

On the Parameterized Complexity of Compact Set Packing

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Abstract. The SET PACKING problem is, given a collection of sets \mathcal{S} over a ground set \mathcal{U} , to find a maximum collection of sets that are pairwise disjoint. The problem is among the most fundamental NP-hard optimization problems that have been studied extensively in various computational regimes. The focus of this work is on parameterized complexity, PARAMETERIZED SET PACKING (PSP): Given $r \in \mathbb{N}$, is there a collection $\mathcal{S}' \subseteq \mathcal{S} : |\mathcal{S}'| = r$ such that the sets in \mathcal{S}' are pairwise disjoint? Unfortunately, the problem is not fixed parameter tractable unless $W[1] = FPT$, and, in fact, an “enumerative” running time of $|\mathcal{S}|^{\Omega(r)}$ is required unless the exponential time hypothesis (ETH) fails. This paper is a quest for tractable instances of SET PACKING from parameterized complexity perspectives. We say that the input $(\mathcal{U}, \mathcal{S})$ is “compact” if $|\mathcal{U}| = f(r) \cdot \Theta(\text{poly}(\log |\mathcal{S}|))$, for some $f(r) \geq r$. In the COMPACT PSP problem, we are given a compact instance of PSP. In this direction, we present a “dichotomy” result of PSP: When $|\mathcal{U}| = f(r) \cdot o(\log |\mathcal{S}|)$, PSP is in FPT, while for $|\mathcal{U}| = r \cdot \Theta(\log(|\mathcal{S}|))$, the problem is $W[1]$ -hard; moreover, assuming ETH, COMPACT PSP does not admit $|\mathcal{S}|^{o(r/\log r)}$ time algorithm even when $|\mathcal{U}| = r \cdot \Theta(\log(|\mathcal{S}|))$. Although certain results in the literature imply hardness of compact versions of related problems such as SET r -COVERING and EXACT r -COVERING, these constructions fail to extend to COMPACT PSP. A novel contribution of our work is the identification and construction of a gadget, which we call Compatible Intersecting Set System pair, that is crucial in obtaining the hardness result for COMPACT PSP. Finally, our framework can be extended to obtain improved running time lower bounds for COMPACT r -VECTORSUM.

1 Introduction

Given a graph $G = (V, E)$, the problem of finding a maximum-size subset of disjoint edges (matching) is tractable, but its generalization to hypergraphs, even when the edge length is 3, is NP-hard. This general problem is known as the Hypergraph Matching problem. The hyper-graph $H = (W, F)$ can be equivalently viewed as a set system $(\mathcal{U}, \mathcal{S})$, where the universe (or the ground set) \mathcal{U} corresponds to the vertex set W and \mathcal{S} corresponds to the collection of hyperedges F . Then finding a maximum matching in H is equivalent to finding maximum number of pairwise disjoint sets (packing) in \mathcal{S} . Hence the Hypergraph Matching problem is also known as the SET PACKING problem, which is a fundamental problem in combinatorial optimization with numerous applications. While this

problem captures many classical combinatorial problems such as maximum independent set (or maximum clique), k -dimensional matching and also, some graph packing problems [9,16], this generalization also makes it intractable in several regimes. One computational regime in which SET PACKING has been explored extensively is approximation algorithms. Since SET PACKING generalizes the maximum independent set problem [1], it inherits the inapproximability of the latter problem [17]. This immediately implies that the trivial approximation of picking simply one set in the packing is roughly the best to hope for. Furthermore, approximations in terms of $|\mathcal{U}|$ are also not hopeful since the result also implies inapproximability bound of $|\mathcal{U}|^{1/2-\epsilon}$, which is matched by [15]. To combat these intractabilities, various restrictions of SET PACKING have been studied. Particularly, a restriction where the size of the sets in \mathcal{S} is bounded by some integer k , which is known as k -Set Packing, is also a well-studied problem. However, k -Set Packing captures the independent set problem in bounded degree graphs, which again is a notoriously hard problem to approximate beyond the “trivial” bound [2,3]. While [18] improves the lower bound for k -Set Packing to $\Omega(k/\ln k)$, the best known approximation is $(k+1+\epsilon)/3$ [8,11], yielding a logarithmic gap between the bounds. Besides approximation algorithms, SET PACKING has also been studied from the parameterized complexity perspectives (with the standard parameter on the size of an optimal packing solution). In this problem, known as the PARAMETERIZED SET PACKING (PSP) problem, we are given an instance $(\mathcal{U}, \mathcal{S}, r)$ and the task is to decide if there exists a packing of size r . Unfortunately, even PSP remains intractable and is, actually, W[1]-complete [13]. In fact, *Exponential Time Hypothesis* (ETH) implies that the trivial *enumerative algorithm* running in $O^*(|\mathcal{S}|^r)$ time to find an r -packing is asymptotically our best hope [12]. The algorithmic outlook for PSP worsens further due to [7], which rules out $o(r)$ -FPT-approximation algorithm assuming the *Gap Exponential Time Hypothesis* (Gap-ETH). Assuming a weaker hypothesis of $\text{FPT} \neq \text{W}[1]$, very recently [24] showed that there is no FPT algorithm for PSP problem that finds a packing of size $r/r^{1/H(r)}$, for any increasing function $H(\cdot)$, when given a promise that there is an r -packing in the instance. Thus, the flurry of these negative results make it likely that SET PACKING problem is intractable in all computational regimes.

In this paper, we consider PSP on *compact* instances. We say that an instance $(\mathcal{U}, \mathcal{S}, r)$ of PSP is compact if $|\mathcal{U}| = f(r) \cdot \Theta(\text{poly}(\log |\mathcal{S}|))$, for some function $f(r) \geq r$, that is, the universe is relatively small compared to the number of sets¹. Besides the algorithmic motivation, compact instances have recently been used as an “intermediate step” to prove FPT inapproximability results of the (non-compact) classical problems (see, e.g., [4,20] where the compact instances were used in proving FPT-inapproximability of the k -EvenSet and Dominating Set). We hope that studying COMPACT PSP would lead to some ideas that would be

¹ In fact there is another way to define compactness: when $|\mathcal{S}| = f(r) \cdot \Theta(\text{poly}(\log |\mathcal{U}|))$. However in this case, the *enumerative algorithm* running in time $O^*(|\mathcal{S}|^r)$ is already fixed parameter tractable [6]. Thus, the interesting case is when the universe is compact, which is the case we will be focusing on.

useful in proving tight FPT inapproximability of PSP (that is, to weaken the Gap-ETH assumption used in [7]).

1.1 Our Results

Our main result is the following dichotomy of PARAMETERIZED SET PACKING.

Theorem 1 (Dichotomy). *The following dichotomy holds for PSP.*

- If $|\mathcal{U}| = f(r) \cdot o(\log |\mathcal{S}|)$, for any f , then PSP is in FPT.
- PSP remains W[1]-hard even when $|\mathcal{U}| = r \cdot \Theta(\log |\mathcal{S}|)$.

The algorithmic result follows from well-known dynamic programming based algorithms [5,12] that run in time $O^*(2^{|\mathcal{U}|})$, and observing that this running time is fixed parameter tractable [6] when $|\mathcal{U}| = f(r)o(\log |\mathcal{S}|)$. However, for completeness, in Appendix A we present a simple dynamic program algorithm based on finding longest path in a DAG running in time $O^*(2^{|\mathcal{U}|})$. The main contribution of our work is the W[1]-hardness of PSP even when $|\mathcal{U}| = r \cdot \Theta(\log |\mathcal{S}|)$. Towards this, we show an FPT-reduction from SUBGRAPH ISOMORPHISM (SGI) to COMPACT PSP. The hardness result follows since SGI is W[1]-hard. In fact, our hardness result can be strengthened assuming Exponential Time Hypothesis (ETH) [12] to obtain the following result.

Theorem 2. *COMPACT PSP requires time $|\mathcal{S}|^{\Omega(r/\log r)}$ even when $|\mathcal{U}| = r \cdot \Theta(\log |\mathcal{S}|)$, unless ETH fails.*

The result of Theorem 2 follows from the ETH-hardness result of SGI due to [22], and from the fact that the hardness reduction of Theorem 1 is parameter preserving up to a multiplicative constant. Note that since PSP can be trivially solved by enumeration in time $O^*(|\mathcal{S}|^r)$, the above result says that, even for the compact instances this is essentially our best hope, up to a log factor in the exponent. An interesting consequence of the dichotomy theorem coupled with Theorem 2 is the fact that, as soon as instances get asymptotically smaller, not only we beat the enumerative algorithm, but we actually obtain an FPT algorithm. We would like to remark that the universe size in Theorem 2 is tight (up-to $\log r$ factor) since having $|\mathcal{U}| = o(r/\log r) \cdot \Theta(\log |\mathcal{S}|)$ would already allow $|\mathcal{S}|^{o(r/\log r)}$ time algorithm. Further, note that for W[1]-hardness, it is sufficient to have $|\mathcal{U}| = f(r)\Theta(\log |\mathcal{S}|)$, for some f , since we can add $f(r) - r$ new sets each with a unique dummy element and inflate the parameter to $f(r)$. However, this is not true for ETH based running time lower bounds as such inflation fail to transfer the lower bounds asymptotically.

We would like to remark that, after our article was made public, Huairui Chu [10] improved the lower bound of Theorem 2 to rule out $|\mathcal{S}|^{\Omega(r)}$ time.

Finally, we extend our construction framework (Theorem 3) to improve the running time lower bound (matching the trivial upper bound up to a log factor in the exponent) for the compact version of r -VECTORSUM: Given a collection \mathcal{C} of N vectors in \mathbb{F}_2^d , and a target vector $\vec{b} \in \mathbb{F}_2^d$, r -VECTORSUM asks if there are r vectors in \mathcal{C} that sum to \vec{b} . COMPACT r -VECTORSUM is defined when $d = f(r) \cdot \Theta(\text{poly}(\log N))$, for some $f(r) \geq r$.

Theorem 3. COMPACT r -VECTORSUM requires time $N^{\Omega(r/\log r)}$, even when $d = r \cdot \Theta(\log N)$, unless ETH fails.

The present bound of [4] rules out $N^{o(\sqrt{r})}$ time under ETH. The proof of this theorem is present in Appendix B.

1.2 Our contributions and comparison to existing works

In this section, we compare our contribution with existing works to highlight its significance. To our best knowledge, the compact version of combinatorial problems has not previously been formalized and investigated. However, several existing reductions already imply the hardness of compact version of some of the combinatorial problems. Here we review and compare the related results.

Our Contribution. As far as we know, there are no results showing W[1]-hardness of COMPACT PSP, and hence the corresponding dichotomy (Theorem 1). The key contribution of this paper is to show the hardness result for COMPACT PSP. On the way, we also show an ETH-based almost tight running time lower bound for COMPACT PSP, with tight (up-to $\log r$ factor) universe size $|\mathcal{U}| = r \cdot \Theta(\log |\mathcal{S}|)$. Interestingly, we show both of these results with a single FPT reduction. In addition, we extend our framework to improve the running time lower bounds for COMPACT r -VECTORSUM.

Next we survey some known hardness results for SET r -COVERING in the compact regime and argue their limitations in extending them to PSP. In particular, [19, Lemma 25] shows a reduction from SGI to a variant of SET r -COVERING called COMPACT EXACT r -COVERING, where we want to find an r -packing that is also a covering (in fact they show hardness for COMPACT SET r -COVERING. But a closer inspection of their construction shows that the intended set cover is also a packing). The high level idea of the construction is similar to ours: first assign each vertex of G a logarithm length binary pattern vector. Then, create two kinds of sets: V -sets that capture the mapping of the vertices and E -sets that capture the mapping of edges. The idea is to use the pattern vectors to create these sets so that there is an isomorphic copy of H in G if and only if there are $|V_H|$ many V -sets and $|E_H|$ many E -sets covering the universe exactly once. However, if we consider the Soundness (No case) proof of this reduction, then it crucially relies on the fact that no candidate solution can cover the entire universe exactly once. In fact, it is quite easy to find r sets that are mutually disjoint but do not form a cover. Therefore, it fails to yield hardness for COMPACT PSP. The heart of our construction lies in ensuring that in the No case, any r sets intersect. To this end, we construct a combinatorial gadget called Compatible Intersecting Set System (ISS) pair. This gadget is a pair of set systems $(\mathcal{A}, \mathcal{B})$ over a universe U that guarantees two properties: First, every pair of sets within each set system intersects, and second, for any set $a \in \mathcal{A}$, there exists $b \in \mathcal{B}$ such that a intersects every set in \mathcal{B} except b . Further, we present a simple greedy algorithm that finds such compatible ISS pair $(\mathcal{A}, \mathcal{B})$ over a universe of size N , each having roughly $2^{\Omega(N)}$ sets. Note that this

gadget, which we use to build our compact hard instance, also has a “compact” universe. While, on the other hand, [25] shows COMPACT SET r -COVERING is W[1]-hard using a reduction from k -CLIQUE to SET r -COVERING with $r = \Theta(k^2)$ and $|\mathcal{U}| = r^{3/2} \cdot \Theta(\log |\mathcal{S}|)$, but does not yield a tight ETH-based running time lower bound. In contrast, [23] shows such tight ETH lower bound for COMPACT SET r -COVERING: requiring time $|\mathcal{S}|^{\Omega(r)}$, which can be easily modified to obtain a similar running time lower bound for COMPACT EXACT r -COVERING (by reducing from 1-IN-3-SAT, instead from 3-SAT).

1.3 Overview of Techniques

In this section we sketch the main ideas of our hardness proof of Theorem 1. To this end, we present a reduction from SGI, which asks, given a graph G on n vertices and another graph H with k edges, if there is a subgraph of G isomorphic (not necessarily induced) to H , with parameter k . The reduction produces an instance $\mathcal{I} = (\mathcal{U}, \mathcal{S}, r)$ of PSP in FPT time such that $r = \Theta(k)$ and $|\mathcal{U}| = \Theta(r \log |\mathcal{S}|)$. We remark that the classical reduction given in [13] also has parameter $r = \Theta(k)$, but $|\mathcal{U}|$ is linear in the size of G , which is the size of $|\mathcal{S}|$. Below we present a reduction that constructs a compact instance, but falls short in achieving its goal. However, it illustrates some of the main ideas that form the basis of the actual hardness proof. This failed attempt also highlights the crucial properties of the gadget that are necessary for the correct reduction.

Our reduction constructs the instance $\mathcal{I} = (\mathcal{U}, \mathcal{S}, r)$ of PSP using a special set system gadget – which we call the Intersecting Set System (ISS) gadget. A set system $\mathcal{A} = (U_A, S_A)$ with M sets over N elements is called an (M, N) -Intersecting set system, if every pair $s^i, s^j \in S_A$ intersects (i.e., s^i, s^j has a non-empty intersection). We show how to efficiently construct an (M, N) -ISS $\mathcal{A} = (U_A, S_A)$ with $M = 2^{N-1}$. Let $U_A = \{1, 2, \dots, N+1\}$. Then, for every subset $s \subseteq \{2, 3, \dots, N+1\}$, add the set $s' := \{1\} \cup s$ to S_A . Note that \mathcal{A} has a compact universe since $|U_A| = \log_2 M + 1 = \log_2 |S_A| + 1$, which is crucial in constructing a compact instance of PSP. We are now ready to present the reduction using this compact ISS gadget. Let the given instance of SGI be $\mathcal{J} = (G = (V_G, E_G), H = (V_H, E_H), k)$. Let $\ell := |V_H|$, $n := |V_G|$ and $m := |E_G|$. Further, let $V(G) = \{1, \dots, n\}$. Note that $\ell \leq 2k$, since isomorphic sub-graph in G to H is not necessarily induced. Let $\mathcal{A} = (U_A, S_A)$ be the (M, N) -ISS gadget specified above with $N = \lceil \log n \rceil + 1$. Since $M \geq n$, assume $S_A = \{s^\alpha\}_{\alpha \in V(G)}$ by arbitrarily labeling sets in S_A and ignoring the sets $s^\alpha, \alpha > n$. We construct an instance $\mathcal{I} = (\mathcal{U}, \mathcal{S}, r)$ of COMPACT PSP as follows. For every $v \in V_H$, and $w \in N_H(v)$, let $\mathcal{A}_{v,w} = (U_{v,w}, S_{v,w})$ be a distinct copy of ISS \mathcal{A} (that is, the universes $\{U_{v,w}\}_{w \in N_H(v)}$ are disjoint) with the same labeling of sets as that of S_A . Note that $\mathcal{A}_{v,w}$ and $\mathcal{A}_{w,v}$ are distinct copies of \mathcal{A} . Let $U_v := \cup_{w \in N(v)} U_{v,w}$. The universe \mathcal{U} in \mathcal{I} is defined as $\mathcal{U} = \cup_{v \in V_H} U_v$. Now for \mathcal{S} , we will construct two types of sets, that we call V -sets and E -sets. For every $\alpha \in V(G)$ and $v \in V_H$, add the set $S_{\alpha \mapsto v} := \cup_{w \in N(v)} s_{v,w}^\alpha$ to \mathcal{S} . These sets are referred as V -sets. For each edge $(\alpha, \beta) \in E_G$ and each edge $(v, w) \in E_H$, add the set $S_{(\alpha, \beta) \mapsto (v, w)} := \bar{s}_{v,w}^\alpha \cup \bar{s}_{w,v}^\beta$ to \mathcal{S} . These sets are called E -sets. Finally, setting the parameter

$r = \ell + k$, concludes the construction of PSP instance $\mathcal{I} = (\mathcal{U}, \mathcal{S}, r)$. First, note that for the base ISS gadget $\mathcal{A} = (U_A, S_A)$, we have that $|U_A| = \Theta(\log n)$. Hence, $|\mathcal{U}| = \sum_{i \in [\ell]} \sum_{j \in [d(v_i)]} |U_A| = \Theta(k \log n)$, where as $|\mathcal{S}| = \Theta(mk + n\ell) = \Theta(n^2k)$. Since $r = \Theta(k)$, we have $|\mathcal{U}| = \Theta(r \log |\mathcal{S}|)$, yielding a COMPACT PSP instance.

To illustrate the main ideas, we analyze the completeness and discuss how the soundness fails. In the completeness case, let us assume that there exists an injection $\phi : V_H \rightarrow V_G$ which specifies the isomorphic subgraph in G . Consider $T \subseteq \mathcal{S}$ as $T = \{S_{\phi(v) \mapsto v}\}_{v \in V_H} \cup \{S_{(\phi(v), \phi(w)) \mapsto (v, w)}\}_{(v, w) \in E_H}$. Notice that we have chosen $\ell + k$ sets from \mathcal{S} . To see that T forms a packing, fix $v, w \in V_H$ such that $w \in N(v)$. Let $T|_{U_{v, w}}$ be the restriction of T on $U_{v, w}$ (Formally, $T|_{U_{v, w}} := \{t \cap U_{v, w} : t \in T\}$). Then note that $T|_{U_{v, w}}$ forms a packing since $T|_{U_{v, w}} = \{s_{v, w}^{\phi(v)}, \bar{s}_{v, w}^{\phi(v)}\}$.

For soundness, we show the proof for a simpler case. Suppose $T \subseteq \mathcal{S}$ with $|T| = r$ is a packing with at most one V -set from each vertex of G . Further, assume $|T_V| = \ell$ and $|T_E| = k$, where T_V and T_E denote the V -sets and E -sets of T respectively. Finally, we also assume that T covers \mathcal{U} . Let $V_H = \{v_1, \dots, v_\ell\}$. Relabel the sets in T_V as $T_V = \{T_V^i : \exists S_{\alpha \mapsto v_i} \in T_V, \text{ for some } \alpha \in V_G, v_i \in V_H\}$. Now consider $V' := \{\alpha \mid T_V^i = S_{\alpha \mapsto v_i}\} \subseteq V_G$, and relabel the vertices of V' as $V' = \{\alpha'_1, \dots, \alpha'_\ell\}$, where $\alpha'_i = \alpha$ such that $T_V^i = S_{\alpha \mapsto v_i}$. We claim that $G[V']$ is isomorphic to H with injection $\phi : V_H \rightarrow V_G$ such that $\phi(v_i) = \alpha'_i$. To this end, we show $(v_i, v_j) \in E_H \implies (\phi(v_i), \phi(v_j)) \in E_{G[V']}$. Note that since \mathcal{A}_{v_i, v_j} is an ISS, $T_V|_{U_{v_i, v_j}} = T_V^i|_{U_{v_i, v_j}} = s_{v_i, v_j}^{\alpha'_i}$. Further, combining the fact that T is a packing covering \mathcal{U} with the fact $|T_E| = k$, we have that $T_E|_{U_{v_i, v_j}} = \bar{s}_{v_i, v_j}^{\alpha'_i}$. Similarly, it holds that $T_V|_{U_{v_j, v_i}} = T_V^j|_{U_{v_j, v_i}} = s_{v_j, v_i}^{\alpha'_j}$, and hence $T_E|_{U_{v_j, v_i}} = \bar{s}_{v_j, v_i}^{\alpha'_j}$. But this implies that $S_{(\alpha'_i, \alpha'_j) \mapsto (v_i, v_j)} \in T_E$, which means $(\phi(v_i), \phi(v_j)) = (\alpha'_i, \alpha'_j) \in E_{G[V']}$, as desired. However, for the general case, we would require a gadget that enforces all the above assumptions in any candidate packing.

1.4 Open problems.

An interesting direction is FPT approximating COMPACT PSP: Given a promise that there is an r -packing, is it possible to find a packing of size $\omega(1)$ in FPT time? Note that for the general PSP problem, there is no $o(r)$ FPT-approximation, assuming Gap-ETH. However, recent results [21, 24] use a weaker assumption of $W[1] \neq \text{FPT}$ but also obtain weaker FPT-inapproximability. It is also interesting to show such hardness of approximation for COMPACT PSP.

2 Preliminaries

2.1 Notations

For $q \in \mathbb{N}$, denote by $[q]$, the set $\{1, \dots, q\}$. For a finite set $[q]$ and $i \in [q]$, we overload '+' operator and denote by $i+1$ as the (cyclic) successor of i in $[q]$. Thus,

the successor of q is 1 in $[q]$. All the logs are in base 2. For a graph $G = (V, E)$ and a vertex $v \in V$, denote by $N(v)$, the set of vertices adjacent to v . Further, $d(v)$ denotes the degree of v , i.e., $d(v) := |N(v)|$. For a finite universe U and $s \subseteq U$, denote by \bar{s} as the complement of s under U , i.e., $\bar{s} := U \setminus s$. Similarly, for a family of sets $S = \{s_1, \dots, s_M\}$ over U , we denote by $\text{comp}(S) = \{\bar{s}_1, \dots, \bar{s}_M\}$. Further, for a subset $s \subseteq U$ and a sub-universe $U' \subseteq U$, denote by $s|_{U'}$ as the restriction of s on sub-universe U' , i.e., $s|_{U'} := s \cap U'$. Similarly, for a family of sets $S = \{s_1, \dots, s_M\}$ over U , denote by $S|_{U'}$ as the restriction of every set of S on U' , i.e., $S|_{U'} := \{s_1|_{U'}, \dots, s_M|_{U'}\}$. For a set system $A = (U_A, S_A)$, we denote the complement set system by $\bar{A} = (U_A, \text{comp}(S_A))$. For $s, t \subseteq U$, we say s and t intersects if $s \cap t \neq \emptyset$.

2.2 Parameterized Complexity

The parameterized complexity theory concerns computational aspects of languages (L, κ) over a fixed and finite alphabet Σ , where $L \subseteq \Sigma^*$, and $\kappa : \Sigma^* \rightarrow \mathbb{N}$, called the parameter, is a polynomial time computable function. Thus a parameterized problem is a classical problem together with a parameter κ . As an example consider the following classical NP-complete problem.

CLIQUE
Instance: A graph G and $k \in \mathbb{N}$
Problem: Decide if G has a clique of size k

Now consider a parameterized version of CLIQUE defined by $\kappa(G, k) := k$.

k -CLIQUE
Instance: A graph G and $k \in \mathbb{N}$
Parameter: k
Problem: Decide if G has a clique of size k

When the parameter κ represents the size of solution, then it is called *natural* parameter.

Definition 1 (Fixed Parameter Tractable). A parameterized problem (L, κ) is called *fixed parameter tractable* if there is an algorithm A , a constant c and a computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that on all inputs $y = (x, k)$, A decides whether x is in L and runs in time at most $f(k) \cdot |x|^c$.

The complexity class FPT is the set of all fixed parameter tractable problems. In this paper, we consider parameterized problems with natural parameter i.e. κ represents the size of solution. Once we define the class FPT, the next natural thing is to define *parameterized reduction* or *FPT-reduction* with the intention that such a reduction from parameterized problem Q to another parameterized problem Q' allows converting an FPT algorithm of Q' to an FPT algorithm of Q .

Definition 2 (FPT-reduction). An *FPT-reduction* from $Q \subseteq \Sigma^* \times \mathbb{N}$ to $Q' \subseteq \Sigma^* \times \mathbb{N}$ is an algorithm R mapping from $\Sigma^* \times \mathbb{N}$ to $\Sigma^* \times \mathbb{N}$ such that for all $y = (x, k)$, $R(y) \in Q'$ if and only if $y \in Q$, and for some computable function

f and a constant c , $R(y)$ runs in time $f(k) \cdot |x|^c$ and $R(y) = (x', k')$, where $k' \leq g(k)$ for some computable function g .

An extensive treatment of the subject can be found in [12,13,14].

2.3 Problem definitions

Definition 3 (PARAMETERIZED SET PACKING (PSP)). *Given a collection of sets $\mathcal{S} = \{S_1, \dots, S_m\}$ over an universe $\mathcal{U} = \{e_1, \dots, e_n\}$ and an integer r , the PSP problem asks if there is a collection of sets $\mathcal{S}' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| = r$ and, $S_i \cap S_j = \emptyset$ for every $S_i \neq S_j \in \mathcal{S}'$. An instance of PSP is denoted as $(\mathcal{U}, \mathcal{S}, r)$.*

COMPACT PSP is defined when the instances have $|\mathcal{U}| = f(r) \cdot \Theta(\text{poly}(\log |\mathcal{S}|))$, for some function $f(r) \geq r$.

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a homomorphism from H to G is a map $\phi : V_H \rightarrow V_G$ such that if $(v_i, v_j) \in E_H$ then $(\phi(v_i), \phi(v_j)) \in E_G$.

Definition 4 (SUBGRAPH ISOMORPHISM (SGI)). *Given a graph $G = (V_G, E_G)$ and a smaller graph $H = (V_H, E_H)$ with $|E_H| = k$, the SGI problem asks if there is an injective homomorphism from H to G . An instance of SGI is denoted as $(G = (V_G, E_G), H = (V_H, E_H), k)$.*

The parameterized version of SGI has parameter $\kappa = |E_H| = k$. Without loss of generality, we assume $|V_H| \leq 2k$, and every vertex of H has degree at most k .

3 Dichotomy of PSP

In this section we prove the hardness part of the dichotomy theorem (Theorem 1). First in Section 3.1, we identify the gadget and its associated properties that are crucial for the reduction. Then, in Section 3.2, using this gadget, we show an FPT-reduction from SGI to COMPACT PSP.

3.1 Compatible Intersecting Set System Pair

A set system $\mathcal{A} = (U_A, S_A)$ is called an (M, N) -Intersecting set system (ISS), if it contains M sets over N elements such that every pair $s, t \in S_A$ intersects.

Definition 5 (Compatible ISS pair). *Given two ISS $\mathcal{A} = (U, S_A)$ and $\mathcal{B} = (U, S_B)$ on a universe U , we say that $(\mathcal{A}, \mathcal{B})$ is a compatible ISS pair if there exists an efficiently computable bijection $f : S_A \rightarrow S_B$ such that*

- (Complement partition) $\forall s \in S_A$, s and $f(s)$ forms a partition of U , and
- (Complement exchange) $\forall s \in S_A$, $\mathcal{A}_s := (U, (S_A \setminus \{s\}) \cup \{f(s)\})$ is an ISS.

Since f is as bijection, we have $|S_A| = |S_B|$, and $\forall t \in S_B$, the set system $\mathcal{B}_t := (U, (S_B \setminus \{t\}) \cup \{f^{-1}(t)\})$ is also an ISS. Also, for $(s, t) \in (S_A, S_B)$ if $s \cup t = U$, then $t = f(s)$. The following lemma efficiently constructs a compatible (M, N) -ISS pair, which is a key ingredient in our hardness proof.

Lemma 1. *For even $N \geq 2$, we can compute a compatible (M, N) -ISS pair $(\mathcal{A}, \mathcal{B})$ with $M \geq 2^{N/2-1}$ in time polynomial in M and N . Further, $\mathcal{B} = \bar{\mathcal{A}}$.*

Proof. Given even $N \geq 2$, we construct two set systems $\mathcal{A} = (U, S_A)$ and $\mathcal{B} = (U, S_B)$ greedily as follows. First set $U = [N]$, and unmark all the subsets $s \subseteq U$ of size $N/2$. Then, for every subset $s \subseteq U$, $|s| = N/2$ that is unmarked, add s to S_A and \bar{s} to S_B , and mark both s and \bar{s} . Note that $|S_A| = |S_B|$, and $\mathcal{B} = \bar{\mathcal{A}}$. First, we claim that both \mathcal{A} and \mathcal{B} are $(|S_A|, N)$ -ISS. Indeed, observe that any $s, t \in S_A$ ($s, t \in S_B$) intersects since $|s| = |t| = N/2$ and $t \neq \bar{s}$. Next, for lower bounding M , we have $|S_A| = |S_B| = \frac{1}{2} \binom{N}{N/2} \geq 2^{N/2-1}$. The total time to construct $(\mathcal{A}, \mathcal{B})$ is $2^N \text{poly}(N) = \text{poly}(M, N)$. Finally, to see that $(\mathcal{A}, \mathcal{B})$ is a compatible ISS pair, consider the bijection $f : S_A \mapsto S_B$ with $f(s) = \bar{s}$, for $s \in S_A$, and note that for $s \in S_A$, $\mathcal{A}_s = (U, (S_A \setminus \{s\}) \cup \{f(s)\})$ is an ISS. This is because for $s \in S_A$ and any $t \in S_A \setminus \{s\}$, since $|\bar{s}| = |\bar{t}| = \frac{N}{2}$ and $\bar{t} \neq \bar{s}$, we have $\bar{s} \setminus \bar{t} \neq \emptyset$. But $\bar{s} \cap \bar{t} = \bar{s} \setminus \bar{t} \neq \emptyset$. Hence, $f(s)$ intersects t . \square

3.2 Hardness of COMPACT PSP

Our hardness result follows from the following FPT-reduction from SGI that yields compact instances of PSP and the fact that SGI is W[1]-hard.

Theorem 4. *There is an FPT-reduction that, for every instance $\mathcal{I} = (G = (V_G, E_G), H = (V_H, E_H), k)$ of SGI with $|V_G| = n$ and $|E_G| = m$, computes $\mu = O(k!)$ instances $\mathcal{J}_p = (\mathcal{U}_p, \mathcal{S}_p, r)$, $p \in [\mu]$ of PSP with $|\mathcal{U}_p| = \Theta(k \log n)$, $|\mathcal{S}_p| = \Theta(n^2 k + mk)$, and $r = \Theta(k)$, such that there is a subgraph of G isomorphic to H if and only if there is an r -packing in at least one of the instances $\{\mathcal{J}_p\}_{p \in [\mu]}$.*

Proof. The construction follows the approach outlined in Section 1.3. Let $V_G = \{1, \dots, n\}$. Let $(\mathcal{A}, \bar{\mathcal{A}})$ be the compatible (M, N) -ISS pair given by Lemma 1, for $N = 2 \lceil \log(n+1) \rceil + 2$. We call $\mathcal{A} = (U_A, S_A)$ and $\bar{\mathcal{A}} = (U_A, \text{comp}(S_A))$ as the base ISS gadgets. Further, assume an arbitrary ordering on $S_A = \{s^1, \dots, s^M\}$. Since $M \geq 2^{N/2-1} > n$, every $\alpha \in V_G$ can be identified by the set $s^\alpha \in S_A$ corresponding to the index $\alpha \in [M]$. For each ordering $p : V_H \rightarrow [\ell]$, create an instance $\mathcal{J}_p = (\mathcal{U}_p, \mathcal{S}_p, r)$ of COMPACT PSP as follows. Rename the vertices of V_H as $\{v_1, \dots, v_\ell\}$ with $v_i := v \in V_H$ such that $p(v) = i$. For each $v_i \in V_H$, create a collection \mathcal{C}_{v_i} of $d(v_i) + 1$ many different copies of base ISS gadget \mathcal{A} (i.e., each has its own distinct universe) as: $\mathcal{C}_{v_i} := \{\mathcal{A}_{v_i,0}, \{\mathcal{A}_{v_i,w}\}_{w \in N(v_i)}\}$, where $\mathcal{A}_{v_i,0} = (U_{v_i,0}, S_{v_i,0})$ and $\mathcal{A}_{v_i,w} = (U_{v_i,w}, S_{v_i,w})$. Let $U_{v_i} = \cup_{w \in N(v)} U_{v_i,w}$. For each \mathcal{C}_{v_i} , let $U_{\mathcal{C}_i} = U_{v_i,0} \cup U_{v_i}$. Now, we define the universe \mathcal{U}_p of \mathcal{J}_p as $\mathcal{U}_p = \bigcup_{i \in [\ell]} U_{\mathcal{C}_i}$.

The sets in \mathcal{S}_p are of two types: V -sets and E -sets as defined below. For $\alpha \in V_G$ and $v \in V_H$, denote by $S_v^\alpha = \cup_{w \in N(v)} s_{v,w}^\alpha$. Recall that for $\alpha \in V_G$ and $(v, w) \in E_H$, the set $s_{v,w}^\alpha$ is the α^{th} set in $S_{v,w}$ of ISS $\mathcal{A}_{v,w} = (U_{v,w}, S_{v,w})$.

V -sets: For each $\alpha \in V_G$, for each $v_i \in \{v_1, \dots, v_{\ell-1}\}$, and for each $\beta \in V_G$, $\beta > \alpha$, add a set $S_{\alpha \mapsto v_i, \beta}$ to \mathcal{S}_p such that

$$S_{\alpha \mapsto v_i, \beta} := s_{v_i,0}^\alpha \cup S_{v_i}^\alpha \cup \bar{s}_{v_{i+1},0}^\beta$$

Further, for each $\alpha \in V_G$, and for each $\beta \in V_G, \beta < \alpha$, add a set $S_{\alpha \mapsto v_\ell, \beta}$ to \mathcal{S}_p such that

$$S_{\alpha \mapsto v_\ell, \beta} := s_{v_\ell, 0}^\alpha \cup S_{v_\ell}^\alpha \cup \bar{s}_{v_1, 0}^\beta$$

E-sets: For each edge $(\alpha, \beta) \in E_G$ and each edge $(v_i, v_j) \in E_H$, add a set $S_{(\alpha, \beta) \mapsto (v_i, v_j)}$ to \mathcal{S}_p such that

$$S_{(\alpha, \beta) \mapsto (v_i, v_j)} := \bar{s}_{v_i, v_j}^\alpha \cup \bar{s}_{v_j, v_i}^\beta$$

Parameter: Set $r := k + \ell$.

This concludes the construction. Before we prove its correctness, we note the size of the constructed instance \mathcal{J}_p . First, $r = \Theta(k)$, since $\ell \leq 2k$. Then, $|\mathcal{U}_p| = \sum_{i=1}^\ell |\mathcal{U}_{C_i}| = \sum_{i=1}^\ell (d(v_i) + 1)N = \Theta(k \log n)$, and $|\mathcal{S}_p| = \Theta(n^2 \ell + mk) = \Theta(n^2 k)$.

Yes case. Suppose there is a subgraph $G' = (V_{G'}, E_{G'})$ of G that is isomorphic to H with injection $\phi : V_H \rightarrow V_{G'}$. Let $V_{G'} = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\} \subseteq [n]$ such that $\alpha_1 < \alpha_2 < \dots < \alpha_\ell$. Relabel the vertices of H as $\{v_1, \dots, v_\ell\}$, where $v_i := \phi^{-1}(\alpha_i), i \in [\ell]$. Now, consider the ordering p of V_H such that $p(v_i) = i$, for $i \in [\ell]$, and fix the corresponding instance $\mathcal{J}_p = (\mathcal{U}_p, \mathcal{S}_p, r)$. Consider the following collection of V -sets and E -sets: $T_V := \bigcup_{i \in [\ell]} S_{\alpha_i \mapsto v_i, \alpha_{i+1}}$ and $T_E := \bigcup_{(v_i, v_j) \in E_H} S_{(\alpha_i, \alpha_j) \mapsto (v_i, v_j)}$. Let $T = T_V \cup T_E$. Note that for this choice of p , we have $T_V \subseteq \mathcal{S}_p$ due to construction, and $T_E \subseteq \mathcal{S}_p$ due to ϕ , and hence $T \subseteq \mathcal{S}_p$. Further, $|T| = |T_V| + |T_E| = \ell + k = r$, as required. Now, we claim that T forms a packing in \mathcal{J}_p . Towards this goal, note that it is sufficient to show that the sets in $T|_{\mathcal{U}_{C_i}}$ are mutually disjoint, for all $i \in [\ell]$. To this end, it is sufficient to show that both $T|_{U_{v_i, 0}}$ and $T|_{U_{v_i}}$ are packing, for all $i \in [\ell]$. Fix $i \in [\ell]$, and consider the following cases:

1. $T|_{U_{v_i, 0}}$: Since $T_E|_{U_{v_i, 0}} = \emptyset$ by construction, we focus on $T_V|_{U_{v_i, 0}}$. But, $T_V|_{U_{v_i, 0}}$ is a packing since,

$$T_V|_{U_{v_i, 0}} = \begin{cases} \{S_{\alpha_{i-1} \mapsto v_{i-1}, \alpha_i}|_{U_{v_i, 0}}, S_{\alpha_i \mapsto v_i, \alpha_{i+1}}|_{U_{v_i, 0}}\} = \{\bar{s}_{v_i, 0}^{\alpha_i}, s_{v_i, 0}^{\alpha_i}\}, & \text{if } i \neq 1 \\ \{S_{\alpha_\ell \mapsto v_\ell, \alpha_1}|_{U_{v_1, 0}}, S_{\alpha_1 \mapsto v_1, \alpha_2}|_{U_{v_1, 0}}\} = \{\bar{s}_{v_1, 0}^{\alpha_1}, s_{v_1, 0}^{\alpha_1}\}, & \text{if } i = 1. \end{cases}$$

2. $T|_{U_{v_i}}$: It is sufficient to show that $T|_{U_{v_i, v_j}}$ is a packing, $\forall v_j \in N(v_i)$. But this follows since, $\forall v_j \in N(v_i)$,

$$\begin{aligned} T|_{U_{v_i, v_j}} &= \{T_V|_{U_{v_i, v_j}}, T_E|_{U_{v_i, v_j}}\} \\ &= \{S_{\alpha_i \mapsto v_i, \alpha_{i+1}}|_{U_{v_i, v_j}}, S_{(\alpha_i, \alpha_j) \mapsto (v_i, v_j)}|_{U_{v_i, v_j}}\} = \{s_{v_i, v_j}^{\alpha_i}, \bar{s}_{v_i, v_j}^{\alpha_i}\}. \end{aligned}$$

No case. Suppose there is an r -packing $T \subseteq \mathcal{S}_p$ in some instance $\mathcal{J}_p, p \in [\mu]$, then we show that there is a subgraph G_T of G that is isomorphic to H . First note that $p \in [\mu]$ gives a labeling $\{v_1, \dots, v_\ell\}$ of V_H such that $v_i = p^{-1}(i)$, for $i \in [\ell]$. Next, partition T into T_V and T_E , such that T_V and T_E correspond to the V -sets and E -sets of T respectively. This can be easily done since $t \in T$ is a V -set if and only if $t|_{U_{v_i, 0}} = s_{v_i, 0}^\alpha \in \mathcal{A}_{v_i, 0}$, for some $\alpha \in V_G, v_i \in V_H$. Let $U_0 = \{U_{v_i, 0}\}_{v_i \in V_H}$ and $U_1 = \{U_{v_i}\}_{v_i \in V_H}$. We claim the following.

Lemma 2. $|T_V| = \ell$ and $|T_E| = k$.

Proof. Note that for $t \in T_V$, we have $t|_{U_0} = \{s_{v_i,0}^\alpha, \bar{s}_{v_{i+1},0}^\beta\}$, for some $\alpha, \beta \in V_G$ and $v_i, v_{i+1} \in V_H$. Hence, it follows that $|t|_{U_0}| = N$. Since $|U_0| = \ell N$ and T_V is a packing, we have $|T_V| \leq \ell$. For bounding $|T_E|$, consider $t \in T_E$, and note that $t|_{U_1} = \{\bar{s}_{v_i,v_j}^\alpha, \bar{s}_{v_j,v_i}^\beta\}$, for some $(\alpha, \beta) \in E_G$ and $(v_i, v_j) \in E_H$. But also note that we have $\bar{s}_{v_i,v_j}^\alpha \in \bar{\mathcal{A}}_{v_i,v_j}$ and $\bar{s}_{v_j,v_i}^\beta \in \bar{\mathcal{A}}_{v_j,v_i}$. Hence, by the virtue of T_E being a packing and using the facts that U_1 is the union of universes of $2k$ many base ISS $\{\bar{\mathcal{A}}_{v,w}\}_{v \in V_H, w \in N(v)}$, and each $t \in T_E$ contains sets from two of such ISS, it follows $|T_E| \leq k$. Finally, $|T| = r = \ell + k$ implies $|T_V| = \ell$ and $|T_E| = k$. \square

For $i \in [\ell]$, as $\mathcal{A}_{v_i,0} = (U_{v_i,0}, S_{v_i,0})$ is an ISS, we can relabel the sets in T_V as $T_V = \{T_V^1, \dots, T_V^\ell\}$, where $T_V^i := t \in T_V$ such that $t|_{U_0} \ni s_{v_i,0}^\alpha$, for some $s_{v_i,0}^\alpha \in S_{v_i,0}$. The following lemma is our key ingredient.

Lemma 3. T covers the whole universe \mathcal{U}_p .

Proof. Since $\mathcal{U}_p = U_0 \cup U_1$, we will show that $T|_{U_j}$ covers U_j , for $j = \{0, 1\}$. For U_0 , note that $T|_{U_0} = T_V|_{U_0}$ by construction. For $T_V^i \in T_V$, we have $|T_V^i|_{U_0}| = N$ due to complement partition axiom of $(\mathcal{A}_{v_i,0}, \bar{\mathcal{A}}_{v_i,0})$. Since T_V forms a packing, we have that $|\bigcup_{i \in [\ell]} T_V^i|_{U_0}| = \ell N = |U_0|$, as desired. Next, we have $|U_1| = 2kN$. Consider $T_V^i \in T_V$ and notice $|T_V^i|_{U_1}| = \frac{N}{2}d(v_i)$ since $T_V^i|_{U_1} = S_{v_i}^\alpha$, for some $\alpha \in V_G$. Since T_V forms a packing, we have $|\bigcup_{i \in [\ell]} T_V^i|_{U_1}| = \sum_{i=1}^\ell |T_V^i|_{U_1}| = kN$. Now consider $t = S_{(\alpha,\beta) \mapsto (v_i,v_j)} \in T_E$, for some $(\alpha, \beta) \in E_G$ and $(v_i, v_j) \in E_H$. Since, $t|_{U_1} = \{\bar{s}_{v_i,v_j}^\alpha, \bar{s}_{v_j,v_i}^\beta\}$, we have $|t|_{U_1}| = N$. As T_E forms a packing, we have $|\bigcup_{t \in T_E} t|_{U_1}| = \sum_{t \in T_E} |t|_{U_1}| = kN$. Finally, T being a packing, we have $|\bigcup_{\tau \in T} \tau|_{U_1}| = |\bigcup_{i \in [\ell]} T_V^i|_{U_1}| + |\bigcup_{t \in T_E} t|_{U_1}| = 2kN = |U_1|$ as desired. \square

Let $\alpha_i = \alpha \in V_G$ such that $T_V^i|_{U_{v_i,0}} \ni s_{v_i,0}^\alpha$, for $i \in [\ell]$. Let $V_T = \{\alpha_i\}_{i \in [\ell]}$. The following lemma asserts that $|V_T| = \ell$.

Lemma 4. For each vertex $\alpha \in V_G$, there is at most one V -set $S_{\alpha \mapsto v_i, \beta}$ in T_V , for some $v_i \in V_H$ and $\beta \in V_G$.

Proof. It is sufficient to show $\alpha_i < \alpha_{i+1}$, for $i \in [\ell-1]$. Fix such i and consider the universe $U_{v_i,0}$ of $\mathcal{A}_{v_{i+1},0}$. Then, note that only T_V^i and T_V^{i+1} contain elements of $U_{v_i,0}$. Let $T_V^i = S_{\alpha_i \mapsto v_i, \beta}$ for $\beta > \alpha_i$, and let $T_V^{i+1} = S_{\alpha_{i+1} \mapsto v_{i+1}, \gamma}$, for $\gamma > \alpha_{i+1}$. As T covers $U_{v_i,0}$ (Lemma 3), and using the complement partition property of the compatible ISS pair $(\mathcal{A}_{v_{i+1},0}, \bar{\mathcal{A}}_{v_{i+1},0})$, we have that $\alpha_{i+1} = \beta > \alpha_i$. \square

Lemma 5. For every edge $(\alpha, \beta) \in E_G$, there is at most one E -set $S_{(\alpha,\beta) \mapsto (v_i,v_j)}$ in T_E , for some $(v_i, v_j) \in E_H$.

Proof. Suppose there are two sets $S_{(\alpha,\beta) \mapsto (v_i,v_j)}, S_{(\alpha,\beta) \mapsto (v'_i,v'_j)} \in T_E$, for some $(\alpha, \beta) \in E_G$. Without loss of generality assume $v_i \neq v'_i$. Then, we will show that $S_{\alpha \mapsto v_i, \gamma}, S_{\alpha \mapsto v'_i, \delta} \in T_V$, for some $\gamma, \delta \in V_G$, contradicting Lemma 4. Since $S_{(\alpha,\beta) \mapsto (v_i,v_j)}, S_{(\alpha,\beta) \mapsto (v'_i,v'_j)} \in T_E$, it holds that $T_E|_{U_{v_i,v_j}} = \bar{s}_{v_i,v_j}^\alpha$, and $T_E|_{U_{v'_i,v'_j}} =$

$\bar{s}_{v'_i, v'_j}^\alpha$. As T covers \mathcal{U}_p , in particular, T covers U_{v_i, v_j} , it must be that $T_V \upharpoonright_{U_{v_i, v_j}} = s_{v_i, v_j}^\alpha$ as $(\mathcal{A}_{v_i, v_j}, \bar{\mathcal{A}}_{v_i, v_j})$ is a compatible ISS pair. By similar reasoning for $U_{v'_i, v'_j}$, it must be that $T_V \upharpoonright_{U_{v'_i, v'_j}} = s_{v'_i, v'_j}^\alpha$. This implies that $T_V \upharpoonright_{U_{v_i}} = S_{v_i}^\alpha$ and $T_V \upharpoonright_{U_{v'_i}} = S_{v'_i}^\alpha$. Thus, $S_{\alpha \mapsto v_i, \gamma}, S_{\alpha \mapsto v'_i, \delta} \in T_V$ for $v_i \neq v'_i$, for some $\gamma, \delta \in V_G$. \square

Let $G_T = G[V_T] = (V_T, E_T)$, be the induced subgraph of G on V_T . To finish the proof, we claim that G_T is isomorphic to H with the injective homomorphism $\phi : V_H \rightarrow V_T$ given by $\phi(v_i) = \alpha_i$, for $i \in [\ell]$. To this end, we will show that for any $(v_i, v_j) \in E_H$, it holds that $(\phi(v_i), \phi(v_j)) = (\alpha_i, \alpha_j) \in E_T$. Consider the universe U_{v_i, v_j} , and note that $T_V^i \upharpoonright_{U_{v_i, v_j}} = s_{v_i, v_j}^{\alpha_i}$. As T covers U_{v_i, v_j} , it holds that $T_E \upharpoonright_{U_{v_i, v_j}} = \bar{s}_{v_i, v_j}^{\alpha_i}$ since $(\mathcal{A}_{v_i, v_j}, \bar{\mathcal{A}}_{v_i, v_j})$ is a compatible ISS pair. Hence $S_{(\alpha_i, \beta) \mapsto (v_i, v_j)} \in T_E$, for some $(\alpha_i, \beta) \in E_G$. This implies that $T_E \upharpoonright_{U_{v_j, v_i}} = \bar{s}_{v_j, v_i}^\beta$. By similar arguments for U_{v_j, v_i} , we have that $\beta = \alpha_j$ as $T_V^j \upharpoonright_{U_{v_j, v_i}} = s_{v_j, v_i}^{\alpha_j}$. Hence $(\alpha_i, \alpha_j) = (\phi(v_i), \phi(v_j)) \in E_G$. \square

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References

1. Giorgio Ausiello, Alessandro D’Atri, and Marco Protasi. Structure preserving reductions among convex optimization problems. *Journal of Computer and System Sciences*, 21(1):136–153, 1980.
2. Per Austrin, Subhash Khot, and Muli Safra. Inapproximability of vertex cover and independent set in bounded degree graphs. In *2009 24th Annual IEEE Conference on Computational Complexity*, pages 74–80. IEEE, 2009.
3. Nikhil Bansal, Anupam Gupta, and Guru Guruganesh. On the lovász theta function for independent sets in sparse graphs. *SIAM Journal on Computing*, 47(3):1039–1055, 2018.
4. Arnab Bhattacharyya, Ameet Gadekar, Suprovat Ghoshal, and Rishi Saket. On the Hardness of Learning Sparse Parities. In Piotr Sankowski and Christos Zaroliagis, editors, *24th Annual European Symposium on Algorithms (ESA 2016)*, volume 57 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 11:1–11:17, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
5. Andreas Björklund, Thore Husfeldt, and Mikko Koivisto. Set partitioning via inclusion-exclusion. *SIAM J. Comput.*, 39(2):546–563, July 2009.
6. Liming Cai and David Juedes. Subexponential parameterized algorithms collapse the w-hierarchy. In *International Colloquium on Automata, Languages, and Programming*, pages 273–284. Springer, 2001.
7. Parinya Chalermsook, Marek Cygan, Guy Kortsarz, Bundit Laekhanukit, Pasin Manurangsi, Danupon Nanongkai, and Luca Trevisan. From gap-eth to fpt-inapproximability: Clique, dominating set, and more. In *2017 IEEE 58th Annual*

- Symposium on Foundations of Computer Science (FOCS)*, pages 743–754. IEEE, 2017.
8. Yuk Hei Chan and Lap Chi Lau. On linear and semidefinite programming relaxations for hypergraph matching. *Mathematical programming*, 135(1-2):123–148, 2012.
 9. Frédéric Chataigner, G Manić, Yoshiko Wakabayashi, and Raphael Yuster. Approximation algorithms and hardness results for the clique packing problem. *Discrete Applied Mathematics*, 157(7):1396–1406, 2009.
 10. Huairui Chu. A tight lower bound for compact set packing, 2023.
 11. Marek Cygan. Improved approximation for 3-dimensional matching via bounded pathwidth local search. In *2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 509–518. IEEE, 2013.
 12. Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Daniel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer Publishing Company, Incorporated, 1st edition, 2015.
 13. Rodney G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer Publishing Company, Incorporated, 2012.
 14. J. Flum and M. Grohe. *Parameterized Complexity Theory (Texts in Theoretical Computer Science. An EATCS Series)*. Springer-Verlag, Berlin, Heidelberg, 2006.
 15. Magnús M Halldórsson, Jan Kratochvíl, and Jan Arne Telle. Independent sets with domination constraints. *Discrete Applied Mathematics*, 99(1-3):39–54, 2000.
 16. Refael Hassin and Shlomi Rubinfeld. An approximation algorithm for maximum triangle packing. *Discrete Applied Mathematics*, 154(6):971–979, 2006.
 17. J. Hastad. Clique is hard to approximate within $n^{1-\epsilon}$. In *Proceedings of the 37th Annual Symposium on Foundations of Computer Science, FOCS '96*, page 627, USA, 1996. IEEE Computer Society.
 18. Elad Hazan, Shmuel Safra, and Oded Schwartz. On the complexity of approximating k -set packing. *computational complexity*, 15(1):20–39, 2006.
 19. Mark Jones, Daniel Lokshtanov, M. S. Ramanujan, Saket Saurabh, and Ondřej Suchý. Parameterized complexity of directed steiner tree on sparse graphs. In Hans L. Bodlaender and Giuseppe F. Italiano, editors, *Algorithms – ESA 2013*, pages 671–682, Berlin, Heidelberg, 2013. Springer Berlin Heidelberg.
 20. Bingkai Lin. A Simple Gap-Producing Reduction for the Parameterized Set Cover Problem. In Christel Baier, Ioannis Chatzigiannakis, Paola Flocchini, and Stefano Leonardi, editors, *46th International Colloquium on Automata, Languages, and Programming (ICALP 2019)*, volume 132 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 81:1–81:15, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.
 21. Bingkai Lin. Constant approximating k -clique is $w[1]$ -hard. In Samir Khuller and Virginia Vassilevska Williams, editors, *STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21-25, 2021*, pages 1749–1756. ACM, 2021.
 22. Dániel Marx. Can you beat treewidth? In *48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07)*, pages 169–179. IEEE, 2007.
 23. Mihai Pătraşcu and Ryan Williams. On the possibility of faster sat algorithms. In *Proceedings of the twenty-first annual ACM-SIAM symposium on Discrete Algorithms*, pages 1065–1075. SIAM, 2010.
 24. Karthik C. S. and Subhash Khot. Almost polynomial factor inapproximability for parameterized k -clique, 2021.

25. Karthik C. S., Bundit Laekhanukit, and Pasin Manurangsi. On the parameterized complexity of approximating dominating set. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2018, page 1283–1296, New York, NY, USA, 2018. Association for Computing Machinery.

A A simple algorithm for PSP

In this section we show an algorithm for SET PACKING that runs in time $O^*(2^{|\mathcal{U}|})$. The main idea is to exploit the fact that $|\mathcal{U}|$ is small and convert the problem to the path finding problem in directed acyclic graphs (DAG). To this end, we enumerate all the subsets of \mathcal{U} by creating a vertex for each subset. Then, we add a directed edge from subset T_1 to subset T_2 if there is a set $S_i \in \mathcal{S}$ such that $T_1 \cap S_i = \emptyset$ and $T_1 \cup S_i = T_2$. Intuitively, the edge (T_1, T_2) in the DAG captures the fact that, if the union of our present solution is T_1 , then we can improve it by including S_i to get a solution whose union is T_2 . Thus finding a maximum sized packing reduces to finding a longest path in the DAG which can be found efficiently by standard dynamic programming technique.

Theorem 5. *There is an algorithm for SET PACKING running in time $O^*(2^{|\mathcal{U}|})$.*

Proof. Given an instance $(\mathcal{U}, \mathcal{S})$ of SET PACKING, the idea is to construct a graph $G = (V, E)$ such that there is a vertex in G for every subset of \mathcal{U} . For a subset $T \subseteq \mathcal{U}$, let v_T be the corresponding vertex in G . Then, add a labeled directed edge from vertex v_{T_i} to v_{T_j} with label $S_k \in \mathcal{S}$ if there is $S_k \in \mathcal{S}$ such that $T_i \cup S_k = T_j$ and $T_i \cap S_k = \emptyset$. In other words, adding the set S_k to our present solution, whose union is denoted by T_i , is safe and results into a new solution whose union is T_j . First note that G can be constructed from $(\mathcal{U}, \mathcal{S})$ in time $O(2^{|\mathcal{U}|}|\mathcal{S}||\mathcal{U}|)$. Now we claim that there are ℓ pairwise disjoint sets in $(\mathcal{U}, \mathcal{S})$ if and only if there is a path of length ℓ starting at vertex v_\emptyset in G .

For one direction, suppose there is a path $P = \{v_\emptyset, v_{T_1}, v_{T_2}, \dots, v_{T_\ell}\}$ of length ℓ in G . Then, there are ℓ sets in \mathcal{S} labeled by the edges of P that are pairwise disjoint by construction. For the other direction, let $S = \{S_1, S_2, \dots, S_\ell\}$ be pairwise disjoint sets in \mathcal{S} . Now, fix some order on the sets of S and consider the following sets, for $i \in [\ell]$, $T_i := \bigcup_{j=1}^i S_j$. Now consider the collection of vertices $P = \{v_{T_0}, v_{T_1}, \dots, v_{T_\ell}\}$, where $T_0 := \emptyset$. It is easy to see that P is a path in G since there is an edge from v_{T_i} to $v_{T_{i+1}}$ in G , labeled by set $S_{i+1} \in \mathcal{S}$, for every $i \in \{0, 1, \dots, \ell - 1\}$.

Next we show how to find an ℓ length path in G starting v_\emptyset in time $O^*(2^{|\mathcal{U}|})$. First note that G is a directed acyclic graph(DAG) since every directed edge (v_{T_i}, v_{T_j}) in G implies $|T_j| > |T_i|$. Now we use standard dynamic program to find an ℓ length path at v_\emptyset . Rename the vertices of G to $\{0, 1, \dots, N - 1\}$, where $N = 2^{|\mathcal{U}|}$, such that the vertex v_{T_j} is renamed to $i \in \{0, 1, \dots, N - 1\}$ where i is the number whose binary representation corresponds to the characteristic vector χ_{T_j} . Next, define a two dimensional bit array \mathbf{A} such that for $0 \leq i \leq N - 1$ and $1 \leq j \leq \ell$,

$$\mathbf{A}[i, j] := \begin{cases} 1 & \text{if there is a path from vertex 0 to vertex } i \text{ of length } j \\ 0 & \text{otherwise} \end{cases}$$

Computing bottom up, we can fill \mathbf{A} in time $O(\ell(N + M))$, where $M := 2^{|\mathcal{U}|}|\mathcal{S}|$ is the number of edges in G , and then look for entry 1 in the array $\mathbf{A}[\cdot, \ell]$. Further,

to find a path, we can modify the algorithm such that, instead of storing a bit, $\mathbf{A}[i, j]$ now stores one of the paths (or just the preceding vertex of a path). Thus we can find an optimal solution of SET PACKING in time $O^*(2^{|\mathcal{U}|})$. \square

B Hardness of COMPACT r -VECTORSUM

In this section, we give a proof sketch of Theorem 3. First, we define the problem.

Definition 6 (r -VECTORSUM). *Given a collection $\mathcal{C} = \{\vec{\eta}_1, \dots, \vec{\eta}_N\}$ of d dimensional vectors over \mathbb{F}_2 , a vector $\vec{b} \in \mathbb{F}_2^d$, and an integer k , the r -VECTORSUM problem asks if there is $I \subseteq [N]$, $|I| = k$ such that $\sum_{j \in I} \vec{\eta}_j = \vec{b}$, where the sum is over \mathbb{F}_2^d . An instance of r -VECTORSUM is denoted as $(\mathcal{C}, \vec{b}, d, r)$.*

COMPACT r -VECTORSUM is defined when $d = f(r) \cdot \Theta(\text{poly}(\log N))$, for some $f(r) \geq r$. The hardness of COMPACT r -VECTORSUM follows from the following theorem.

Theorem 6. *There is an FPT-reduction that, for every instance $\mathcal{I} = (G = (V_G, E_G), H = (V_H, E_H), k)$ of SGI with $|V_G| = n$ and $|E_G| = m$, computes $\mu = O(k!)$ instances $\mathcal{L}_p = (\mathcal{C}_p, \vec{b}, d, r)$, $p \in [\mu]$, of r -VECTORSUM with the following properties:*

- $d = \Theta(k \log n)$
- $|\mathcal{C}_p| = \Theta(n^2 k + mk)$
- $r = \Theta(k)$

such that there is a subgraph of G isomorphic to H if and only if there exists $p \in [\mu]$ such that there are at most r vectors in the instance \mathcal{J}_p that sum to \vec{b} .

Proof Sketch. The first part of the reduction is, in fact, same as that described in Theorem 4, with a simple observation that any optimal packing in the instance generated by Theorem 4 is also a covering. This holds true in No case due to Lemma 3, and it holds true in Yes case due to the complement exchange property of the compatible ISS-pair gadget used in the construction. We call such solution as *exact cover*. In the second part, we transform this instance of Theorem 4 to an instance of r -VECTORSUM. The following definition is useful for the transformation.

Definition 7 (Characteristic vector). *Let U be a universe of q elements. Fix an order on the elements of $U = (e_1, \dots, e_q)$. For any set $S \subseteq U$, define the characteristic vector $\vec{\chi}_S \in \mathbb{F}_2^q$ of S as follows. The t^{th} co-ordinate of $\vec{\chi}_S$,*

$$\vec{\chi}_S(t) := \begin{cases} 1 & \text{if } e_t \in S, \\ 0 & \text{if } e_t \notin S. \end{cases}$$

For every instance $\mathcal{J}_p = (\mathcal{U}_p, \mathcal{S}_p, r)$, $p \in [\mu]$ generated by Theorem 4, we create an instance $\mathcal{L}_p = (\mathcal{C}_p, \vec{b}, d, r)$ of r -VECTORSUM as follows. Rename the vertices of V_H as $\{v_1, \dots, v_\ell\}$ such that $v_i := v \in V_H$ such that $p(v) = i$. Note that this induces an ordering on V_H as $v_1 < \dots < v_\ell$. Thus, for $v_i \in V_H$, we have an ordering on $N(v_i) = \{v'_1, \dots, v'_{d(v_i)}\}$ as $v'_1 < \dots < v'_{d(v_i)}$. Hence, for $\lambda \in [d(v_i)]$, we call v'_λ as the λ^{th} neighbour of v_i . Now, for $v_i \in V_H$, we define $\Gamma_i : N(v_i) \mapsto [d(v_i)]$ as $\Gamma_i(v_j) := \lambda \in [d(v_i)]$, such that v_j is the λ^{th} neighbour of v_i . Next, we construct vectors corresponding to the sets in \mathcal{S}_p . For every V -set $S_{\alpha \rightarrow v_i}$, for $v_i \in V_H$ and $\alpha \in V_G$, of \mathcal{S}_p , define $|\mathcal{U}_p| + \ell + 2k$ length vector $\vec{\chi}'_{S_{\alpha \rightarrow v_i}}$ as follows.

$$\vec{\chi}'_{S_{\alpha \rightarrow v_i}}(t) := \begin{cases} \vec{\chi}_{S_{\alpha \rightarrow v_i}}(t) & \text{if } t \in [|\mathcal{U}_p|], \\ 1 & \text{if } t = |\mathcal{U}_p| + i, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, for every E -set $S_{(\alpha, \beta) \mapsto (v_i, v_j)}$, for $(\alpha, \beta) \in E_G$ and $(v_i, v_j) \in E_H$, define $|\mathcal{U}_p| + \ell + 2k$ length vector $\vec{\chi}'_{S_{(\alpha, \beta) \mapsto (v_i, v_j)}}$ as follows.

$$\vec{\chi}'_{S_{(\alpha, \beta) \mapsto (v_i, v_j)}}(t) := \begin{cases} \vec{\chi}_{S_{(\alpha, \beta) \mapsto (v_i, v_j)}}(t) & \text{if } t \in [|\mathcal{U}_p|], \\ 1 & \text{if } t = |\mathcal{U}_p| + \ell + \sum_{\rho=1}^{i-1} d(v_\rho) + \Gamma_i(j), \\ 1 & \text{if } t = |\mathcal{U}_p| + \ell + \sum_{\rho=1}^{j-1} d(v_\rho) + \Gamma_j(i), \\ 0 & \text{otherwise.} \end{cases}$$

Now consider the instance $\mathcal{L}_p = (\mathcal{C}_p, \vec{b}, d, r)$, where

- $d := |\mathcal{U}_p| + \ell + 2k$
- $\mathcal{C}_p := \{\vec{\chi}'_S\}_{S \in \mathcal{S}_p}$
- $\vec{b} := \vec{1}$, the all ones vector.

Before we prove the correctness, we define some notations. The first ℓ bits that we appended to $\vec{\chi}_S$ are called V -indicator bits, and the next $2k$ bits are called E -indicator bits. The universe corresponding to V -indicator bits and E -indicator bits is denoted as U_2 and U_3 respectively.

Yes Case: From Theorem 4, there is $p \in [\mu]$ such that $\mathcal{J}_p = (\mathcal{U}_p, \mathcal{S}_p, r)$ has a r -packing $\mathcal{S}'_p \subseteq \mathcal{S}_p$ that covers \mathcal{U}_p . Then, consider the corresponding instance $\mathcal{L}_p = (\mathcal{C}_p, \vec{b}, d, r)$ of r -VECTORSUM. Then, note that

$$\sum_{S \in \mathcal{S}'_p} \vec{\chi}_S = \vec{1} = \vec{b}$$

since \mathcal{S}'_p is a packing covering \mathcal{U}_p . Hence $\{\vec{\chi}_S\}_{S \in \mathcal{S}'_p} \subseteq \mathcal{C}_p$ is a solution to \mathcal{L}_p .

No Case: Let $W \subseteq \mathcal{C}_p$, $|W| = r$, be a solution of \mathcal{L}_p , for some $p \in [\mu]$. Let $T \subseteq \mathcal{S}_p$ be the corresponding collection of sets in \mathcal{J}_p to W . Let T_V and T_E be the V -sets and E -sets of T respectively. Let W_V and W_E be the set of vectors corresponding to T_V and T_E respectively. We say vectors in W_V and W_E as V -vectors and E -vectors respectively. Note that $W = W_V \dot{\cup} W_E$ as $T = T_V \dot{\cup} T_E$. The following lemma is equivalent to Lemma 2.

Lemma 6. $|W_V| = \ell$ and $|W_E| = k$.

Proof. First consider W_V , and note that only V -vectors have V -indicator bits set to 1. Since a V -vector has at most one V -indicator bit set to 1, and there are ℓ V -indicator bits set to 1 in \vec{b} , it follows that $|W_V| \geq \ell$. Now consider W_E , and note that only E -vectors have E -indicator bits set to 1. Since a E -vector has at most two E -indicator bit set to 1, and there are $2k$ E -indicator bits set to 1 in \vec{b} , it follows that $|W_E| \geq k$. Since, $|W| = \ell + k$, we have $|W_V| = \ell$ and $|W_E| = k$. \square

Now we claim that T is an exact r -cover in \mathcal{J}_p . It is sufficient to show T is an r -packing since T covers \mathcal{U}_p . To this end, note that Lemma 6 implies that $|T_V| = \ell$ and $|T_E| = k$. This implies that that $T|_{U_2}$ and $T|_{U_3}$ is a packing. Hence, $T|_{U_2 \cup U_3}$ is a packing. Let $U'_0 := U_0 \cup U_2 \cup U_3$ and $U'_1 := U_1 \cup U_2 \cup U_3$. Next consider T_V and note that each set in T_V covers exactly N elements of U_0 . Thus, the total number of elements covered by T_V is at most ℓN . But then, since T_V covers U_0 and $|U_0| = \ell N$, it follows that the every set in T_V must cover different elements of U_0 . Hence, $T_V|_{U'_0}$ is a packing. On the other hand, T_V covers at most kN elements of U_1 as each set in T_V covers exactly $\frac{N}{2}d(v_i)$ elements of U_1 , for some $v_i \in V_H$. Since, T covers U_1 , it must be that T_E must cover at least kN elements of U_1 , as $|U_1| = 2kN$. Since each set in T_E covers N elements of U_1 , from Lemma 6 it follows that T_E covers at most kN elements of U_1 . Thus, the sets in T_E must cover different elements of U_1 , and hence $T_E|_{U'_1}$ is a packing. However, $T_E|_{U'_1} = T_E$ since sets in T_E only contain elements of U'_1 . Hence, T_E is a packing. This means that T_V must cover at least kN elements of U_1 . Then, it follows that $T_V|_{U'_1 \cup T_E|_{U'_1}} = T|_{U'_1}$ is a packing since we observed above that T_V covers at most kN elements of U_1 . Thus, T is a packing as $T|_{U_0} = T_V|_{U_0} \cup T_E|_{U_0} = T_V|_{U_0}$ is also a packing.

Now since T is an exact r -cover of \mathcal{U}_p , we can use the No case of Theorem 4 to recover the isomorphic subgraph of G to H , which finishes the proof. \square