Parametrized Modal Logic II: The Unidimensional Case

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Abstract. We consider a syntax and semantics of modal logics based on parametrized modal connectives with ∃∀-satisfaction definitions, we axiomatically introduce different parametrized modal logics, we prove their completeness with respect to appropriate classes of parametrized relational structures and we show the decidability of some related satisfiability problems. AQ1 AQ2

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1 Introduction

The connective $\Diamond$ of arity 1 usually considered in the propositional modal language has a ∃-satisfaction definition: in relational models of the form (W, R, V) where R is a binary relation on a nonempty set W of possible worlds, V interprets formulas in such a way that for all formulas φ,

– the possible world s is in $V(\Diamond\varphi)$ exactly when for some possible world t, sRt and $t \in V(\varphi)$.

Within the context of temporal reasoning, the until connective $\mathcal{U}$ of arity 2 has been considered in order to increase the expressive power of the propositional modal language, its ∃∀-satisfaction definition being such that in models (W, R, V) as above, for all formulas φ, ψ,

– the possible world s is in $V(\varphi\mathcal{U}\psi)$ exactly when for some possible world u, sRu, $u \in V(\psi)$ and for every possible world t, if sRt and tRu then $t \in V(\varphi)$.

As is well-known, the use of the until connective $\mathcal{U}$ of arity 2 allows to characterize more classes of relational structures than we can characterize in the propositional modal language based on the connective $\Diamond$ of arity 1 [4, Chapter 7]. Moreover, the use of the until connective $\mathcal{U}$ has no dramatic consequence on the computational complexity of the satisfiability problem, this problem being **PSPACE**-complete in the most popular classes of models usually considered for applications of temporal reasoning [22,23].

Therefore, a question naturally arises: without dramatically affecting the computational complexity of the satisfiability problem, are there other ways to increase the

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expressive power of propositional modal languages by considering other connectives with complex satisfaction definitions? Let us consider a propositional modal language based on a connective $\lozenge$ of arity 2. Traditionally, its relational models are of the form (W, R, V) where R is a ternary relation on a nonempty set W of possible worlds and V interprets formulas in such a way that for all formulas φ, ψ,

– the possible world s is in $V(\varphi\lozenge\psi)$ exactly when for some possible world u, $u \in V(\psi)$ and for some possible world t, $t \in V(\varphi)$ and $sR(t, u)$.

On the pattern of the until connective $\mathcal{U}$ and its $\exists\forall$-satisfaction definition, let us consider a propositional modal language based on a new connective $\blacklozenge$ of arity 2 and such that in models (W, R, V) as above, for all formulas φ, ψ,

– the possible world s is in $V(\varphi\blacklozenge\psi)$ exactly when for some possible world u, $u \in V(\psi)$ and for every possible world t, if $t \in V(\varphi)$ then $sR(t, u)$.

With such syntax and semantics at hand, can we characterize more classes of relational structures than we can characterize in the propositional modal language based on the connective $\lozenge$ of arity 2? And what is the price to pay in terms of the computational complexity of the satisfiability problem?

Obviously, given a ternary relation R on a nonempty set W, we can naturally consider the function $\mathbf{R} : \wp(W) \longrightarrow \wp(W \times W)$ such that for all subsets A of W and for every s, u in W, $s\mathbf{R}(A)u$ exactly when for every t in W, if $t \in A$ then $sR(t, u)$. Obviously, the main property of such function is that for all subsets A of W, $\mathbf{R}(A) = \bigcap\{\mathbf{R}(\{t\}) : t \in A\}$. Reciprocally, given a nonempty set W and a function $\mathbf{R} : \wp(W) \longrightarrow \wp(W \times W)$ satisfying the above property, we can naturally consider the ternary relation R on W such that for every s, t, u in W, $sR(t, u)$ exactly when $s\mathbf{R}(\{t\})u$. This suggests us to consider a propositional modal language based on a connective $\blacklozenge$ of arity 2 and such that in models of the form $(W, \mathbf{R}, V)$ where W is a nonempty set of possible worlds and $\mathbf{R} : \wp(W) \longrightarrow \wp(W \times W)$ is a function satisfying the above property, for all formulas φ, ψ,

– the possible world s is in $V(\varphi\blacklozenge\psi)$ exactly when for some possible world u, $u \in V(\psi)$ and $s\mathbf{R}(V(\varphi))u$.

In this paper, with such syntax (Sect. 2) and semantics (Sect. 3) at hand, we axiomatically introduce different modal logics (Sect. 4), we prove their completeness with respect to appropriate classes of relational structures (Sects. 5 and 6) and we show the decidability of some related satisfiability problems (Sect. 7). When our results are immediate consequences of our definitions, their proofs are not included in the paper.

2 Syntax

Let $\mathcal{P}$ be a countably infinite set (with typical members denoted p, q, etc.). Members of $\mathcal{P}$ will be called *atomic formulas*. A *tip* is a set Σ of finite words over the alphabet $\mathcal{P} \cup \{\bot, \neg, \vee, \blacklozenge, (,)\}$ (with typical members denoted φ, ψ, etc.). Let $\mathcal{L}$ be the least tip such that $\mathcal{P} \subseteq \mathcal{L}$ and for all finite words φ, ψ,

- $\bot \in \mathcal{L}$,
- if $\varphi \in \mathcal{L}$ then $\neg\varphi \in \mathcal{L}$,
- if $\varphi, \psi \in \mathcal{L}$ then $(\varphi \vee \psi) \in \mathcal{L}$,
- if $\varphi, \psi \in \mathcal{L}$ then $(\varphi\blacklozenge\psi) \in \mathcal{L}$.

Members of $\mathcal{L}$ will be called *formulas*. The Boolean connectives $\top$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined as the usual abbreviations. For all $\varphi, \psi \in \mathcal{L}$, anticipating the fact that the roles of φ and ψ in $(\varphi\blacklozenge\psi)$ are not symmetric, let $(\varphi\blacksquare\psi)$ be an abbreviation of $\neg(\varphi\blacklozenge\neg\psi)$. We adopt the standard rules for omission of the parentheses. A tip Σ is *readable* if $\Sigma \subseteq \mathcal{L}$. A readable tip Σ is *closed* if for all $\varphi, \psi \in \mathcal{L}$,

- if $\neg\varphi \in \Sigma$ then $\varphi \in \Sigma$,
- if $\varphi \vee \psi \in \Sigma$ then $\varphi, \psi \in \Sigma$,
- if $\varphi\blacklozenge\psi \in \Sigma$ then $\varphi, \psi \in \Sigma$.

As a result, for all closed readable tips Σ and for all $\varphi, \psi \in \mathcal{L}$, if $\varphi\blacksquare\psi \in \Sigma$ then $\varphi, \psi \in \Sigma$. For all $\varphi \in \mathcal{L}$, let Σ_φ be the least closed readable tip containing φ. For all $\varphi \in \mathcal{L}$, let $\|\varphi\|$ be the length of φ.

Lemma 1. *For all $\varphi \in \mathcal{L}$, $\texttt{Card}(\Sigma_\varphi) \leq \|\varphi\|$.*

From now on in this paper, for all $\varphi, \psi \in \mathcal{L}$, we will write "$\langle\varphi\rangle\psi$" instead of "$\varphi\blacklozenge\psi$" and "$[\varphi]\psi$" instead of "$\varphi\blacksquare\psi$". For all $\varphi \in \mathcal{L}$ and for all readable tips Σ, let $[\varphi]\Sigma$ be the set of all $\psi \in \mathcal{L}$ such that $[\varphi]\psi \in \Sigma$. From now on in this paper, readable tips will be called *sets of formulas*.

3 Relational Semantics

A *frame* is a couple (W, R) where W is a nonempty set and $R : \wp(W) \longrightarrow \wp(W \times W)$. A frame (W, R) is *conjunctive* if for all $A \in \wp(W)$, $R(A) = \bigcap\{R(\{s\}) : s \in A\}$. A frame (W, R) is *preconjunctive* if $R(\emptyset) = W \times W$ and for all $A, B \in \wp(W)$, $R(A \cup B) = R(A) \cap R(B)$. A frame (W, R) is *paraconjunctive* if $R(\emptyset) = W \times W$ and for all $A, B \in \wp(W)$, if $A \subseteq B$ then $R(A) \supseteq R(B)$.

Lemma 2. *Every conjunctive frame is preconjunctive.*

Example 1. There exists preconjunctive nonconjunctive frames. For instance, the frame (W, R) where $W = \mathbb{N}$ and for all $A \in \wp(\mathbb{N})$, if A is finite then $R(A) = \mathbb{N} \times \mathbb{N}$ else $R(A) = \emptyset$. Obviously, this frame is preconjunctive. Indeed, $R(\emptyset) = \mathbb{N} \times \mathbb{N}$. Moreover, for all $A, B \in \wp(\mathbb{N})$, $R(A \cup B) = R(A) \cap R(B)$, seeing that $A \cup B$ is finite if and only if A is finite and B is finite. However, it is not conjunctive, seeing that $R(\mathbb{N}) = \emptyset$ and $\bigcap\{R(\{s\}) : s \in \mathbb{N}\} = \mathbb{N} \times \mathbb{N}$.

Lemma 3. *Every preconjunctive frame is paraconjunctive.*

Example 2. There exists paraconjunctive nonpreconjunctive frames. For instance, the frame (W, R) where $W = \mathbb{N}$ and for all $A \in \wp(\mathbb{N})$, if $\texttt{Card}(A) < 2$ then $R(A) = \mathbb{N} \times \mathbb{N}$ else $R(A) = \emptyset$. Obviously, this frame is paraconjunctive. Indeed, $R(\emptyset) = \mathbb{N} \times \mathbb{N}$. Moreover, for all $A, B \in \wp(\mathbb{N})$, if $A \subseteq B$ then $R(B) \subseteq R(A)$, seeing that if $A \subseteq B$ then $\texttt{Card}(A) \leq \texttt{Card}(B)$. However, it is not preconjunctive, seeing that $R(\{0, 1\}) = \emptyset$ and $R(\{0\}) \cap R(\{1\}) = \mathbb{N} \times \mathbb{N}$.

A *frame of indiscernibility* is a frame (W, R) such that for all $A \in \wp(W)$, $R(A)$ is an equivalence relation on W.

Example 3. Let $\mathbf{Ob}$ be a nonempty set of *objects,* $\mathbf{At}$ be a nonempty set of *attributes,* $\mathbf{Val}$ be a nonempty set of *values* and $m : \mathbf{Ob} \times \mathbf{At} \longrightarrow \wp(\mathbf{Val})$. In the 4-tuple $(\mathbf{Ob}, \mathbf{At}, \mathbf{Val}, m)$, the objects s and t are *equivalent for the attribute* u if $m(s, u) = m(t, u)$ whereas the attributes s and t are *equivalent for the object* u if $m(u, s) = m(u, t)$. Such systems have been introduced and developed by Orłowska and Pawlak within the context of analysis of data and representation of nondeterministic information [18,20]. The frame (W, R) where

- $W = \mathbf{Ob} \cup \mathbf{At}$,
- for all $A \in \wp(\mathbf{Ob} \cup \mathbf{At})$, $R(A)$ is the binary relation on $\mathbf{Ob} \cup \mathbf{At}$ such that for all $s, t \in \mathbf{Ob} \cup \mathbf{At}$, $sR(A)t$ if and only if either $A = \emptyset$, or $s, t \in \mathbf{Ob}$ and for all $u \in \mathbf{At}$, if $u \in A$ then $m(s, u) = m(t, u)$, or $s, t \in \mathbf{At}$ and for all $u \in \mathbf{Ob}$, if $u \in A$ then $m(u, s) = m(u, t)$,

is a conjunctive frame of indiscernibility.

A *valuation on a frame* (W, R) is a $V : \mathcal{L} \longrightarrow \wp(W)$ such that for all $\varphi, \psi \in \mathcal{L}$,

- $V(\bot) = \emptyset$,
- $V(\neg\varphi) = W \setminus V(\varphi)$,
- $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
- $V(\langle\varphi\rangle\psi) = \{s \in W : \exists t \in W\ (sR(V(\varphi))t\ \&\ t \in V(\psi))\}$.

As a result, for all valuations $V : \mathcal{L} \longrightarrow \wp(W)$ on the frame (W, R) and for all $\varphi, \psi \in \mathcal{L}$, $V([\varphi]\psi) = \{s \in W : \forall t \in W\ (sR(V(\varphi))t \Rightarrow t \in V(\psi))\}$. A *model* is a triple consisting of a frame and a valuation on that frame. A model is *conjunctive (resp., preconjunctive, paraconjunctive)* if it is based on a conjunctive (resp., preconjunctive, paraconjunctive) frame. A *model of indiscernibility* is a model based on a frame of indiscernibility.

Example 4. The frame (W, R) where

- $W = \mathbb{R}^2$,
- for all $A \in \wp(\mathbb{R}^2)$, $R(A)$ is the binary relation on $\mathbb{R}^2$ such that for all $s, t \in \mathbb{R}^2$, $sR(A)t$ if and only if for all $u \in A$, $d(s, t) \leq d(s, u)$ where $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ is the distance function in $\mathbb{R}^2$,

is conjunctive. For all valuations V on (W, R) and for all $\varphi, \psi \in \mathcal{L}$, if $V(\varphi) \neq \emptyset$ then $V(\langle\varphi\rangle\psi)$ is the set of all s in $\mathbb{R}^2$ such that for some t in $\mathbb{R}^2$, t is in $V(\psi)$ and the open disc with center s and radius $d(s, t)$ does not intersect $V(\varphi)$.

Example 5. The frame (W, R) where

- $W = \mathbb{R}^3$,
- for all $A \in \wp(\mathbb{R}^3)$, $R(A)$ is the binary relation on $\mathbb{R}^3$ such that for all $s, t \in \mathbb{R}^3$, $sR(A)t$ if and only if for all $u \in A$, not $L(s, t, u)$ where $L \subseteq \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ is the collinearity relation in $\mathbb{R}^3$,

is conjunctive. For all valuations V on (W, R) and for all $\varphi, \psi \in \mathcal{L}$, if $V(\varphi) \neq \emptyset$ then $V(\langle\varphi\rangle\psi)$ is the set of all s in $\mathbb{R}^3$ such that for some t in $\mathbb{R}^3$, t is in $V(\psi)$ and the line passing through s and t does not intersect $V(\varphi)$.

A formula φ is *satisfiable in a model* (W, R, V) if $V(\varphi) \neq \emptyset$. A formula φ is *true in a model* (W, R, V) (in symbols $(W, R, V) \models \varphi$) if $V(\varphi) = W$. A formula φ is *satisfiable on a frame* (W, R) if there exists a (W, R)-valuation V such that φ is satisfiable in (W, R, V). A formula φ is *valid on a frame* (W, R) (in symbols $(W, R) \models \varphi$) if for all (W, R)-valuations V, $(W, R, V) \models \varphi$. A formula φ is *satisfiable on a class $\mathcal{C}$ of frames* if there exists a frame (W, R) in $\mathcal{C}$ such that φ is satisfiable on (W, R). A formula φ is *valid on a class $\mathcal{C}$ of frames* (in symbols $\mathcal{C} \models \varphi$) if for all frames (W, R) in $\mathcal{C}$, $(W, R) \models \varphi$.

Example 6. On the class of all paraconjunctive frames, the following formulas are valid: $[\bot]\varphi \to \varphi$ and $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$.

Example 7. On the class of all paraconjunctive frames, the following formulas are valid: $[\bot](\varphi \to \psi) \to ([\varphi]\chi \to [\psi]\chi)$.

Example 8. On the class of all frames of indiscernibility, the following formulas are valid: $[\varphi]\psi \to \psi$ and $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

The *satisfiability problem on a class $\mathcal{C}$ of frames* is the following decision problem:

input: a formula φ,
output: determine whether there exists a model (W, R, V) based on a frame in $\mathcal{C}$ such that $V(\varphi) \neq \emptyset$.

A *bounded morphism from a frame* (W, R) *to a frame* (W', R') is a $f : W \longrightarrow W'$ such that

Forward condition: for all $s, t \in W$ and for all $A \in \wp(W)$, if $sR(A)t$ then $f(s) R'(f[A])f(t)$,
Backward condition: for all $s \in W$, for all $t' \in W'$ and for all $A \in \wp(W)$, if $f(s)R'(f[A])t'$ then there exists $t \in W$ such that $sR(A)t$ and $f(t) = t'$.

Lemma 4. *For all frames (W, R) and (W', R') and for all bounded morphisms f from (W, R) to (W', R'), if f is surjective then for all valuations V' on (W', R'), the $V : \mathcal{L} \longrightarrow \wp(W)$ such that for all $\varphi \in \mathcal{L}$, $V(\varphi) = f^{-1}[V'(\varphi)]$ is a valuation on (W, R).*

Lemma 5. *For all frames (W, R) and (W', R') and for all bounded morphisms f from (W, R) to (W', R'), if f is surjective then for all formulas φ, if $(W, R) \models \varphi$ then $(W', R') \models \varphi$.*

4 Axiomatizations

A *unidimensional parametrized modal logic (UPML)* is a set of formulas containing the following formulas:

$(\mathbf{A}_1)$ all formulas obtained from propositional tautologies after having uniformly replaced their atomic formulas by arbitrary formulas,
$(\mathbf{A}_2)$ $[\varphi](\psi \to \chi) \to ([\varphi]\psi \to [\varphi]\chi)$,

and closed under the following rules:

$(\mathbf{R}_1)$ $\frac{\varphi,\ \varphi\to\psi}{\psi}$,
$(\mathbf{R}_2)$ $\frac{\varphi}{[\psi]\varphi}$,
$(\mathbf{R}_3)$ $\frac{\varphi\leftrightarrow\psi}{[\varphi]\chi\leftrightarrow[\psi]\chi}$.

A UPML is *conjunctive* if it contains the following formulas:

$(\mathbf{A}_3)$ $[\bot]\varphi \to \varphi$, $\langle\bot\rangle\varphi \to [\bot]\langle\bot\rangle\varphi$,
$(\mathbf{A}_4)$ $[\bot](\varphi \to \psi) \to ([\varphi]\chi \to [\psi]\chi)$.

Let $\mathbf{K_g}$ (resp., $\mathbf{K_c}$) be the least UPML (resp., the least conjunctive UPML). Let $\mathbf{S5_g}$ (resp., $\mathbf{S5_c}$) be the least UPML (resp., the least conjunctive UPML) containing the following formulas:

$(\mathbf{A}_5)$ $[\varphi]\psi \to \psi$, $\langle\varphi\rangle\psi \to [\varphi]\langle\varphi\rangle\psi$.

For all UPMLs $\mathbf{L}$ and for all sets Σ of formulas, let $\mathbf{L} + \Sigma$ be the least UPML containing $\mathbf{L} \cup \Sigma$. A UPML $\mathbf{L}$ is *consistent* if $\mathbf{L} \neq \mathcal{L}$. For all UPMLs $\mathbf{L}$, we will say that a set s of formulas is $\mathbf{L}$-*consistent* if for all $n \in \mathbb{N}$ and for all $\varphi_1, \ldots, \varphi_n \in s$, $\neg(\varphi_1 \wedge \ldots \wedge \varphi_n) \notin \mathbf{L}$. Notice that for all consistent UPMLs $\mathbf{L}$, $\mathbf{L}$ is a $\mathbf{L}$-consistent set of formulas.

Lemma 6. *For all UPMLs $\mathbf{L}$ and for all $\mathbf{L}$-consistent sets s of formulas, there exists a maximal $\mathbf{L}$-consistent set t of formulas such that $s \subseteq t$.*

Lemma 7. *For all UPMLs $\mathbf{L}$, for all maximal $\mathbf{L}$-consistent sets s of formulas and for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in s$ then $[\varphi]s \cup \{\psi\}$ is a $\mathbf{L}$-consistent set of formulas.*

Table 1. .

UPMLs	Classes of frames
$\mathbf{K_g}$	All frames
$\mathbf{S5_g}$	All frames of indiscernibility
$\mathbf{K_c}$	All paraconjunctive frames
	All preconjunctive frames
	All conjunctive frames
$\mathbf{S5_c}$	All paraconjunctive frames of indiscernibility
	All preconjunctive frames of indiscernibility
	All conjunctive frames of indiscernibility

For all formulas φ, let $\widehat{\varphi}$ be the set of all maximal $\mathbf{L}$-consistent sets of formulas containing φ. A UPML $\mathbf{L}$ is *sound with respect to a class $\mathcal{C}$ of frames* if for all formulas φ, if $\varphi \in \mathbf{L}$ then $\mathcal{C} \models \varphi$. A UPML $\mathbf{L}$ is *complete with respect to a class $\mathcal{C}$ of frames* if for all formulas φ, if $\mathcal{C} \models \varphi$ then $\varphi \in \mathbf{L}$. The proofs of the soundness statements expressed in Proposition 1 are as expected.

Proposition 1. *In Table 1, the UPMLs listed in the left column are sound with respect to the corresponding classes of frames listed in the right column.*

As for the proofs of the corresponding completeness statements, they are not so obvious, especially when the considered UPMLs are conjunctive. Indeed, the problem with conjunctive UPMLs is that the operation of intersection — which is used in conjunctive frames for the interpretation of the modalities — is not modally definable [1, 19]. AQ3

5 Completeness: The General Case

From now on in this section, we will assume that **L** is a consistent UPML. Let (W_g, R_g) be the couple where

- W_g is the set of all maximal **L**-consistent sets of formulas,
- $R_g : \wp(W_g) \longrightarrow \wp(W_g \times W_g)$ is such that for all $A \in \wp(W_g)$ and for all $s, t \in W_g$,
  - $sR_g(A)t$ if and only if for all formulas φ, if $\widehat{\varphi} = A$ then $[\varphi]s \subseteq t$,

  where $\widehat{\varphi}$ denotes the set of all $u \in W_g$ such that $\varphi \in u$.

Lemma 8. *For all formulas φ, ψ, if $\widehat{\varphi} = \widehat{\psi}$ then $\varphi \leftrightarrow \psi \in$ **L**.*

Since **L** is a **L**-consistent set of formulas, by Lemma 6, W_g is nonempty.

Lemma 9. *(W_g, R_g) is a frame.*

The couple (W_g, R_g) will be called *general canonical frame for* **L**.

Lemma 10. *If* **L** *contains* $\mathbf{S5_g}$ *then the general canonical frame for* **L** *is a frame of indiscernibility.*

Let $V_g : \mathcal{L} \longrightarrow \wp(W_g)$ be such that for all formulas φ, $V_g(\varphi) = \widehat{\varphi}$. The triple (W_g, R_g, V_g) will be called *general canonical model for* **L**.

Lemma 11 (Truth Lemma: the general case). *The general canonical model for* **L** *is a model.*

Proposition 2 is a consequence of Lemmas 6, 9, 10 and 11.

Proposition 2.
- $\mathbf{K_g}$ *is complete with respect to the class of all frames,*
- $\mathbf{S5_g}$ *is complete with respect to the class of all frames of indiscernibility.*

6 Completeness: The Conjunctive Case

From now on in this section, we will assume that **L** is a consistent conjunctive UPML.

Lemma 12. *For all maximal* **L***-consistent sets s, t, u of formulas, $[\bot]s \subseteq s$ and if $[\bot]s \subseteq t$ and $[\bot]s \subseteq u$ then $[\bot]t \subseteq u$.*

Let s_0 be a maximal **L**-consistent set of formulas. Let (W_c, R_c) be the couple where

- W_c is the set of all maximal **L**-consistent sets s of formulas such that $[\bot]s_0 \subseteq s$,
- $R_c : \wp(W_c) \longrightarrow \wp(W_c \times W_c)$ is such that for all $A \in \wp(W_c)$ and for all $s, t \in W_c$,

- $sR_c(A)t$ if and only if for all formulas φ, if $\widehat{\varphi} \subseteq A$ then $[\varphi]s \subseteq t$,

where $\widehat{\varphi}$ denotes the set of all $u \in W_c$ such that $\varphi \in u$.

Lemma 13. *For all formulas φ, ψ,*

– *if $\widehat{\varphi} \subseteq \widehat{\psi}$ then for all $s \in W_c$, $[\bot](\varphi \to \psi) \in s$,*
– *if $\widehat{\varphi} = \emptyset$ then for all $s, t \in W_c$, $[\varphi]s \subseteq t$.*

Since s_0 is a maximal $\mathbf{L}$-consistent set of formulas, by Lemma 12, W_c is nonempty.

Lemma 14. *(W_c, R_c) is a paraconjunctive frame.*

The couple (W_c, R_c) will be called *conjunctive canonical frame for* $\mathbf{L}$.

Lemma 15. *If $\mathbf{L}$ contains $\mathbf{S5_c}$ then the conjunctive canonical frame for $\mathbf{L}$ is a paraconjunctive frame of indiscernibility.*

Let $V_c : \mathcal{L} \longrightarrow \wp(W_c)$ be such that for all formulas φ, $V_c(\varphi) = \widehat{\varphi}$. The triple (W_c, R_c, V_c) will be called *conjunctive canonical model for* $\mathbf{L}$.

Lemma 16 (Truth Lemma: the paraconjunctive case). *The conjunctive canonical model for $\mathbf{L}$ is a model.*

Proposition 3 is a consequence of Lemmas 6, 14, 15 and 16.

Proposition 3. – $\mathbf{K_c}$ *is complete with respect to the class of all paraconjunctive frames,*
– $\mathbf{S5_c}$ *is complete with respect to the class of all paraconjunctive frames of indiscernibility.*

Now, let us turn to the completeness of $\mathbf{K_c}$ with respect to the class of all preconjunctive frames and the class of all conjunctive frames and the completeness of $\mathbf{S5_c}$ with respect to the class of all preconjunctive frames of indiscernibility and the class of all conjunctive frames of indiscernibility. In this respect, Lemmas 17 and 18 will be our key results.

Lemma 17. *Let (W, R) be a paraconjunctive frame. There exists a conjunctive frame (W', R') and a surjective bounded morphism from (W', R') to (W, R).*

Proof. This proof ends after Claim 6. Let Λ be the set of all $\tau : \wp(W) \times W \longrightarrow \{0, 1\}$. Let (W', R') be the couple where

– $W' = W \times \Lambda$,
– $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $(s, \sigma), (t, \tau) \in W'$,
  - $(s, \sigma)R'(A')(t, \tau)$ if and only if for all $A \in \wp(W)$,
    * if $A' \cap (A \times \Lambda) \neq \emptyset$ then $sR(A)t$ if and only if for all $u \in A$, $\sigma(A, u) = \tau(A, u)$,
    * for all $(u, v) \in A' \cap (A \times \Lambda)$, $\sigma(A, u) = \tau(A, u)$.

Claim. For all $A' \in \wp(W')$, $R'(A') = \bigcap\{R'(\{(u, v)\}) : (u, v) \in A'\}$.

Proof. Let $A' \in \wp(W')$. We demonstrate $R'(A') \supseteq \bigcap\{R'(\{(u,\upsilon)\}) : (u,\upsilon) \in A'\}$, the "$\subseteq$" direction being left as an exercise for the reader. Arguing by contradiction, suppose $R'(A') \not\supseteq \bigcap\{R'(\{(u,\upsilon)\}) : (u,\upsilon) \in A'\}$. Hence, there exists $(s,\sigma),(t,\tau) \in W'$ such that not $(s,\sigma)R'(A')(t,\tau)$ and for all $(u,\upsilon) \in A'$, $(s,\sigma)R'(\{(u,\upsilon)\})(t,\tau)$. Thus, for all $(u,\upsilon) \in A'$ and for all $A \in \wp(W)$,

- if $\{(u,\upsilon)\} \cap (A \times \Lambda) \neq \emptyset$ then $sR(A)t$ if and only if for all $v \in A$, $\sigma(A,v) = \tau(A,v)$,
- for all $(v,\omega) \in \{(u,\upsilon)\} \cap (A \times \Lambda)$, $\sigma(A,v) = \tau(A,v)$.

Consequently, for all $A \in \wp(W)$,

- if $A' \cap (A \times \Lambda) \neq \emptyset$ then $sR(A)t$ if and only if for all $v \in A$, $\sigma(A,v) = \tau(A,v)$,
- for all $(v,\omega) \in A' \cap (A \times \Lambda)$, $\sigma(A,v) = \tau(A,v)$.

Hence, $(s,\sigma)R'(A')(t,\tau)$: a contradiction.

Claim 6 is a consequence of Claim 6.

Claim. (W',R') is a conjunctive frame.

Let $f : W' \longrightarrow W$ be such that for all $(s,\sigma) \in W'$, $f(s,\sigma) = s$.

Claim. $f : W' \longrightarrow W$ is surjective.

Notice that for all $A \in \wp(W)$, $f^{-1}[A] = A \times \Lambda$.

Claim. For all $(s,\sigma),(t,\tau) \in W'$ and for all $A' \in \wp(W')$, if $(s,\sigma)R'(A')(t,\tau)$ then $sR(f[A'])t$.

Proof. Let $(s,\sigma),(t,\tau) \in W'$ and $A' \in \wp(W')$. Suppose $(s,\sigma)R'(A')(t,\tau)$. Arguing by contradiction, suppose not $sR(f[A'])t$. Hence, $f[A'] \neq \emptyset$. Thus, $A' \cap (f[A'] \times \Lambda) \neq \emptyset$. Since $(s,\sigma)R'(A')(t,\tau)$, $sR(f[A'])t$ if and only if for all $v \in f[A']$, $\sigma(f[A'],v) = \tau(f[A'],v)$. Moreover, for all $(u,\upsilon) \in A' \cap (f[A'] \times \Lambda)$, $\sigma(f[A'],u) = \tau(f[A'],u)$. Consequently, for all $u \in f[A']$, $\sigma(f[A'],u) = \tau(f[A'],u)$. Since $sR(f[A'])t$ if and only if for all $v \in f[A']$, $\sigma(f[A'],v) = \tau(f[A'],v)$, $sR(f[A'])t$: a contradiction.

Claim. For all $(s,\sigma) \in W'$, for all $t \in W$ and for all $A' \in \wp(W')$, if $sR(f[A'])t$ then there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$.

Proof. Let $(s,\sigma) \in W'$, $t \in W$ and $A' \in \wp(W')$. Suppose $sR(f[A'])t$. We demonstrate there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$. Indeed, we are looking for a $\tau : \wp(W) \times W \longrightarrow \{0,1\}$ such that for all $B \in \wp(W)$,

$(\mathbf{C_1})$ if $A' \cap (B \times \Lambda) \neq \emptyset$ then $sR(B)t$ if and only if for all $v \in B$, $\sigma(B,v) = \tau(B,v)$,
$(\mathbf{C_2})$ for all $(v,\omega) \in A' \cap (B \times \Lambda)$, $\sigma(B,v) = \tau(B,v)$.

For all $B \in \wp(W)$, let $\tau^B : W \longrightarrow \{0,1\}$ be defined as follows:

Case "$sR(B)t$"**:** for all $v \in W$, let $\tau^B(v) = \sigma(B,v)$,

Case "not $sR(B)t$"**:** let $v^B \in W$ be such that $v^B \in B$ and $v^B \not\in f[A']$ (such v^B exists for otherwise $B \subseteq f[A']$ and not $sR(f[A'])t$) and for all $v \in W$,

– if $v \neq v^B$ then let $\tau^B(v) = \sigma(B, v)$,
– otherwise, let $\tau^B(v) = 1 - \sigma(B, v)$.

Let $\tau : \wp(W) \times W \longrightarrow \{0, 1\}$ be such that for all $B \in \wp(W)$ and for all $v \in W$, $\tau(B, v) = \tau^B(v)$. Now, we just have to verify that for all $B \in \wp(W)$, $(\mathbf{C_1})$ and $(\mathbf{C_2})$ hold. Let $B \in \wp(W)$. About $(\mathbf{C_1})$, suppose $A' \cap (B \times \Lambda) \neq \emptyset$ and consider the following two cases: "$sR(B)t$" and "not $sR(B)t$". In the former case, for all $v \in W$, $\tau^B(v) = \sigma(B, v)$. Hence, for all $v \in W$, $\sigma(B, v) = \tau(B, v)$. Since $sR(B)t$, $(\mathbf{C_1})$ holds. In the latter case, $\tau^B(v) = \sigma(B, v)$ for every $v \in W$ except when $v = v^B$. Thus, $\sigma(B, v) = \tau(B, v)$ for every $v \in W$ except when $v = v^B$. Since not $sR(B)t$, $(\mathbf{C_1})$ holds. As for $(\mathbf{C_2})$, it holds, seeing that for all $v \in W$, if $v \in B$ and $v \in f[A']$ then $\tau^B(v) = \sigma(B, v)$.

Claim 6 is a consequence of Claims 6 and 6.

Claim. $f : W' \longrightarrow W$ is a bounded morphism from (W', R') to (W, R).

Lemma 18. *Let (W, R) be a paraconjunctive frame of indiscernibility. There exists a conjunctive frame of indiscernibility (W', R') and a surjective bounded morphism from (W', R') to (W, R).*

Proof. This proof ends after Claim 6. Let $\det : \wp(W) \times W \times W \longrightarrow \wp(W)$ be such that for all $A \in \wp(W)$ and for all $s, t \in W$, $\det(A, s, t) = [s]_{R(A)} \oplus [t]_{R(A)}$ where $[s]_{R(A)}$ and $[t]_{R(A)}$ are the equivalence classes of s and t modulo $R(A)$ and $\oplus$ is the operation of symmetric difference in $\wp(W)$. Notice that for all $A \in \wp(W)$ and for all $s, t \in W$, $\det(A, s, t) = \emptyset$ if and only if $sR(A)t$. Let Λ be the set of all $\tau : \wp(W) \times W \longrightarrow \wp(W)$ such that for all $A \in \wp(W)$, $\{s \in W : \tau(A, s) \neq \emptyset\}$ is finite. Let (W', R') be the couple where

– $W' = W \times \Lambda$,
– $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $(s, \sigma), (t, \tau) \in W'$,
  - $(s, \sigma)R'(A')(t, \tau)$ if and only if for all $A \in \wp(W)$,
    - * if $A' \cap (A \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(A, u) \oplus \tau(A, u) : u \in A\} = \det(A, s, t)$,
    - * for all $(u, v) \in A' \cap (A \times \Lambda)$, $\sigma(A, u) \oplus \tau(A, u) = \emptyset$,

    where $\bigoplus\{\sigma(A, u) \oplus \tau(A, u) : u \in A\}$ denotes $\sigma(A, u_1) \oplus \tau(A, u_1) \oplus \ldots \oplus \sigma(A, u_N) \oplus \tau(A, u_N)$, $(u_1, \ldots, u_N)$ being the list of all $u \in A$ such that $\sigma(A, u) \neq \tau(A, u)$.

Claim. For all $A' \in \wp(W')$, $R'(A') = \bigcap\{R'(\{(u, v)\}) : (u, v) \in A'\}$.

Proof. Let $A' \in \wp(W')$. We demonstrate $R'(A') \supseteq \bigcap\{R'(\{(u, v)\}) : (u, v) \in A'\}$, the "$\subseteq$" direction being left as an exercise for the reader. Arguing by contradiction, suppose $R'(A') \not\supseteq \bigcap\{R'(\{(u, v)\}) : (u, v) \in A'\}$. Hence, there exists $(s, \sigma), (t, \tau) \in W'$ such that not $(s, \sigma)R'(A')(t, \tau)$ and for all $(u, v) \in A'$, $(s, \sigma)R'(\{(u, v)\})(t, \tau)$. Thus, for all $(u, v) \in A'$ and for all $A \in \wp(W)$,

– if $\{(u, v)\} \cap (A \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(A, v) \oplus \tau(A, v) : v \in A\} = \det(A, s, t)$,
– for all $(v, \omega) \in \{(u, v)\} \cap (A \times \Lambda)$, $\sigma(A, v) \oplus \tau(A, v) = \emptyset$.

Consequently, for all $A \in \wp(W)$,

- if $A' \cap (A \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(A,v) \oplus \tau(A,v) : v \in A\} = \det(A,s,t)$,
- for all $(v,\omega) \in A' \cap (A \times \Lambda)$, $\sigma(A,v) \oplus \tau(A,v) = \emptyset$.

Hence, $(s,\sigma)R'(A')(t,\tau)$: a contradiction.

Claim 6 is a consequence of Claim 6 and of the fact that for all $A \in \wp(W)$ and for all $s,t,u \in W$, $\det(A,s,s) = \emptyset$ and $\det(A,s,t) \oplus \det(A,s,u) = \det(A,t,u)$.

Claim. (W',R') is a conjunctive frame of indiscernibility.

Let $f : W' \longrightarrow W$ be such that for all $(s,\sigma) \in W'$, $f(s,\sigma) = s$.

Claim. $f : W' \longrightarrow W$ is surjective.

Notice that for all $A \in \wp(W)$, $f^{-1}[A] = A \times \Lambda$.

Claim. For all $(s,\sigma),(t,\tau) \in W'$ and for all $A' \in \wp(W')$, if $(s,\sigma)R'(A')(t,\tau)$ then $sR(f[A'])t$.

Proof. Let $(s,\sigma),(t,\tau) \in W'$ and $A' \in \wp(W')$. Suppose $(s,\sigma)R'(A')(t,\tau)$. Arguing by contradiction, suppose not $sR(f[A'])t$. Hence, $f[A'] \neq \emptyset$. Thus, $A' \cap (f[A'] \times \Lambda) \neq \emptyset$. Since $(s,\sigma)R'(A')(t,\tau)$, $\bigoplus\{\sigma(f[A'],u) \oplus \tau(f[A'],u) : u \in f[A']\} = \det(f[A'],s,t)$. Moreover, for all $(u,v) \in A' \cap (f[A'] \times \Lambda)$, $\sigma(f[A'],u) \oplus \tau(f[A'],u) = \emptyset$. Consequently, for all $u \in f[A']$, $\sigma(f[A'],u) \oplus \tau(f[A'],u) = \emptyset$. Hence, $\bigoplus\{\sigma(f[A'],u) \oplus \tau(f[A'],u) : u \in f[A']\} = \emptyset$. Since $\bigoplus\{\sigma(f[A'],u) \oplus \tau(f[A'],u) : u \in f[A']\} = \det(f[A'],s,t)$, $\det(f[A'],s,t) = \emptyset$. Thus, $sR(f[A'])t$: a contradiction.

Claim. For all $(s,\sigma) \in W'$, for all $t \in W$ and for all $A' \in \wp(W')$, if $sR(f[A'])t$ then there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$.

Proof. Let $(s,\sigma) \in W'$, $t \in W$ and $A' \in \wp(W')$. Suppose $sR(f[A'])t$. We demonstrate there exists $\tau \in \Lambda$ such that $(s,\sigma)R'(A')(t,\tau)$. Indeed, we are looking for a $\tau : \wp(W) \times W \longrightarrow \wp(W)$ such that for all $B \in \wp(W)$,

$(\mathbf{C_0})$ $\{u \in W : \tau(B,u) \neq \emptyset\}$ is finite,
$(\mathbf{C_1})$ if $A' \cap (B \times \Lambda) \neq \emptyset$ then $\bigoplus\{\sigma(B,u) \oplus \tau(B,u) : u \in B\} = \det(B,s,t)$,
$(\mathbf{C_2})$ for all $(v,\omega) \in A' \cap (B \times \Lambda)$, $\sigma(B,v) \oplus \tau(B,v) = \emptyset$.

For all $B \in \wp(W)$, let $\tau^B : W \longrightarrow \wp(W)$ be defined as follows:

Case "$B \subseteq f[A']$"**:** for all $v \in W$, let $\tau^B(v) = \sigma(B,v)$,

Case "$B \not\subseteq f[A']$"**:** let $v^B \in W$ be such that $v^B \in B$ and $v^B \not\in f[A']$ and for all $v \in W$,

- if $v \neq v^B$ then let $\tau^B(v) = \sigma(B,v)$,
- otherwise, let $\tau^B(v) = \sigma(B,v) \oplus \det(B,s,t)$.

Let $\tau : \wp(W) \times W \longrightarrow \wp(W)$ be such that for all $B \in \wp(W)$ and for all $v \in W$, $\tau(B,v) = \tau^B(v)$. Now, we just have to verify that for all $B \in \wp(W)$, $(\mathbf{C_0})$, $(\mathbf{C_1})$ and $(\mathbf{C_2})$ hold. Let $B \in \wp(W)$. Concerning $(\mathbf{C_0})$, it holds, seeing that $\tau^B(v) = \sigma(B,v)$ for every $v \in W$ except when $B \not\subseteq f[A']$ and $v = v^B$. About $(\mathbf{C_1})$, suppose $A' \cap (B \times A) \neq \emptyset$ and consider the following two cases: "$B \subseteq f[A']$" and "$B \not\subseteq f[A']$". In the former case, since $sR(f[A'])t$, $sR(B)t$. Hence, $\det(B,s,t) = \emptyset$. Since $B \subseteq f[A']$, for all $v \in W$, $\tau^B(v) = \sigma(B,v)$. Thus, for all $w \in W$, $\sigma(B,v) \oplus \tau(B,v) = \emptyset$. Consequently, $\bigoplus\{\sigma(B,v) \oplus \tau(B,v) : v \in B\} = \emptyset$. Since $\det(B,s,t) = \emptyset$, $(\mathbf{C_1})$ holds. In the latter case, $\tau^B(v) = \sigma(B,v)$ for every $v \in W$ except when $v = v^B$. Hence, $\bigoplus\{\sigma(B,v) \oplus \tau(B,v) : v \in B\} = \sigma(B,v^B) \oplus \tau(B,v^B)$. Since $\tau^B(v^B) = \sigma(B,v^B) \oplus \det(B,s,t)$, $(\mathbf{C_1})$ holds. As for $(\mathbf{C_2})$, it holds, seeing that for all $v \in W$, if $v \in B$ and $v \in f[A']$ then $\tau^B(v) = \sigma(B,v)$.

Claim 6 is a consequence of Claims 6 and 6.

Claim. $f : W' \longrightarrow W$ is a bounded morphism from (W', R') to (W, R).

Proposition 4 is a consequence of Lemmas 5, 17 and 18 and Proposition 3.

Proposition 4. – $\mathbf{K_c}$ *is complete with respect to the class of all preconjunctive frames and the class of all conjunctive frames,*
– $\mathbf{S5_c}$ *is complete with respect to the class of all preconjunctive frames of indiscernibility and the class of all conjunctive frames of indiscernibility.*

7 Filtrations

The *equivalence setting determined by a model* (W, R, V) *and a closed set* Σ *of formulas* is the equivalence relation $\bowtie$ on W defined by

– $s \bowtie t$ if and only if for all formulas φ in Σ, $s \in V(\varphi)$ if and only if $t \in V(\varphi)$.

For all models (W, R, V), for all closed sets Σ of formulas and for all $s \in W$, the equivalence class of s modulo $\bowtie$ will be denoted $[s]$. For all models (W, R, V), for all closed sets Σ of formulas and for all $A \in \wp(W)$, the quotient of A modulo $\bowtie$ will be denoted $A/\bowtie$. A model (W', R', V') is a *filtration of a model* (W, R, V) *with respect to a closed set* Σ *of formulas* if

– $W' = W/\bowtie$,
– for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$,
  • if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$,
  • if $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$,
– for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Lemma 19. *If the model* (W', R', V') *is a filtration of the model* (W, R, V) *with respect to a closed set* Σ *of formulas then for all formulas* φ*, if* $\varphi \in \Sigma$ *then* $V'(\varphi) = V(\varphi)/\bowtie$*.*

Now, let us turn to the decidability of the satisfiability problem on the class of all frames, the class of all frames of indiscernibility, the class of all conjunctive frames and the class of all conjunctive frames of indiscernibility. In this respect, Lemmas 20–23 will be our key results.

Lemma 20. *Let Σ be a closed set of formulas and (W, R, V) be a model. There exists a model (W', R', V') such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. This proof ends after Claim 7. Let (W', R', V') be a model such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$,
  - $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie = A'$ then there exists $u, v \in W$ such that $s \bowtie u$, $t \bowtie v$ and $uR(V(\varphi))v$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Claim. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$. Let $s, t \in W$. Suppose $sR(V(\varphi))t$. We demonstrate $[s]R'(V(\varphi)/\bowtie)[t]$. Arguing by contradiction, suppose not $[s]R'(V(\varphi)/\bowtie)[t]$. Hence, there exists formulas φ', ψ' such that $\langle\varphi'\rangle\psi' \in \Sigma$, $V(\varphi')/\bowtie = V(\varphi)/\bowtie$ and for all $u, v \in W$, if $s \bowtie u$ and $t \bowtie v$ then not $uR(V(\varphi'))v$. Thus, $V(\varphi') = V(\varphi)$. Moreover, not $sR(V(\varphi'))t$. Consequently, not $sR(V(\varphi))t$: a contradiction.

Claim. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$. Let $s, t \in W$. Suppose $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$. We demonstrate $s \in V(\langle\varphi\rangle\psi)$. Since $[s]R'(V(\varphi)/\bowtie)[t]$, there exists $u, v \in W$ such that $s \bowtie u$, $t \bowtie v$ and $uR(V(\varphi))v$. Since $t \in V(\psi)$, $v \in V(\psi)$. Since $uR(V(\varphi))v$, $u \in V(\langle\varphi\rangle\psi)$. Since $s \bowtie u$, $s \in V(\langle\varphi\rangle\psi)$.

Lemma 21. *Let Σ be a closed set of formulas and (W, R, V) be a model of indiscernibility. There exists a model (W', R', V') of indiscernibility such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. This proof ends after Claim 7. Let (W', R', V') be a model of indiscernibility such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$,
  - $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie = A'$ then $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Claim. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $sR(V(\varphi))t$ then $[s]R'(V(\varphi)/\bowtie)[t]$. Let $s, t \in W$. Suppose $sR(V(\varphi))t$. We demonstrate $[s]R'(V(\varphi)/\bowtie)[t]$. Arguing by contradiction, suppose not $[s]R'(V(\varphi)/\bowtie)[t]$. Hence, there exists formulas φ', ψ' such that $\langle\varphi'\rangle\psi' \in \Sigma$, $V(\varphi')/\bowtie = V(\varphi)/\bowtie$ and either $s \in V(\langle\varphi'\rangle\psi')$ and $t \notin V(\langle\varphi'\rangle\psi')$, or $s \notin V(\langle\varphi'\rangle\psi')$ and $t \in V(\langle\varphi'\rangle\psi')$. Without loss of generality, suppose $s \in V(\langle\varphi'\rangle\psi')$ and $t \notin V(\langle\varphi'\rangle\psi')$. Thus, there exists $u \in W$ such that $sR(V(\varphi'))u$ and $u \in V(\psi')$. Since $V(\varphi')/\bowtie = V(\varphi)/\bowtie$, $V(\varphi') = V(\varphi)$. Since $t \notin V(\langle\varphi'\rangle\psi')$ and $u \in V(\psi')$, not $tR(V(\varphi'))u$. Since $sR(V(\varphi'))u$, not $sR(V(\varphi'))t$. Since $V(\varphi') = V(\varphi)$, not $sR(V(\varphi))t$: a contradiction.

Claim. For all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ then for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)$ $[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$.

Proof. Let φ, ψ be formulas. Suppose $\langle\varphi\rangle\psi \in \Sigma$. We demonstrate for all $s, t \in W$, if $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$ then $s \in V(\langle\varphi\rangle\psi)$. Let $s, t \in W$. Suppose $[s]R'(V(\varphi)/\bowtie)[t]$ and $t \in V(\psi)$. We demonstrate $s \in V(\langle\varphi\rangle\psi)$. Since $[s]R'(V(\varphi)/\bowtie)$ $[t]$, $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$. Since $t \in V(\psi)$, $t \in V(\langle\varphi\rangle\psi)$. Since $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$, $s \in V(\langle\varphi\rangle\psi)$.

Lemma 22. *Let Σ be a closed set of formulas and (W, R, V) be a paraconjunctive model. There exists a paraconjunctive model (W', R', V') such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. Let (W', R', V') be a model such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$,
  - $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie \subseteq A'$ then there exists $u, v \in W$ such that $s \bowtie u$, $t \bowtie v$ and $uR(V(\varphi))v$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Now, the rest of the proof is similar to the corresponding rest of the proof of Lemma 20, the main difference being that one has to verify here that (W', R', V') is a paraconjunctive model, an exercise that we leave for the reader.

Lemma 23. *Let Σ be a closed set of formulas and (W, R, V) be a paraconjunctive model of indiscernibility. There exists a paraconjunctive model (W', R', V') of indiscernibility such that (W', R', V') is a filtration of (W, R, V) with respect to Σ.*

Proof. Let (W', R', V') be a model of indiscernibility such that

- $W' = W/\bowtie$,
- $R' : \wp(W') \longrightarrow \wp(W' \times W')$ is such that for all $A' \in \wp(W')$ and for all $s, t \in W$,
  - $[s]R'(A')[t]$ if and only if for all formulas φ, ψ, if $\langle\varphi\rangle\psi \in \Sigma$ and $V(\varphi)/\bowtie \subseteq A'$ then $s \in V(\langle\varphi\rangle\psi)$ if and only if $t \in V(\langle\varphi\rangle\psi)$,
- for all atomic formulas p, if $p \in \Sigma$ then $V'(p) = V(p)/\bowtie$.

Now, the rest of the proof is similar to the corresponding rest of the proof of Lemma 21, the main difference being that one has to verify here that (W', R', V') is a paraconjunctive model, an exercise that we leave for the reader.

Proposition 5 is a consequence of [4, Theorem 6.7] and Lemmas 5, 17, 18 and 20–23.

Proposition 5. *The satisfiability problem is decidable on the following classes of frames:*

- *the class of all frames,*
- *the class of all frames of indiscernibility,*
- *the class of all conjunctive frames,*
- *the class of all conjunctive frames of indiscernibility.*

By Ladner's Theorem, for the propositional modal language based on the connective $\Diamond$ of arity 1, the satisfiability problem is $\mathbf{PSPACE}$-hard on the class of all Kripke models [4, Theorem 6.50].

Proposition 6. *For the language $\mathcal{L}$, the satisfiability problem is $\mathbf{PSPACE}$-hard on the class of all frames.*

Proof. We prove this by giving a reduction of the satisfiability problem on the class of all Kripke models for the propositional modal language based on the connective $\Diamond$ of arity 1. Let q be a fixed atomic formula. Let $\mathtt{tr}_q$ be the translation from the propositional modal language based on the connective $\Diamond$ of arity 1 to the language $\mathcal{L}$ defined as follows:

- $\mathtt{tr}_q(p) = p$,
- $\mathtt{tr}_q(\bot) = \bot$,
- $\mathtt{tr}_q(\neg\varphi) = \neg\mathtt{tr}_q(\varphi)$,
- $\mathtt{tr}_q(\varphi \vee \psi) = \mathtt{tr}_q(\varphi) \vee \mathtt{tr}_q(\psi)$,
- $\mathtt{tr}_q(\Diamond\varphi) = \langle q\rangle\mathtt{tr}_q(\varphi)$.

As the reader may easily verify, for all q-free formulas φ of the propositional modal language based on the connective $\Diamond$ of arity 1, φ is satisfiable in the class of all Kripke models if and only if $\mathtt{tr}_q(\varphi)$ is satisfiable in the class of all frames. Since, as mentioned above, the satisfiability problem on the class of all Kripke models for the propositional modal language based on the connective $\Diamond$ of arity 1 is $\mathbf{PSPACE}$-hard, the satisfiability problem on the class of all frames for the language $\mathcal{L}$ is $\mathbf{PSPACE}$-hard.

Conjecture 1. We believe the satisfiability problem is in $\mathbf{PSPACE}$ for the language $\mathcal{L}$ on the class of all frames and the class of all frames of indiscernibility.

As is well-known, for the propositional modal language based on the connective $\Diamond$ of arity 1 and enriched with the global modality $\exists$ of arity 1, the satisfiability problem is $\mathbf{EXPTIME}$-hard on the class of all Kripke models [4, Exercise 6.8.1].

Proposition 7. *For the language $\mathcal{L}$, the satisfiability problem is $\mathbf{EXPTIME}$-hard on the class of all conjunctive frames.*

Proof. We prove this by giving a reduction of the satisfiability problem on the class of all Kripke models for the propositional modal language based on the connective $\Diamond$ of arity 1 and enriched with the global modality $\exists$ of arity 1. Let q be a fixed atomic formula. Let $\mathtt{tr}_q^{\exists}$ be the translation from the propositional modal language based on the connective $\Diamond$ of arity 1 and enriched with the global modality $\exists$ of arity 1 to the language $\mathcal{L}$ defined as follows:

- $\mathtt{tr}_q^{\exists}(p) = p$,
- $\mathtt{tr}_q^{\exists}(\bot) = \bot$,
- $\mathtt{tr}_q^{\exists}(\neg\varphi) = \neg\mathtt{tr}_q^{\exists}(\varphi)$,
- $\mathtt{tr}_q^{\exists}(\varphi \vee \psi) = \mathtt{tr}_q^{\exists}(\varphi) \vee \mathtt{tr}_q^{\exists}(\psi)$,
- $\mathtt{tr}_q^{\exists}(\Diamond\varphi) = \langle q\rangle\mathtt{tr}_q^{\exists}(\varphi)$,
- $\mathtt{tr}_q^{\exists}(\exists\varphi) = \langle\bot\rangle\mathtt{tr}_q^{\exists}(\varphi)$.

As the reader may easily verify, for all q-free formulas φ of the propositional modal language based on the connective $\Diamond$ of arity 1 and enriched with the global modality $\exists$ of arity 1, φ is satisfiable in the class of all Kripke models if and only if $\mathtt{tr}_q^{\exists}(\varphi)$ is satisfiable in the class of all conjunctive frames. Since, as mentioned above, the satisfiability problem on the class of all Kripke models for the propositional modal language based on the connective $\Diamond$ of arity 1 and enriched with the global modality $\exists$ of arity 1 is $\mathbf{EXPTIME}$-hard, the satisfiability problem on the class of all conjunctive frames for the language $\mathcal{L}$ is $\mathbf{EXPTIME}$-hard.

Conjecture 2. We believe the satisfiability problem is in $\mathbf{EXPTIME}$ for the language $\mathcal{L}$ on the class of all conjunctive frames and the class of all conjunctive frames of indiscernibility.

8 Conclusion

Our motive for introducing UPMLs lies in the fact that in an application domain such as reasoning about knowledge where states and agents have been identified as the primitive entities of interest, with respect to many situations, one would like to use relational structures of the form $(S, A, \equiv, \triangleright)$ where one can find

- a nonempty set S of states,
- a nonempty set A of agents,
- a function $\equiv$ associating an equivalence relation $\equiv_a$ on S to every element a of A,
- a function $\triangleright$ associating a binary relation $\triangleright_s$ on A to every element s of S.

Which situations? Situations where relationships between states and relationships between agents such as the following ones have to be taken into account: "agent a cannot distinguish between states s and t", "agent a trusts agent b in state s", etc [10,15,16,21]. In these situations,

- for all $a \in A$, two states s and t are related by $\equiv_a$ exactly when agent a cannot distinguish between s and t,
- for all $s \in S$, two agents a and b are related by $\triangleright_s$ exactly when a trusts b in state s.

Moreover, one will naturally assume that

- $\equiv$ is also a function associating an equivalence relation $\equiv_B$ on S to every $B \in \wp(A)$ in such a way that for all $B \in \wp(A)$, $\equiv_B = \bigcap\{\equiv_a\colon\ a \in B\}$,
- $\triangleright$ is also a function associating a binary relation $\triangleright_T$ on A to every $T \in \wp(S)$ in such a way that for all $T \in \wp(S)$, $\triangleright_T = \bigcap\{\triangleright_s :\ s \in T\}$.

The modal language interpreted over relational structures of the form $(S, A, \equiv, \triangleright)$ will naturally consist of two types of formulas: state-formulas — to be interpreted by sets of states — and agent-formulas — to be interpreted by sets of agents. State-formulas will be constructed over the Boolean connectives and the modal connectives $[\alpha]$ — α ranging over the set of all agent-formulas — whereas agent-formulas will be constructed over the Boolean connectives and the modal connectives $[\varphi]$ — φ ranging over the set of all state-formulas. In some model,

- the state-formula $[\alpha]\varphi$ will be true in a state s if the state-formula φ is true in every state of that model that can be distinguished from state s by no α-agents,
- the agent-formula $[\varphi]\alpha$ will be true in an agent a if the agent-formula α is true in every agent of that model that is trusted by agent a at all φ-states.

Within the context of a two-typed parametrized modal language, Balbiani and Fernández González [3] have defined a parametrized modal logic (PML) as a couple whose components are sets of formulas containing, in their respective types, all propositional tautologies and the distribution axiom and closed, in their respective types, under modus ponens, uniform substitution and generalization. They have axiomatically introduced different two-typed PMLs and they have proved their completeness with respect to appropriate classes of two-typed frames by means of an adaptation of the canonical model construction. The UPMLs introduced in this paper constitute the unidimensional version of the PMLs introduced in [3]. Interesting avenues of research about UPMLs and PMLs might consist in

- importing first-order ideas into UPMLs and PMLs (constructs of hybrid logics [2,8], the difference operator [4, Section 7.1], etc.),
- developing the model theory of UPMLs and PMLs (classical definition of bisimulations [4, Section 2.2], classical definition of saturated models [4, Section 2.6], etc.),
- elaborating the correspondence theory of UPMLs and PMLs (analogue of Sahlqvist Correspondence Theorem [4, Section 3.6], analogues of Chagrova's Theorems [5,7], etc.),
- investigating the computability of the satisfiability problem in such-and-such class of frames and developing automatic procedures for solving it (filtration method [6, Chapter 5], tableaux-based approach [13], etc.),
- comparing UPMLs and PMLs with other forms of modal logics based on parametrized connectives (knowledge representation logics [9,17,24], Boolean modal logic [11,12], etc.),
- constructing the duality theory of UPMLs and PMLs (standard definition of Boolean algebras with operators [14, Section 2.2], standard definition of general frames [14, Section 4.6], etc.).

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Appendix

This Appendix includes the proofs of some of our results. Most of these proofs are relatively simple and we have included them here just for the sake of the completeness.

Proof of Lemma 4. Similar to the proof of Bounded Morphism Lemma [4, Proposition 2.14].
Proof of Lemma 5. Consequence of Lemma 4.

Proof of Lemma 6. Similar to the proof of Lindenbaum's Lemma [6, Lemma 5.1].

Proof of Lemma 7. Similar to the proof of Existence Lemma [14, Proposition 2.8.4].

Proof of Lemma 8. Consequence of Lemma 6.

Proof of Lemma 10. Consequence of the fact that $\mathbf{S5_g}$ contains all formulas of the form $[\varphi]\psi \rightarrow \psi$ and $\langle\varphi\rangle\psi \rightarrow [\varphi]\langle\varphi\rangle\psi$.

Proof of Lemma 11. The proof that V_g satisfies the conditions for $\bot$, $\neg$ and $\vee$ is as expected. We only show that V_g satisfies the condition for $\langle\cdot\rangle$. Let φ, ψ be a formulas. Let $s \in W_g$. We only demonstrate $s \in V_g(\langle\varphi\rangle\psi)$ only if there exists $t \in W_g$ such that $sR_g(V_g(\varphi))t$ and $t \in V_g(\psi)$, the "if" direction being left as an exercise for the reader. Suppose $s \in V_g(\langle\varphi\rangle\psi)$. We demonstrate there exists $t \in W_g$ such that $sR_g(V_g(\varphi))t$ and $t \in V_g(\psi)$. Since $s \in V_g(\langle\varphi\rangle\psi)$, $\langle\varphi\rangle\psi \in s$. Let $t_0 = [\varphi]s \cup \{\psi\}$. Notice that $[\varphi]s \subseteq t_0$ and $\psi \in t_0$. By Lemma 7, t_0 is a $\mathbf{L}$-consistent set of formulas. Hence, by Lemma 6, let t be a maximal $\mathbf{L}$-consistent set of formulas such that $t_0 \subseteq t$. Since $[\varphi]s \subseteq t_0$ and $\psi \in t_0$, $[\varphi]s \subseteq t$ and $\psi \in t$. Thus, $t \in V_g(\psi)$.

Claim. $sR_g(V_g(\varphi))t$.

Proof. We demonstrate for all formulas φ', if $\widehat{\varphi'} = V_g(\varphi)$ then $[\varphi']s \subseteq t$. Let φ' be a formula. Suppose $\widehat{\varphi'} = V_g(\varphi)$. We demonstrate $[\varphi']s \subseteq t$. Let ψ' be a formula. Suppose $[\varphi']\psi' \in s$. We demonstrate $\psi' \in t$. Since $\widehat{\varphi'} = V_g(\varphi)$, by Lemma 8, $\varphi' \leftrightarrow \varphi \in \mathbf{L}$. Hence, $[\varphi']\psi' \leftrightarrow [\varphi]\psi' \in \mathbf{L}$. Since $[\varphi']\psi' \in s$, $[\varphi]\psi' \in s$. Since $[\varphi]s \subseteq t$, $\psi' \in t$.

Proof of Lemma 12. Consequence of the fact that $\mathbf{L}$ contains all formulas of the form $[\bot]\varphi \rightarrow \varphi$ and $\langle\bot\rangle\varphi \rightarrow [\bot]\langle\bot\rangle\varphi$.

Proof of Lemma 13. Let φ, ψ be formulas.

Suppose $\widehat{\varphi} \subseteq \widehat{\psi}$. We demonstrate for all $s \in W_c$, $[\bot](\varphi \rightarrow \psi) \in s$. Let $s \in W_c$. We demonstrate $[\bot](\varphi \rightarrow \psi) \in s$. Arguing by contradiction, suppose $[\bot](\varphi \rightarrow \psi) \notin s$. Hence, $\langle\bot\rangle(\varphi \wedge \neg\psi) \in s$. Let $u_0 = [\bot]s \cup \{\varphi, \neg\psi\}$. Notice that $[\bot]s \subseteq u_0$, $\varphi \in u_0$ and $\neg\psi \in u_0$. By Lemma 7, u_0 is a $\mathbf{L}$-consistent set of formulas. Thus, by Lemma 6, let u be a maximal $\mathbf{L}$-consistent set of formulas such that $u_0 \subseteq u$. Since $[\bot]s \subseteq u_0$, $\varphi \in u_0$ and $\neg\psi \in u_0$, $[\bot]s \subseteq u$, $\varphi \in u$ and $\neg\psi \in u$. Since $[\bot]s_0 \subseteq s$, by Lemma 12, $[\bot]s_0 \subseteq u$. Consequently, $u \in W_c$. Since $\varphi \in u$ and $\neg\psi \in u$, $u \in \widehat{\varphi}$ and $\psi \notin u$. Since $\widehat{\varphi} \subseteq \widehat{\psi}$, $u \in \widehat{\psi}$. Hence, $\psi \in u$: a contradiction.

Suppose $\widehat{\varphi} = \emptyset$. We demonstrate for all $s, t \in W_c$, $[\varphi]s \subseteq t$. Let $s, t \in W_c$. We demonstrate $[\varphi]s \subseteq t$. Let χ be a formula. Suppose $[\varphi]\chi \in s$. We demonstrate $\chi \in t$. Since $\widehat{\varphi} = \emptyset$, by the previous item, $[\bot](\varphi \rightarrow \bot) \in s$. Thus, $[\varphi]\chi \rightarrow [\bot]\chi \in s$. Since $[\varphi]\chi \in s$, $[\bot]\chi \in s$. Since $[\bot]s_0 \subseteq s$, $\langle\bot\rangle[\bot]\chi \in s_0$. Consequently, $[\bot]\chi \in s_0$. Since $[\bot]s_0 \subseteq t$, $\chi \in t$.

Proof of Lemma 14. Indeed, $R_c(\emptyset) = W_c \times W_c$. Why? Simply because by Lemma 13, for all $s, t \in W_c$ and for all formulas φ, if $\widehat{\varphi} = \emptyset$ then $[\varphi]s \subseteq t$. Hence, for all $s, t \in W_c$, $sR_c(\emptyset)t$. Moreover, for all $A, B \in \wp(W_c)$, if $A \subseteq B$ then $R_c(A) \supseteq R_c(B)$. Why? Simply because for all $A, B \in \wp(W_c)$, if $A \subseteq B$ then for all formulas φ, if $\widehat{\varphi} \subseteq A$ then $\widehat{\varphi} \subseteq B$. Thus, for all $A, B \in \wp(W_c)$, if $A \subseteq B$ then for all $t, u \in W_c$, if $tR_c(B)u$ then $tR_c(A)u$.

Proof of Lemma 15. Consequence of the fact that $\mathbf{S5_c}$ contains all formulas of the form $[\varphi]\psi \rightarrow \psi$ and $\langle\varphi\rangle\psi \rightarrow [\varphi]\langle\varphi\rangle\psi$.

Proof of Lemma 16. The proof that V_c satisfies the conditions for $\bot$, $\neg$ and $\vee$ is as expected. We only show that V_c satisfies the condition for $\langle\cdot\rangle$. Let φ, ψ be formulas. Let $s \in Wc$. We only demonstrate $s \in V_c(\langle\varphi\rangle\psi)$ only if there exists $t \in W_c$ such that $sR_c(V_c(\varphi))t$ and $t \in V_c(\psi)$, the "if" direction being left as an exercise for the reader. Suppose $s \in V_c(\langle\varphi\rangle\psi)$. We demonstrate there exists $t \in W_c$ such that $sR_c(V_c(\varphi))t$ and $t \in V_c(\psi)$. Since $s \in V_c(\langle\varphi\rangle\psi)$, $\langle\varphi\rangle\psi \in s$. Let $t_0 = [\varphi]s \cup \{\psi\}$. Notice that $[\varphi]s \subseteq t_0$ and $\psi \in t_0$. By Lemma 7, t_0 is a $\mathbf{L}$-consistent set of formulas. Hence, by Lemma 6, let t be a maximal $\mathbf{L}$-consistent set of formulas such that $t_0 \subseteq t$. Since $[\varphi]s \subseteq t_0$ and $\psi \in t_0$, $[\varphi]s \subseteq t$ and $\psi \in t$. Thus, $t \in V_c(\psi)$.

Claim. $sR_c(V_c(\varphi))t$.

Proof. We demonstrate for all formulas φ', if $\widehat{\varphi'} \subseteq V_c(\varphi)$ then $[\varphi']s \subseteq t$. Let φ' be a formula. Suppose $\widehat{\varphi'} \subseteq V_c(\varphi)$. We demonstrate $[\varphi']s \subseteq t$. Let ψ' be a formula. Suppose $[\varphi']\psi' \in s$. We demonstrate $\psi' \in t$. Since $\widehat{\varphi'} \subseteq V_c(\varphi)$, by Lemma 13, $[\bot](\varphi' \rightarrow \varphi) \in s$. Hence, $[\varphi']\psi' \rightarrow [\varphi]\psi' \in s$. Since $[\varphi']\psi' \in s$, $[\varphi]\psi' \in s$. Since $[\varphi]s \subseteq t$, $\psi' \in t$.

Proof of Lemma 19. Similar to the proof of Filtration Theorem [4, Theorem 2.39].

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