# <span id="page-0-0"></span>Algorithms that get old : the case of generative deep neural networks

Gabriel Turinici<sup>[0000–0003–2713–006X]</sup>

CEREMADE, Universit´e Paris Dauphine - PSL, CNRS, Paris, France gabriel.turinici@dauphine.fr <https://turinici.com>

Abstract. Generative deep neural networks used in machine learning, like the Variational Auto-Encoders (VAE), and Generative Adversarial Networks (GANs) produce new objects each time when asked to do so with the constraint that the new objects remain similar to some list of examples given as input. However, this behavior is unlike that of human artists that change their style as time goes by and seldom return to the style of the initial creations.

We investigate a situation where VAEs are used to sample from a probability measure described by some empirical dataset.

Based on recent works on Radon-Sobolev statistical distances, we propose a numerical paradigm, to be used in conjunction with a generative algorithm, that satisfies the two following requirements: the objects created do not repeat and evolve to fill the entire target probability distribution.

Keywords: variational auto-encoder · generative adversarial network · statistical distance · vector quantization · deep neural network · measure compression

# 1 Motivation

Consider a distribution  $\mu$  and  $\mu_e = \frac{1}{M} \sum_{\ell=1}^{M} \delta_{x_\ell}$  an empirical measure sampling this distribution given by a collection of objects  $x_m$ ,  $m = 1,..., M$  where  $x_m \in$  $\mathbb{R}^N$  are independent and follow the law  $\mu$ ; in some sense to be defined latter (cf. discussion on statistical distances)  $\mu_e$  is close to  $\mu$ ; we focus on generative deep neural network architectures that, given  $\mu_e$  can produce samples from the distribution  $\mu$ . One such neural network class are the Variational Auto Encoders (cf. [\[8,](#page-7-0)[15\]](#page-8-0) for an introduction) that, after some training, output two functions (that in practice are implemented as neural networks): the encoder function  $E: x \in \mathbb{R}^N \mapsto z \in \mathbb{R}^L$  and the decoder function  $D: z \in \mathbb{R}^L \mapsto y \in \mathbb{R}^N$ ; the decoder function has the property that the image of a multi-dimensional Gaussian on the latent space  $\mathbb{R}^L$  through E is close to  $\mu_e$  thus to  $\mu$ . Some recent proposals to construct such a VAE are presented in [\[16\]](#page-8-1), which will also be our inspiration for the statistical distance used in this work (see also [\[14\]](#page-8-2)). The quality of a VAE is given

- 2 G. Turinici
- 1. by the proximity of the  $D \circ E$  to the identity operator (at least on the support of the target measure  $\mu$ );
- 2. and the small distance between target distribution  $\mu$  and  $D(\mathcal{N}(O, I_L))$  (here  $\mathcal N$  is the *L*-dimensional standard Gaussian);

However, although in general the VAEs (same thing applies to the Generative Adversarial Networks - GANs ) obtain very good quality results by the previous criteria, the sampling performed at the exploitation phase is, because of the construction, done in an independent way: each time a new  $y \sim \mu$  is required, a  $z \simeq \mathcal{N}_L$  is sampled and  $D(z)$  computed. But such a procedure is at odds with what we observe in real life: the painters do not paint the same landscape again (but still paint pictures), the musical composers' productions vary in style over the years, etc, in general some evolution is witnessed with time. Such a phenomena is probably due to taking into account the objects previously created. Our goal is to be able to mimic such an evolution and propose a generative algorithm that

- 1. is able to create new objects from some target distribution  $\mu$  (that for VAEs and GANs is the latent distribution);
- 2. is able to "recall" having created previous objects; This second point will therefore synthetically induce an "artificial age" for an AI because the process is irreversible.

A non-aging generative algorithm, when asked to produce, e.g. one new result will likely produce the same object (or similar) over and over again: think of the situation of a standard 1D-Gaussian: most likely the origin will be drawn over and over again, one has to wait a long time to obtain let's say, a value at 3 standard deviations from the mean. The main goal of this paper is to speed up this waiting time. The advantages of such a process is to allow some "maturation" for the results i.e. to be able to create new results, not the same ones again; this comes at the price of a irreversibility and additional computation cost.

## 1.1 Relation to previous literature

Technically, our proposal has some similarities with different areas in computational statistics: first one can invoke the "vector quantization" procedures (see [\[9,](#page-7-1)[10](#page-7-2)[,7\]](#page-7-3) and references therein) that, given a distribution, find a set of objects that represent it as a sum of Dirac masses. However, there the technical solution (Voronoi diagrams for instance) is naturally oriented to use for probability measures (or more generally finite positive measures) which is not our situation (our effort involves signed measures); in the same vein see also [\[3\]](#page-7-4) in the context of machine learning algorithms. On the other hand some efforts have been made to generalize the quantile idea to multi-dimensional distributions; in a one-dimensional situation our procedure and these techniques give similar results but they diverge as soon as the dimension is increased, see [\[6,](#page-7-5)[5,](#page-7-6)[12\]](#page-8-3).

On another hand and from a completely different perspective, the notion of "age" of a task in a queue is used in scheduling to ensure execution of low priority processes, see [\[13\]](#page-8-4).

#### 1.2 Technical goal of the paper

Continuing the works above, and given the discussion on the generative algorithms, we need a procedure that can incrementally find a good representation of a target measure  $\mu$  as a number of K Dirac masses (K is given and fixed) centered at some  $x_k$ ,  $k = 1, ..., K$  while taking into account a set of points  $Y = (y_j)_{j=1}^{K_p}$ already available. The points Y are called **historical points**. To put it otherwise we want to find the multi-point  $X = (x_k)_{k=1}^K \in \mathbb{R}^{N \times K}$   $(k = 1, ..., K)$  that minimizes the distance from the total empirical distribution  $\frac{\sum_{k=1}^{K_p} \delta_{y_k} + \sum_{l=1}^{K} \delta_{x_k}}{K_p + K}$ to the target measure  $\mu$  (here the points  $y_k$  are not submitted to optimization); this can be written as minimizing the distance  $d(\delta_X, \eta)^2$  from the distribution  $\delta_X = \frac{1}{K} \sum_{l=1}^K \delta_{x_k}$  to the signed measure

<span id="page-2-3"></span>
$$
\eta = \frac{(K_p + K)\mu - K_p \delta_Y}{K_P + K}, \text{ where } \delta_Y = \frac{1}{K_p} \sum_{k=0}^{K_p} \delta_{y_k}.
$$
 (1)

We present in section [2](#page-2-0) our choice of distance d and a theoretical result ensuring that, under appropriate hypotheses, the minimum with respect to X exits. The algorithm to find such a minimum is presented in section [3](#page-4-0) together with some numerical results. Final remarks are the object of section [4.](#page-6-0)

# <span id="page-2-0"></span>2 Theoretical results

In order to present the theoretical framework we need to define the distance  $d$ between signed measures  $\zeta$  and  $\eta$ . Note that in fact  $\zeta$  is a probability measure and the total mass of both is set to 1.

We will take a kernel-based metric given as follows: choose  $h(\cdot)$  a conditionally negative definite kernel (see [\[11\]](#page-8-5) for the precise definition and an introduction), taken here to be  $\sqrt{a^2 + |x|^2} - a$  for some given constant  $a \ge 0$ , see [\[4](#page-7-7)[,16\]](#page-8-1) for some use cases in machine learning; for any  $\eta_1$ ,  $\eta_2$  signed measures such that  $\int (1 + |X|) \eta_i(dX) < \infty$   $(i = 1, 2)$  we define :

<span id="page-2-1"></span>
$$
d(\eta_1, \eta_2) = \sqrt{\int \int -h(|X - Y|)(\eta_1 - \eta_2)(dX)(\eta_1 - \eta_2)(dY)}.
$$
 (2)

The fact that the quantity inside the square root is positive is a consequence of the fact that  $h$  is a conditionally negative definite kernel.

Note that in particular, if both  $\eta_1$  and  $\eta_2$  are sums of (signed) Dirac masses such that  $\eta_1 - \eta_2 = \sum_{k=1}^K p_k \delta_{z_k}$  (with  $K < \infty$ ) then equation [\(2\)](#page-2-1) can be written  $(see [16])$  $(see [16])$  $(see [16])$ :

<span id="page-2-2"></span>
$$
d(\eta_1, \eta_2)^2 = -\sum_{k,\ell=1}^K p_k p_\ell h(|z_k - z_\ell|). \tag{3}
$$

Once the distance is defined, a legitimate question is whether, given a target signed measure  $\eta$  one can indeed find a uniform sum of Dirac masses  $\zeta$  that minimizes  $d(\zeta, \eta)^2$ . This question is settled in the following

## 4 G. Turinici

**Proposition 1.** Suppose K is a fixed positive integer. Let  $\eta$  be a signed measure such that  $\int (1 + |X|) \eta(dX) < \infty$ . For any vector  $Z = (z_j)_{j=1}^J \in \mathbb{R}^{N \times J}$  denote

$$
\delta_Z := \frac{1}{J} \sum_{j=1}^{J} \delta_{z_j}, \ f(Z) := d(\delta_Z, \eta)^2.
$$
 (4)

Then the minimization problem :

<span id="page-3-1"></span>
$$
\inf_{X = (x_k)_{k=1}^K \in \mathbb{R}^{N \times K}} f(X)
$$
\n(5)

admits at least one solution.

Remark 1. The previous result only states the existence of a solution, the uniqueness is not necessarily true as one can observe by taking e.g. a rotation invariant measure: any solid rotation of a minimum will still be a minimum.

Proof. Let us denote

$$
m_{\eta} := \inf_{X = (x_k)_{k=1}^K \in \mathbb{R}^{N \times K}} f(X).
$$
 (6)

Take a point X such that  $f(X) \leq m_{\eta} + 1$  (the existence of X is guaranteed by the definition of  $m_{\eta}$ ). Then (denoting by 0 the null vector in  $\mathbb{R}^{N \times K}$ ):

$$
m_{\eta} + 1 \ge f(X) = d(\delta_X, \eta)^2 \ge \frac{d(\delta_X, \delta_0)^2 - 2d(\delta_0, \eta)^2}{2},
$$
 (7)

which implies

$$
d(\delta_X, \delta_0)^2 \le 2(m_\eta + 1) + 2d(\delta_0, \eta)^2.
$$
 (8)

But, using equation [\(3\)](#page-2-2), we obtain :

<span id="page-3-0"></span>
$$
d(\delta_X, \delta_0)^2 = \frac{2}{K} \sum_{k=1}^K h(|x_k|) - \frac{1}{K^2} \sum_{k,k'=1}^K h(|x_k - x_{k'}|)
$$
(9)

$$
\geq \frac{2}{K} \sum_{k=1}^{K} h(|x_k|) - \frac{1}{K^2} \sum_{k,k'=1,k\neq k'}^{K} [a + h(|x_k|) + h(|x_{k'}|)] \tag{10}
$$

$$
\geq \frac{2}{K^2} \sum_{k=1}^{K} h(|x_k|) - a,\tag{11}
$$

where for the passage from [\(9\)](#page-3-0) to [\(11\)](#page-3-0) we used the inequality  $h(|x-y|) \le$  $h(|x|) + h(|y|) + a$  true for any  $x, y \in \mathbb{R}^N$ . We obtain that, when  $f(X) \leq$  $m_{\eta}+1$  there exists a constant  $C_0 = K^2 \left(a/2 + m_{\eta} + 1 + d(\delta_0, \eta)^2\right)$  such that  $\sum_{k=1}^{K} h(|x_k|) \leq C_0$ ; therefore any minimizing sequence  $(X_n)_{n\geq 1}$  (that is, any sequence such that  $\lim_{n\to\infty} f(X_n) = m$  is bounded. This sequence will have a sub-sequence  $(X_{n_k})_{k\geq 1}$  that is convergent to some  $X^*$ ; but since the distance is continuous, we obtain that  $f(X^*) = m$  which means that  $\delta_{X^*}$  is a solution of the minimization problem  $(5)$ .

# <span id="page-4-0"></span>3 Algorithm and numerical results

#### 3.1 Algorithm formulation

Consider now a target distribution  $\mu$  and a set of previously constructed points Y (of cardinal  $K_p$ ); these **historical points** are explicitly known; we will propose an algorithm that, given a number  $K$  of points to be constructed, will find a multi-point  $X \in \mathbb{R}^{N \times K}$  such that the overall measure  $\delta_{X \cup Y}$  minimizes the distance to the target measure  $\mu$ .

<span id="page-4-1"></span>Algorithm A1History aware (signed measure) compression algorithm : HAW-C 1: procedure HAW-C 2: • set batch size B, parameter  $a = 10^{-6}$ , 3: • load the historical points  $y_k$ ,  $k = 1, ..., K_p$ 4: • initialize points  $x_k$ ,  $k = 1, ..., K$  sampled at random from  $\mu$ , denote  $X =$  $(x_k)_{k=1}^K$  (considered as vector in  $\mathbb{R}^{N \times K}$ ) 5: while (max iteration not reached) do 6: • sample  $z_1, ..., z_B \sim \mu$  (i.i.d). 7:  $\bullet$  compute the global loss  $\frac{1}{1}$  $\frac{1}{1}$  $\frac{1}{1}$  using formula [\(3\)](#page-2-2):  $L(X) := d \left( \frac{1}{K} \sum_{l=1}^{K} \delta_{x_k}, \frac{K_p+1}{B} \sum_{b=1}^{B} \delta_{z_b} - \sum_{j=1}^{K_p} \delta_{y_j} \right)^2;$ 8: • backpropagate the loss  $L(X)$  in order to minimize  $L(X)$  and update X. 9: end while 10: end procedure

## 3.2 Numerical results

A Python code implementing the algorithm in both history unaware and history aware compression modes can be consulted at [\[2\]](#page-7-8).

## History unaware compression of a 2D Gaussian mix distribution

We first test the algorithm without any historical points i.e.,  $K_p = 0$ . When the target measure is positive, the HAW-C algorithm [A1](#page-4-1) allows to compress any given (probability) measure, as illustrated in figure [1](#page-5-0) for the situation of a uniform mixture of 16 lattice-centered 2D normal variables. Good results are obtained: without any previous knowledge, the algorithm can unveil the mixing structure and allow a coherent compression.

## History aware multi-dimensional Gaussian compression and application to generative algorithms

We move now to a test where incremental compression is performed: we consider a 2D Gaussian centered at origin. First we compress it with a single point



<span id="page-5-0"></span>Fig. 1. Test without any historical points,  $K_p = 0$ . An example of compression for an uniform Gaussian mixture of 16 Gaussians centered on points of a  $4\times 4$  grid (red points are the centers of the Gaussians, blue points are the compressed points). We used  $K$ points to summarize the distribution :  $K = 48$  (left image) or  $K = 3$  (right image). Good quality results are obtained as the algorithm "understands" the mixing structure: for instance for  $K = 48$  the algorithm allocates precisely 3 points per Gaussian mixture term.

 $u_1$ ; then we use  $K_p = 1$  and  $y_1 = u_1$  as history and compress the signed measure : initial Gaussian minus the first obtained point  $u_1$ , as detailed in equation [\(1\)](#page-2-3), with another supplementary point  $u_2$ ; then consider  $K_p = 2$  and  $y_i = u_i$  $(i = 1, 2)$  and compress the Gaussian measure minus the sum of Dirac masses in  $u_i$  with another point  $u_3$ ; the procedure is then continued recursively, each step being an application of the algorithm [A1.](#page-4-1) The results are presented in figure [2.](#page-5-1)



<span id="page-5-1"></span>Fig. 2. An example of recursive compression of a 2D standard Gaussian (see text for details). Left image: the result of the compression after 10 iterations. Each point  $u_i$ is labeled by its corresponding index i when it was found.

In order to test these results on a practical case, we used the CVAE procedure from the Tensorflow documentation [\[1\]](#page-7-9) with default parameters (except that we used 100 epochs instead of 10 because the results with 10 epochs are very fuzzy). The code was executed once in order to construct the encoder/ decoder networks and then the sampling was done in the latent space using either a multi-dimensional sampling of 10 objects or an incremental sampling; once the sampling is done the data is propagated through the decoder network and the resulting images are presented in figure [3.](#page-6-1) We note that the history aware sampling retains a good diversity with respect to the uniform sampling and avoids some repetitions:

- the figures 1 and 2 that are repeated in the propagated random samples and only appear once in the incremental sampling;

- the figure 9 appears only very unclearly in the left sample and more clearly in the right sample

- the incremental sampling avoids the symbol in row 2 column 2 (left image) which is not a figure.

<span id="page-6-1"></span>

Fig. 3. Images from [\[1\]](#page-7-9) obtained by taking either a random sampling of 10 points from a 2D Gaussian (left image) or the sampling obtained in figure [2](#page-5-1) (right image). The decoder is the one obtained by a run of the code from [\[1\]](#page-7-9). The right image appears more faithful of the database. Improved quality results are available in figure [4.](#page-7-10)

.

Note that this out-of-the-box C-VAE is not good enough to make figures too precise which explains large numbers of fuzzy images - resembling to a 8, 6 or 9- present in both results). To improve the result, we re-run the CVAE for 20 epochs but increased, as recommenced in the documentation, all 'filters' numbers to 512 (instead of 32 or 64 in the initial setup). We obtain the results in figure [4:](#page-7-10) the quality of the VAE is indeed increased and the same conclusions hold for the comparison of the i.i.d. sampling with the incremental sampling.

## <span id="page-6-0"></span>4 Final remarks

We explored in this work the construction of objects from generative algorithms (like VAEs and GANs); more specifically the construction was incremental in the sense that each new sampling from the latent space considers the previous samples, called the **history** and tries to both respect the desired target distribution of the latent space but also stays away from the points already sampled. We described some theoretical properties of the procedure and tested it both on general data and on a C-VAE benchmark.

<span id="page-7-10"></span>

Fig. 4. Results of the same procedure as in figure [3](#page-6-1) but obtained for an improved network (512 filters everywhere and 20 epochs): left image : random sampling; right image: decoding of the incremental sampling from figure [2.](#page-5-1) The left figure has several repetitions (for instance figure 7) that are absent from the right figure but more importantly, some figures abondant in the database and not present in the left figure appear in the right one, like the figures 1 and 6.

.

More experiments are required to characterize fully the applicability domain of the proposed procedure, but the present results provide encouraging arguments to do so.

## References

- <span id="page-7-9"></span>1. <https://www.tensorflow.org/tutorials/generative/cvae>, cVAE, Tensorflow documentation, retrieved Jan 30, 2022
- <span id="page-7-8"></span>2. Python implementation for measure compression algorithm described in arxiv:2202.03008 [stat.ML] by G. Turinici, [https://doi.org/10.5281/zenodo.](https://doi.org/10.5281/zenodo.6671523) [6671523](https://doi.org/10.5281/zenodo.6671523), doi: 10.5281/zenodo.6671523
- <span id="page-7-4"></span>3. Chazal, F., Levrard, C., Royer, M.: Optimal quantization of the mean measure and applications to statistical learning (2021)
- <span id="page-7-7"></span>4. Deshpande, I., Hu, Y.T., Sun, R., Pyrros, A., Siddiqui, N., Koyejo, S., Zhao, Z., Forsyth, D., Schwing, A.G.: Max-Sliced Wasserstein Distance and Its Use for GANs. In: The IEEE Conference on Computer Vision and Pattern Recognition (CVPR) (June 2019)
- <span id="page-7-6"></span>5. Fraiman, R., Pateiro-Lopez, B.: Quantiles for finite and infinite dimensional data. Journal of Multivariate Analysis 108, 1–14 (2012), [https://www.sciencedirect.](https://www.sciencedirect.com/science/article/pii/S0047259X12000176) [com/science/article/pii/S0047259X12000176](https://www.sciencedirect.com/science/article/pii/S0047259X12000176)
- <span id="page-7-5"></span>6. Glazer, A., Lindenbaum, M., Markovitch, S.: q-OCSVM: A q-Quantile Estimator for High-Dimensional Distributions. In: Burges, C.J.C., Bottou, L., Welling, M., Ghahramani, Z., Weinberger, K.Q. (eds.) Advances in Neural Information Processing Systems. vol. 26. Curran Associates, Inc. (2013), [https://proceedings.](https://proceedings.neurips.cc/paper/2013/file/819f46e52c25763a55cc642422644317-Paper.pdf) [neurips.cc/paper/2013/file/819f46e52c25763a55cc642422644317-Paper.pdf](https://proceedings.neurips.cc/paper/2013/file/819f46e52c25763a55cc642422644317-Paper.pdf)
- <span id="page-7-3"></span>7. Graf, S., Luschgy, H.: Foundations of quantization for probability distributions. Springer (2007)
- <span id="page-7-0"></span>8. Kingma, D.P., Max, W.: An Introduction to Variational Autoencoders. Now Publishers Inc (Nov 2019)
- <span id="page-7-1"></span>9. Kohonen, T.: Learning Vector Quantization. In: Kohonen, T. (ed.) Self-Organizing Maps, pp. 175–189. Springer Berlin Heidelberg, Berlin, Heidelberg (1995)
- <span id="page-7-2"></span>10. R. Gray: Vector quantization. IEEE ASSP Magazine 1(2), 4–29 (Apr 1984). [https:](https://doi.org/10.1109/MASSP.1984.1162229) [//doi.org/10.1109/MASSP.1984.1162229](https://doi.org/10.1109/MASSP.1984.1162229)
- <span id="page-8-5"></span>11. Sejdinovic, D., Sriperumbudur, B., Gretton, A., Fukumizu, K.: Equivalence of distance-based and RKHS-based statistics in hypothesis testing. The Annals of Statistics 41(5), 2263–2291 (2013), <http://www.jstor.org/stable/23566550>
- <span id="page-8-3"></span>12. Serfling, R.: Quantile functions for multivariate analysis: approaches and applications. Statistica Neerlandica 56(2), 214–232 (2002). [https://doi.org/https:](https://doi.org/https://doi.org/10.1111/1467-9574.00195) [//doi.org/10.1111/1467-9574.00195](https://doi.org/https://doi.org/10.1111/1467-9574.00195), [https://onlinelibrary.wiley.com/](https://onlinelibrary.wiley.com/doi/abs/10.1111/1467-9574.00195) [doi/abs/10.1111/1467-9574.00195](https://onlinelibrary.wiley.com/doi/abs/10.1111/1467-9574.00195)
- <span id="page-8-4"></span>13. Silberschatz, A., Gagne, G., Galvin, P.B.: Operating System Concepts. John Wiley  $&$  Sons Inc, 10e édition edn. (Jan 2018)
- <span id="page-8-2"></span>14. Székely, G.J., Rizzo, M.L.: Energy statistics: A class of statistics based on distances. Journal of Statistical Planning and Inference 143(8), 1249–1272 (Aug 2013). <https://doi.org/10.1016/j.jspi.2013.03.018>, [https://www.sciencedirect.](https://www.sciencedirect.com/science/article/pii/S0378375813000633) [com/science/article/pii/S0378375813000633](https://www.sciencedirect.com/science/article/pii/S0378375813000633)
- <span id="page-8-0"></span>15. Tabor, J., Knop, S., Spurek, P., Podolak, I.T., Mazur, M., Jastrzkebski, S.: Cramer-Wold AutoEncoder. CoRR abs/1805.09235 (2018), [http://arxiv.org/](http://arxiv.org/abs/1805.09235) [abs/1805.09235](http://arxiv.org/abs/1805.09235)
- <span id="page-8-1"></span>16. Turinici, G.: Radon–Sobolev Variational Auto-Encoders. Neural Networks 141, 294–305 (Sep 2021). <https://doi.org/10.1016/j.neunet.2021.04.018>, [https:](https://www.sciencedirect.com/science/article/pii/S0893608021001556) [//www.sciencedirect.com/science/article/pii/S0893608021001556](https://www.sciencedirect.com/science/article/pii/S0893608021001556)