Transitivity on subclasses of chordal graphs

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Abstract

Let G=(V,E) be a graph, where V and E are the vertex and edge sets, respectively. For two disjoint subsets A and B of V, we say A dominates B if every vertex of B is adjacent to at least one vertex of A in G. A vertex partition $\pi=\{V_1,V_2,\ldots,V_k\}$ of G is called a transitive k-partition if V_i dominates V_j for all i,j, where $1\leq i< j\leq k$. The maximum integer k for which the above partition exists is called transitivity of G and it is denoted by Tr(G). The Maximum Transitivity Problem is to find a transitive partition of a given graph with the maximum number of partitions. It was known that the decision version of Maximum Transitivity Problem is NP-complete for chordal graphs [Iterated colorings of graphs, Discrete Mathematics, 278, 2004]. In this paper, we first prove that this problem can be solved in linear time for split graphs and for the complement of bipartite chain graphs, two subclasses of chordal graphs. We also discuss Nordhaus-Gaddum type relations for transitivity and provide counterexamples for an open problem posed by J. T. Hedetniemi and S. T. Hedetniemi [The transitivity of a graph, J. Combin. Math. Combin. Comput, 104, 2018]. Finally, we characterize transitively critical graphs having fixed transitivity.

Keywords. Transitivity, Split graphs, Complement of bipartite chain graphs, Nordhaus-Gaddum relations, Transitively critical graphs.

1 Introduction

Graph partitioning is one of the classical problems in graph theory. In a partitioning problem, the goal is to partition the vertex set (or edge set) into some parts with desired properties, such as independence, having minimum edges across partite sets, etc. In this article, we are interested in partitioning the vertex set into some parts such that the partite sets follow some domination relation among themselves. For a graph G = (V, E), the neighbourhood of a vertex $v \in V$ is the set of all adjacent vertices of v and is denoted as $N_G(v)$. The degree of a vertex v in G, denoted as $\deg_G(v)$, is the number of edges incident to v. A vertex v is said to dominate itself and all its neighbouring vertices. A dominating set of G = (V, E) is a subset of vertices D such that every vertex $x \in V \setminus D$ has a neighbour $y \in D$, that is, x is dominated by some vertex y of D. For two disjoint subsets A and B of V, we say A dominates B if every vertex of B is adjacent to at least one vertex of A.

Graph partitioning problems, based on a domination relation among the partite sets, have been extensively studied in literature. Cockayne and Hedetniemi, in 1977, introduced the notion of domatic partition of a graph G = (V, E), where the vertex set is partitioned into k parts, $\pi = \{V_1, V_2, \dots, V_k\}$, such that each V_i is a dominating set of G [CH77]. The maximum order of such a domatic partition is called domatic number of G and it is denoted by d(G). Another similar type of partitioning problem is the Grundy partition. Christen and Selkow introduced a Grundy partition of a graph G = (V, E)in 1979 [CS79]. In the Grundy partitioning problem, the vertex set is partitioned into k parts, $\pi =$ $\{V_1, V_2, \dots, V_k\}$, such that each V_i is an independent set and for all $1 \leq i < j \leq k$, V_i dominates V_i . The maximum order of such a partition is called the Grundy number of G and it is denoted by $\Gamma(G)$. In 2004, Hedetniemi et al. introduced another such partitioning problem, namely upper iterated domination partition [EHLP03]. In an upper iterated domination partition, the vertex set is partitioned into k parts, $\pi = \{V_1, V_2, \dots, V_k\}$, such that for each $1 \leq i \leq k$, V_i is a minimal dominating set of $G \setminus (\bigcup_{i=1}^{i-1} V_i)$. The upper iterated domination number, denoted by $\Gamma^*(G)$, is equal to the maximum order of such a vertex partition. Recently, in 2018, Haynes et al. generalized the idea of domatic partition and introduced the concept of upper domatic partition of a graph G, where the vertex set is partitioned into k parts, $\pi = \{V_1, V_2, \dots, V_k\}$, such that for each i, j, with $1 \le i < j \le k$, either V_i dominates V_j or V_j dominates V_i or both [HHH⁺20]. The maximum order of such an upper domatic partition is called upper

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domatic number of G and it is denoted by D(G). All these problems, domatic number [Cha94], Grundy number [HHB82, Zak06], upper iterated number [EHLP03], upper domatic number [HHH⁺20] have been extensively studied both from an algorithmic and structural point of view.

In this article, we study a similar graph partitioning problem, namely transitive partition. In 2018, Hedetniemi et al. [HH18] have introduced this notion as a generalization of Grundy partition. A transitive k-partition is defined as a partition of the vertex set into k parts, $\pi = \{V_1, V_2, \ldots, V_k\}$, such that for all $1 \leq i < j \leq k$, V_i dominates V_j . The maximum order of such a transitive partition is called transitivity of G and is denoted by Tr(G). The MAXIMUM TRANSITIVITY PROBLEM (MTP) is to find a transitive partition of a given graph with the maximum number of parts. Note that a Grundy partition is a transitive partition with the additional restriction that each partite set must be independent. In a domatic partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G, since each partite set is a dominating set of G, we have domination property in both directions, that is, V_i dominates V_j and V_j dominates V_i for all $1 \leq i < j \leq k$. However, in a transitive partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G, we have domination property in one direction, that is, V_i dominates V_j for $1 \leq i < j \leq k$. In an upper domatic partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G, for all $1 \leq i < j \leq k$, either V_i dominates V_j or V_j dominates V_i or both. The definition of each vertex partitioning problem ensures the following inequalities for any graph G. For any graph G, $1 \leq \Gamma(G) \leq \Gamma^*(G) \leq T(G) \leq D(G) \leq n$.

In the introductory paper, J. T. Hedetniemi and S. T. Hedetniemi [HH18] showed, that the upper bound on the transitivity of a graph G is $\Delta(G)+1$, where $\Delta(G)$ is the maximum degree of G. They also gave two characterizations for graphs with Tr(G)=2 and for graphs with $Tr(G)\geq 3$. They further showed that transitivity and Grundy number are the same for trees. Therefore, the linear-time algorithm for finding the Grundy number of a tree, presented in [HHB82], implies that we can find the transitivity of a tree in linear time as well. Also, for a subclass of bipartite graphs, namely bipartite chain graphs, MTP can be solved in linear time [PS22]. Moreover, for any graph, transitivity is equal to upper iterated domination number, that is, $\Gamma^*(G) = Tr(G)$ [HH18], and the decision version of the upper iterated domination problem is known to be NP-complete for chordal graphs [HHM+04]. Therefore, MTDP is NP-complete for chordal graphs as well. MTDP is also known to be NP-complete for perfect elimination bipartite graphs [PS22]. It is also known that every connected graph G with $Tr(G) = k \geq 3$ has a transitive partition $\pi = \{V_1, V_2, \ldots, V_k\}$ such that $|V_k| = |V_{k-1}| = 1$ and $|V_{k-i}| \leq 2^{i-1}$ for $2 \leq i \leq k-2$ [HHH+19]. This implies that MTP is fixed-parameter tractable [HHH+19]. Also, graphs with transitivity at least t, for some integer t, have been characterized in [PS22].

In this article, we study the computational complexity of the transitivity problem in subclasses of chordal graphs. The organization and main contributions of this article are summarized as follows. Section 2 contains basic definitions and notations that are followed throughout the article. Sections 3 and 4 describe two linear-time algorithms for split and for the complement of bipartite chain graphs, respectively. Section 5 deals with Nordhaus-Gaddum type relations for transitivity. In Section 6, we present a characterization of transitively vertex-edge critical graphs having fixed transitivity. Finally, Section 7 concludes the article.

2 Notation and definition

Let G=(V,E) be a graph with V and E as its vertex and edge sets, respectively. A graph H=(V',E') is said to be a subgraph of a graph G=(V,E), if and only if $V'\subseteq V$ and $E'\subseteq E$. For a subset $S\subseteq V$, the induced subgraph on S of G is defined as the subgraph of G whose vertex set is S and edge set consists of all of the edges in E that have both endpoints in S and it is denoted by G[S]. The complement of a graph G=(V,E) is the graph $\overline{G}=(\overline{V},\overline{E})$, such that $\overline{V}=V$ and $\overline{E}=\{uv|uv\notin E\}$.

A subset of $S \subseteq V$, is said to be an *independent set* of G, if every pair of vertices in S are non-adjacent. A subset of $K \subseteq V$, is said to be a *clique* of G, if every pair of vertices in K are adjacent. The cardinality of a clique of maximum size is called *clique number* of G and it is denoted by $\omega(G)$. A graph G = (V, E) is said to be a *split graph* if V can be partitioned into an independent set S and a clique K.

A graph is called bipartite if its vertex set can be partitioned into two independent sets. A bipartite graph $G = (X \cup Y, E)$ is called a bipartite chain graph if there exists an ordering of vertices of X and Y, say $\sigma_X = (x_1, x_2, \ldots, x_{n_1})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_2})$, such that $N(x_{n_1}) \subseteq N(x_{n_1-1}) \subseteq \ldots \subseteq N(x_2) \subseteq N(x_1)$ and $N(y_{n_2}) \subseteq N(y_{n_2-1}) \subseteq \ldots \subseteq N(y_2) \subseteq N(y_1)$. Such ordering of X and Y is called a chain ordering and it can be computed in linear time [HK07]. A graph G is said to be a $2K_2$ -free, if it does not contain a pair of independent edges as an induced subgraph. It is well-known that the class of bipartite chain graphs and $2K_2$ -free bipartite graphs are the same. An edge between two non-consecutive vertices of a

cycle is called a *chord*. If every cycle in G of length at least four has a chord, then G is called a *chordal* graph.

3 Transitivity in split graphs

In this section, we design a linear-time algorithm for finding the transitivity of a given split graph. To design the algorithm, we first prove that the transitivity of a split graph G can be either $\omega(G)$ or $\omega(G) + 1$, where $\omega(G)$ is the size of a maximum clique in G. Further, we characterize the split graphs with the transitivity equal to $\omega(G) + 1$.

Lemma 1. Let $G = (S \cup K, E)$ be a split graph, where S and K are an independent set and a clique of G, respectively. Also, assume that K is the maximum clique of G, that is, $\omega(G) = |K|$. Then $\omega(G) \leq Tr(G) \leq \omega(G) + 1$. Further, $Tr(G) = \omega(G) + 1$ if and only if every vertex of K has a neighbour in S.

Proof. Note that $Tr(G) \geq \omega(G)$. As we can make a transitive partition $\pi = \{V_1, V_2, \dots, V_{\omega(G)}\}$ of size $\omega(G)$ by considering each V_i contains exactly one vertex from maximum clique and all the other vertices in V_1 . To prove that $Tr(G) \leq \omega(G) + 1$, suppose $Tr(G) \geq \omega(G) + 2$. Let $\pi = \{V_1, V_2, \dots, V_{\omega(G)+2}\}$ be a transitive partition of G. Since $|K| = \omega(G)$, there exist at least two sets in π , say V_i and V_j with i < j, such that V_i and V_j contains only vertices from S. Note that, in this case V_i cannot dominate V_j as S is an independent set of G. Therefore, we have a contradiction. Hence, $\omega(G) \leq Tr(G) \leq \omega(G) + 1$.

Let every vertex of K have a neighbour in S. Now consider a vertex partition of G, say $\pi = \{V_1, V_2, \ldots, V_{\omega(G)+1}\}$, such that $V_1 = S$ and for each i > 1, V_i contains exactly one vertex from K. Since every vertex of K has a neighbour in S, V_1 dominates every other partition in π . Moreover, as K is a clique, each V_i , with i > 1 dominates V_j for all $2 \le i < j \le \omega + 1$. Hence, π is a transitive partition of G. Now, since $Tr(G) \le \omega(G) + 1$, we have $Tr(G) = \omega(G) + 1$.

Conversely, let $Tr(G) = \omega(G) + 1$ and $\pi = \{V_1, V_2, \dots, V_{\omega(G)+1}\}$ be a transitive partition of G. Note that if there exist two sets in π , that contain only vertices from S, then using similar arguments as before, we have a contradiction. Therefore, there exists exactly one set in π , say V_l , that contains only vertices from S as $\omega(G) = |K|$. Hence, each set of π , except V_l , contains exactly one vertex from K. Suppose there exists a vertex, say x, in K that has no neighbour in S and also let $x \in V_p$ for some set V_p in π . Note that there is no edge between the vertices of V_p and V_l . Therefore, neither V_p dominates V_l nor V_l dominates V_p . This contradicts the fact that π is a transitive partition. Hence, every vertex of K has a neighbour in S.

Based on the above lemma, we have the following algorithm for finding transitivity of a given split graph.

Algorithm 1 Transitivity_Split(G)

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Input: A split graph G = (V, E)
Output: The transitivity of G, that is, Tr(G)
Find a vertex partition of V into S and K, where S and K are an independent set and a clique of G, respectively and ω(G) = |K|.
for all v ∈ K do
if v has no neighbour in S then
t = |K|.
break
end if
end for
t = |K| + 1.
return (t)
```

Note that the required vertex partition in line 3 of Algorithm 1 can be computed in linear time [HS81]. Also, the for loop in line 4-7 runs in O(n+m) time. Hence, we have the following theorem:

Theorem 2. The Maximum Transitivity Problem can be solved in linear time for split graphs.

4 Transitivity in the complement of bipartite chain graphs

In this section, we find the transitivity of the complement of a bipartite chain graph, say G, by showing that the transitivity of G is equal to the Grundy number of G. To do that, first, we show two essential properties, one of the transitive partitions and the second of the Grundy partitions for the complement of a bipartite chain graph. The proofs of these lemmas can be found in the Appendix.

Lemma 3. Let $G = (X \cup Y, E)$ be the complement of a bipartite chain graph and Tr(G) = k. Then there exists a transitive partition $\pi = \{V_1, V_2, \dots, V_k\}$ of G such that either $|V_1| = 1$ or $V_1 = \{x, y\}$, where $x \in X$ and $y \in Y$.

Proof. Let $\pi = \{V_1, V_2, \dots, V_k\}$ be a transitive partition of G. If either $|V_1| = 1$ or $V_1 = \{x, y\}$, we are done. Otherwise, exactly one of the following two cases happen:

Case 1. $|V_1| > 2$ and V_1 contains vertices from X and Y

In this case, we can construct another transitive partition $\pi' = \{V'_1, V'_2, \dots, V'_k\}$ as follows: we retain only two vertices in V'_1 , one from X, say x, and another from Y, say y. After that, we put every other vertex of V_1 into V'_2 . Therefore, $V'_1 = \{x, y\}$ for some $x \in X$ and $y \in Y$, $V'_2 = V_2 \cup (V_1 \setminus \{x, y\})$ and $V'_1 = V_2$, for all $i \geq 3$. Note that V_1 dominates every other set in π' as both X and Y induces complete graph in G. Hence, π' is a transitive partition of G.

Case 2. $|V_1| \ge 2$ and V_1 contains only vertices from X or Y

Without loss of generality, let us assume that $V_1 = \{x_{t_1}, x_{t_2}, \dots, x_{t_s}\}$ contains only vertices from $X = \{x_1, x_2, \dots, x_{n_1}\}$. In this case, we show that there exists a vertex in V_1 which is adjacent with every vertex of $Y = \{y_1, y_2, \dots, y_{n_2}\}$. Let x_{t_i} be a vertex in V_1 which has maximum number of neighbours in Y. Let $Y_{t_i} = \{y_{p_1}, y_{p_2}, \dots, y_{p_{s'}}\}$ be the neighbours of x_{t_i} in Y. Suppose $|Y_{t_i}| < |Y|$. Consider any vertex from $(Y \setminus Y_{t_i})$, say y. Since y is not adjacent to x_{t_i} and π is a transitive partition of G, y must be adjacent with some vertex, say x_{t_j} , in V_1 other than x_{t_i} . Now, consider an arbitrary vertex $y' \in Y_{t_i}$. Note that $x_{t_j}y' \in E$ because otherwise $\{x_{t_i}, y, x_{t_j}, y'\}$ induces a $2K_2$ in \overline{G} (see Fig. 1). This contradicts the fact that \overline{G} is a bipartite chain graph. Now, since y' is chosen arbitrarily from Y_{t_i} , we can say that x_{t_j} is adjacent to every vertex of Y_{t_i} . This contradicts the fact that x_{t_i} is a vertex in V_1 which has maximum number of neighbours in Y. Hence, there exists a vertex, say x, in V_1 which is adjacent with every vertex of Y. We consider the partition $\pi' = \{V_1', V_2', \dots, V_k'\}$, where $V_1' = \{x\}, V_2' = V_2 \cup (V_1 \setminus \{x\})$ and $V_i' = V_i$, for all $i \geq 3$. Clearly, π' is a transitive partition of G.

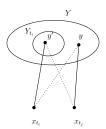


Figure 1: $2K_2$ in \overline{G}

Hence, we can always construct a transitive partition $\pi = \{V_1, V_2, \dots, V_k\}$ of G such that either $|V_1| = 1$ or $V_1 = \{x, y\}$, where $x \in X$ and $y \in Y$.

Lemma 4. Let $G = (X \cup Y, E)$ be the complement of a bipartite chain graph. Also, let $\pi = \{V_1, V_2, \dots, V_k\}$ be a Grundy partition of G with $\Gamma(G) = k$ and $X'_G = \{x \in X | x \in V_i \text{ and } |V_i| = 1\}$ and $Y'_G = \{y \in Y | y \in V_i \text{ and } |V_i| = 1\}$. Then exactly one of the following two cases is true:

- (i) Both $|X'_G|$ and $|Y'_G|$ cannot be empty simultaneously.
- (ii) The graph G is the disjoint union of $K_{|X|}$ and $K_{|Y|}$ and |X| = |Y|.

Proof. As π is a Grundy partition of the complement of a bipartite graph, then every set in π is either size two or size one. If $|X| \neq |Y|$, then we always have either $|X'_G| \neq \phi$ or $|Y'_G| \neq \phi$. So, we assume that |X| = |Y|. Now, if either $|X'_G| \neq \phi$ or $|Y'_G| \neq \phi$, then we are done. Hence, let us assume that $|X'_G| = \phi$ and $|Y'_G| = \phi$. Note that, in this case, each set in π contains exactly one vertex from X and another

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vertex from Y, because each set has to be an independent set. Let for all $1 \le i \le k$, $V_i = \{x_i, y_i\}$. Consider the following cases:

Case 1. There exists $V_p, V_q \in \pi$ such that $x_p y_q \in E$ and $x_q y_p \in E$

In this case, $\{x_p, y_p, x_q, y_q\}$ forms a $2K_2$ in \overline{G} , since V_p and V_q are independent sets. This contradicts the fact that \overline{G} is a bipartite chain graph.

Case 2. There exists $V_p, V_q \in \pi$ such that $x_p y_q \in E$ and $x_q y_p \notin E$

Without loss of generality, let p < q. In this case, we form a new partition $\pi' = \{V_1', V_2', \dots, V_{k+1}'\}$ as follows: $V_i' = V_i$ for all $1 \le i \le (q-1)$ and $i \ne p$, $V_p' = \{x_q, y_p\}$, $V_j' = V_{j+1}$ for $q \le j \le (k-1)$, $V_k' = \{x_p\}$ and $V_{k+1}' = \{y_q\}$. Note that for all $1 \le i \le (k-1)$, as V_i' contains both vertices from X and Y, clearly, V_i' dominates every other partition in π' . Also, since $x_p y_q \in E$, V_k' dominates V_{k+1}' . Therefore, π' is a Grundy partition which is a contradiction to the fact that $\Gamma(G) = k$.

Case 3. There exists $V_p, V_q \in \pi$ such that $x_p y_q \notin E$ and $x_q y_p \in E$

Similar to the previous case.

Case 4. For every pair of sets $V_p, V_q \in \pi$, $x_p y_q \notin E$ and $x_q y_p \notin E$

In this case, there is no edge between X and Y in G. Hence, G is the disjoint union of $K_{|X|}$ and $K_{|Y|}$. Hence, we have the above lemma.

Now, we are ready to show that the transitivity and the Grundy number are equal for the complement of a bipartite chain graph.

Theorem 5. Let $G = (X \cup Y, E)$ be the complement of a bipartite chain graph. Then $\Gamma(G) = Tr(G)$.

Proof. We use induction on n, where n is the number of vertices of G. If n=1, then $\Gamma(G)=Tr(G)=1$ trivially. For n=2, G is either K_2 or \overline{K}_2 and therefore, $\Gamma(G)=Tr(G)$. Let us assume that the induction hypothesis is true, that is, $\Gamma(G) = Tr(G)$ for the complement of every bipartite chain graph having less than n vertices. Let us consider a transitive partition $\pi = \{V_1, V_2, \dots, V_k\}$ of G with Tr(G) = k. By Lemma 3, we can assume that $|V_1| = 1$ or $V_1 = \{x, y\}$ for some $x \in X$ and $y \in Y$. Let $H = G \setminus V_1$. Note that H is also the complement of a bipartite chain graph, since deleting a vertex from X (or Y) does not change the chain ordering of the remaining vertices. By induction hypothesis, we have $\Gamma(H) = Tr(H)$. Moreover, note that Tr(H) = k - 1. Hence, we have $\Gamma(H) = Tr(H) = k - 1$. Let $\pi' = \{V'_1, V'_2, \dots, V'_{k-1}\}$ be a Grundy partition of H. Now, if $V_1 = \{x\}$ (or $\{y\}$), then x (correspondingly y) is adjacent to every vertex of G because π is a transitive partition of G. Therefore, $\pi'' = \{V_1, V_1', V_2', \dots, V_{k-1}'\}$ forms a Grundy partition of G which implies $\Gamma(G) \geq k = Tr(G)$. Also, for any graph we know that $\Gamma(G) \leq Tr(G)$, hence $\Gamma(G) = Tr(G)$. So, let us assume that $V_1 = \{x, y\}$ for some $x \in X$ and $y \in Y$. Now, if $xy \notin E$, that is, V_1 is an independent set, then from a Grundy partition of H of order (k-1) we can construct a Grundy partition of G of order k by appending V_1 . Then by similar argument, we have $\Gamma(G) = Tr(G)$. So, we assume that $xy \in E$. Since, $H = G \setminus V_1$ is the complement of a bipartite chain graph, by induction hypothesis $\Gamma(H) = Tr(H)$. Now, by Lemma 4, we can assume that H has a Grundy partition, say $\pi' = \{V'_1, V'_2, \dots, V'_{k-1}\}$, such that either $|X'_H| \neq \phi$ or $|Y'_H| \neq \phi$, where X'_H and Y'_H is defined in a similar way as in Lemma 4 or H is the disjoint union of $K_{|X_H|}$ and $K_{|Y_H|}$ and $|X_H| = |Y_H|$, where $X_H = X \cap H$ and $Y_H = Y \cap H$. If H is the disjoint union of $K_{|X_H|}$ and $K_{|Y_H|}$ and $|X_H| = |Y_H|$, then $\pi'' = \{V'_1, V'_2, \dots, V'_{k-1}, \{x\}, \{y\}\}\$ forms a transitive partition of G of order (k+1). This is a contradiction to the fact that Tr(G) = k. So, we assume that H has a Grundy partition, say $\pi' = \{V'_1, V'_2, \dots, V'_{k-1}\}$, such that either $|X'_H| \neq \phi$ or $|Y'_H| \neq \phi$. The remaining proof is done by dividing into the following four

Case 1. Every vertex of X'_H and Y'_H are adjacent to y and x, respectively

In this case, consider the vertex partition $\pi'' = \{V'_1, V'_2, \dots, V'_{k-1}, \{x\}, \{y\}\}$. Clearly, π'' is a transitive partition of G of order (k+1). This is a contradiction to the fact that Tr(G) = k.

Case 2. Every vertex of X'_H is adjacent to y but there exists a vertex $y_t \in Y'_H$ such that $xy_t \notin E$

Let $y_t \in V_p'$. In this case, let us consider the vertex partition $\pi'' = \{U_1, U_2, \dots, U_k\}$, where $U_1 = \{x, y_t\}$, $U_i = V_{i-1}'$, for $2 \le i \le p$, $U_{p+1} = \{y\}$ and $U_j = V_{j-1}'$, for all $p+2 \le j \le k$. Clearly, the partition π'' forms a Grundy partition of G which implies $\Gamma(G) \ge k = Tr(G)$. Since, for any graph $\Gamma(G) \le Tr(G)$, therefore, $\Gamma(G) = Tr(G)$.

Case 3. There exists a vertex $x_s \in X'_H$ such that $yx_s \notin E$ but every vertex of Y'_H is adjacent to x. This case is similar to Case 2.

Case 4. There exists a vertex $x_s \in X_H'$ such that $yx_s \notin E$ and there exists a vertex $y_t \in Y_H'$ such that $xy_t \notin E$

In this case, $\{x, y_t, y, x_s\}$ induces a $2K_2$ in \overline{G} . This is a contradiction to the fact that \overline{G} is a bipartite chain graph.

Hence, for the complement of a bipartite chain graph G, $\Gamma(G) = Tr(G)$.

It was proved in [Zak06] that for the complement of a bipartite graph, $\Gamma(G) = n - p$, where n is the number of vertices of G and p is the cardinality of a minimum edge dominating set of \overline{G} . We also know that the minimum edge dominating set of a bipartite chain graph can be computed in linear time [VP20]. Therefore, we have the following corollary:

Corollary 6. The transitivity of the complement of bipartite chain graphs can be computed in linear time.

Remark 1. Identifying graphs with equal transitivity and Grundy number was posed as an open question in [HH18]. Theorem 5 partially answers this question by showing that the complement of bipartite chain graphs form such a graph class.

5 Nordhaus–Gaddum type bounds for transitivity

Let G=(V,E) be a simple graph. A proper k-coloring of G is a function c from V to $\{1,2,\ldots,k\}$ such that $c(u) \neq c(v)$ if and only if $uv \in E$. The minimum value of k for which a proper coloring exists is called *chromatic number* of G and it is denoted by $\chi(G)$. In 1956, Nordhaus and Gaddum [NG56] studied the chromatic number of a graph G and its complement \overline{G} . They established lower and upper bound for the product and the sum of $\chi(G)$ and $\chi(\overline{G})$ in terms of the number of vertices of G. Since then, any bound on the sum or the product of a parameter of a graph G and its complement \overline{G} is called a Nordhaus–Gaddum type inequality. In this section, we study Nordhaus and Gaddum type relations for transitivity.

From [PS22], it is known that for a bipartite chain graph G, Tr(G) = t + 1, where t is the maximum integer such that G contains either $K_{t,t}$ or $K_{t,t} - \{e\}$ as an induced subgraph. Let $\sigma_X = (x_1, x_2, \dots, x_{n_1})$ and $\sigma_Y = (y_1, y_2, \dots, y_{n_2})$ be the chain ordering of G. Because of this chain ordering if $x_p y_p \in E$ for some p, then $\{x_1, x_2, \dots, x_p\}$ and $\{y_1, y_2, \dots, y_p\}$ induces a complete bipartite graph. Therefore, it follows that if j is the maximum index such that $x_j y_j \in E$, then

$$Tr(G) = \begin{cases} j+2 & x_{j+1}y_j, x_jy_{j+1} \in E\\ j+1 & \text{otherwise} \end{cases}$$

We know that for the complement of a bipartite graph G, $\Gamma(G) = n - p$, where n is the number of vertices of G and p is the cardinality of a minimum edge dominating set of \overline{G} [Zak06]. Therefore, from Theorem 5 we have, for a bipartite chain graph G, $Tr(\overline{G}) = n - p$, where n is the number of vertices of G and p is the cardinality of a minimum edge dominating set of G. Also from [VP20], we know that for a bipartite chain graph, p is equal to the maximum index p such that p

Theorem 7. Let $G = (X \cup Y, E)$ be a bipartite chain graph with the ordering $\sigma_X = (x_1, x_2, \dots, x_{n_1})$ and $\sigma_Y = (y_1, y_2, \dots, y_{n_2})$ as its chain ordering, that is, $N(x_{n_1}) \subseteq N(x_{n_1-1}) \subseteq \dots \subseteq N(x_2) \subseteq N(x_1)$ and $N(y_{n_2}) \subseteq N(y_{n_2-1}) \subseteq \dots \subseteq N(y_2) \subseteq N(y_1)$. Also assume that j is the maximum index such that $x_j y_j \in E$. Then,

$$Tr(G) + Tr(\overline{G}) = \begin{cases} n+2 & x_{j+1}y_j, x_jy_{j+1} \in E\\ n+1 & otherwise \end{cases}$$

Based on Lemma 1 and the fact that the complement of a split graph is also a split graph, we have the following Nordhaus-Gaddum relation for transitivity in split graphs, whose proof can be found in Appendix.

Theorem 8. Let $G = (S \cup K, E)$ be a split graph, where S and K are the independent set and clique of G, respectively. Also, assume that K is the maximum clique of G, that is, $\omega(G) = |K|$. Then,

$$Tr(G) + Tr(\overline{G}) = \begin{cases} n+2 & \text{if, in } G, \text{ every vertex of } K \text{ has a neighbour in } S \\ n+1 & \text{otherwise} \end{cases}$$

Proof. Let us assume that, in G, every vertex of K has a neighbour in S. Then by Lemma 1, Tr(G) = |K|+1. Note that, in \overline{G} , S forms a clique. Moreover, none of the vertices in K is adjacent to every vertex of S as, in G, every vertex of K has a neighbour in K. Therefore, $\omega(\overline{G}) = |S|$. Also, note that, in \overline{G} , every vertex of S has a neighbour in K. If not, let us assume that there is a vertex in \overline{G} , say $x \in S$, which has no neighbour in K. Then $(K \cup \{x\})$ forms a clique in G, which contradicts the fact that $\omega(G) = |K|$. Therefore, the complement of G, that is, $\overline{G} = (K \cup S, \overline{E})$ is a split graph, where K is an independent set and S is a clique with $\omega(\overline{G}) = |S|$. Also, in \overline{G} , every vertex of S has a neighbour in K. Hence by Lemma 1, $Tr(\overline{G}) = |S| + 1$. This implies that, in this case,

$$Tr(G) + Tr(\overline{G}) = |K| + 1 + |S| + 1 = n + 2.$$

On the other hand, let us assume that, in G, there exists a vertex, say x, in K which has no neighbour in S. Then by Lemma 1, Tr(G) = |K|. Let $S' = (S \cup \{x\})$ and $K' = K \setminus \{x\}$. Note that, in \overline{G} , S' forms a maximum clique in \overline{G} , that is, $\omega(\overline{G}) = |S'|$. Also, since K forms an independent set in \overline{G} , x is not adjacent to any vertex of K'. Therefore, the complement of G, that is, $\overline{G} = (K' \cup S', \overline{E})$ is a split graph, where K' is an independent set and S' is a clique with $\omega(\overline{G}) = |S'|$. Also, in \overline{G} , there exists a vertex, namely x, in S' which has no neighbour in K'. Hence by Lemma 1, $Tr(\overline{G}) = |S'|$. This implies that, in this case,

$$Tr(G) + Tr(\overline{G}) = |K| + |S'| = |K| + |S| + 1 = n + 1.$$

Hence, the Nordhaus-Gaddum relation holds for split graphs.

Remark 2. In [HH18], Hedetniemi and Hedetniemi posed the following open question about the sum of Tr(G) and $Tr(\overline{G})$: for any graph G, is $Tr(G) + Tr(\overline{G}) = n + 1$ if and only if $G = K_n$ or $G = \overline{K_n}$? Theorem 7 and 8 show the existence of some bipartite chain graph and split graph, respectively, for which $Tr(G) + Tr(\overline{G}) = n + 1$. Moreover, also for $K_{n,n}$, $Tr(G) + Tr(\overline{G}) = 2n + 1$ which shows another counter example for the above mentioned open question.

6 Transitively critical graphs

The concept of transitively critical graph was introduced by Haynes et al. in [HHH⁺19]. A graph G = (V, E) is said to be transitively vertex critical (transitively edge critical) if deleting any vertex from V (respectively, edge from E) results in a graph whose transitivity is less than Tr(G). A transitively vertex critical (transitively edge critical) graph G with Tr(G) = k is called by Tr_k^v -critical (respectively, Tr_k^e -critical). Characterizations of vertex critical graph have been studied in [HHH⁺19]for some small values of k. In this section, we introduce a generalization of transitively critical graphs, namely transitively vertex-edge critical graphs and give characterization of such graphs for every fixed value of k. Using this characterization, we then characterize transitively edge critical graphs for every fixed value of k.

A transitively vertex-edge critical graph is basically a graph which is both transitively vertex and edge critical. The formal definition is as follows:

Definition 9. A graph G = (V, E) is said to be a transitively vertex edge-critical graph if deleting any element from $V \cup E$ results in a graph whose transitivity is less than Tr(G). A transitively vertex edge-critical graph G with Tr(G) = k is called $Tr_k^{(v,e)}$ -critical.

Note that unlike transitively edge critical graphs, every transitively vertex-edge critical graph is connected. The graph K_1 is the only connected graph with Tr(G) = 1 and it is both transitively edge and vertex critical. Therefore, the only $Tr_1^{(v,e)}$ -critical is K_1 .

The following proposition characterizes the $Tr_2^{(v,e)}$ -critical graphs.

Proposition 10. The only $Tr_2^{(v,e)}$ -critical graph is K_2 .

Proof. Clearly, transitivity of K_2 is 2. Also, if we remove any edge or any vertex from K_2 , we are left with \overline{K}_2 or K_1 and transitivity of those graphs is 1. Hence, K_2 is a $Tr_2^{(v,e)}$ -critical.

Let G be a $Tr_2^{(v,e)}$ -critical graph with n vertices. Since Tr(G) = 2, G is a disjoint union of stars which is shown by Hedetniemi et al. [HH18]. As G is a vertex critical graph, which implies G must be a connected graph [HHH⁺19]. Therefore, G is a star. Now, if G contains more than one edge, then removal of that edge from G does not decrease the transitivity, which contradicts the fact that G is a transitively edge-critical. Therefore, G can only be K_2 . Hence, the only $Tr_2^{(v,e)}$ -critical graph is K_2 .

Next, we generalize the characterization for $Tr_k^{(v,e)}$ -critical graphs for $k \geq 3$. To this end, we recall the concept of t-atom, which was introduced by Zaker in [Zak06]. For the sake of completeness, we give the definition of t-atom here.

Definition 11 ([Zak06]). A t-atom is defined in a recursive way as follows:

- 1. The only 1-atom is K_1 .
- 2. Let H = (V, E) be any (t 1)-atom with n vertices. Consider an independent set I_r on r vertices for any $r \in \{1, 2, ... n\}$. For that fixed r, consider a r vertex subset W of V and add a perfect matching between the vertices of I_r and W. Then join an edge between each vertex of $V \setminus W$ and an (and to only one) arbitrary vertex of I_r . The resultant graph G is a t-atom.

The set of t-atoms is denoted by A_t . The following lemma describes the transitively vertex edge-critical graph with transitivity k.

Lemma 12. If G is $Tr_k^{(v,e)}$ -critical, then $G \in \mathcal{A}_k$.

Proof. For an integer t, $Tr(G) \ge t$ if and only if G contains a t-atom as a subgraph, which is shown by Paul and Santra [PS22]. Since the transitivity of G is k, therefore, G contains a k-atom as a subgraph. Let $H \in \mathcal{A}_k$ and G contains H as a subgraph. Since the $Tr(H) \ge k$ and $Tr(G) \ge Tr(H)$, then Tr(H) = k. If G has an edge other than edges of H, then removal of that edge from G does not decrease the transitivity, which contradicts the fact that G is a transitively edge-critical. Also, G cannot contain more vertex than H, as G is a transitively vertex critical graph too. Therefore, G = H.

The only 3-atoms are K_3 and P_4 . Also the graphs K_3 and P_4 are both Tr_3^v -critical and Tr_3^e -critical. Therefore, the converse of Lemma 12 is true for k=3. Hence, we have the following corollary.

Corollary 13. The only $Tr_3^{(v,e)}$ -critical graphs are K_3 or P_4 .

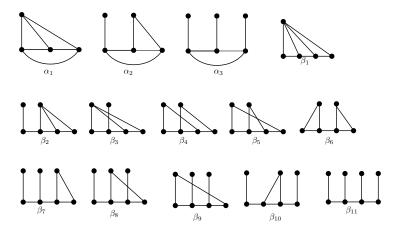


Figure 2: The class \mathcal{A}_4 .

For k=4, the converse of Lemma 12 is not true. The class of graphs \mathcal{A}_4 is illustrated in Figure 2. Note that every graph in \mathcal{A}_4 , has transitivity equal to 4 but only β_2 is not transitively edge-critical. Therefore, we have the following corollary.

Corollary 14. A graph G is $Tr_4^{(v,e)}$ -critical if and only if $G \in \mathcal{A}'_4 = (\mathcal{A}_4 \setminus \{\beta_2\})$.

Generalizing this result, we have the following main theorem.

Theorem 15. Let A_k be the set of all k-atoms and B_k be the set of k-atoms which are neither $Tr_k^{(v,e)}$ -critical nor have transitivity equal to k. A graph G is $Tr_k^{(v,e)}$ -critical if and only if $G \in \mathcal{A}'_k = (\mathcal{A}_k \setminus \mathcal{B}_k)$.

Proof. Let G be a $Tr_k^{(v,e)}$ -critical. Since the transitivity of G is k, then G contains a k-atom as a subgraph, as for an integer t, $Tr(G) \ge t$ if and only if G contains a t-atom as a subgraph, which is shown by Paul and Santra [PS22]. Let H be a k-atom and G contains H as a subgraph. Since the $Tr(H) \ge k$ and $Tr(G) \ge Tr(H)$, then Tr(H) = k. If G has an edge other than edges of H, then removal of that edge from G does not decrease the transitivity, which contradicts the fact that G is a transitively edge critical. Also, G cannot contain more vertex than H, as G is also a transitively vertex critical graph. Hence, $G \in \mathcal{A}_k' = \mathcal{A}_k \setminus \mathcal{B}_k$.

Next, we characterize the Tr_k^e -critical graphs for a fixed value of k. For this characterization, we first show the following relation between $Tr_k^{(v,e)}$ -critical and Tr_k^e -critical graphs. The proof can be found in the Appendix.

Theorem 16. A graph G with n vertices and Tr(G) = k is Tr_k^e -critical if and only if $G = H \cup \overline{K}_{n-n_H}$, where H is a $Tr_k^{(v,e)}$ -critical graph having n_H vertices.

Proof. Let $G = H \cup \overline{K}_{n-n_H}$, where H is a $Tr_k^{(v,e)}$ -critical graph having n_H vertices. Since H is a $Tr_k^{(v,e)}$ -critical, then removal of any edge from H decreases its transitivity. Therefore, G is a transitively edge-critical graph.

Conversely, let G be a Tr_k^e -critical graph. Also, let $G = C_1 \cup C_2 \cup \ldots \cup C_t$, where C_i are the connected components of G. Since $Tr(G) = \max\{Tr(C_i)|1 \leq i \leq t\}$, then we may assume $Tr(G) = Tr(C_1)$. If there is an edge in C_i , for any $i \geq 2$, then removal of that edge from G does not decrease the transitivity, which contradicts the fact that G is a transitively edge critical. Therefore, $C_i = K_1$, for all $i \geq 2$. Next, we prove that C_1 is a $Tr_k^{(v,e)}$ -critical. Clearly, C_1 is a transitively edge critical as G is a transitively edge critical. If C_1 is not a transitively vertex critical, then there exists a vertex in C_1 , say x such that $Tr(C_1) = Tr(C_1 \setminus \{x\}) = k$. Let e be an edge incident to x in C_1 and consider the graph $G' = C_1 \setminus \{x\}$ and $G'' = C_1 \setminus \{e\}$. As, $G' \subseteq G'' \subseteq C_1$, so $k = Tr(G') \leq Tr(G'') \leq Tr(C_1) = k$. Therefore, Tr(G'') = k, which contradicts the fact that G is a transitively edge critical. So, C_1 is a transitively vertex critical graph. Hence, C_1 is a $Tr_k^{(v,e)}$ -critical graph. Therefore, $G = C_1 \cup \overline{K}_{n-nc_1}$.

The characterization of Tr_k^e -critical graphs follows immediately from the above theorem.

Corollary 17. Let A_k be the set of k-atoms and B_k be the set of k-atoms which are neither $Tr_k^{(v,e)}$ -critical nor have transitivity equal to k. A graph G with n vertices, is Tr_k^e -critical if and only if $G = H \cup \overline{K}_{n-n_H}$, where $H \in \mathcal{A}'_k = \mathcal{A}_k \setminus \mathcal{B}_k$.

7 Conclusion

In this paper, we have proved that the transitivity of a given split and the complement of bipartite chain graphs can be computed in linear time. Then, we have discussed Nordhaus-Gaddum type relations for transitivity in split graphs and bipartite chain graphs and have given counter-examples to an open question posed in [HH18]. We have also studied transitively vertex-edge critical graphs. It would be interesting to investigate the complexity status of this problem in other subclasses of chordal graphs. Designing an approximation algorithm for this problem would be another challenging open problem.

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