for P and P' , with the exception of nodes on the \mathcal{S} cookie in tr of P (if the cookie exists) and the lowest \mathcal{W} cookie of P . The switch of sq shortens the \mathcal{W} cookie by two units and lengthens the \mathcal{S} cookie by two units (or grows a \mathcal{S} cookie of length 2 in tr if none exists in P). Thus each node lies on an internal path of an appropriate type with respect to P' . To the left of η_1 , path $P' = \text{path } P$ above δ_2 . The two segments δ_1 and δ_2 of P that were in the lowest \mathcal{W} cookie of P have been reconfigured in P' , and the lowest node on $Col(-1)$ that lies on a segment of P' extending to \mathcal{W} is two units higher than for P . \square

Lemma 5. *If P has no \mathcal{W} cookies, then $k^\perp = 1$, and switching square sq_1 on l_z yields a new simple s, t Hamiltonian path P' that fills l_a and l_z with cross separators joined by an edge on \mathcal{N} . The path P' is simple s, t Hamiltonian and has $k' = k + 2$ cross separators.*

Proof. Using Lemma 3 and Observation 1, sq_1 is switchable and switching it yields a s, t Hamiltonian path P' . Furthermore, switching sq_1 gives P' the two edges on l_z and $l_a = Col(-1)$ (i.e., the non-edge on l_z incident to the center of sq , and the non-edge of P on l_a incident to b_j) that were missing in P , but does not remove any path edges from those lines. Thus P' has two cross separators in l_z and l_a , joined by an edge on \mathcal{N} . To complete the proof, we now show that P' is simple by considering the two possible internal paths of u in P : u on an \mathcal{N} cookie and on μ_{j-1} .

If u lies on an \mathcal{N} cookie, then switching sq_1 creates a corner separator whose vertical segment lies one unit west of the new straight separator of P' and whose horizontal segment remains in the same row. This forms the last corner separator μ'_j in P' . The other corner separators are the same in P and P' . The internal paths for nodes on μ_j and the \mathcal{N} cookie with respect to P are now on μ'_j and the new cross separator of P' in l_b . The internal paths for internal nodes that were on an \mathcal{S} cookie in tr with respect to P are now on cross separators. No other internal paths of P are changed by the switch of sq , so P' is simple in this case.

If u lies on μ_{j-1} , then switching c_{fl} creates a \mathcal{W} cookie in $p_{s,t}$ that ends on $Col(-3) = l_b$. Internal nodes with internal path μ_{j-1} or μ_j with respect to P now have internal paths that are either the new \mathcal{W} cookie in P' or the new cross separator in l_b . Switching sq_1 does not change any other internal paths of P , which therefore remain the same in P' . Thus P' is simple in this case.

This completes the proof of the statement of the lemma. \square

Similar to the previous subsection, we define a zip operation for simple s, t Hamiltonian paths that have $\mathcal{S} - \mathcal{N}$ straight separators but are not in canonical or almost canonical form.

Definition 3 (Zip operation \mathcal{S} to \mathcal{N}). *Let P be a simple s, t Hamiltonian path visiting α before β , where P is neither canonical nor almost canonical, and let l_z and l_a be Cols -1 and -2 , respectively, directed \mathcal{S} to \mathcal{N} . Then the zip operation $Z = \text{zip}(\mathbb{G}, P, l_z, l_a)$ applies switches to the squares sq_i , $1 \leq i \leq k^\perp$ and i odd, in order from $i = 1$ to k^\perp .*

We summarize the running times of the two zip operations in the following observation.

Observation 6. Each square-switch can be performed in $O(1)$ time. The zip operation \mathcal{S} to \mathcal{N} takes time $\Theta(m)$, and the zip operation \mathcal{W} to \mathcal{E} takes $\Theta(n)$ time.

Proof. Paths can be stored for example as lists of bit vectors for rows and columns. Zip $Z = zip(\mathbb{G}, P, l_z, l_a)$ can be performed with the following steps: i) read up l_z to find the first non-edge (e.g., the first 0 of the bit vector for l_z). The upper endpoint of this non-edge is the center p_5 of sq_1 , which determines the row index of δ_1 . ii) Switch sq_1 by changing in constant time the bit vectors for its sides in l_a , l_z , and l_b and in the rows at and one above and below δ_1 . iii) while the grid edge above the center of the current square is a non-edge of P , advance 2 units along l_z and repeat step ii). Output the new simple s, t Hamiltonian path.

4 Reconfiguration Algorithm

In this section, we give an algorithm to reconfigure any simple path P to another simple path P' , maintaining the simplicity of the intermediate path after each application of square-switch. The algorithm reconfigures P and P' to canonical paths \mathbb{P} and \mathbb{P}' , respectively. If $\mathbb{P} \neq \mathbb{P}'$, i.e., one is $\mathcal{N}\text{-}\mathcal{S}$ and the other is $\mathcal{E}\text{-}\mathcal{W}$, the algorithm reconfigures \mathbb{P} to \mathbb{P}' and then reverses the steps taken from P' to \mathbb{P}' to complete the reconfiguration.

4.1 Reconfiguring P to \mathbb{P}

We give an algorithm RECONFIGSIMP to reconfigure any simple path $P = P_{s,t}$ (straight separators assumed to be $\mathcal{N}\text{-}\mathcal{S}$) to a canonical path \mathbb{P} , where the resulting \mathbb{P} might be either $\mathcal{N}\text{-}\mathcal{S}$ or $\mathcal{E}\text{-}\mathcal{W}$. The algorithm runs in three steps:

Step (a): Reconfigure the initial subpath of $P_{s,t}$ up to η_1 : take Column (-2) as the zipline l_z , Column (-1) as the line l_a , and apply zip from \mathcal{S} to \mathcal{N} to get another simple path P_1 that contains two straight separators in l_z and l_a . If $x(\eta_1) \leq 2$ in P_1 , move to Step (b). Otherwise, take Column (-2) of P_1 as the l_z , in effect shifting the previous l_z two units to the \mathcal{W} . Apply zip from \mathcal{S} to \mathcal{N} on P_1 similar to the zip on P . Repeat this process until a simple path P_2 is reached such that $x(\eta_1) \leq 2$, and then move to Step (b).

Step (b): We rotate the grid graph 180° about its center, and exchange the roles of s and t in P_2 . Apply the same process as in Step (a) until a path P_3 is reached that has $x(\eta_1) \leq 2$. Path P_3 either is a canonical path or an almost canonical path. If P_3 is a canonical path, then terminate. Otherwise, P_3 is an almost canonical path, so move to Step (c).

Step (c): P_3 must have at least one run of unit size \mathcal{E} or \mathcal{W} cookies. Take Row 1 as the zipline l_z , the \mathcal{N} boundary as l_a , and apply zip from \mathcal{W} to \mathcal{E} . Let P_4

be the path obtained after the zip. Then l_a and l_z are segments in P_4 . Move each of the lines l_a , l_z , and l_b two rows down, and perform the next \mathcal{W} to \mathcal{E} zip. Repeatedly zip and move downward until reaching an \mathcal{E} - \mathcal{W} canonical path of \mathbb{G} .

We now prove that the correctness and time complexity of RECONFIGSIMP.

Theorem 1. *Algorithm RECONFIGSIMP reconfigures a simple path in a rectangular grid graph \mathbb{G} to a canonical path of \mathbb{G} in $O(|\mathbb{G}|)$ time by switching at most $|\mathbb{G}|/2$ squares. Each square-switch produces a simple path.*

Proof. For Steps (a) and (b), each of the squares on the zipline l_z in Column (-2) is switchable by Lemma 3. By Lemma 4 and 5, each square switching gives a simple s, t path, and by Lemma 5, l_z and l_a are covered by two new straight separators after the zip. Each zip in these steps increases the number of straight separators by 2, and we end up with a canonical or almost canonical path. Since the zipline is moved two columns after each zip, the squares that are switched do not overlap in cells. Therefore, at most $|\mathbb{G}|/4$ squares are switched. In Step (c), the squares are switchable by Lemma 1, and after each square-switch we obtain a simple path by Lemma 2. Since no two squares contain a common cell, at most $|\mathbb{G}|/4$ squares are switched. The total number of square-switches is $|\mathbb{G}|/2$. \square

4.2 Reconfiguring \mathbb{P} to \mathbb{P}'

This step is similar to Step (c) of RECONFIGSIMP. If \mathbb{P} is \mathcal{N} - \mathcal{S} , we grow horizontal straight separators by sweeping the zipline downward. Otherwise, we transpose the grid with the embedded path, and apply the same technique as above. We call this algorithm RECONFIGCANONICAL. We now prove its correctness.

Theorem 2. *Let \mathbb{P} and \mathbb{P}' be two different canonical paths of \mathbb{G} . Then RECONFIGCANONICAL reconfigures \mathbb{P} to \mathbb{P}' in $O(|\mathbb{G}|)$ time by switching at most $|\mathbb{G}|/4$ squares.*

Proof. To check whether \mathbb{P} contains vertical separators, we just check in $O(1)$ time whether the first edge on \mathbb{P} is vertical or horizontal. As in the proof of Theorem 1 we can prove that a total of at most $|\mathbb{G}|/4$ squares are switched, which takes $O(|\mathbb{G}|)$ time. \square

4.3 Main Result

We summarize our main algorithmic result in the following theorem.

Theorem 3. *Let P and P' be two simple paths of a rectangular grid graph \mathbb{G} . Then P can be reconfigured to P' in $O(|\mathbb{G}|)$ time by at most $5|\mathbb{G}|/4$ square-switches, where each square-switch produces a simple path.*

Proof. By Theorem 1, P can be reconfigured to a canonical path \mathbb{P} in $O(|\mathbb{G}|)$ time by switching $|\mathbb{G}|/2$ squares. Similarly, P' can be reconfigured to a canonical path \mathbb{P}' in $O(|\mathbb{G}|)$ time by at most $|\mathbb{G}|/2$ square-switches. Reconfiguring \mathbb{P} to \mathbb{P}'

takes $O(|\mathbb{G}|)$ time and $|\mathbb{G}|/4$ square-switches by Theorem 2. If needed, reversing the steps of reconfiguring P' to \mathbb{P}' takes $O(|\mathbb{G}|)$ time. Hence the total time to reconfigure P to P' is $O(|\mathbb{G}|)$, where at most $5|\mathbb{G}|/4$ squares are switched. All square-switches produce simple paths. \square

We observe that reconfiguring a $\mathcal{N}\text{-}\mathcal{S}$ canonical path \mathbb{P} to a $\mathcal{E}\text{-}\mathcal{W}$ canonical path \mathbb{P}' requires at least $|\mathbb{G}|/4$ square-switch operations as each such operation can only produce 4 edges of \mathbb{P}' . This observation together with above theorem immediately implies the following result.

Theorem 4. *The Hamiltonian path graph \mathcal{G} of \mathbb{G} for simple s, t Hamiltonian paths is connected with respect to the operation square-switch, and the diameter of \mathcal{G} is $\Theta(|\mathbb{G}|)$ and indeed at most $5|\mathbb{G}|/4$.*

5 Conclusion and Open Problems

In this paper, we introduced a *square-switch* operation, and gave a linear time algorithm that uses at most $5|\mathbb{G}|/4$ square-switches to reconfigure any simple s, t Hamiltonian path in a rectangular grid graph \mathbb{G} to any other such path. We ensured that each square-switch made by the algorithm yields a simple path. This result proves the connectivity of the Hamiltonian path graph \mathcal{G} of \mathbb{G} for simple paths with respect to the square-switch operation, and shows that the diameter of \mathcal{G} is linear in the size of the grid graph \mathbb{G} . We defined a very restricted notion of square-switch to achieve our results. We propose that the square-switch, or a generalization of it, can be used to solve a reconfiguration problems for a variety of other families of s, t Hamiltonian paths in the same or other settings.

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