

Queue Layouts of Two-Dimensional Posets

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Abstract. The queue number of a poset is the queue number of its cover graph when the vertex order is a linear extension of the poset. Heath and Pemmaraju conjectured that every poset of width w has queue number at most w . The conjecture has been confirmed for posets of width $w = 2$ and for planar posets with 0 and 1. In contrast, the conjecture has been refuted by a family of general (non-planar) posets of width $w > 2$. In this paper, we study queue layouts of two-dimensional posets. First, we construct a two-dimensional poset of width $w > 2$ with queue number $2(w - 1)$, thereby disproving the conjecture for two-dimensional posets. Second, we show an upper bound of $w(w + 1)/2$ on the queue number of such posets, thus improving the previously best-known bound of $(w - 1)^2 + 1$ for every $w > 3$.

Keywords: poset · queue number · width · dimension · linear extension

1 Introduction

Let G be a simple, undirected, finite graph with vertex set V and edge set E , and let σ be a total order of V . For a pair of distinct vertices u and v , we write $u <_{\sigma} v$ (or simply $u < v$), if u precedes v in σ . We also write $[v_1, v_2, \dots, v_k]$ to denote that v_i precedes v_{i+1} for all $1 \leq i < k$; such a subsequence of σ is called a *pattern*. Two edges $(u, v) \in E$ and $(a, b) \in E$ *nest* if $u <_{\sigma} a <_{\sigma} b <_{\sigma} v$. A k -queue layout of G is a total order of V and a partition of E into subsets E_1, E_2, \dots, E_k , called *queues*, such that no two edges in the same set E_i nest. The *queue number* of G , $\text{qn}(G)$, is the minimum k such that G admits a k -queue layout. Equivalently, the queue number is the minimum k such that there exists an order σ containing no $(k + 1)$ -rainbow, that is, a set of edges $\{(u_i, v_i); i = 1, 2, \dots, k + 1\}$ forming pattern $[u_1, \dots, u_{k+1}, v_{k+1}, \dots, v_1]$ in σ .

Queue layouts can be studied for partially ordered sets (or simply *posets*). A poset over a finite set of elements X is a transitive and asymmetric binary relation $<$ on X . The main idea is that given a poset, one should lay it out respecting the relation. Two elements a, b of a poset, $P = (X, <)$, are called *comparable* if $a < b$ or $b < a$, and *incomparable*, denoted by $a \parallel b$, otherwise. Posets are visualized by their diagrams: Elements are placed as points in the plane and whenever $a < b$ in the poset and there is no element c with $a < c < b$, there is a curve from a to b going upwards (that is y -monotone); see Fig. 1a. Such relations, denoted by $a \prec b$, are known as *cover relations*; they are essential in the sense that they are not implied by transitivity. The directed graph implicitly

defined by such a diagram is the cover graph G_P of the poset P . Given a poset P , a linear extension L of P is a total order on the elements of P such that $a <_L b$, whenever $a <_P b$. Finally, the queue number of a poset P , denoted by $\text{qn}(P)$, is the smallest k such that there exists a linear extension L of P for which the resulting layout of G_P contains no $(k + 1)$ -rainbow; see Fig. 1c.

Queue layouts of posets were first studied by Heath and Pemmaraju [5], who provided bounds on the queue number of posets in terms of their *width*, that is, the maximum number of pairwise incomparable elements. In particular, they observed that the size of a rainbow in a queue layout of a poset of width w cannot exceed w^2 , and therefore, $\text{qn}(P) \leq w^2$ for every poset P . Furthermore, Heath and Pemmaraju conjectured that $\text{qn}(P) \leq w$ for a width- w poset P . The study of the conjecture received a notable attention in the recent years. Knauer, Micek, and Ueckerdt [6] confirmed the conjecture for posets of width $w = 2$ and for planar posets with 0 and 1. Later Alam et al. [1] constructed a poset of width $w \geq 3$ whose queue number is $w + 1$, thus refuting the conjecture for general non-planar posets. In the same paper Alam et al. improved the upper bound by showing that $\text{qn}(P) \leq (w - 1)^2 + 1$ for all posets P of width w . Finally, Felsner, Ueckerdt, and Wille [4] strengthened the lower bound by presenting a poset of width $w > 3$ with $\text{qn}(P) \geq w^2/8$.

In this short paper we refine our knowledge on queue layouts of posets by improving the known upper and lower bounds of the queue number of two-dimensional posets. Recall that the *dimension* of poset P is the least positive integer d for which there are d linear extensions (*realizers*) L_1, \dots, L_d of P so that $a < b$ in P if and only if $a < b$ in L_i for every $i \in \{1, \dots, d\}$. *Two-dimensional* posets are described by realizers L_1 and L_2 and often represented by *dominance drawings* in which the coordinates of the elements are their positions in L_1 and L_2 ; see Fig. 1b. We emphasize that the existing lower bound constructions [1, 4] are not two-dimensional. Thus, Felsner et al. [4] asked whether the conjecture of Heath and Pemmaraju holds for posets with dimension 2. Our first result answers the question negatively.

Theorem 1. *There exists a two-dimensional poset P of width $w > 1$ with $\text{qn}(P) \geq 2(w - 1)$.*

Observe that our construction and the proof of Theorem 1 for $w = 3$ is arguably much simpler than the one of Alam et al. [1], which is based on a tedious case analysis. Thus, it can be interesting on its own right.

Next we study the upper bound on the queue number of two-dimensional posets. Our result is the following theorem, which is an improvement over the known $(w - 1)^2 + 1$ bound of Alam et al. [1] for every $w > 3$.

Theorem 2. *Let P be a two-dimensional poset with realizers L_1, L_2 . Then there is a layout of P in at most $w(w + 1)/2$ queues using either L_1 or L_2 as the vertex order.*

The paper is structured as follows. In Section 3 we prove Theorem 1 and in Section 2 we prove Theorem 2. Section 4 concludes the paper with interesting open questions.

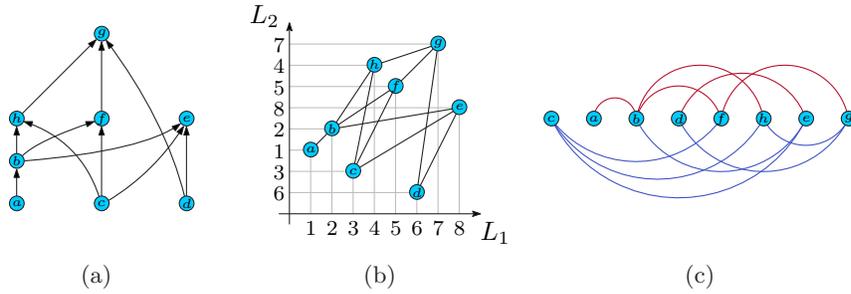


Fig. 1: A two-dimensional poset of width 3, its dominance drawing, and a 2-queue layout

2 An Upper Bound

Consider a two-dimensional poset $P = (X, <)$ of width $w \geq 1$ with realizers L_1 and L_2 . In this section we study queue layouts of P using vertex orders L_1 or L_2 , which we call *realizer-based*. It is well-known that the elements of P can be partitioned into w *chains*, that is, subsets of pairwise comparable elements. We fix such a partition and treat it as a function $\mathcal{C} : X \rightarrow \{1, \dots, w\}$ such that if $\mathcal{C}(u) = \mathcal{C}(v)$ and $u \neq v$, then either $u < v$ or $v < u$.

We start with a property of a linear extension of a poset, whose proof follows directly from the absence of transitive edges in G_P . Recall that \prec indicates cover relations of P , that is, edges of G_P .

Proposition 1 *A linear extension of a poset P with chain partition \mathcal{C} does not contain pattern $[b_1, b_2, b_3]$, where $\mathcal{C}(b_1) = \mathcal{C}(b_2) = \mathcal{C}(b_3)$ and $b_1 \prec b_3$.*

The next observation, whose proof is immediate, provides a crucial property of realizer-based linear extensions of two-dimensional posets. In fact, a poset, P , admits a linear extension with such a property if and only if P has dimension 2; see for example [3] where such linear extensions are called *non-separating*.

Proposition 2 *Consider a two-dimensional poset P with realizers L_1, L_2 and chain partition \mathcal{C} . Let $[a_1, b, a_2]$ be a pattern in L_1 (or L_2) with $\mathcal{C}(a_1) = \mathcal{C}(a_2)$. Then either $a_1 < b$ or $b < a_2$.*

The next useful property in the section holds for realizer-based linear extensions of two-dimensional posets.

Proposition 3 *Consider a two-dimensional poset P with realizers L_1, L_2 and chain partition \mathcal{C} . Then L_1 (or L_2) does not contain pattern $[a_1, b_2, a, a_2, b_1]$, where $\mathcal{C}(a_1) = \mathcal{C}(a_2) = \mathcal{C}(a)$, $\mathcal{C}(b_1) = \mathcal{C}(b_2)$, and $a_1 \prec b_1$, $b_2 \prec a_2$.*

Proof. For the sake of contradiction, assume that $[a_1, b_2, a, a_2, b_1]$ is in L_1 , with $\mathcal{C}(a_1) = \mathcal{C}(a_2) = \mathcal{C}(a)$, $\mathcal{C}(b_1) = \mathcal{C}(b_2)$, and $a_1 \prec b_1$, $b_2 \prec a_2$. Notice that $a_1 \parallel b_2$,

as otherwise we have $a_1 < b_2 < b_1$ and the edge (a_1, b_1) is transitive. Hence by [Proposition 2](#) applied to $[a_1, b_2, a]$, $b_2 < a$. Therefore, it holds that $b_2 < a < a_2$, which contradicts to non-transitivity of edge (b_2, a_2) .

Now we ready to prove the main result of the section.

Proof of [Theorem 2](#). Assume that poset P is partitioned into w chains, and consider a maximal rainbow, denoted T , induced by the order L_1 . We need to prove that $|T| \leq w(w+1)/2$.

First observe that the rainbow, T , does not contain two distinct edges (a_1, b_1) and (a_2, b_2) with $\mathcal{C}(a_1) = \mathcal{C}(a_2)$ and $\mathcal{C}(b_1) = \mathcal{C}(b_2)$. Otherwise, the former edge nests the latter one and we have $a_1 < a_2 < b_2 < b_1$, which violates non-transitivity of (a_1, b_1) . Therefore, we already have $|T| \leq w^2$. (This is the argument of Heath and Pemmaraju for their original upper bound in [\[5\]](#))

Next we show two more configurations that are absent in T :

- (i) For every pair of distinct chains, the rainbow does not contain edges (a_1, b_1) , (b_2, a_2) , and (a_3, a_4) with $\mathcal{C}(a_1) = \mathcal{C}(a_2) = \mathcal{C}(a_3) = \mathcal{C}(a_4)$ and $\mathcal{C}(b_1) = \mathcal{C}(b_2)$. For a contradiction, assume the rainbow contains the three edges. By [Proposition 1](#), edge (a_3, a_4) cannot cover elements a_1 or a_2 . Thus, L_1 contains pattern $[a_1, b_2, a_3, a_4, a_2, b_1]$ or $[b_2, a_1, a_3, a_4, b_1, a_2]$. Both patterns violate [Proposition 3](#).
- (ii) For every triple of distinct chains, the rainbow does not contain edges (a_1, b_1) , (b_2, a_2) , (a_3, c_3) , and (c_4, a_4) with $\mathcal{C}(a_1) = \mathcal{C}(a_2) = \mathcal{C}(a_3) = \mathcal{C}(a_4)$, $\mathcal{C}(b_1) = \mathcal{C}(b_2)$, and $\mathcal{C}(c_3) = \mathcal{C}(c_4)$.

For a contradiction, assume T contains the four edges. Consider the innermost edge in the rainbow; without loss of generality, assume the edge is (a_1, b_1) . Vertex a_1 is covered by two edges, (a_3, c_3) and (c_4, a_4) , forming the pattern of [Proposition 3](#); a contradiction.

Now observe that T may contain at most w *uni-colored* edges (that is, (u, v) such that $\mathcal{C}(u) = \mathcal{C}(v)$) and at most $w(w-1)$ *bi-colored* edges (that is, (u, v) such that $\mathcal{C}(u) \neq \mathcal{C}(v)$).

On the one hand, if T contains exactly w uni-colored edges and $|T| > w(w+1)/2$, then it must contain at least one pair of bi-colored edges (a_1, b_1) , (b_2, a_2) with $\mathcal{C}(a_1) = \mathcal{C}(a_2)$, $\mathcal{C}(b_1) = \mathcal{C}(b_2)$. Together with the uni-colored edge from chain $\mathcal{C}(a_1)$, the triple forms the forbidden configuration (i).

On the other hand, if T contains at most $w-1$ uni-colored edges and $|T| > w(w+1)/2$, then T contains two pairs of bi-colored edges, as in configuration (ii); a contradiction.

This completes the proof of the theorem. □

Notice that the bound of [Theorem 2](#) is worst-case optimal, as we show next.

Lemma 1. *There exists a two-dimensional poset of width $w \geq 1$, denoted R_w , with realizers L_1, L_2 such that its layout with vertex order L_1 contains a $(w(w+1)/2)$ -rainbow.*

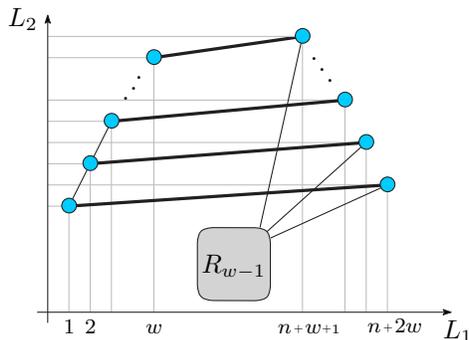


Fig. 2: A 2-dimensional poset of width $w \geq 1$, R_w , with a realizer-based order containing a $(w(w + 1)/2)$ -rainbow, which is comprised of w thick edges that nest all edges of R_{w-1}

Proof. The poset R_w is built recursively. For $w = 1$, the poset consists of two comparable elements. For $w > 1$, we assume that R_{w-1} is constructed and described by realizers L_1^{w-1} and L_2^{w-1} . The poset R_w is constructed from R_{w-1} by adding $2w$ elements. Assume $|L_1^{w-1}| = n$ and the elements of R_{w-1} are indexed by $w + 1, \dots, w + n$. We set L_1^w to the identity permutation and use

$$L_2^w = L_2^{w-1} \cup (1, n + 2w, 2, n + 2w - 1, \dots, w, n + w + 1),$$

where \cup denotes the concatenation of the two orders. Fig. 2 illustrates the construction. It is easy to verify that the width of the new poset is exactly w . Observe that in the layout of R_w with order L_1^w , edges $(1, n + 2w), \dots, (w, n + w + 1)$ form a w -rainbow and nest all edges of R_{w-1} . Therefore, the layout contains a $(w(w + 1)/2)$ -rainbow, as claimed.

We remark that Lemma 1 provides a poset whose queue layout with one of its realizers contains a $(w(w + 1)/2)$ -rainbow. It is straightforward to extend the construction (by concatenating R_w with its dual) so that both realizer-based vertex orders yield a rainbow of that size. However, the queue number of the poset (and the proposed extension) is at most w , which is achieved with a different, non-realizer-based, vertex order. Thus, a more delicate construction is needed to force a larger rainbow in every linear extension of a poset.

3 A Lower Bound

In this section we provide a new counter-example to the conjecture of Heath and Pemmaraju [5] by describing a two-dimensional poset of width $w \geq 3$ whose queue number exceeds w . The poset, denoted P_w , is constructed recursively. The base case, P_2 , is a four-element poset with $L_1 = (1, 2, 3, 4)$ and $L_2 = (2, 1, 4, 3)$; see Fig. 3b. The step of the construction is illustrated in Fig. 3c. Poset P_w

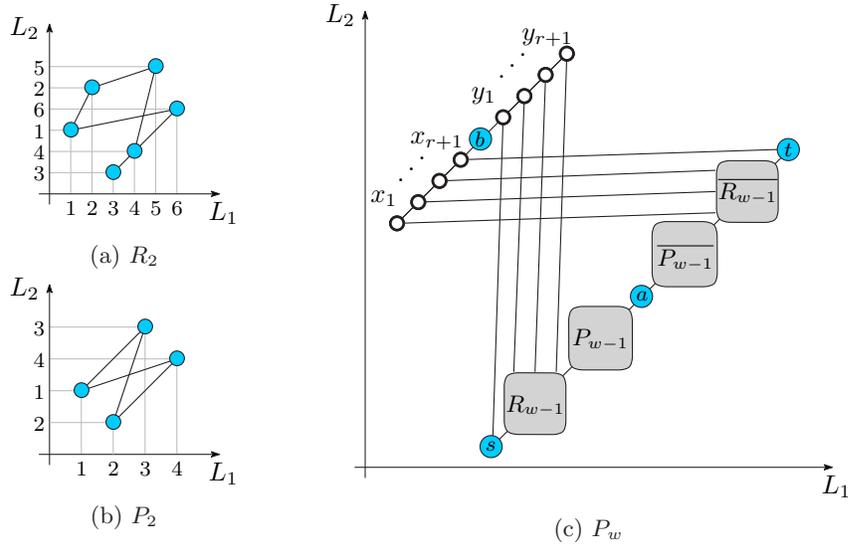


Fig. 3: A counter-example to the conjecture of Heath and Pemmaraju [5]: A two-dimensional poset, P_w , of width $w \geq 3$ with queue number exceeding w

consists of a copy of P_{w-1} , a copy of the poset R_{w-1} utilized in Lemma 1, the duals of the two posets, and a chain of additional elements. Recall that the *dual* of a poset, P , is the poset, \overline{P} , on the same set of elements such that $x < y$ in P if and only if $y < x$ in \overline{P} for every pair of the elements x and y .

We now formally describe the construction. Denote by $L_1(P), L_2(P)$ the two realizers of a two-dimensional poset P . Let \cup denote the concatenation of two sequences, and let $(x_1, x_2, \dots) \uplus (y_1, y_2, \dots)$ denote the *interleaving* of two equal-length sequences, that is, $(x_1, y_1, x_2, y_2, \dots)$. Assume that R_{w-1} contains r elements. Then we set

$$\begin{aligned}
 L_1(P_w) &= (x_1, \dots, x_{r+1}) \cup b \cup s \cup y_1 \cup (L_1(R_{w-1}) \uplus (y_2, \dots, y_{r+1})) \cup \\
 &\quad L_1(P_{w-1}) \cup a \cup L_1(\overline{P_{w-1}}) \cup L_1(\overline{R_{w-1}}) \cup t, \text{ and} \\
 L_2(P_w) &= s \cup L_2(R_{w-1}) \cup L_2(P_{w-1}) \cup a \cup L_2(\overline{P_{w-1}}) \cup \\
 &\quad ((x_1, \dots, x_r) \uplus L_2(\overline{R_{w-1}})) \cup x_{r+1} \cup t \cup b \cup (y_1, \dots, y_{r+1}).
 \end{aligned}$$

We refer to Fig. 3 for the illustration of the construction and to Fig. 5 for the instance of P_3 . Now we prove that the constructed poset has queue number at least $2w - 2$.

Proof of Theorem 1. It is easy to verify that the constructed poset, P_w , is two-dimensional and has width exactly w . Furthermore, the poset is dual to itself, that is, $P_w = \overline{P_w}$ with a and b being the fixed points. Thus, we may assume that in the linear extension corresponding to the optimal queue layout of the poset, element a precedes b and we have $s < \dots < a < b < y_1 < \dots < y_{r+1}$. Next

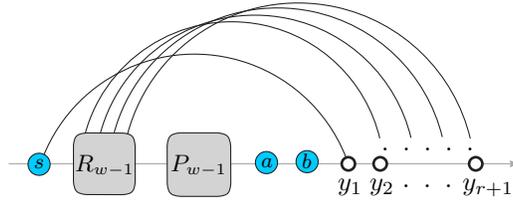


Fig. 4: A linear extension of poset P_w for the proof of Theorem 1 in which $a < b$

we consider the queue layout induced by the elements $s, R_{w-1}, P_{w-1}, a,$ and y_1, \dots, y_{r+1} ; see Fig. 4.

We prove the theorem by induction. For $w = 2$, the claim holds trivially. For $w > 2$, we assume that $\text{qn}(P_{w-1}) \geq 2(w - 2)$ and distinguish two cases depending on the size of the maximum rainbow, T , formed by edges $(s, y_1), (v_1, y_2), \dots, (v_r, y_{r+1})$, where $v_i, 1 \leq i \leq r$ are elements of R_{w-1} :

- if $|T| \geq 2$, then $\text{qn}(P_w) \geq \text{qn}(P_{w-1}) + |T| \geq 2(w - 1)$, as all edges of P_{w-1} are nested by edges of T ;
- if $|T| = 1$, then the elements of R_{w-1} must appear in the order induced by $L_1(R_{w-1})$, since otherwise at least two of the edges of T nest. By Lemma 1, the edges of R_{w-1} form a $(w(w - 1)/2)$ -rainbow. The rainbow is covered by edge (s, y_1) , which yields $\text{qn}(P_w) \geq (w(w - 1)/2) + 1 \geq 2(w - 1)$ for $w \geq 3$.

This completes the proof of Theorem 1. □

4 Conclusions

We disproved the conjecture of Heath and Pemmaraju for two-dimensional posets and answered a question posed by Felsner et al. [4]. A number of intriguing problems in the area remain unsolved.

- Is it possible to get a subquadratic upper bound on the queue number of two-dimensional posets of width w ? A poset of Felsner et al. [4] that requires $w^2/8$ queues in every linear extension is not two-dimensional, which leaves a hope for an asymptotically stronger result than the one given by Theorem 2.
- What is the queue number of two-dimensional posets of width 3? By Theorem 1 and the result of Alam et al. [1], the value is either 4 or 5.
- Queue layouts of graphs are closely related to so-called *track layouts*, which are connected with the existence of low-volume three-dimensional graph drawings [2, 7]. In particular, every t -track (undirected) graph has a $(t - 1)$ -queue layout, and every q -queue (undirected) graph has track number at most $4q \cdot 4q^{(2q-1)(4q-1)}$. We think it is interesting to study the relationship between the two concepts for directed graphs and posets.

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Additional Illustrations

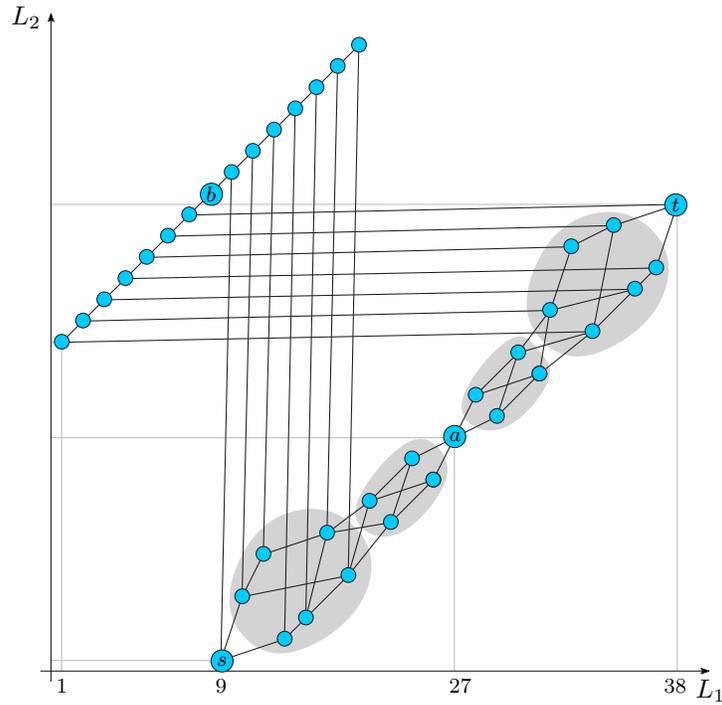


Fig. 5: A two-dimensional poset with 38 elements and width 3. The queue number of the poset is exactly 4; the lower bound is shown in [Theorem 2](#), and the upper bound is verified computationally via an open source SAT-based solver available at <http://be.cs.arizona.edu>