

Empty Triangles in Generalized Twisted Drawings of K_n

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Abstract. Simple drawings are drawings of graphs in the plane or on the sphere such that vertices are distinct points, edges are Jordan arcs connecting their endpoints, and edges intersect at most once (either in a proper crossing or in a shared endpoint). Simple drawings are generalized twisted if there is a point O such that every ray emanating from O crosses every edge of the drawing at most once and there is a ray emanating from O which crosses every edge exactly once. We show that all generalized twisted drawings of K_n contain exactly $2n - 4$ empty triangles, by this making a substantial step towards proving the conjecture that this is the case for every simple drawing of K_n .

Keywords: simple drawings · simple topological graphs · empty triangles

1 Introduction

Simple drawings are drawings of graphs in the plane or on the sphere such that vertices are distinct points, edges are Jordan arcs connecting their endpoints, and edges intersect at most once either in a proper crossing or in a shared endpoint. The edges and vertices of a drawing partition the plane into regions, which are called the *cells* of the drawing. A *triangle* in a simple drawing D is a subdrawing of D which is a drawing of K_3 . By the definition of simple drawings, any triangle is crossing free and thus splits the plane (or the sphere) in two connected regions. We call those regions the *sides* of the triangle. If one side of a triangle does not contain any vertices of D , that side is called an *empty side* of the triangle, and



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the triangle is called *empty triangle*. Note that empty (sides of) triangles might be intersected by edges. We observe that simple drawings of K_3 consist of exactly one triangle, which has two empty sides. Triangles in simple drawings of graphs with $n \geq 4$ vertices have at most one empty side.

In this work, we study the number of empty triangles in simple drawings of K_n . Note that simple drawings are a topological generalization of straight-line drawings. The number $h_3(n)$ of empty triangles that every straight-line drawing of K_n contains has been subject of intensive research. It is easy to see that $h_3(n) = \Omega(n^2)$. The currently best known bounds are $n^2 - \frac{32}{7}n + \frac{22}{7} \leq h_3(n) \leq 1.6196n^2 + o(n^2)$ [2,6].

For simple drawings of complete graphs, the situation changes drastically. Harborth [9] showed in 1989 that there are simple drawings of K_n that contain only $2n - 4$ empty triangles; see Fig. 1b. This especially implies that most edges in these drawings are not incident to any empty triangles. On the other hand, Harborth observed that every vertex in these drawings is incident to at least two empty triangles, a property he conjectured to be true in general. This conjecture has been proven in 2013 [8,13]. The currently best lower bound on the number of empty triangles in simple drawings of K_n is n [4]. Further, it is conjectured that Harborth's upper bound should actually be the true lower bound.

Conjecture 1 ([4]). *For any $n \geq 4$, every simple drawing of K_n contains at least $2n - 4$ empty triangles.*

The drawings that Harborth used for his upper bound are now well known as *twisted drawings* [12] and have received considerable attention [1,5,7,10,11,12,14]. A generalization of twisted drawings was introduced in [3] as a special type of *c-monotone drawings*. A simple drawing D in the plane is *c-monotone* if there is a point O such that any ray emanating from O intersects any edge of D at most once. A *c-monotone drawing* D is *generalized twisted* if there exists a ray r emanating from O that intersects every edge of D .

As twisted drawings and the upper bound obtained by them are crucial in the study of empty triangles, it is natural to ask about the number of triangles in their generalization. The initial goal of this work was to prove Conjecture 1 for generalized twisted drawings. As the number of such drawings is exponential (in the number of vertices), one might expect that they contain different numbers of empty triangles. However, we show that surprisingly, the conjectured bound is tight for all of them.

Theorem 1. *For any $n \geq 4$, every generalized twisted drawing of K_n contains exactly $2n - 4$ empty triangles.*

Outline. In Section 2, we introduce some properties of generalized twisted drawings and empty triangles in general simple drawings. Then, in Section 3, we show several results about empty triangles in generalized twisted drawings, which we finally put together to obtain a proof of Theorem 1.

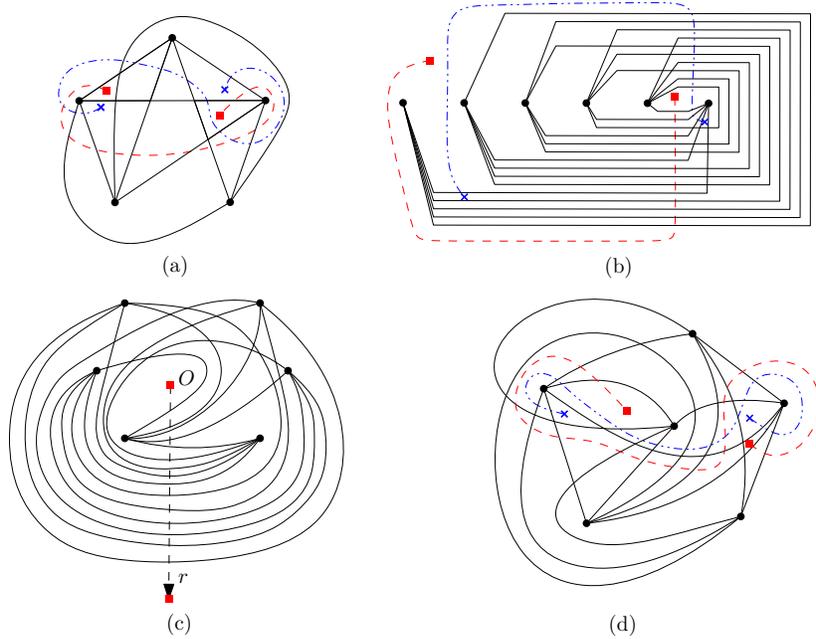


Fig. 1: All (up to weak isomorphism) generalized twisted drawings of K_5 and K_6 . O and Z have to lie in cells marked with red squares or in cells with blue crosses. Curves OZ are drawn dashed or dash-dotted (and as ray in c).

2 Preliminaries

Two simple drawings are weakly isomorphic if the same pairs of edges cross. It is well-known that the weak isomorphism class of a drawing of K_n completely determines which triangles are empty. To prioritize readability, several of our figures show drawings that are weakly isomorphic to generalized twisted (sub-)drawings rather than a generalized twisted drawing.

For a generalized twisted drawing D of K_n , we put a point Z into the unbounded cell of D , on the ray r that crosses everything. Similarly, for every drawing that is weakly isomorphic to a generalized twisted drawing, there exists a simple curve OZ corresponding to the part of the ray r from O to Z ; see [3]. Note that, given a simple drawing D of K_n , there might be several cell pairs where O and Z could be placed such that D is weakly isomorphic to a generalized twisted drawing with the corresponding cells for O and Z . For instance, Fig. 1 shows all [3] generalized twisted drawings of K_5 and K_6 up to weak isomorphism, together with all possible cell pairs for O and Z and some curve OZ for each pair. With this addition of O and Z , we will use the following properties of generalized twisted drawings, which have been shown in [3].

Lemma 1 ([3]). *Let D be a simple drawing in the plane that is weakly isomorphic to a generalized twisted drawing of K_n . Then the following holds:*

1. For each triangle of D , the cell containing O and the cell containing Z lie on different sides. In particular, this implies that D does not contain three interior-disjoint triangles.
2. The cells containing O and Z each have at least one vertex on their boundary.
3. Every subdrawing of D induced by four vertices contains a crossing. If p is such a crossing, then O and Z lie in two different cells that are incident to p and opposite to each other (see Fig. 2a for an illustration).

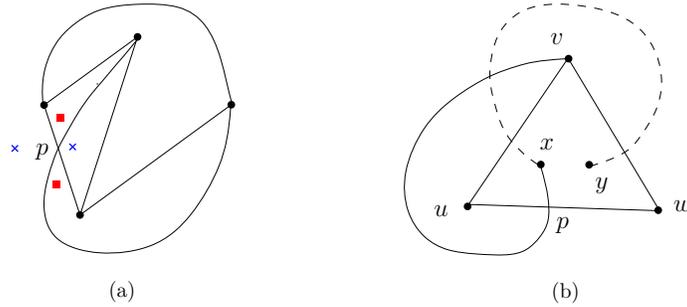


Fig. 2: (a) O and Z have to lie in cells marked with red squares or in cells with blue crosses. (b) The edge (x, y) cannot cross (v, u) and (v, w) simultaneously.

We will also use the following technical lemma for simple drawings.

Lemma 2. *Let D be a simple drawing of K_n . Let Δ be a triangle of D with vertices u, v, w . Let x, y be two vertices on the same side of Δ . If the edge (x, v) crosses (u, w) , then the edge (x, y) can cross at most one of (v, u) and (v, w) .*

Proof. Assume that (x, y) crosses both (v, u) and (v, w) . Since x and y are on the same side of Δ , the edge (x, y) must cross the boundary of Δ an even number of times. Thus, if (x, y) crosses (v, u) and (v, w) , it cannot cross (u, w) . Let p be the crossing point between (x, v) and (u, w) . See Fig. 2b for an illustration. Suppose that (x, y) crosses first (v, u) and then (v, w) . After crossing (v, u) , the edge (x, y) and the vertex y are in different regions defined by the closed curve C consisting of (v, u) , the part of the edge (u, w) from u to p and the part of the edge (x, v) from p to v . Then, after crossing (v, u) , the edge (x, y) must cross C to reach y , and this is not possible without violating the simplicity of the drawing. Therefore, (x, y) cannot cross (v, u) and (v, w) simultaneously. An analogous analysis can be done if (x, y) crosses first (v, w) and then (v, u) .

In addition to the properties of generalized twisted drawings, we will use the concept of star triangles as introduced in [4]. A triangle Δ with vertices x, y, z is a *star triangle* at x if yz is not crossed by any edges incident to x . We will use the following properties of star triangles in simple drawings of K_n .

Lemma 3. *Let D be a simple drawing of K_n in the plane and x be a vertex of D . Then the following holds:*

1. *There are at least two empty star triangles at x .*
2. *A star triangle xyz at a vertex x is an empty triangle if and only if the vertices y and z are consecutive in the rotation around x .*
3. *For any two different empty star triangles at x , $xy'z'$ and xyz , the empty sides of $xy'z'$ and xyz are disjoint.*

Proof. Properties 1 and 2 have been shown in [4]. To prove Property 3, consider the boundary edges of the triangles xyz and $xy'z'$. Of these edges, the only pair that could cross is (y, z) and (y', z') . However, y' and z' lie on the same side of the triangle xyz , so (y', z') has to cross the boundary of xyz an even number of times, which is not possible if exactly (y, z) and (y', z') cross. Thus, no edges on the boundary of the star triangles cross, and therefore their empty sides are disjoint.

3 Proof of Theorem 1

In the following, we derive several lemmata about empty triangles in generalized twisted drawings. These lemmata put together will give the proof of Theorem 1.

Lemma 4. *Let D be a generalized twisted drawing of K_n in the plane with $n \geq 4$ and x be a vertex of D . Then x is incident to exactly two empty star triangles, one has O on the empty side and the other has Z on the empty side. Further, these star triangles have disjoint empty sides.*

Proof. By Lemma 3(1 and 3), for every vertex x there are at least two empty star triangles at x and the empty sides of these triangles are disjoint. By Lemma 1(1), any triangle of a generalized twisted drawing has O on one side and Z on the other side, and D cannot contain three interior-disjoint triangles. Thus, for any vertex x in a generalized twisted drawing it holds that: (i) one empty star triangle at x has O on the empty side, (ii) another empty star triangle at x has Z on the empty side, and (iii) there cannot be a third empty star triangle at x .

Lemma 5. *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Let C_O be the cell of D containing O and let v be a vertex on the boundary of C_O . Let Δ be an empty triangle in D that has O on the empty side. Then the following holds:*

1. *The vertex v is a vertex of Δ , that is, $\Delta = xyv$ for some x, y .*
2. *The triangle $\Delta = xyv$ is an empty star triangle at x or y or both.*
3. *If $\Delta = xyv$ is a star triangle at both x and y , then all edges emanating from v cross (x, y) . Hence, Δ is a star triangle for at most two of its vertices.*

Proof. Since Δ has O on its empty side and since C_O is a cell of D , Δ also has C_O on the empty side. Therefore, since v belongs to C_O and Δ is empty, necessarily v must be one of the vertices of Δ and Property 1 is fulfilled.

To prove Property 2, assume to the contrary that $\Delta = xyv$ is not a star triangle at x or y . Then at least one edge (x, x') must cross (v, y) at a point q' and at least an edge (y, y') must cross (v, x) at a point q (see Fig. 3a for an illustration). Note that since Δ is empty and any edge incident to x or y can cross Δ at most once, (x, x') and (y, y') must emanate from x and y , respectively, at the empty side of Δ , and cross at a point p on that side of Δ . Without loss of generality, we may assume that O is very close to v . Consider the subdrawing D' induced by x, x', y and y' . Observe that since (x, y') cannot cross (x, v) or (y, y') , the cell of D' defined by x, p and y' cannot contain O , regardless of the shape of (x, y') . Thus, by Lemma 1(3) applied to D' , O is in the region defined by x', p and y' , and Z is in the region defined by x, p and y , contradicting that O and Z lie on different sides of Δ .

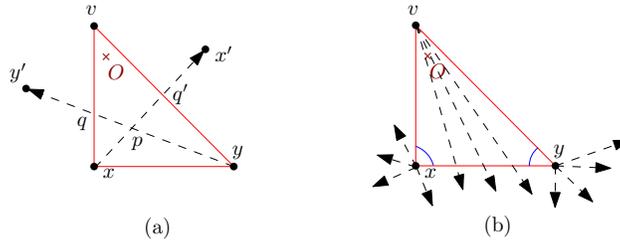


Fig. 3: (a) Illustrating the proof of Lemma 5(2). (b) Any empty triangle of D that has O on the empty side cannot be a star triangle at three vertices.

To prove Property 3, take a vertex w that is not a vertex of Δ . By Lemma 1(3), the subdrawing induced by v, x, y and w has a crossing. As Δ is a star triangle at x and y , any edge incident to x or y emanates from x or y on the non-empty side of Δ , so neither (x, w) nor (y, w) can cross Δ . Then (v, w) must cross (x, y) , emanating from v on the empty side of Δ . Therefore, Property 3 follows.

Note that by Lemma 1(2), the cell containing O always has a vertex on its boundary. Hence, by Lemma 5, any empty triangle with O on the empty side is a star triangle at one or two vertices. The following lemma proves that there are exactly two such triangles that are star triangles at two vertices.

Lemma 6. *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Then D contains exactly two empty triangles with O on the empty side that are star triangles at two vertices.*

Proof. Let C_O be the cell containing O and v be a vertex on the boundary of C_O , which exists by Lemma 1(2). By Lemma 4, there is an empty star triangle $\Delta = vuv$ at v that has C_O on the empty side. By Lemma 5, Δ is a star triangle at exactly one of u or w , say w . Thus, Δ is an empty star triangle at two vertices with O on the empty side.

On the other hand, by Lemma 5(3), all edges emanating from u cross (v, w) . Among all edges crossing (v, w) , we choose the edge (a, b) that crosses (v, w) closest to v . Let p be the crossing point between (a, b) and (v, w) , and consider the triangle vab . We will show that the triangle vab is a star triangle at a and b and that the side F of it in which O lies is empty.

If $a \neq u$, then (a, b) crosses (v, u) at a point q ; see Fig. 4. Note that F is partitioned into three triangular shapes vqa , vqp , and vpb , where O lies in vqp . Assume, for a contradiction, that a vertex x lies in vbp . Since (a, x) can cross neither (a, b) nor (v, p) , it has to cross (v, b) . As no simple drawing of the K_4 can contain more than one crossing, the edges xv and xb have to stay completely in F . Since (v, p) is crossing-free, the edges (x, v) and (x, b) must be in vbp , one side of the triangle vxb is contained in vpb . As O is not on the side of vxb contained in F , then Z has to be on that side. This implies that both O and Z are in F , a contradiction. Therefore, vbp is empty. Using a similar argument, one can prove that vqa is also empty, so F is empty. Besides, any edge incident to a or b must emanate outside F because (v, p) is crossing free. As a consequence, F is the empty side of the triangle vab , which is a star triangle at a and b .

The reasoning for $a = u$ is similar (with two triangular shapes vup and vpb).

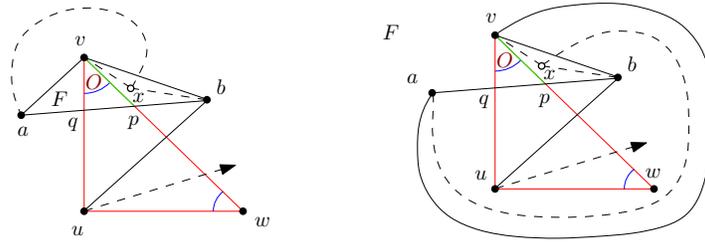


Fig. 4: Illustrating the proof of Lemma 6.

What remains to show is that there is no third empty triangle with O on the empty side that is a star triangle at two vertices. Assume for a contradiction that such a triangle $x'y'z'$ exists. By Lemma 5(1), one of x', y' and z' must be v , say $z' = v$. As v, a , and b are incident to at most one empty star triangle with O on the empty side, by Lemma 4, x' and y' are different from a and b , and $vx'y'$ is a star triangle at x' and y' . Consider the triangle vab . Since x' and y' are on the same side of vab and (v, x') crosses (a, b) by Lemma 5(3), the edge (x', y') cannot cross (v, a) and (v, b) by Lemma 2. But all edges emanating from v must cross (x', y') by Lemma 5(3), a contradiction. Hence $x'y'z'$ cannot exist.

We note that the lemmata above and their proofs hold for every choice of C_O (if there are many) and any vertex v on the boundary C_O . However, whether a triangle is empty and at how many vertices it is a star triangle does not change between weakly isomorphic drawings. As a consequence, the empty star triangles obtained in the previous lemmata and proofs must be the same, regardless of

the choice of C_O and the vertex v on the boundary of C_O . We also note that for empty triangles having Z on the empty side, the reasoning in Lemmas 5 and 6 works to prove that these triangles are star triangles at one or two vertices, and that exactly two of them are star triangles at two vertices. By Lemma 6 and these observations, we get the following lemma.

Lemma 7. *Let D be a generalized twisted drawing of K_n with $n \geq 4$. Then D contains exactly four empty triangles that are star triangles at two vertices.*

Now, we can prove our main theorem.

Proof (of Theorem 1). When summing up the number of empty star triangles over all vertices, we obtain $2n$ empty star triangles by Lemma 4 (n triangles with O on the empty side and n with Z on the empty side). By Lemma 5, all empty triangles have been counted this way, but the triangles that are empty star triangles at two vertices have been counted twice. By Lemma 7, there are exactly four triangles that are empty star triangles at two vertices. Thus, there are exactly four triangles that have been counted exactly twice and the precise number of empty triangles in D is $2n - 4$.

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