# Bit catastrophes for the Burrows-Wheeler Transform 

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#### Abstract

A bit catastrophe, loosely defined, is when a change in just one character of a string causes a significant change in the size of the compressed string. We study this phenomenon for the Burrows-Wheeler Transform (BWT), a string transform at the heart of several of the most popular compressors and aligners today. The parameter determining the size of the compressed data is the number of equal-letter runs of the BWT, commonly denoted $\boldsymbol{r}$. We exhibit infinite families of strings in which insertion, deletion, resp. substitution of one character increases $\boldsymbol{r}$ from constant to $\Theta(\log \boldsymbol{n})$, where $\boldsymbol{n}$ is the length of the string. These strings can be interpreted both as examples for an increase by a multiplicative or an additive $\Theta(\log n)$-factor. As regards multiplicative factor, they attain the upper bound given by Akagi, Funakoshi, and Inenaga [Inf \& Comput. 2023] of $\mathcal{O}(\log n \log r)$, since here $r=\mathcal{O}(1)$. We then give examples of strings in which insertion, deletion, resp. substitution of a character increases $\boldsymbol{r}$ by a $\boldsymbol{\Theta}(\sqrt{\boldsymbol{n}})$ additive factor. These strings significantly improve the best known lower bound for an additive factor of $\boldsymbol{\Omega}(\log \boldsymbol{n})$ [Giuliani et al., SOFSEM 2021].


Keywords: Burrows-Wheeler transform, equal-letter run, repetitiveness measure, sensitivity.

## 1 Introduction

The Burrows-Wheeler Transform (BWT) [7] is a reversible transformation of a string, consisting of a permutation of its characters. It can be obtained by sorting all of its rotations and then concatenating their last characters. The BWT is present in several compressors, such as bzip [31]. It also lies at the heart of some of the most powerful compressed indexes in terms of query time and functionality, such as the well-known FM-index [11], and the more recent $R L B W T$ [25] and r-index [2, 4, 5, 13, 14]. Some of the most commonly used bioinformatics tools such as bwa [23], bowtie [22], and SOAP2 [21] also use the BWT at their core.

Given a string $w$, the measure $r=r(w)$ is defined as $r(w)=\operatorname{runs}(\operatorname{BWT}(w))$, where $\operatorname{runs}(v)$ denotes the number of maximal equal-letter runs of a string $v$. It is well known that $r$ tends to be small on repetitive inputs. This is because, on texts with many repeated substrings, the BWT tends to create long runs of equal characters (so-called clustering effect) [29]. Due to the widespread use of runlength-compressed BWT-based data structures, the BWT can thus be viewed as a compressor, with the number of runs $r$ the compression size. On the other hand, $r$ is also increasingly being used as a repetitiveness measure, i.e. as a parameter of the input text itself. In his recent survey [28], Navarro explored the relationships between many repetitiveness measures, among these $r$. In particular, all repetitiveness measures considered are lower bounded by $\delta$ [19], a measure closely related to the factor complexity function [24]. It was further shown in [17] that $r$ is within a $\operatorname{poly} \log (n)$ factor of $z$, the number of phrases of the LZ77-compressor [32].

Giuliani et al. [15] studied the ratio of the runs of the BWT of a string and of its reverse. The authors gave an infinite families of strings for which this ratio is $\Theta(\log n)$, where $n$ is the length of the string. This family can also serve as an example of strings in which appending one character can cause $r$ to increase from $\Theta(1)$ to $\Theta(\log n)$. In this paper we further explore this effect, extending it to the other edit operations of deletion and substitution, for which we also give examples of a change from $\Theta(1)$ to $\Theta(\log n)$. Note that this attains the known upper bound of $\mathcal{O}(\log r \log n)$ [1].

Akagi et al. [1] explored the question of how changes of just one character affect the compression ratio of known compressors; they refer to this as the compressors' sensitivity. More precisely, the sensitivity of a compressor is the maximum multiplicative factor by which a single-character edit operation can change the size of the output of the compressor. In addition, they also study the maximum additive factor an edit operation may cause in the output. Our second family of strings falls in this category: these are strings with $r$ in $\Theta(\sqrt{n})$ on which an edit operation (insertion, deletion, or substitution) can cause $r$ to increase by a further additive factor of $\Theta(\sqrt{n})$. This is a significant improvement over the previous lower bound of $\Omega(\log n)[15]$.

Lagarde and Perifel in [20] show that Lempel-Ziv 78 (LZ78) compression [33] suffers from the so-called "one-bit catastrophe": they give an infinite family of strings for which prepending a character causes a previously compressible string to become incompressible. They also show that this "catastrophe" is not a "tragedy": they prove that this can only happen when the original string was already poorly compressible.

Here we use the term "one-bit catastrophe" in a looser meaning, namely simply to denote the effect that an edit operation may change the compression size significantly,
i.e. increase it such that $r\left(w_{n}^{\prime}\right)=\omega\left(r\left(w_{n}\right)\right)$, for an infinite family $\left(w_{n}\right)_{n>0}$, where $w_{n}^{\prime}$ is the word resulting from applying a single edit operation to $w_{n}$. For a stricter terminology we would need to decide for one of the different definitions of BWTcompressibility currently in use. In particular, a string may be called compressible with the BWT if $r$ is in $\mathcal{O}(n / \operatorname{polylog}(n))$ [17], or if $r(w) / r u n s(w) \rightarrow 0$ [12], or even as soon as $\operatorname{runs}(w)>r(w)[26]$.

Some of our bit catastrophes can also be thought of as "tragedies", since the example families of the first group are precisely those with the best possible compression: their BWT has 2 runs. In this sense our result on the BWT is even more surprising than that of [20] on LZ78.

Note that, in contrast to Lempel-Ziv compression, for the BWT, appending, prepending, and inserting are equivalent operations, since the BWT is invariant w.r.t. conjugacy. This means that, if there exists a word $w$ and a character $c$ s.t. appending $c$ to $w$ causes a certain change in $r$, then this immediately implies the existence of equivalent examples for prepending and inserting character $c$. This is because $r(w c) / r(w)=x$ (appending) implies that $r(c w) / r(w)=x$ (prepending), as well as $r(u c v) / r(u v)=x$, for every conjugate $u v$ of $w=v u$ (insertion).

Finally, the BWT comes in two variants: in one, the transform is applied directly on the input string: this is the preferred variant in literature on combinatorics on words, and the one we concentrate on in most of the paper. In the other, the input string is assumed to have an end-of-string marker, usually denoted $\$$ : this variant is common in the string algorithms and data structures literature. We show that there can be a multiplicative $\Theta(\log n)$, or an additive $\Theta(\sqrt{n})$ factor difference between the two transforms. It is interesting to note that the previous remark about the equivalence of insertion in different places in the text does not extend to the variant with the final dollar. We show, however, that our results regarding the $\Theta(\sqrt{n})$ additive factor apply also to this variant, for all three edit operations, and that appending a character at the end of the string-i.e., just before the $\$$-character - can result in a multiplicative $\Theta(\log n)$ increase. This is in stark contrast with the known fact that prepending a character can change the number of runs of the $\$$-variant by at most 2 [1].

This work is an extended version of our conference article with the same title, published in the proceedings of DLT 2023 [16].

## 2 Preliminaries

In this section, we give the necessary definitions and terminology used throughout the paper.

### 2.1 Basics on words

Let $\Sigma$ be a finite ordered alphabet, of cardinality $\sigma$. The elements of $\Sigma$ are called characters or letters. A word (or string) over $\Sigma$ is a finite sequence $w=w[0] w[1] \cdots w[n-1]=$ $w[0 . . n-1]$ of characters from $\Sigma$. We denote by $|w|=n$ the length of $w$, with $\epsilon$ the unique word of length 0 . The set of words of length $n$ is denoted $\Sigma^{n}$, and $\Sigma^{*}=\cup_{n \geq 0} \Sigma^{n}$ is the set of all words over $\Sigma$. Given a word $w=w[0 . . n-1]$, its reverse is the word $\operatorname{rev}(s)=w[n-1] w[n-2] \cdots w[0]$. For a non-empty word $w=w[0 . . n-1]$, we denote by
$\widehat{w}$ the word $w[0 . . n-2]$, i.e. $w$ without its last character. We use the notation $\prod_{i=1}^{k} v_{i}$ for the concatenation of the words $v_{1}, v_{2}, \ldots, v_{k}$. In particular, $v^{k}$ for $k \geq 1$ stands for the $k$-fold concatenation of the word $v$.

Let $w$ be a word over $\Sigma$. If $w=u x v$ for some (possibly empty) words $u, x, v \in \Sigma^{*}$, then $u$ is called a prefix, $x$ a substring (or factor), and $v$ a suffix of $w$. A proper prefix, substring, or suffix of $w$ is one that does not equal $w$. If $x$ is a substring of $w$, then there exist $i, j$ such that $x=w[i] \cdots w[j]=w[i . . j]$, where $w[i . . j]=\epsilon$ if $i>j$. If $w[i . . j]=x$, then $i$ is called an occurrence of $x$.

Let $u, v \in \Sigma^{*}$. If $w=u v$ and $w^{\prime}=v u$, then $w$ and $w^{\prime}$ are called conjugates or rotations of one another. Equivalently, $w^{\prime}$ is a conjugate of $w$ if there is $0 \leq i \leq|w|$ such that $w^{\prime}=w[i . .|w|] w[0 . . i-1]$. In this case, we write $w^{\prime}=\operatorname{conj}_{i}(w)$. A word $u$ is a circular factor of a word $w$ if it is a prefix of $\operatorname{conj}_{i}(w)$ for some $0 \leq i<|w|$, and $i$ is called an occurrence of $u$. If a word $w$ can be written as $w=v^{k}$ for some $k>1$, then $w$ is called a power, otherwise $w$ is called primitive. It is easy to see that $w$ is primitive if and only if it has $|w|$ distinct conjugates.

Given two words $v, w$, the longest common prefix of $v$ and $w$, denoted $\operatorname{lcp}(v, w)$, is the unique word $u$ such that $u$ is a prefix of both $v$ and $w$, and $v[|u|] \neq w[|u|]$ if neither of the two words is prefix of the other. The lexicographic order on $\Sigma^{*}$ is defined as follows: $v \leq_{\text {lex }} w$ if $v=\operatorname{lcp}(v, w)$, or else if $v[|u|]<w[|u|]$, where $u=\operatorname{lcp}(v, w)$. A word is called a Lyndon word if it is lexicographically strictly smaller than all of its conjugates.

Finally, an equal-letter run (or simply run) is a maximal substring consisting of the same character, and $\operatorname{runs}(v)$ is the number of equal-letter runs in the word $v$. For example, the word catastrophic has 12 runs, while the word mississippi has 8 runs.

### 2.2 The Burrows-Wheeler Transform

Let $w \in \Sigma^{*}$. The conjugate array $\mathrm{CA}=\mathrm{CA}_{w}$ of $w$ is a permutation of $\{0,1, \ldots, n-1\}$ defined by: $\mathrm{CA}[i]<\mathrm{CA}[j]$ if (i) $\operatorname{conj}_{i}(w)<_{\operatorname{lex}} \operatorname{conj}_{j}(w)$, or $(\mathrm{ii}) \operatorname{conj}_{i}(w)=\operatorname{conj}_{j}(w)$ and $i<j$. So CA $[k]$ contains the index of the $k$ th conjugate of $w$ in lexicographic order. (Note that the conjugate array is the circular equivalent of the suffix array.) For example, if $w=$ catastrophic, then $\mathrm{CA}_{w}=[3,1,0,11,9,10,7,8,6,4,2,5]$.

The Burrows-Wheeler Transform (BWT) of the word $w$ is a permutation of the characters of $w$, usually denoted $L=\operatorname{BWT}(w)$, defined as $L[i]=w[n-1]$ if $\mathrm{CA}[i]=0$, and $L[i]=w[\mathrm{CA}[i]-1]$ otherwise. For example, the BWT of the word catastrophic is tcciphrotaas. It follows from the definition that $w$ and $w^{\prime}$ are conjugates if and only if $\operatorname{BWT}(w)=\operatorname{BWT}\left(w^{\prime}\right)$.

We denote with $r(w)=\operatorname{runs}(\operatorname{BWT}(w))$ the number of runs in the BWT of the word $w$. For example, $r$ (catastrophic $)=\operatorname{runs}($ tcciphrotaas $)=10$.

In the context of string algorithms and data structures, it is usually assumed that each string terminates with an end-of-string symbol (denoted by $\$$ ), not occurring elsewhere in the string; the $\$$-symbol is smaller than all other symbols in the alphabet. In fact, with this assumption, sorting the conjugates of $w \$$ can be reduced to lexicographically sorting its suffixes. Note that appending the character $\$$ to the word $w$ changes the output of BWT. We denote by $r_{\$}(w)=\operatorname{runs}(\mathrm{BWT}(w \$))$. For example, BWT $($ catastrophic $\$)=$ ctci\$phrotaas and $r_{\$}($ catastrophic $)=12$.

### 2.3 Standard words

Given an infinite sequence of integers $\left(d_{0}, d_{1}, d_{2}, \ldots\right)$, with $d_{0} \geq 0$, and $d_{i}>0$ for all $i>0$, called a directive sequence, define a sequence of words $s_{i}$ with $i \geq 0$ of increasing length as follows: $s_{0}=\mathrm{b}, s_{1}=\mathrm{a}, s_{i+1}=s_{i}^{d_{i-1}} s_{i-1}$, for $i \geq 1$. The words $s_{i}$ are called standard words. The index $i$ is referred to as the order of $s_{i}$. Without loss of generality, here we can assume that $d_{0}>0$ (otherwise, the role of b and a is exchanged.). It is known that for $i \geq 2$, every standard word $s_{i}$ can be written as $s_{i}=x_{i}$ ab if $i$ is even, $s_{i}=x_{i} \mathrm{ba}$ if $i$ is odd, where the factor $x_{i}$ is a palindrome [9].

Standard words are a well studied family of binary words with a lot of interesting combinatorial properties and characterizations and appear as extreme cases in many contexts $[8,18,27,30]$. In particular, in [27], it was shown that the BWT of a binary word has exactly two runs if and only if it is a conjugate of a standard word or a conjugate of a power of a standard word.

Fibonacci words are a particular case of standard words, with directive sequence consisting of only ones. More precisely, Fibonacci words can be defined as follows: $s_{0}=\mathrm{b}, s_{1}=\mathrm{a}, s_{i+1}=s_{i} s_{i-1}$, for $i \geq 1$. It is easy to see that for all $i \geq 0,\left|s_{i}\right|=$ $F_{i}$, where $\left(F_{i}\right)_{i \geq 0}$ is the Fibonacci sequence $1,1,2,3,5,8,13,21, \ldots$, defined by the recurrence $F_{0}=F_{1}=1$, and $F_{i+1}=F_{i}+F_{i-1}$ for $i \geq 1$. Using the well-known fact that the Fibonacci sequence grows exponentially in $i$, we have that $i=\Theta\left(\log \left|s_{i}\right|\right)$. Moreover, for all $k \geq 1, s_{2 k}=x_{2 k} \mathrm{ab}$ and $s_{2 k+1}=x_{2 k+1} \mathrm{ba}$, where $x_{2 k}$ and $x_{2 k+1}$ are palindromes (in particular, $x_{2}=\epsilon$ ). These words $x_{h}$, for $h \geq 2$, are also referred to as central words. The recursive structure of the words $x_{2 k}$ and $x_{2 k+1}$ is also known [10]: $x_{2 k}=x_{2 k-1} \mathrm{ba} x_{2 k-2}=x_{2 k-2} \mathrm{ab} x_{2 k-1}$ and $x_{2 k+1}=x_{2 k} \mathrm{ab} x_{2 k-1}=x_{2 k-1} \mathrm{ba} x_{2 k}$.

## 3 Increasing $r$ by a $\Theta(\log n)$-factor

In this section we give infinite families of words on which a single edit operation, such as insertion, deletion or substitution of a character, can cause an increase of $r$ from constant to $\Theta(\log n)$, where $n$ is the length of the word. The impact of the three edit operations on the BWT of the word is shown in Figure 1.

### 3.1 Inserting a character

First we recall a result from [15], namely that appending a character to the reverse of a Fibonacci word can increase the number of runs by a logarithmic factor [15]. This result was shown using the following proposition:

Proposition 1 ([15], Prop. 3). Let $v$ be the reverse of a Fibonacci word $s$. If $s$ is of even order $2 k$, then $r(v \mathrm{~b})=2 k$. If $s$ is of odd order $2 k+1$, then $r(v \mathrm{a})=2 k$.

A well-known property of each standard word is that its reverse is one of its rotations [9]. Since $s$ is a Fibonacci word, and thus a standard word, its reverse $v$ has the same BWT as $s$.Since $s$ is a standard word, $r(s)=2$, and therefore, also $r(v)=2$. Using the fact that the length of the $i$ th Fibonacci word is $F_{i}$, and that the Fibonacci sequence $\left(F_{i}\right)_{i \geq 0}$ grows exponentially in $i$, it follows that by appending a final b , the

|  | CA | rotations of <br> abaababaabaab |  |
| ---: | ---: | :--- | :--- |
| 0 | 7 | aabaababaabab | b |
| 1 | 2 | aababaabaabab | b |
| 2 | 10 | aababaababaab | b |
| 3 | 5 | abaabaababaab | b |
| 4 | 0 | abaababaabaab | b |
| 5 | 8 | abaababaababa | a |
| 6 | 3 | ababaabaababa | a |
| 7 | 11 | ababaababaaba | a |
| 8 | 6 | baabaababaaba | a |
| 9 | 1 | baababaabaaba | a |
| 10 | 9 | baababaababaa | a |
| 11 | 4 | babaabaababaa | a |
| 12 | 12 | babaababaabaa | a |

(a) Fibonacci word of order 6

|  | CA | rotations of <br> abaababaabaa |  |
| ---: | ---: | :--- | :--- |
| 0 | 10 | aaabaababaab | b |
| 1 | 7 | aabaaabaabab | b |
| 2 | 11 | aabaababaaba | a |
| 3 | 2 | aababaabaaab | b |
| 4 | 8 | abaaabaababa | a |
| 5 | 5 | abaabaaabaab | b |
| 6 | 0 | abaababaabaa | a |
| 7 | 3 | ababaabaaaba | a |
| 8 | 9 | baaabaababaa | a |
| 9 | 6 | baabaaabaaba | a |
| 10 | 1 | baababaabaaa | a |
| 11 | 4 | babaabaaabaa | a |
|  |  |  |  |

(c) Deletion

|  | CA | rotations of <br> abaababbaabaab |  |
| ---: | ---: | :--- | :--- |
| 0 | 8 | aabaababaababb | b |
| 1 | 11 | aababaababbaab | b |
| 2 | 2 | aababbaabaabab | b |
| 3 | 9 | abaababaababba | a |
| 4 | 0 | abaababbaabaab | b |
| 5 | 12 | ababaababbaaba | a |
| 6 | 3 | ababbaabaababa | a |
| 7 | 5 | abbaabaababaab | b |
| 8 | 7 | baabaababaabab | b |
| 9 | 10 | baababaababbaa | a |
| 10 | 1 | baababbaabaaba | a |
| 11 | 13 | babaababbaabaa | a |
| 12 | 4 | babbaabaababaa | a |
| 13 | 6 | bbaabaababaaba | a |

(b) Insertion

|  | CA | rotations of <br> abaababaabaaa |  |
| ---: | ---: | :--- | :--- |
| 0 | 10 | aaaabaababaab | b |
| 1 | 11 | aaabaababaaba | a |
| 2 | 7 | aabaaaabaabab | b |
| 3 | 12 | aabaababaabaa | a |
| 4 | 2 | aababaabaaaab | b |
| 5 | 8 | abaaaabaababa | a |
| 6 | 5 | abaabaaaabaab | b |
| 7 | 0 | abaababaabaaa | a |
| 8 | 3 | ababaabaaaaba | a |
| 9 | 9 | baaaabaababaa | a |
| 10 | 6 | baabaaaabaaba | a |
| 11 | 1 | baababaabaaaa | a |
| 12 | 4 | babaabaaaabaa | a |

(d) Substitution

Fig. 1: The BWT matrix of the Fibonacci word of order 6 (a), and that of the result for 3 bit-catastrophes: (b) inserting a character in position $6=F_{6-1}-2$, (c) deleting the last character, (d) substituting the last character.
number of runs of the BWT is increased by a logarithmic factor in $n=|v|$, namely from $2=\mathcal{O}(1)$ to either $i$ (if $i=2 k$, for some $k$ ) or $i-1$ (if $i=2 k+1$, for some $k$ ), which are both $\Theta(\log n)$.

Similarly to Prop. 1, we will prove that adding a character x greater than b and not present in the word has the same effect as adding a further b at the end of the reverse of a Fibonacci word of even order. Intuitively, this is because in both cases a new factor is introduced in the word, namely bb respectively $x$. Both these factors are greater than all the other factors of the word, and they are the only change in the word. Adding a further a to the reverse of a Fibonacci word of odd order, or a character smaller than a, have a similar effect. We formalize this in the following proposition:

Proposition 2. Let $v$ be the reverse of the Fibonacci word $s$, with $|v| \geq 2$.

1. If $s$ is of even order $2 k$ and $\mathrm{x}>\mathrm{b}$, then $r(v \mathrm{x})=2 k+1$.
2. If $s$ is of odd order $2 k+1$ and $\mathrm{x}<\mathrm{a}$, then $2 k+2 \leq r(v \mathrm{x}) \leq 2 k+3$.

Proof. Recall that $\operatorname{BWT}(v \mathrm{x})=\mathrm{BWT}(\mathrm{x} v)$. For the proof we consider the conjugate $\mathrm{x} v$.
Let us consider firstly the case in which the Fibonacci word $s$ is of even order $2 k$. Let us consider the word $v^{\prime}$ obtained by prepending the character b to $v$, i.e. $v^{\prime}=\mathrm{b} v$. If we denote by $n=\left|v^{\prime}\right|$, then $v=v^{\prime}[1 . . n-1]$, and $\mathrm{x} v$ and $v^{\prime}$ differs only for the characters in position 0 . Moreover, $\mathrm{x} v[0]=\mathrm{xb}$ is the lexicographically greatest 2-length substring of $v$, and $v^{\prime}[0 . .1]=\mathrm{bb}$ is the lexicographically greatest 2 -length substring of $v^{\prime}$. In particular, the relative lexicographic order of the conjugates $\operatorname{conj}_{h}(\mathrm{x} v)$ and $\operatorname{conj}_{h}\left(v^{\prime}\right)$ with $0 \leq h \leq n-1$ is the same. Therefore the BWT of $\mathrm{x} v$ and $v^{\prime}$ differs only by the character preceding the conjugates starting in position 1. By [15, Proposition 4], the character preceding $\operatorname{conj}_{1}\left(v^{\prime}\right)$ is the last character of a run of b's, therefore the character x which precedes $\operatorname{conj}_{1}(\mathrm{x} v)$ adds only one run, namely the 1-length run of x . Since the $r\left(v^{\prime}\right)=2 k$, then $\operatorname{BWT}(\mathrm{x} v)$ has the same $2 k$ runs, plus the further run consisting of the single x . Therefore $r(v \mathbf{x})=r(\mathrm{x} v)=2 k+1$.

Let us consider now the case in which $s$ is of odd order $2 k+1$. Let us consider the word $v^{\prime}=\mathrm{a} v$ of length $n$. If $k=1$, it is easy to verify that $r$ (abaa) $=2$ and $r(\operatorname{abax})=4$, so $r(v \mathrm{x})=2 k+2$. Let us suppose $k>1$. From [15, Prop. 3], we know that $\operatorname{BWT}(\mathrm{a} v)=\mathrm{b}^{F_{2 k-2}} \mathrm{aab}^{F_{2 k-4}} \cdots \mathrm{~b}^{F_{2}} \mathrm{ab}^{F_{0}} \mathrm{a}^{F_{2 k}-k+1}$. Let us consider also the conjugate $\mathrm{x} v$ having the same BWT as $v \mathrm{x}$. Since $v=v^{\prime}[1 . . n-1]$, then $\mathrm{x} v$ and $v^{\prime}$ differs only for the characters in position 0 . Note that xa and aaa are the lexicographically smallest substrings of $\mathrm{x} v$ and $v^{\prime}$, respectively. It follows that the relative lexicographic order of the conjugates $\operatorname{conj}_{h}(\mathrm{x} v) \operatorname{conj}_{h}\left(v^{\prime}\right)$ with $1 \leq h \leq n-1$ is the same. The additional rotation $\mathrm{x} v$ starts with x , and it is now the smallest among all rotations, therefore it will be at the very beginning of the matrix. Since $\mathrm{x} v$ ends with a and the lexicographically following rotations end with b , it increments the number of runs by 1 with respect to the BWT of $v^{\prime}$. On the other hand, the rotation conj $j_{0}\left(v^{\prime}\right)$ ends with a and follows all the rotations that start with aa and end with b , and precedes the rotation $\operatorname{conj}_{n-3}\left(v^{\prime}\right)$ that starts with abaaa and ends with a. Additionally, the rotation ending with x contributes to $r$ with at most 2 more runs. This is because it either falls in between runs of two distinct characters, or within a run of a single character. In total, $\operatorname{BWT}(\mathrm{x} v)$ has at most $2 k+3$ runs, and $2 k+2 \leq r(\mathrm{x} v)=r(v \mathrm{x}) \leq 2 k+3$.

The following proposition can be deduced from Prop. 1 and shows that there exist at least two positions in a Fibonacci word of even order where adding a character causes a logarithmic increment of $r$. In Fig. 2 these two positions are shown.

Proposition 3. Let $s$ be the Fibonacci word of even order $2 k$, and $n=|s|$. Let $s^{\prime}$ be the word resulting from inserting $a \mathrm{~b}$ at position $F_{2 k-1}-2$, and $s^{\prime \prime}$ the word resulting from inserting an a at position $F_{2 k}-2$. Then $B W T\left(s^{\prime}\right)$ and $B W T\left(s^{\prime \prime}\right)$ have $\Theta(\log n)$ runs.

Proof. It is known that each standard word and its reverse are conjugate. Let us consider $s=x_{2 k} \mathrm{ab}$, where $x_{2 k}$ is palindrome. Moreover, from known properties on Fibonacci words, $x_{2 k}=x_{2 k-1}$ ba $x_{2 k-2}=x_{2 k-2} \mathrm{ab} x_{2 k-1}$ with $x_{2 k-1}$ and $x_{2 k-2}$ palindrome, as well. Let $v=\mathrm{ba} x_{2 k}$ be the reverse of $s$. From Proposition 1, we know that $r(v \mathrm{~b})=2 k$. One can verify that appending a b to $v$ is equivalent to inserting b at


Fig. 2: Example of adding a b and an a within the Fibonacci word of even order at positions $F_{2 k-1}-2$ and $F_{2 k}-2$, respectively. It causes a logarithmic increment of the number of BWT-runs.
position $F_{2 k-1}-2$ of $s$. On the other hand, the word $t=x_{2 k}$ ba is a conjugate of both $v$ and $s$. Moreover, by using properties of directive sequences, $t$ is a standard word of odd order $2 k-1$ [3]. It is easy to verify that $\operatorname{rev}(t)=\operatorname{conj}_{F_{2 k}-2}(s)$. By [15, Proposition 8], $r(\operatorname{rev}(t) \mathrm{a})=2 k-2=\Theta(\log n)$, and for similar considerations as above, appending an a to $\operatorname{rev}(t)$ is equivalent to inserting a at position $F_{2 k}-2$ in $s$.

An analogous result to Prop. 3 can be proved for Fibonacci words of odd order.

### 3.2 Deleting a character

We next show that deleting a character can result in a logarithmic increment in $r$. In particular, we consider a Fibonacci word of even order and compute the form of its BWT, as shown in the following proposition.

Proposition 4. Let $s$ be the Fibonacci word of even order $2 k>4$ and $\widehat{s}=s[0 . . n-2]$, where $n=|s|$. Then $B W T(\widehat{s})$ has the following form:

$$
B W T(\widehat{s})=\mathrm{b}^{k-1} \mathrm{ab}^{F_{2 k-3}-k+1} \mathrm{ab}^{F_{2 k-5}} \cdots \mathrm{~b}^{F_{5}} \mathrm{ab}^{F_{3}} \mathrm{aba}^{F_{2 k-1}-k+1}
$$

Therefore, $B W T(\widehat{s})$ has $2 k$ runs.
To give the proof, we divide the BWT matrix of the word $\widehat{s}$ in three parts: top, middle and bottom part, showing the form of each part separately:

$$
\begin{aligned}
\operatorname{BWT}_{\text {top }}(\widehat{s}) & =\mathrm{b}^{k-1} \mathrm{ab}^{F_{2 k-3}-k+1}, \text { consisting of } 3 \text { runs, } \\
\operatorname{BWT}_{\text {mid }}(\widehat{s}) & =\mathrm{ab}^{F_{2 k-5}} \mathrm{ab}^{F_{2 k-7}} \cdots \mathrm{ab}, \text { consisting of } 2(k-2) \mathrm{runs}, \\
\operatorname{BWT}_{\mathrm{bot}}(\widehat{s}) & =\mathrm{a}^{F_{2 k-1}-k+1}, \text { consisting of } 1 \text { run. }
\end{aligned}
$$

Altogether, we then have $3+2(k-2)+1=2 k$ runs.
In order to describe the structure of the matrix, we start with the following lemma that provides information on the structure of $s$.

Lemma 5. Let $s$ be the Fibonacci word of even order $2 k>4$ and $n=|s|$. Then, $s$ can be factorized as follows:

$$
s=x_{2 k-1} \mathrm{ba} x_{2 k-3} \mathrm{ba} \cdots x_{7} \mathrm{ba} x_{5} \mathrm{ba} s_{4}=x_{2 k-2} \mathrm{ab} x_{2 k-3} \mathrm{ba} x_{2 k-4} \mathrm{ab} \cdots x_{4} \mathrm{ab} x_{3} \mathrm{ba} s_{4}
$$

where $x_{i}$ denotes the central word of order $i$, with $3 \leq i \leq 2 k-1$ and $s_{4}=x_{4} \mathrm{ab}=$ abaab is the Fibonacci word of order 4.

Proof. It follows by using the induction on $k$ and the fact that $s=s_{2 k-1} s_{2 k-2}$ and $s=x_{2 k} \mathrm{ab}$, where $x_{2 k}=x_{2 k-1} \mathrm{ba} x_{2 k-2}=x_{2 k-2} \mathrm{ab} x_{2 k-1}$ and $x_{2 k-1}=x_{2 k-3} \mathrm{ba} x_{2 k-2}=$ $x_{2 k-2} \mathrm{ab} x_{2 k-3}$. In fact, the equality $s_{6}=x_{5} \mathrm{ba} x_{4} \mathrm{ab}$ holds since $s_{4}=x_{4} \mathrm{ab}$. On the other hand, $s_{6}=x_{4} \mathrm{ab} x_{5} \mathrm{ab}=x_{4} \mathrm{ab} x_{3} \mathrm{ba} x_{4} \mathrm{ab}$. Since $s_{2 k+2}=s_{2 k+1} s_{2 k}$ and $s_{2 k+1}=x_{2 k+1} \mathrm{ba}$, we have that $s_{2 k+2}=x_{2 k+1}$ bax $x_{2 k-1}$ ba $x_{2 k-3} \mathrm{ba} \cdots x_{7} \mathrm{ba} x_{5} \mathrm{ba} s_{4} b$. On the other hand, $s_{2 k+2}=s_{2 k+1} x_{2 k-2} \mathrm{ab} x_{2 k-3} \mathrm{ba} x_{2 k-4} \mathrm{ab} \cdots x_{4} \mathrm{ab} x_{3} \mathrm{ba} s_{4}$. The thesis follows from the fact that $s_{2 k+1}=x_{2 k+1} \mathrm{ba}=x_{2 k} \mathrm{ab} x_{2 k-1} \mathrm{ba}$.

We identify the following 3 conjugates of the word $\widehat{s}=s[0 . . n-2]$ of length $n-1$ that delimit the 3 parts of the BWT matrix of the word:

$$
\begin{aligned}
\operatorname{conj}_{n-3}(\widehat{s}) & =\operatorname{aa} x_{2 k-1} \mathrm{ba} x_{2 k-3} \mathrm{ba} \cdots x_{5} \mathrm{baab} \\
\operatorname{conj}_{n-5}(\widehat{s}) & =x_{4} \mathrm{a} x_{2 k-1} \cdots x_{5} \mathrm{ba} \\
\operatorname{conj}_{0}(\widehat{s}) & =x_{2 k} \mathrm{a}
\end{aligned}
$$

The structure of these 3 conjugates follows from Lemma 5 . It is easy to see that $\operatorname{conj}_{n-3}(\widehat{s})<\operatorname{conj}_{n-5}(\widehat{s})<\operatorname{conj}_{0}(\widehat{s})$. The rotation $\operatorname{conj}_{n-3}(\widehat{s})$, starting with aa $x_{2 k-1}$ is the smallest rotation in the matrix due to the unique aaa prefix. The rotation $\operatorname{conj}_{n-5}(\widehat{s})$, starting with $x_{4}=$ aba indicates the beginning of the middle part, and it is the smallest rotation starting with ab. Finally, the word itself $\widehat{s}=x_{2 k}$ a determines the beginning of the bottom part, namely the last long run of a's in the BWT.

The top part of the matrix consists of all rotations of the word starting with aa. We give first the following lemma characterizing all occurrences of the factor aa in $\widehat{s}$.

Lemma 6. Let $\widehat{s}=s[0 . . n-2]$, where $s$ is the Fibonacci word of order $2 k>4$ and $n=|s|$. Every occurrence of the circular factor aa in $\widehat{s}$ is an occurrence of a $x_{2 h}$ for some $2 \leq h \leq k$. For every $2 \leq h \leq k-1$ there is exactly one occurrence of a $x_{2 h}$ followed by aa.

Proof. By using Lemma 5. we have that $\widehat{s}=x_{2 k-1}$ ba $x_{2 k-3}$ ba $\cdots x_{7}$ ba $x_{5}$ ba $x_{4}$ a. It follows from the recursive construction of $s$ that occurrences of aa are generated whenever we create $s_{2 h}=s_{2 h-1} s_{2 h-2}=x_{2 h-1} \mathrm{ba} x_{2 h-2} \mathrm{ab}$. This is because each central word $x_{i}$ starts with ab and ends with ba. Therefore, it follows from the structure of $\hat{s}$ that we can partition all its occurrences of aa into four disjoint sets: the occurrence of a $x_{2 k}$, the occurrences of a $x_{2 h}$ followed by ab, the ones followed by ba, the ones followed by aa, with $4 \leq h \leq k-1$. Since $x_{2 h}$ ends with a and the factor aaa occurs only at position $n-3$ in $\widehat{s}$, we can state that the last set of occurrences corresponds to the occurrences of a $x_{2 h}$ aa (with $h=2, \ldots, k-1$ ). We have only one occurrence for each of these factors. In fact, $\mathrm{ax}_{4}$ aa occurs at position $n-6$. Moreover, for $h=3, \ldots, k-1$,
every a $x_{2 h-1}$ in the previous factorization of $\widehat{s}$ is an occurrence of a $\widehat{s_{2 h}}$, then it is an occurrence of the circular factor a $x_{2 h}$ aa.

We are now going to show that only one of the rotations of $\widehat{s}$ starting with aa ends with an a, and we show where the a in the BWT of the top part breaks the run of b's.

Lemma 7 (Top part). Given $\widehat{s}=s[0 . . n-2]$, where $s$ is the Fibonacci word of order $2 k$ and $n=|s|$, then the first $k$ rotations in the BWT matrix are aaa $\cdots \mathrm{b}<\mathrm{a} x_{4} \mathrm{aa} \cdots \mathrm{b}<$ $\mathrm{a} x_{6} \mathrm{aa} \cdots \mathrm{b}<\ldots<\mathrm{a} x_{2 k-2} \mathrm{aa} \cdots \mathrm{b}<\mathrm{a} x_{2 k}$. All other $F_{2 k-3}-k+1$ rotations starting with aa end with $a \mathrm{~b}$.

Proof. There are $F_{2 k-3}+1$ occurrences of aa. In fact, $\widehat{s}$ has $F_{2 k-1}$ occurrences of a's and $F_{2 k-2}-1$ occurrences of b's. Since bb does not occur in $\widehat{s}$, it follows that $F_{2 k-2}-1$ a's are followed by a b. Therefore there are $F_{2 k-1}-F_{2 k-2}+1=F_{2 k-3}+1$ occurrences of a followed by an a.

Among all rotations starting by aa the $k$ smallest ones are those starting with a $x_{h}$ aa for each even $h$ in increasing order of $h$. This is because of the single occurrence of aaa, consisting in the rightmost occurrence of $x_{4}$ followed by aa. Finally, only a $x_{2} k$ is preceded by an a, therefore the $k$ smallest rotations of $\widehat{s}$ are all preceded by a b except for the largest of them which is preceded by an a. This shows that the $k$ smallest rotations in the BWT matrix form two runs: $\mathrm{b}^{k-1} \mathrm{a}$. All the remaining $F_{2 k-3}-k+1$ rotations starting with aa correspond to some occurrence of a $x_{2 h}$ for some $2 \leq h<k$, by using Lemma 6 . By using the properties of central words, each of these occurrences is followed by ba. Then the correspondent rotation is lexicographically greater than $\mathrm{a} x_{2 k}$, which is prefixed by a $x_{2 h} \mathrm{ab}$. Since there is a unique occurrence of aaa, all these rotations starting with aa are preceded by by construction. Therefore, we have $\mathrm{b}^{k-1} \mathrm{ab}^{F_{2 k-3}-k+1}$ in the top part of the BWT matrix of $\widehat{s}$.

Fig. 3 displays the structure of the middle part of the BWT of $\widehat{s}$. To determine the number of runs in this middle part of the matrix, it is crucial first to consider that all the rotations starting with $x_{h}$, with $h=4, \ldots, 2 k-1$, are grouped together in the BWT matrix. Specifically, these occur in blocks where rotations starting with $x_{h} \mathrm{a}, h$ odd, are immediately preceded by the unique rotation starting with $x_{h-1}$ aa, and immediately followed by the rotation starting with $x_{h+1}$ aa, as illustrated in Fig. 3. This is proved in the following lemma.

Lemma 8. For $4 \leq h \leq 2 k-1$, rotations starting with some $x_{h}$ aa are smaller then rotations starting with $x_{h+1} \mathrm{a}$. In particular, the word $\widehat{s}=x_{2 k} \mathrm{a}$ is greater than any of the aforementioned rotations. Moreover, if $h$ is odd, rotations starting with some $x_{h}$ a are smaller then rotations starting with $x_{h+1}$.
Proof. Every $x_{h}$ is a prefix of $x_{h+1}$. Since there is exactly one circular occurrence of aaa in $\widehat{s}$, then $x_{h+1} \mathrm{a}$ is either prefixed by $x_{h} \mathrm{ab}$ or by $x_{h}$ ba, i.e. the aaa factor occurs earlier in $x_{h}$ aa. In both cases, the first claim holds. Finally, the thesis follows by observing that if $h$ is odd, $x_{h}$ ba is a prefix of $x_{h+1}$.


Fig. 3: The middle part $B W T_{m i d}(\widehat{s})$ of the BWT matrix for the deletion of the last character of a Fibonacci word of even order $2 k$ is shown.

Lemma 9. There are $F_{2 h}-1$ occurrences of $\mathrm{b} x_{2(k-h)} \mathrm{a}$ and $F_{2 h+1}$ occurrences of $\mathrm{b} x_{2(k-h)-1} \mathrm{a}$ as circular factors in $\widehat{s}=x_{2 k} \mathrm{a}$.
Proof. The claim can be proved by induction. For $h=0$, the statement follows from the fact that there is one occurrence of $\mathrm{b} x_{2 k} \mathrm{a}$ in the Fibonacci word of order $2 k$, therefore there are $F_{0}-1=0$ occurrences in $\widehat{s}$ because of the missing b at the end. There are $F_{1}=1$ occurrences of $\mathrm{b} x_{2 k-1} \mathrm{a}$ in both words. Note that the occurrence of any $\mathrm{b} x_{2 h} \mathrm{a}$ in position $F_{2 k}-1$ of the Fibonacci word of order $2 k$ is missing in $\widehat{s}$ due to the missing b at the end of the word.

Let us suppose the statement holds for all $i \leq h$. The factor $\mathrm{b} x_{2(k-h)-2} \mathrm{a}$ appears as a prefix of $\mathrm{b} x_{2(k-h)-1} \mathrm{a}$ and as a prefix of $\mathrm{b} x_{2(k-h)} \mathrm{a}$. Moreover, the mentioned occurrences are distinct because $\mathrm{b} x_{2(k-h)-1} \mathrm{a}$ is not a prefix of $\mathrm{b} x_{2(k-h)} \mathrm{a}$. Therefore, by induction, the number of occurrences of $\mathrm{b} x_{2(k-h)} \mathrm{a}$ is equal to the sum of the number of occurrences of $\mathrm{b} x_{2(k-h)-2} \mathrm{a}$ and those of $\mathrm{b} x_{2(k-h)-1} \mathrm{a}: F_{2 h}-1+F_{2 h+1}=$ $F_{2 h+2}-1$. On the other, $\mathrm{b} x_{2(k-h)-3} \mathrm{a}$ appears as a suffix of $\mathrm{b} x_{2(k-h)-2} \mathrm{a}$ and as a suffix of $\mathrm{b} x_{2(k-h)-1} \mathrm{a}$. Moreover, the mentioned occurrences are distinct because $\mathrm{b} x_{2(k-h)-2} \mathrm{a}$ is not a suffix of $\mathrm{b} x_{2(k-h)-1} \mathrm{a}$. Finally, $\mathrm{b} x_{2(k-h)-3} \mathrm{a}$ appears once also as suffix of $\mathrm{a} x_{2(k-h)-2} \mathrm{a}$, starting in position $F_{2 k}-2$ of $\widehat{s}$. Therefore, by induction, the number of occurrences of $\mathrm{b} x_{2(k-h)-3} \mathrm{a}$ is equal to the sum of the number of occurrences of $\mathrm{b} x_{2(k-h)-2} \mathrm{a}$ and those of $\mathrm{b} x_{2(k-h)-1} \mathrm{a}$ plus one: $F_{2 h+2}-1+F_{2 h+1}+1=F_{2 h+3}$.

Lemma 10 (Middle part). The middle part contributes to $r(v)$ with $2(k-2)$ runs in the following form: $\mathrm{ab}^{F_{2 k-5}} \mathrm{ab}^{F_{2 k-7}} \cdots \mathrm{ab}^{F_{3}} \mathrm{ab}$.

Proof. By construction, all the rotations starting with ab are prefixed by a central word $x_{h}$, for some $h<2 k$. For all $\operatorname{conj}_{i}(\widehat{s})$ such that $\operatorname{lcp}\left(\widehat{s}, \operatorname{conj}_{i}(\widehat{s})\right)=x_{h}$ for some odd $h$, $\operatorname{conj}_{i}(\widehat{s})$ is prefixed by $x_{h}$ a since $\widehat{s}$ is prefixed by $x_{h} \mathrm{~b}$, then $\operatorname{conj}_{i}(\widehat{s})$ is in the middle part (i.e. smaller than $\widehat{s}$ by Lemma 8 ) since the word $\widehat{s}$ separates the middle and the bottom part. For all $\operatorname{conj}_{i}(\widehat{s})$ such that $\operatorname{lcp}\left(\widehat{s}, \operatorname{conj}_{i}(\widehat{s})\right)=x_{h}$ for some even $h, \operatorname{conj}_{i}(\widehat{s})$ is prefixed by $x_{h} \mathrm{~b}$ since $\widehat{s}$ is prefixed by $x_{h} \mathrm{a}$, then $\operatorname{conj}_{i}(\widehat{s})$ is in the bottom part (i.e. greater than $\widehat{s}$ by Lemma 8) By using Lemma 5 , it follows that the shortest nonempty central word starting with ab that appears in $\widehat{s}$ as a circular factor is $x_{4}$. One can prove that, among rotations starting with the same $x_{h}, h \geq 4$ even, the smallest one is preceded by a. In fact, it starts with $x_{4}$ aa. All the following $F_{2 k-h-1}$ rotations starting with $x_{h+1}$ a are preceded by b (Lemma 9). The fact that there exist exactly $k-2$ such $x_{h}$ proves the claim.

The rotations that divide the middle part from the bottom part are the two rotations prefixed by the two occurrences of $x_{2 k-1}$. By properties of Fibonacci words, one rotation is prefixed by $x_{2 k-1} \mathrm{a}$ (end middle part) and the other by $x_{2 k-1} \mathrm{~b}$ (beginning bottom part). The latter follows the first in lexicographic order. Note that the rotation starting with $x_{2 k-1} \mathrm{~b}$ is $\widehat{s}$, namely $x_{2 k-1} \mathrm{ba} x_{2 k-2} \mathrm{a}$.

Lemma 11 (Bottom part). All rotations greater than $v=x_{2 k-1}$ bax $x_{2 k-2}$ a end with a .
Proof. From Lemma 7 we have that $k-1+F_{2 k-3}-k+1$ rotations ending with b have already appeared in the matrix, and from Lemma $10 F_{2 k-5}+\ldots+F_{3}+F_{1}$ rotations ending with b have already appeared in the matrix. Summing the number of b's we have $k-1+F_{2 k-3}-k+1+F_{2 k-5}+\ldots+F_{3}+F_{1}=F_{2 k-3}+F_{2 k-5}+\ldots+F_{3}+F_{1}$. We can decompose each odd Fibonacci number $F_{2 x+1}$ in the sum $F_{2 x}+F_{2 x-1}$. Therefore, the previous sum becomes $F_{2 k-4}+F_{2 k-5}+F_{2 k-6}+F_{2 k-7} \ldots+F_{2}+F_{1}+F_{1}$. For every Fibonacci number $F_{x}$, it holds that $F_{x}=F_{x-2}+F_{x-3}+F_{x-4}+\ldots+F_{2}+F_{1}+2$. Therefore, $F_{2 k-4}+F_{2 k-5}+F_{2 k-6}+F_{2 k-7} \ldots+F_{2}+F_{1}+F_{1}=F_{2 k-2}-1$, which is exactly the number of b's in $\widehat{s}$. Therefore all the remaining rotations end with a.

In the context of repetitiveness measures of words, a measure $\lambda$ is called monotone if, for each word $v \in \Sigma^{*}$ and for each letter $\mathbf{x} \in \Sigma$, it holds that $\lambda(v) \leq \lambda(v \mathbf{x})$. Since we have shown that appending or deleting a single character can substantially increase the parameter $r$, the following known result on the monotonicity of $r$ can be derived:

Corollary 12. The measure $r$ is not monotone.

### 3.3 Substituting a character

In this subsection, we show how to increment $r$ by a logarithmic factor by substituting a character. Consider a Fibonacci word $s$ of even order in which we replace the last b by an a. Denoting this word by $s^{\prime}$, we will prove that $\operatorname{BWT}\left(s^{\prime}\right)$ has $\Theta(\log n)$ runs, where $n$ is the length of the word. We start with the following lemma in which we assess how the number of BWT-runs changes when we append or prepend to a Lyndon word a character that is smaller than or equal to the smallest character appearing in the word itself.

Lemma 13. Let $v \in \Sigma^{*}$ be a Lyndon word containing at least two distinct letters and let $\mathrm{x} \in \Sigma$ be smaller than or equal to the smallest character occurring in $v$. Then, $r(v) \leq r(\mathrm{x} v)=r(v \mathrm{x}) \leq r(v)+2$.

Proof. We can obtain the lexicographic order of the rotations of $\mathrm{x} v$, or equivalently $v \mathrm{x}$, from the order of the rotations of $v$. To do so, we show that given two rotations $\operatorname{conj}_{i}(v)<\operatorname{conj}_{j}(v)$, with $i \neq j$, if $\operatorname{conj}_{i}(v)<\operatorname{conj}_{j}(v)$ then $v[i . .|v|-1] \mathrm{x} v[0 . . i-1]<$ $v[j . .|v|-1] \mathbf{x} v[0 . . j-1]$.

Note that $v$ is the smallest rotation in its BWT matrix. Let us denote by $\operatorname{conj}_{h}(v)$, for some $h$, the second rotation in the BWT matrix. Since $v$ is primitive, there exist a unique circular factor $u$ smaller than all the other circular factors having the same length. In fact, if $t=\left|\operatorname{lcp}\left(v, \operatorname{conj}_{h}(v)\right)\right|$, then $u=v[0 . . t]$. Moreover, for all $\ell<|u|$, $u[0 . . \ell-1]$ is the smallest circular factor of length $\ell$ occurring in $v$. We can then distinguish two cases.

The first case is when $\left|\operatorname{lcp}\left(\operatorname{conj}_{i}(v), \operatorname{conj}_{j}(v)\right)\right|<\min \{|v|-i+1,|v|-j+1\}$. Under this condition, it follows that the insertion of the x does not affect the relative order between $v[i . .|v|-1] \mathrm{x} v[0 . . i-1]$ and $v[j . .|v|-1] \mathrm{x} v[0 . . j-1]$.

Otherwise, if $\left|\operatorname{lcp}\left(\operatorname{conj}_{i}(v), \operatorname{conj}_{j}(v)\right)\right| \geq \min \{|v|-i+1,|v|-j+1\}$, note that $i>j$, i.e. $|v[i . .|v|-1]|<|v[j . .|v|-1]|$. This follows by observing that both $v[i . .|v|-1]$ and $v[j . .|v|-1]$ are (circularly) followed by $u$ that is unique, and by contradiction if $i<j$ then $u$ would circularly occur before in $\operatorname{conj}_{j}(v)$ with respect to $\operatorname{conj}_{i}(v)$, which contradicts $\operatorname{conj}_{i}(v)<\operatorname{conj}_{j}(v)$. We can now further distinguish between two subcases: when either (i) $u$ is a prefix of $v[0 . . i-1]$ or (ii) $v[0 . . i-1]$ is a proper prefix of $u$.

For the subcase (i), as $\left|\operatorname{lcp}\left(\operatorname{conj}_{i}(v), \operatorname{conj}_{j}(v)\right)\right| \geq|v[i . .|v|-1]|$, and the factor $u$ is a prefix of $v[0 . . i-1]$, the first distinct character between $\operatorname{conj}_{i}(v)$ and $\operatorname{conj}_{j}(v)$ lies within the unique occurrence of $u$ in $\operatorname{conj}_{i}(v)$. After the letter x is inserted, $\operatorname{conj}_{i}(v)$ becomes $v[i . .|v|-1] \mathbf{x} v[0 . . i-1]$, yielding a factor $\mathbf{x} u$ occurring at position $|v|-i+1$ that is also unique and smallest among all the factors of length $|\mathrm{x} u|$ in $\mathrm{x} v$. Whatever factor appears in $v[j . .|v|-1] \mathrm{x} u$ at position $|v|-i+1$, has to be greater than $\mathrm{x} u$, and the order is preserved. For the subcase (ii), recall that since $v[0 . . i-1]$ is a proper prefix of $u, v[0 . . i-1]$ is also the smallest $i$-length circular factor in lexicographical order occurring in $v$, but differently from $u$ the circular factor $v[0 . . i-1]$ is repeated (otherwise $|u| \leq|v[0 . . i-1]|$, contradiction). By primitivity of $v$, the first distinct character between $\operatorname{conj}_{i}(v)$ and $\operatorname{conj}_{j}(v)$ lies within $v[0 . . i-1]$, i.e., within $\operatorname{conj}_{i}(v)[|v|-i+1 . .|v|-1]$. After the insertion of the symbol $x$ the analogous behavior of subcase (i) is observed.

We conclude the proof by observing that, with respect to the original BWT, we have one extra rotation, and one rotation for which the letter in the BWT has changed, which are $\mathrm{x} v$ and $v \mathrm{x}$ respectively. Observe that by construction, $\mathrm{x} v$ is now the smallest among all of its rotation, which ends with the last letter of $v$. On the other hand, $v \mathrm{x}$ is now the second smallest rotation and it ends with x . Hence, $\operatorname{BWT}(\mathrm{x} v)=\operatorname{BWT}(v)[0]$. $\mathrm{x} \cdot \operatorname{BWT}(v)[1 . .|v|-1]$, and the thesis follows.

Proposition 14. Let $s$ be the Fibonacci word of even order $2 k>4$, and $n=|s|$. Let $s^{\prime}$ be the word resulting from substituting $a \mathrm{~b}$ by an a at position $F_{2 k}-1$. Then $B W T\left(s^{\prime}\right)$ has $2 k+2$ runs.

Proof. Observe that $s^{\prime}=\widehat{s}$. By Proposition 4, it holds $r(\widehat{s})=2 k$. By Lemma 7, we know that $\operatorname{conj}_{n-3}(\widehat{s})$ is the smallest rotation of $\widehat{s}$. By Lemma $5, \operatorname{conj}_{n-3}(\widehat{s})=$ aa $x_{2 k-1} \cdots \widehat{x_{4}}$. By Lemma 13, it holds $2 k \leq r\left(\operatorname{aconj}_{n-3}(\widehat{s})\right) \leq 2 k+2$. More precisely, it is $2 k+2$ since aconj $j_{n-3}(\widehat{s})$ is the smallest rotation of its BWT matrix, $\operatorname{conj}_{n-3}(\widehat{s})$ ais the second smallest rotation, and the relative order among the rotations of aconj ${ }_{n-3}(\widehat{s})$ coincide with that of the rotations of $\widehat{s}$, using the same argument from the proof of Lemma 13. This means that to obtain $B W T\left(s^{\prime}\right)$, it is enough to insert an a between the first two b's in $B W T(\widehat{s})$. As $r\left(\operatorname{aconj}_{n-3}(\widehat{s})\right)=r(\mathrm{a} \widehat{s})=r(\widehat{s} \mathrm{a})$, the thesis follows.

## 4 Additive $\Theta(\sqrt{n})$ factor

In the previous section we proved that a single edit operation can cause a multiplicative increase by a logarithmic factor in the number of runs. In this section, we will exhibit an infinite family of words on which a single edit operation can cause an additive increment of $r$ by $\Theta(\sqrt{n})$ (see Def. 16 below).

As we saw in the previous section, there exist infinite families of words such that $r=\Theta(\log n)$, where $n$ is the length of the word. Other families with logarithmic number of runs of the BWT are also known from the literature, e.g. the Thue-Morse words [6]. Moreover, there exist words such that $r$ is maximal, i.e., $r(w)=|w|$. For instance, if $w=$ aaaabbababbbbaabab, then $r(w)=18=|w|[26]$. Next we show that there is no gap between these two scenarios, i.e., it is possible to construct infinite families of words $w$ such that $r(w)=\Theta\left(n^{1 / k}\right)$, for any $k>1$.

Proposition 15. Let $k$ be a positive integer. There exists an infinite family $T_{k}$ of binary words such that $r(w)=\Theta\left(n^{1 / k}\right)$, for any $w \in T_{k}$.

Proof. We can define the set $T_{k}=\left\{w_{i, k}=\prod_{j=1}^{i} \mathrm{ab}^{j^{k}} \mid i \geq 1\right\}$. We can state that $\left|w_{i, k}\right|=\Theta\left(i^{k+1}\right)$. Moreover, $r\left(w_{i, k}\right)=\Theta(i)$ In fact, each a in $\operatorname{BWT}\left(w_{i, k}\right)$ corresponds to the last letter of one of the rotations having prefix $\mathrm{b}^{j^{k}} \mathrm{a}$, for some $1 \leq j \leq i$. On the other hand, all the rotations with prefix ab, as well as the remaining rotations with prefix $\mathrm{b}^{\ell} \mathrm{a}$ for all $1 \leq \ell \leq i^{k}$, end with b . It follows then that whenever $k \geq 2$, all the a's in $\operatorname{BWT}\left(w_{i, k}\right)$ are separated by an equal-letter run of $\mathrm{b}^{\prime} s$, leading to $r\left(w_{i, k}\right)=$ $2 i=\Theta(i)$. However, note that for a fixed $\ell$, the rotations starting with $b^{\ell}$ a are sorted according to the length of the maximal run of b's following the common prefix. Thus, even for $k=1$, there is only one run of consecutive a's in $\operatorname{BWT}\left(w_{i, k}\right)$, while the remaining are separated. More in detail, $\operatorname{BWT}\left(w_{i, 1}\right)=\mathrm{bb}^{i} \mathrm{ab}^{i-1} \mathrm{a} \cdots \mathrm{ab}^{3} \mathrm{ab}^{2} \mathrm{aa}$. Hence, $r\left(w_{i, 1}\right)=2 i-2=\Theta(i)$. The claim follows by observing that for the family of words $w_{i, k}$ it holds that $r\left(w_{i, k}\right)=\Theta(i)=\Theta\left(n^{\frac{1}{k+1}}\right)$, where $n=\left|w_{i, k}\right|$.

We will show that the following family of words satisfies that a single edit operation can cause an additive increment of $r$ by $\Theta(\sqrt{n})$.

Definition 16. For any $k>5$, let $s_{i}=\mathrm{ab}^{i} \mathrm{aa}$ and $e_{i}=\mathrm{ab}^{i} \mathrm{aba}^{i-2}$ for all $2 \leq i \leq k-1$, and $q_{k}=\mathrm{ab}^{k} \mathrm{a}$. Then,

$$
w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}=\left(\prod_{i=2}^{k-1} \mathrm{ab}^{i} \mathrm{aaab}^{i} \mathrm{aba}^{i-2}\right) \mathrm{ab}^{k} \mathrm{a} .
$$

The length of these words can be easily deduced from their definition.
Observation 17. Let $n=\left|w_{k}\right|$ for some $k>5$. It holds that $n=\sum_{i=2}^{k-1}(3 i+4)+$ $(k+2)=\left(3 k^{2}+7 k-18\right) / 2$. Moreover, it holds that $k=\Theta(\sqrt{n})$.

The following lemma will be used to show how the rotations of $w_{k}$ can be sorted according to the factorization $s_{2} e_{2} \cdots s_{k-1} e_{k-1} q_{k}$.

Lemma 18. Let $k>5$ be an integer. Then, $s_{2}<e_{2}<s_{3}<e_{3}<\ldots<s_{k-1}<e_{k-1}<$ $q_{k}$. Moreover the set $\mathcal{U}=\bigcup_{i=2}^{k-1}\left\{s_{i}, e_{i}\right\} \cup\left\{q_{k}\right\}$ is prefix-free.
Proof. For the first claim, note from the definition of the words $e_{i}, s_{i}$ and $q_{k}$ that for $i \in[2, k-1]$ it holds $s_{i}<e_{i}$, for $i \in[2, k-2]$ it holds $e_{i}<s_{i+1}$, and it holds $e_{k-1}<q_{k}$. For the second claim, observe that for any two distinct strings $x$ and $y$ in the set $\mathcal{U}$ starting with $\mathrm{ab}^{j} a$ and $a b^{j^{\prime}}$ a respectively, there are two possible cases. If $j=j^{\prime}$ then $x$ and $y$ are $s_{i}$ and $e_{i}$ respectively, and none of them is a prefix of the other. Otherwise, w.l.o.g. $j<j^{\prime}$, so $x=\mathrm{ab}^{j} \mathrm{a} x^{\prime}$ and $y=\mathrm{ab}^{j} \mathrm{~b} y^{\prime}$ for some $x^{\prime}$ and $y^{\prime}$. Hence $x[j+2] \neq y[j+2]$ and none of them is a prefix of the other. Thus, the set $\mathcal{U}$ is prefix-free.

### 4.1 Characterizing the BWT of $\boldsymbol{w}_{\boldsymbol{k}}$

In order to characterize the BWT of the word

$$
w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}=\left(\prod_{i=2}^{k-1} \mathrm{ab}^{i} \mathrm{aa} \cdot \mathrm{ab}^{i} \mathrm{aba}^{i-2}\right) \cdot \mathrm{ab}^{k} \mathrm{a}
$$

we divide its BWT matrix into disjoint ranges of consecutive rotations sharing the same (specific) prefixes, and characterize the substring of $\operatorname{BWT}\left(w_{k}\right)$ corresponding to each one of these prefixes.

Definition 19. Given $x, w \in \Sigma^{*}$, we denote by $\beta(x, w)$ the substring of $B W T(w)$ corresponding to the range of contiguous rotations prefixed by $x$. We omit the second parameter of $\beta(x, w)$ when it is clear from the context.

The structure of the whole BWT matrix of $w_{k}$ is summarized in Table 1. The following series of lemmas characterize the substring of $\operatorname{BWT}\left(w_{k}\right)$ corresponding to each range to be considered.

Lemma $20\left(\beta\left(\mathrm{a}^{k-2} \mathrm{~b}\right)\right)$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, the first rotation in the $B W T$ matrix is $\mathrm{a}^{k-3} q_{k} \cdots \mathrm{~b}$.

Proof. The first rotation in lexicographic order must start with the longest run of a's. By definition of $w_{k}$, the longest run of a's has length $k-2$, and it is obtained by concatenating the suffix $\mathrm{a}^{k-3}$ of $e_{k-1}$ with $q_{k}$, which is preceded by a b (otherwise we could extend the run of a's).

Lemma $21\left(\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)\right.$ for $\left.4 \leq i \leq k-3\right)$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, and an integer $4 \leq i \leq k-3$, the rotations in the BWT matrix starting with $\mathrm{a}^{i} \mathrm{~b}$ are $\mathrm{a}^{i-1} s_{i+2} \cdots \mathrm{~b}<\mathrm{a}^{i-1} s_{i+3} \cdots \mathrm{a}<\ldots<\mathrm{a}^{i-1} s_{k-1} \cdots \mathrm{a}<\mathrm{a}^{i-1} q_{k} \cdots \mathrm{a}$.
Proof. One can notice that, for all $4 \leq i \leq k-3$, the (circular) factor $\mathrm{a}^{i} \mathrm{~b}$ can only be obtained, for all $i+2 \leq j \leq k$, from the concatenation of the suffix a ${ }^{i-1}$ of $e_{j-1}$, with either the prefix ab of $s_{j}$, if $i+2 \leq j \leq k-1$, or the prefix ab of $q_{k}$, if $j=k$. By Lemma 18 , we can sort these rotations according to the lexicographic order of $\bigcup_{j=i}^{k-1}\left\{s_{j}\right\} \cup\left\{q_{k}\right\}$. Note that all these rotations end with an a, with the exception of the rotation starting with $\mathrm{a}^{i-1} s_{i+2}$, since it is where the only occurrence of $\mathrm{ba}^{i} \mathrm{~b}$ can be found.

Lemma 22 ( $\beta(\mathrm{aaab}))$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, the first five rotations in the BWT matrix starting with aaab are $\mathrm{aae}_{2} \cdots \mathrm{~b}<\mathrm{aa} e_{3} \cdots \mathrm{~b}<$ $\mathrm{aa} e_{4} \cdots \mathrm{~b}<\mathrm{aa}_{5} \cdots \mathrm{~b}<\mathrm{aa}_{5} \cdots \mathrm{~b}$, while the remaining are aas $s_{6} \cdots \mathrm{a}<\mathrm{aae}_{6} \cdots \mathrm{~b}<$ $\ldots<$ aa $_{k-1} \cdots \mathrm{a}<$ aa $_{k-1} \cdots \mathrm{~b}<$ aa $_{k} \cdots \mathrm{a}$.
Proof. Analogously to the proof of Lemma 21, some of the rotations starting with aaab can be obtained, for all $5 \leq j \leq k$, from the concatenation of the suffix aa of $e_{j-1}$, with either the prefix ab of $s_{j}$, if $5 \leq j \leq k-1$, or the prefix ab of $q_{k}$, if $j=k$. However, in this case we have more rotations starting with aaab, that are those rotations starting with the suffix aa of $s_{j^{\prime}}$ concatenated with the prefix ab of $e_{j^{\prime}}$, for all $2 \leq j^{\prime} \leq k-1$. Thus, all the rotations starting with aaab are sorted according to the lexicographic order of the words in $\bigcup_{j=5}^{k-1}\left\{s_{j}\right\} \cup \bigcup_{j^{\prime}=2}^{k-1}\left\{e_{j^{\prime}}\right\} \cup\left\{q_{k}\right\}$. Note that all the rotations starting either with aas $s_{j}$, for all $6 \leq j \leq k-1$, or with aa $q_{k}$, end with a. On the other hand, the rotations starting either with aa $s_{5}$ or with aa $e_{j}$, for all $2 \leq j \leq k-1$, end with a b .
Lemma 23 ( $\beta(\mathrm{aab}))$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, the first five rotations in the BWT matrix starting with aab are $\mathrm{a} s_{2} \cdots \mathrm{~b}<\mathrm{a} e_{2} \cdots \mathrm{a}<$ $\mathrm{a} e_{3} \cdots \mathrm{a}<\mathrm{a} s_{4} \cdots \mathrm{~b}<\mathrm{a} e_{4} \cdots \mathrm{a}$, while the remaining are $\mathrm{a} s_{5} \cdots \mathrm{a}<\mathrm{a} e_{5} \cdots \mathrm{a}<\ldots<$ $\mathrm{a} s_{k-1} \cdots \mathrm{a}<\mathrm{a} e_{k-1} \cdots \mathrm{a}<\mathrm{a} q_{k} \cdots \mathrm{a}$.
Proof. Each of the rotations starting with aaab from Lemma 22 induces a rotation starting with aab, obtained by shifting on the left one character a. It follows that all of these rotations end with an $a$. The other rotations starting with aab are the one obtained by concatenating the suffix a of $e_{3}$ and the prefix ab of $s_{4}$, and the one obtained by concatenating the suffix a of $q_{k}$ and the prefix ab of $s_{2}$. Moreover, both the rotations end with a b . The thesis follows by sorting the rotations according to the lexicographic order of the words in $\left\{s_{2}\right\} \cup \bigcup_{j=4}^{k-1}\left\{s_{j}\right\} \cup \bigcup_{j^{\prime}=2}^{k-1}\left\{e_{j^{\prime}}\right\} \cup\left\{q_{k}\right\}$.

Lemma $24(\beta(\mathrm{ab}))$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, the first $k-2$ rotations in the BWT matrix starting with ab are $\mathrm{aba}^{k-3} q_{k} \cdots \mathrm{~b}<\mathrm{aba}^{k-4} s_{k-1} \cdots \mathrm{~b}<$ $\ldots<\mathrm{ab} s_{3} \cdots \mathrm{~b}$, the following four rotations are $s_{2} \cdots \mathrm{a}<e_{2} \cdots \mathrm{a}<s_{3} \cdots \mathrm{~b}<e_{3} \cdots \mathrm{a}$, and the remaining are $s_{4} \cdots \mathrm{a}<e_{4} \cdots \mathrm{a}<\ldots<s_{k-1} \cdots \mathrm{a}<e_{k-1} \cdots \mathrm{a}<q_{k} \cdots \mathrm{a}$.
Proof. For any two distinct integers $i, i^{\prime} \geq 0$, we have that $\mathrm{aba}^{i} \mathrm{~b}<\mathrm{aba}^{i^{\prime}} \mathrm{b}$ if and only if $i>i^{\prime}$. Thus, the first rotation in lexicographic order starting with ab is the one which is followed by the longest run of a's. The smallest of these rotations can be found by concatenating the suffix aba ${ }^{k-3}$ of $e_{k-1}$ with the prefix ab of $q_{k}$, followed by the suffix aba ${ }^{i-2}$ of $e_{i-1}$ concatenated with the prefix ab of $s_{i}$, for all $3 \leq i \leq k-1$ taken in decreasing order. By construction of $e_{i}$, for all $3 \leq i \leq k-1$, these rotations must end with a b.

The remaining rotations starting with ab are exactly those rotations having as prefix either $s_{i}$ or $e_{i}$, for all $2 \leq i \leq k-1$, or $q_{k}$. Note that all of these rotations are obtained by shifting on the left one character a from the rotations starting with aab from Lemma 23, with the exception of the one starting with $s_{3}$. It follows that the latter ends with $\mathrm{a} b$, while all the other rotations with an a .
Lemma 25 ( $\beta(\mathrm{ba}))$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, the first $k-5$ rotations in the BWT matrix starting with ba are $\mathrm{ba}^{k-3} q_{k} \cdots \mathrm{a}<\mathrm{ba}^{k-4} s_{k-1} \cdots \mathrm{a}<$ $\ldots<\mathrm{ba}^{3} s_{6} \cdots \mathrm{a}$, followed by baae $e_{2} \cdots \mathrm{~b}<$ baae $_{3} \cdots \mathrm{~b}<$ baae $_{4} \cdots \mathrm{~b}<$ baa $s_{5} \cdots \mathrm{a}<$ baae ${ }_{5} \cdots \mathrm{~b}$, then by baae $e_{6} \cdots \mathrm{~b}<$ baae $_{7} \cdots \mathrm{~b}<\ldots<$ baae $_{k-1} \cdots \mathrm{~b}<$ ba $s_{2} \cdots \mathrm{~b}<$ ba $s_{4} \cdots \mathrm{a}$, and finally by $\mathrm{baba}^{k-3} q_{k} \cdots \mathrm{~b}<\mathrm{baba}^{k-4} s_{k-1} \cdots \mathrm{~b}<\ldots<\mathrm{bab} s_{3} \cdots \mathrm{~b}<$ $\mathrm{b} s_{3} \cdots \mathrm{a}$.

Proof. One can notice that we have as many circular occurrences of ba as the number of maximal (circular) runs of b's in $w_{k}$. Then, for all $2 \leq i \leq k-1$, we have (i) one run of b's in $s_{i}$, and (ii) two runs in $e_{i}$, and (iii) one run in $q_{k}$.

For the case (i), we have one rotation starting with baae $e_{i}$, for each $2 \leq i \leq k-1$. Since each run of b's within each word from $\bigcup_{i=2}^{k-1}\left\{s_{i}\right\}$ is of length at least 2, all rotations in (i) end with a b.

For the case (ii), for all $2 \leq i \leq k-1$, we can distinguish between two sub-cases, based on where ba starts: if either (ii.a) from the first run of b's in $e_{i}$, or (ii.b) from the second one. For the case (ii.a), we can see that these rotations are of the type baba ${ }^{i-2} s_{i+1}$, if $2 \leq i<k-2$, and $\mathrm{baba}^{k-3} q_{k}$. Analogously to the case (i), each rotations for case (ii.a) end with a b. Each rotation in (ii.b) is obtained by shifting two characters on the right each rotation in (ii.a). Therefore, all of these rotations end with an a and have prefixes of the type $\mathrm{ba}^{i-2} s_{i+1}$, if $2 \leq i<k-2$, or $\mathrm{ba}^{k-3} q_{k}$.

For the case (iii), the rotation starting with ba in $q_{k}$ has bas $s_{2}$ as prefix, and it is preceded by a b.

Observe that only for (ii.b) we have rotations starting with baaaa. Hence, the first rotation in lexicographic order is the one starting with $\mathrm{ba}^{k-3} q_{k}$, followed by those starting with $\mathrm{ba}^{k-4} s_{k-1}<\mathrm{ba}^{k-5} s_{k-2}<\ldots<$ baaa $s_{6}$.

Among the remaining rotations, those having prefix baaa either start with baas $s_{5}$ from (ii.b), or baae $e_{i}$ from (i), for all $2 \leq i \leq k-1$. Thus, by Lemma 18, we can sort them according to the order of the words in $\left\{s_{5}\right\} \cup \bigcup_{i=2}^{k-1}\left\{e_{i}\right\}$. Then, the remaining rotations with prefix baa are those starting with ba $s_{2}$ from (iii), and ba $s_{4}$ from (ii.b). Finally,
let us focus on the rotations from case (ii.a). These rotations are sorted according to the length of the run of a's following the common prefix bab, similarly to the sorting of the rotations from the case (ii.b). The last rotation left is the one starting with $\mathrm{b} s_{3}$ from case (ii.b). Since this rotation is greater than each word from case (ii.a), this is the greatest rotation of $w_{k}$ starting with ba and the thesis follows.
Lemma $26\left(\beta\left(\mathbf{b}^{j} \mathrm{a}\right)\right.$ for all $\left.2 \leq j \leq k-1\right)$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, and an integer $2 \leq i \leq k-2$, the first $k-i$ rotations in the BWT matrix starting with $\mathrm{b}^{i} \mathrm{a}$ are $\mathrm{b}^{i} \mathrm{aae}_{i} \cdots \mathrm{a}<\mathrm{b}^{i} \mathrm{aa}_{i+1} \cdots \mathrm{~b}<\ldots<\mathrm{b}^{i} \mathrm{aae}_{k-1} \cdots \mathrm{~b}<\mathrm{b}^{i} \mathrm{a} s_{2} \cdots \mathrm{~b}$, followed by $\mathrm{b}^{i} \mathrm{aba}^{k-3} q_{k} \cdots \mathrm{~b}<\mathrm{b}^{i} \mathrm{aba}^{k-4} s_{k-1} \cdots \mathrm{~b}<\ldots<\mathrm{b}^{i} \mathrm{aba}^{i-1} s_{i+2} \cdots \mathrm{~b}<$ $\mathrm{b}^{i} \mathrm{aba}^{i-2} s_{i+1} \cdots \mathrm{a}$.

Proof. All runs of b's of length at least $2 \leq i \leq k-2$, either appear in (i) $s_{j}$ or (ii) $e_{j}$, for all $i \leq j \leq k-1$, or in (iii) $q_{k}$. Let us consider the three cases separately. For all $i \leq j \leq k-1$, the rotation starting within $s_{j}$ (i) has as prefix $\mathrm{b}^{i} \mathrm{a} \mathrm{a}_{j}$. For all $i \leq j \leq k-2$, the rotation starting within $e_{j}$ (ii) has as prefix $\mathrm{b}^{i} \mathrm{aba}^{j-2} s_{j+1}$, and for $j=k-1$ we have the rotation with prefix $\mathrm{b}^{i} \mathrm{aba}^{k-3} q_{k}$. Finally, the rotation starting within $q_{k}$ (iii) has as prefix $\mathrm{b}^{i} \mathrm{a} s_{2}$.

By construction, we can see that first we have all the rotations from case (i) sorted according to the lexicographic order of the words in $\bigcup_{j=i}^{k-1}\left\{e_{i}\right\}$ (Lemma 18), then we have the rotation from case (iii), and finally the rotation from case (ii), sorted according to the decreasing length of the run of a's following the common prefix $b^{i} a b$.

Moreover, note that only when the run of b's is of length exactly $i$ the rotation end with an a. Thus, the only for the rotations ending with an a are those starting within $s_{i}$ and $e_{i}$, i.e. those with prefix $\mathrm{b}^{i} e_{i}$ and $\mathrm{b}^{i} \mathrm{aba}^{i-2} s_{i+1}$.
Lemma $27\left(\beta\left(\mathrm{~b}^{k} \mathrm{a}\right)\right)$. Given the word $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$ for some $k>5$, the last four rotations of the BWT matrix are $\mathrm{b}^{k-1} \mathrm{aae}_{k-1} \cdots \mathrm{a}<\mathrm{b}^{k-1} \mathrm{a} s_{2} \cdots \mathrm{~b}<$ $\mathrm{b}^{k-1} \mathrm{aba}^{k-3} q_{k} \cdots \mathrm{a}<\mathrm{b}^{k} \mathrm{a} s_{2} \cdots \mathrm{a}$.

Proof. Observe that the only rotations with prefix $\mathrm{b}^{k-1} \mathrm{a}$ either start within $s_{k-1}$, or $q_{k}$, or $e_{k-1}$. These rotations have prefix respectively $\mathrm{b}^{k-1} \mathrm{a} e_{k-1}, \mathrm{~b}^{k-1} \mathrm{a} s_{2}$, and $\mathrm{b}^{k-1} \mathrm{aba}^{k-3} q_{k}$. One can see that these rotations taken in this order are already sorted, and only the rotation starting within $q_{k}$ ends with a b, while the other two with an a. Finally, the only occurrence of $\mathrm{b}^{k}$ is within $q_{k}$. Hence, the last rotation in lexicographic order starts with $\mathbf{b}^{k} \mathrm{a} s_{2}$, and since the run of b's is maximal it ends with an $\mathbf{a}$, and the thesis follows.

The following proposition puts together the BWT computations carried out for all blocks of consecutive rows, highlighting which prefixes are shared.

Proposition 28. Given an integer $k>5$, let $w_{k}=\left(\prod_{i=2}^{k-1} s_{i} e_{i}\right) q_{k}$. Then,

$$
\begin{aligned}
& \beta\left(\mathrm{a}^{i} \mathrm{~b}\right)=\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2, \\
& \beta\left(\mathrm{a}^{3} \mathrm{~b}\right)=\mathrm{b}^{5}(\mathrm{ab})^{k-6} \mathrm{a} \\
& \beta\left(\mathrm{a}^{2} \mathrm{~b}\right)=\mathrm{baaba}^{2 k-8}, \\
& \beta(\mathrm{ab})=\mathrm{b}^{k-2} \mathrm{aaba}^{2 k-6},
\end{aligned}
$$



Table 1: Scheme of the BWT-matrix of a word $w_{k}$ with $k>5$. The block prefix column shows the common prefix shared by all the rotations in a block. The ordering factor column shows the factor following the block prefix of a rotation, which decides its relative order inside its block. The BWT column shows the last character of each rotation. The dashed lines divide sub-ranges of rotations for which the BWT follows distinct patterns.

$$
\begin{aligned}
\beta(\mathrm{ba}) & =\mathrm{a}^{k-5} \mathrm{bbbab}^{k-4} \mathrm{ab}^{k-2} \mathrm{a} \\
\beta\left(\mathrm{~b}^{j} \mathrm{a}\right) & =\mathrm{ab}^{2 k-2 j-1} \mathrm{a} \text { for all } 2 \leq j \leq k-1, \text { and } \\
\beta\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{a}
\end{aligned}
$$

Hence, the BWT of the $w_{k}$ is $B W T\left(w_{k}\right)=\prod_{i=2}^{k-1} \beta\left(\mathbf{a}^{k-i} \mathbf{b}\right) \cdot \prod_{i=1}^{k} \beta\left(\mathrm{~b}^{i} \mathrm{a}\right)$. Moreover, $r\left(w_{k}\right)=6 k-12$.
Proof. The words $\beta\left(\mathrm{a}^{k-2} \mathrm{~b}\right), \beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ for all $4 \leq i \leq k-2, \beta\left(\mathrm{a}^{3} \mathrm{~b}\right), \beta\left(\mathrm{a}^{2} \mathrm{~b}\right), \beta(\mathrm{ab}), \beta(\mathrm{ba})$, $\beta\left(\mathrm{b}^{j} \mathrm{a}\right)$ for all $2 \leq j \leq k-1$, and $\beta\left(\mathrm{b}^{k} \mathrm{a}\right)$, are the concatenations of the last characters of the rotations from Lemma 20, Lemma 21, Lemma 22, Lemma 23, Lemma 24, Lemma 25, Lemma 26, and Lemma 27 respectively. Moreover, every rotation used to build $\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ is smaller than each rotation used to build $\beta\left(\mathrm{a}^{i^{\prime}} \mathrm{b}\right)$, for every $1 \leq i^{\prime}<i \leq k-2$. Symmetrically, every rotation used to build $\beta\left(\mathrm{b}^{j} \mathrm{a}\right)$ is greater than each rotation used to build $\beta\left(\mathrm{b}^{j^{\prime}} \mathrm{a}\right)$, for every $1 \leq j^{\prime}<j \leq k$. Since we have considered all the disjoint ranges of rotations of $w_{k}$ based on their common prefix, the word $\prod_{i=2}^{k-1} \beta\left(\mathrm{a}^{k-i} \mathrm{~b}\right) \cdot \prod_{i=1}^{k} \beta\left(\mathrm{~b}^{i} \mathrm{a}\right)$ is the BWT of $w_{k}$.

With the structure of $\operatorname{BWT}\left(w_{k}\right)$, we can easily derive its number of runs. The word $\prod_{i=2}^{k-4}\left(\beta\left(\mathrm{a}^{k-i} \mathrm{~b}\right)\right)$ has exactly $2(k-6)$ runs: we start with 2 runs from $\beta\left(\mathrm{a}^{k-2} \mathrm{~b}\right) \beta\left(\mathrm{a}^{k-3} \mathrm{~b}\right)=\mathrm{bba}$, and then, concatenating each other $\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ up to $\beta\left(\mathrm{a}^{4} \mathrm{~b}\right)$
adds 2 new runs each. It is easy to see that $\beta(\mathrm{aaab}), \beta(\mathrm{aab})$, and $\beta(\mathrm{ab})$, have $2(k-5), 4$, and 4 runs, respectively. Moreover, the boundaries between these words do not merge, nor with $\beta\left(\mathrm{a}^{4} \mathrm{~b}\right)$ in the case of $\beta(\mathrm{aaab})$. The word $\beta(\mathrm{ba})$ has exactly 7 runs but it merges with $\beta(\mathrm{ab})$ and $\beta(\mathrm{bba})$, hence we only charge 5 runs to this word. The remaining part of the BWT, i.e., $\prod_{i=2}^{k}\left(\beta\left(\mathrm{~b}^{i} \mathrm{a}\right)\right)$, has $2(k-2)+1$ runs: we start with 3 runs from $\beta$ (bba), and then, concatenating each other $\beta\left(\mathrm{b}^{i} \mathrm{a}\right)$ up to $\beta\left(\mathrm{b}^{k-1} \mathrm{a}\right)$ adds 2 new runs each. The word $\beta\left(\mathrm{b}^{k} \mathrm{a}\right)$ does not add new runs, as it consists only of an a that merges with the previous one. Overall, we have $2(k-6)+2(k-5)+4+4+5+2(k-2)+1=$ $6 k-12$, and the claim holds.

### 4.2 BWT of $w_{k}$ after an edit operation

The following lemmas describe the BWT of $w_{k}$ after some specific edit operations are applied. Instead of proving the whole structure of the BWT from the beginning, we compare how the edit operation changes either the relative order or the last character of the rotations of $w_{k}$. To do so, in this part we use the notation $\beta(v)$ and $\beta^{\star}(v)$ to denote the BWT in correspondence of the rotations with prefix $v \in \Sigma^{*}$ of $w_{k}$ and $w_{k}^{\prime}$ respectively, where $w_{k}^{\prime}$ is obtained after applying to $w_{k}$ an specific edit operation. The number of runs in the BWT of $w_{k}^{\prime}$ can easily be derived by comparing its BWT to the BWT of $w_{k}$, for which we explicitly counted the number of runs, so we omit that part of the proofs. All the edit operations on $w_{k}$ we show in this subsection increase the number $r\left(w_{k}\right)$ by a $\Theta(k)$ additive factor. To give an intuition, this increment comes mainly from the $\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right)$ ranges for $2 \leq j \leq k-2$, because for each one of the corresponding ranges $\beta\left(\mathrm{b}^{j} \mathrm{a}\right)=\mathrm{ab}^{2 k-2 j-1} \mathrm{a}$ in $\operatorname{BWT}\left(w_{k}\right)$, one of the b 's is either moved to the top or the bottom of the range, in a consistent manner for each $j$ (it depends on the edit operation if the b goes to the top or the bottom of the range, but it is the same behavior for all the ranges considered). Then, two ranges that originally agreed on their last and first character in $w_{k}$ are now separated by a b , adding this way 2 new runs for each $j$.

Lemma 29 (BWT of $w_{k} \mathrm{a}$ ). Given an integer $k>5$, for $w_{k} \mathrm{a}$ it holds that

$$
\begin{aligned}
\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right) & =\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2 \\
\beta^{\star}\left(\mathrm{a}^{3} \mathrm{~b}\right) & =\mathrm{bb}^{5}(\mathrm{ab})^{k-6} \mathrm{a} \\
\beta^{\star}\left(\mathrm{a}^{2} \mathrm{~b}\right) & =\mathrm{aaaba}^{2 k-8} \\
\beta^{\star}(\mathrm{ab}) & =\mathrm{b}^{k-2} \mathrm{aaba}^{2 k-6} \\
\beta^{\star}(\mathrm{ba}) & =\mathrm{a}^{k-5} \mathrm{bbbbab}^{k-5} \mathrm{ab}^{k-2} \mathrm{a} \\
\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right) & =\mathrm{bab}^{2 k-2 j-2} \mathrm{a} \text { for all } 2 \leq j \leq k-1 \text { and } \\
\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{a}
\end{aligned}
$$

Hence, $B W T\left(w_{k} \mathbf{a}\right)=\prod_{i=2}^{k-1} \beta^{\star}\left(\mathbf{a}^{k-i} \mathbf{b}\right) \cdot \prod_{i=1}^{k} \beta^{\star}\left(\mathrm{b}^{i} \mathbf{a}\right)$. Moreover, it holds that $r\left(w_{k} \mathbf{a}\right)=$ $8 k-20$.

Proof. By Lemmas 20 and 21, we can see that appending an a after $q_{k}$ does not affect the BWT in the range of rotations having $\mathrm{a}^{i} \mathrm{~b}$ as prefix, for all $4 \leq i \leq k-2$. Thus, $\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right)=\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ for all $4 \leq i \leq k-2$.

The rotation starting with aas $s_{2}$, which is not a circular factor of $w_{k}$, ends with a b. By Lemma 22, we can see that such a rotation is the smallest one with prefix aaab in lexicographic order, while the other rotations maintain their relative order. Therefore, $\beta^{\star}(\mathrm{a} a \mathrm{ab})=\mathrm{b} \cdot \beta(\mathrm{aaab})$.

By Lemma 23, the rotation with prefix a $s_{2}$ is still the smallest rotation starting with aab, with the difference that in this case, it ends with the last a of $q_{k}$. It follows that $\beta^{\star}(\mathrm{aab})$ is obtained by replacing the first b of $\beta(\mathrm{aab})$ with an a .

Both the order and the last symbol of all the rotations having as prefix ab described in Lemma 24 is not affected from the insertion of the a , and therefore $\beta^{\star}(\mathrm{ab})=\beta(\mathrm{ab})$.

Let us now consider all the rotations of $w_{k}$ with prefix $\mathrm{b}^{j} \mathrm{a}_{2}$, for all $1 \leq j \leq k$. One can notice that $w_{k}$ a does not have any rotation starting with $\mathrm{b}^{j}$ a $s_{2}$, for all $1 \leq j \leq k$, but instead it has rotations starting with $\mathrm{b}^{j}$ aas $s_{2}$. Thus, for all $1 \leq j \leq k-1$, to obtain $\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right)$ from $\beta\left(\mathrm{b}^{j} \mathrm{a}\right)$ we have to remove the b in correspondence of the rotations starting with $\mathrm{b}^{j} \mathrm{a} s_{2}$, and add ab in correspondence of the rotations $\mathrm{b}^{j} \mathrm{aa} s_{2}$. By Lemmas 25,26 , and 27 , such rotations are placed right before the rotation starting with $\mathrm{b}^{j} \mathrm{aae}_{2}$.

Finally, the last rotation has still the same prefix $b^{k} a$ and ends with an $a$, and the thesis follows.

Lemma 30 (BWT of $\widehat{w_{k}}$ ). Given an integer $k>5$, for $\widehat{w_{k}}$ it holds that

$$
\begin{aligned}
\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right) & =\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2, \\
\beta^{\star}\left(\mathrm{a}^{3} \mathrm{~b}\right) & =\mathrm{b}^{5}(\mathrm{ab})^{k-6} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{a}^{2} \mathrm{~b}\right) & =\mathrm{aaba}^{2 k-8}, \\
\beta^{\star}(\mathrm{ab}) & =\mathrm{b}^{k-2} \mathrm{baba}^{2 k-6} \\
\beta^{\star}(\mathrm{ba}) & =\mathrm{a}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{ab}^{k-2} \mathrm{ba} \\
\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right) & =\mathrm{ab}^{2 k-2 j-2} \mathrm{ab} \text { for all } 2 \leq j \leq k-1 \text { and } \\
\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{a}
\end{aligned}
$$

Hence, $B W T\left(\widehat{w_{k}}\right)=\prod_{i=2}^{k-1} \beta^{\star}\left(\mathrm{a}^{k-i} \mathrm{~b}\right) \cdot \prod_{i=1}^{k} \beta^{\star}\left(\mathrm{b}^{i} \mathrm{a}\right)$. Moreover, it holds that $r\left(\widehat{w_{k}}\right)=8 k-20$.

Proof. Analogously to the previous Lemma, if we look in Lemmas 20, 21, and 22, at the structure of the BWT in correspondence of the rotations starting with $\mathrm{a}^{i} \mathrm{~b}$, for all $3 \leq i \leq k-2$, we can notice that the order or the symbols in the BWT is not affected. Thus, for all $3 \leq i \leq k-2$, we have $\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right)=\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$.

Since the last a of $q_{k}$ is omitted, the circular factor as $s_{2}$ does not appear anymore in $\widehat{w}$. Thus, $\beta^{\star}(\mathrm{aab})$ is obtained by removing the first b from $\beta(\mathrm{aab})$, since by Lemma 23 it is in correspondence of the rotation with prefix a $s_{2}$.

On the other hand we can observe from Lemma 24 that the rotation with prefix $s_{2}$ maintains its relative order also in $\widehat{w_{k}}$, but its last symbol is now a b instead of an a.

For each $1 \leq j \leq k$, the rotation starting with $\mathrm{b}^{j} \mathrm{a} s_{2}$ of $w_{k}$ does not appear in $\widehat{w_{k}}$, but in fact it is replaced by one having $\mathrm{b}^{j} s_{2}$ as prefix and ending in the same way. When $j=1$, by Lemma 25 such a rotation is located between the last two rotations with the prefix ba , which start by $\mathrm{bab} s_{3}$ and $\mathrm{b} s_{3}$ respectively. When $2 \leq j \leq k-1$, by Lemmas 26 and 27 , the rotation starting with $\mathrm{b}^{3} s_{2}$ is greater than all the other rotations with prefix $\mathrm{b}^{j} \mathrm{a}$. Thus, for all $1 \leq j \leq k-1$, we obtain $\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right)$ by moving the b in correspondence of the rotation starting with $\mathrm{ba} s_{2}$ from $\beta\left(\mathrm{b}^{j} \mathrm{a}\right)$ and placing it in correspondence of $\mathrm{b}^{j} s_{2}$. Finally, the last rotation has still the same prefix $\mathrm{b}^{k} \mathrm{a}$ and ends with an a, and the thesis follows.

Lemma 31 (BWT of $\widehat{w_{k}} \mathrm{~b}$ ). Given an integer $k>5$, for $\widehat{w_{k}} \mathrm{~b}$ it holds that

$$
\begin{aligned}
\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right) & =\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2, \\
\beta^{\star}\left(\mathrm{a}^{3} \mathrm{~b}\right) & =\mathrm{b}^{5}(\mathrm{ab})^{k-6} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{a}^{2} \mathrm{~b}\right) & =\mathrm{aaba}^{2 k-8} \\
\beta^{\star}(\mathrm{ab}) & =\mathrm{b}^{k-2} \mathrm{baba}^{2 k-6}, \\
\beta^{\star}(\mathrm{ba}) & =\mathrm{a}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{ab}^{k-2} \mathrm{ba}, \\
\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right) & =\mathrm{ab}^{2 k-2 j-2} \mathrm{ab} \text { for all } 2 \leq j \leq k-1, \\
\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{b} \text { and } \\
\beta^{\star}\left(\mathrm{b}^{k+1} \mathrm{a}\right) & =\mathrm{a} .
\end{aligned}
$$

Hence, $B W T\left(\widehat{w_{k}} \mathbf{b}\right)=\prod_{i=2}^{k-1} \beta^{\star}\left(\mathrm{a}^{k-i} \mathrm{~b}\right) \cdot \prod_{i=1}^{k+1} \beta^{\star}\left(\mathrm{b}^{i} \mathrm{a}\right)$. Moreover, it holds that $r\left(\widehat{w_{k}} \mathrm{~b}\right)=8 k-20$.

Proof. For the rotations in correspondence of the rotations starting with an a, notice that replacing the last a of $w_{k}$ for a b or removing the last a affects the BWT in the same way. Therefore, $\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right)$ is the same as Lemma 30 for all $1 \leq i \leq k-2$.

The same behaviour can be noticed on the rotations with prefix $\mathrm{b}^{j}$ a, for all $1 \leq$ $j \leq k-1$, while the rotation starting with $\mathrm{b}^{k} \mathrm{a}$ is now preceded by a b .

With respect to the other edit operations, we have the range of rotations starting with $\mathrm{b}^{k+1} \mathrm{a}$, which consists solely in $\mathrm{b}^{k+1} s_{2} \cdots \mathrm{a}$.

The structure of the BWT of $w_{k}$ and other words obtained by applying one or more edit operations on $w_{k}$ are summed up in Table 2.

For a given word $w \neq \epsilon$, let $w^{i n s}, w^{d e l}$, and $w^{\text {sub }}$ be the words obtained by applying on $w$ an insertion, a deletion, and a substitution of a character respectively.

We compare the number of runs of $w_{k}$ and its variations and notice that the difference after applying one of the edit operations is $\Theta(k)$ in the three cases.

Proposition 32. There exists an infinite family of words $w$ such that: (i) $r\left(w^{i n s}\right)-$ $r(w)=\Theta(\sqrt{n})$; (ii) $r\left(w^{\text {del }}\right)-r(w)=\Theta(\sqrt{n})$; (iii) $r\left(w^{\text {sub }}\right)-r(w)=\Theta(\sqrt{n})$.

Proof. The family is composed of the words $w_{k}$ with $k>5$. Let $n=\left|w_{k}\right|$. If $w_{k}^{\text {ins }}=w_{k} \mathrm{a}, w_{k}^{\text {del }}=\widehat{w_{k}}$, and $w_{k}^{\text {sub }}=\widehat{w_{k}} \mathrm{~b}$, from Proposition 28, Lemma 29, Lemma


Table 2: BWTs of the word $w_{k}$ and its variants after different edit operations. The word in the intersection of the column $\beta(x)$ with the row $w$ is the range of $\operatorname{BWT}(w)$ corresponding to all the rotations that have $x$ as a prefix. The columns $\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ and $\beta\left(\mathrm{b}^{j} \mathrm{a}\right)$ represent ranges of columns from $i \in[k-2,4]$ (in that order) and $j \in[2, k-1]$, respectively. Note that the prefixes in the columns are disjoint, and cover all the possible ranges for the set of words considered. The BWT of each word is the concatenation of all the words in its row from left to right. In the last column appears the number of BWT runs of each of these words.

30, and Lemma 31, we have that $r\left(w_{k} \mathbf{a}\right)=r\left(\widehat{w_{k}}\right)=r\left(\widehat{w_{k}} \mathbf{b}\right)=r\left(w_{k}\right)+(2 k-8)$. From Observation 17, we have that $2 k-8=\Theta(\sqrt{n})$.

## 5 Bit catastrophes for $\boldsymbol{r}_{\$}$

In this section, we discuss bit catastrophes when the parameter $r_{\$}$ is considered. Recall that for a word $v, r_{\$}(v)=\operatorname{runs}(\operatorname{BWT}(v \$))$.

### 5.1 When there is no bit catastrophe for $\boldsymbol{r}_{\$}$

First let us consider the case where a symbol $c \in \Sigma$ is prepended to a word $v$. As recently noted in [1], it is well known that in this case the value $r_{\$}$ can only vary by a constant value. For the sake of completeness, we include a proof.

Proposition 33. For any $\mathrm{x} \in \Sigma$, we have $r_{\$}(v)-1 \leq r_{\$}(\mathrm{x} v) \leq r_{\Phi}(v)+2$.
Proof. Let us consider the list of lexicographically sorted cyclic rotations or, equivalently, the list of lexicographically sorted suffixes of $x v \$$. (The equivalence follows from the fact that $\$$ is smaller than all other characters.) This list can be obtained from the list of suffixes of $v \$$, to which the suffix $\mathrm{x} v \$$ is added. Note that the relative order of all suffixes other than $x v \$$ remains the same. Moreover, the corresponding symbols in the BWT also remain the same, except that the character x takes the place of $\$$. This
replacement decreases the number of BWT-runs by 0,1 , or 2 , depending on whether this position in the BWT is preceded by a run of $x$, followed by a run of $x$, or both. The symbol corresponding to the new suffix $\mathrm{x} v \$$ (which produces the insertion of $\$$ in the corresponding position in the BWT) increases the number of BWT-runs by 1 (if it is inserted between two existing runs), or by 2 (if it breaks a run).

The following proposition shows that there are some cases in which $r_{\Phi}$ is not affected by any bit catastrophe.

Proposition 34. Let x be smaller than or equal to the smallest character in a word $v$, then $r_{\$}(v) \leq r_{\$}(v \mathrm{x}) \leq r_{\$}(v)+1$.
Proof. The rotations of $v \mathrm{x} \$$ can be viewed as the rotation $\$ v \mathrm{x}$, plus the rotations of $v \$$, where the occurrence of $\$$ has been replaced by $\mathrm{x} \$$. The smallest of these is of course $\$ v \mathrm{x}$, since it starts with $\$$, while all others appear in the same order as before. This is because x is smaller or equal the smallest character of $v$ and greater than $\$$, and therefore, replacing $\$$ by $\mathrm{x} \$$ does not change the lexicograhic order of these rotations. This implies $\operatorname{BWT}(v \mathrm{x} \$)=\mathrm{x} \cdot \mathrm{BWT}(v \$)$, and thus, $r_{\$}(v) \leq r_{\$}(v \mathrm{x}) \leq r_{\$}(v)+1$.

### 5.2 Multiplicative bit catastrophes for $\boldsymbol{r}_{\boldsymbol{s}}$

We can derive from our results in Sec. 3 that there exist families of strings on which an edit operation can result in an increase of $r_{\$}$ by a multiplicative factor of $\log n$.

Proposition 35. Let $v$ be the Lyndon rotation of the Fibonacci word $s$ of even order $2 k>4$, and $n=|v|$. Let $v^{\prime}$ be the word resulting by appending $a \mathrm{~b}$ to $v$. Then $r_{\Phi}\left(v^{\prime}\right)=\Theta(\log n)$.
Proof. Let $s=x_{2 k} \mathrm{ab}=x_{2 k-1} \mathrm{ba} x_{2 k-2} \mathrm{ab}$ be the Fibonacci word of order $2 k$. One can see that $v=\mathrm{a} x_{2 k} \mathrm{~b}=\mathrm{a} x_{2 k-2} \mathrm{ab} x_{2 k-1} \mathrm{~b}$ [3]. Since $v$ is a rotation of $s$, it holds that $r(v)=2$. By using Lemma $13, r_{\$}(v)=\Theta(1)$ since $v$ is a Lyndon word. When we append b to $v$, we obtain $v^{\prime}=\mathrm{a} x_{2 k-2} \mathrm{ab} x_{2 k-1} \mathrm{bb}$. One can note that $v^{\prime}=\mathrm{a} x_{2 k-2} \mathrm{ab} x_{2 k-1} \mathrm{bb}$ is also a Lyndon word. Moreover, appending b to $v$ is equivalent to inserting b in $s$ at position $F_{2 k-1}-2$, implying that $v^{\prime}$ is a rotation of $s^{\prime}$, where $s^{\prime}$ is $s$ with a b inserted in position $F_{2 k-1}-2$. By using Proposition 3, we thus have that $r\left(v^{\prime}\right)=r\left(s^{\prime}\right)=\Theta(\log n)$. Since $v^{\prime}$ is also a Lyndon word, therefore $r_{\$}\left(v^{\prime}\right)=\Theta(\log n)$, using Lemma 13 again.

### 5.3 Additive bit catastrophes for $\boldsymbol{r}_{\boldsymbol{\$}}$

In general, appending, deleting, or substituting with a symbol that is not the smallest of the alphabet can increase the number of runs of a word by an additive factor of $\Theta(\sqrt{n})$.

Lemma 36 (BWT of $w_{k} \$$ ). Given an integer $k>5$, for $w_{k} \$$ it holds that

$$
\begin{aligned}
\beta^{\star}(\$) & =\mathrm{a} \\
\beta^{\star}(\mathrm{a} \$) & =\mathrm{b}
\end{aligned}
$$

$$
\begin{aligned}
\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right) & =\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2, \\
\beta^{\star}\left(\mathrm{a}^{3} \mathrm{~b}\right) & =\mathrm{b}^{5}(\mathrm{ab})^{k-6} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{a}^{2} \mathrm{~b}\right) & =\mathrm{aaba}^{2 k-8}, \\
\beta^{\star}(\mathrm{ab}) & =\mathrm{b}^{k-2} \$ \mathrm{aba}^{2 k-6} \\
\beta^{\star}(\mathrm{ba}) & =\mathrm{ba}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{ab}^{k-2} \mathrm{a} \\
\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right) & =\mathrm{bab}^{2 k-2 j-2} \mathrm{a} \text { for all } 2 \leq j \leq k-1 \text { and } \\
\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{a} .
\end{aligned}
$$

Hence, $B W T\left(w_{k} \$\right)=\beta^{\star}(\$) \cdot \beta^{\star}(\mathrm{a} \$) \cdot \prod_{i=2}^{k-1} \beta^{\star}\left(\mathrm{a}^{k-i} \mathrm{~b}\right) \cdot \prod_{i=1}^{k} \beta^{\star}\left(\mathrm{b}^{i} \mathrm{a}\right)$. Moreover, it holds that $r\left(w_{k} \$\right)=8 k-16$.

Proof. The first rotation of $\operatorname{BWT}\left(w_{k} \$\right)$ is $\$ w_{k}$ and ends with an a because $w_{k}$ ends with an a. Hence, $\beta^{\star}(\$)=\mathrm{a}$. There is also a rotation $\mathrm{a} \$ \widehat{w_{k}}$, which ends with a b because $\widehat{w_{k}}$ ends with a b . Hence, $\beta^{\star}(\mathrm{a} \$)=\mathrm{b}$. It lefts to compare the remaining ranges $\beta^{\star}(v)$ with respect to $\beta(v)$.

It is easy to see from Lemma 20, Lemma 21, and Lemma 22 that $\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right)=\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ for all $3 \leq i \leq k-2$.

The rotation starting with a $s_{2}$ in $w_{k}$ does not exist anymore when $\$$ is appended to $w_{k}$. By Lemma 23 the remaining rotations keep their last symbols and relative order. Therefore, $\beta^{\star}(\mathrm{aab})$ is the same as $\beta(\mathrm{aab})$ but with the first character removed, i.e., $\beta^{\star}(\mathrm{aab})=\mathrm{aaba}^{2 k-8}$.

For the rotations starting with ab , it happens that the rotation that originally started with $s_{2}$ in $w_{k}$, now ends with a $\$$. By Lemma 24 , the remaining rotations do not change their last symbol. Also, all the rotations keep their relative order. Hence, $\beta^{\star}(\mathrm{ab})=\mathrm{b}^{k-2} \$ \mathrm{aba}^{2 k-6}$.

In the case of the rotations starting with ba, the rotation that originally started with ba $s_{2}$ now starts with $\mathrm{ba} \$ s_{2}$ and is the smallest of its range. From Lemma 25 the remaining rotations keep their last symbols and relative order. Hence, $\beta^{\star}(\mathrm{ba})=$ $\mathrm{ba}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{ab}^{k-2} \mathrm{a}$.

For the rotations starting with $\mathrm{b}^{j} \mathrm{a}$ for $2 \leq j \leq k-1$, one can notice that after appending $\$$ to $w_{k}$, the rotation that previously started with $\mathrm{b}^{j}$ a $s_{2}$ and ended with a b , now starts with $\mathrm{b}^{j} \mathrm{a} \$ s_{2}$ and still ends with a b . Moreover, this rotation is smaller than the rotation starting with $\mathrm{b}^{j} \mathrm{aa}_{\mathrm{a}}{ }_{j}$. From Lemma 26 and Lemma 27 we can see that all the other rotations keep their relative order and last symbols. The rotation starting with $\mathrm{b}^{j}$ aa $e_{j}$ still ends with an a, but now is the second smallest of its range. Hence, $\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right)=\mathrm{bab}^{2 k-2 j-2} \mathrm{a}$ for all $2 \leq j \leq k-1$.

Finally, it is clear that $\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right)=\mathrm{a}$, as there is only one maximal run of $k$ symbol b's, and it is not preceded by $\$$.

Lemma 37 (BWT of $w_{k} \mathrm{~b} \$$ ). Given an integer $k>5$, for $w_{k} \mathrm{~b} \$$ it holds that

$$
\begin{aligned}
\beta^{\star}(\$) & =\mathrm{b} \\
\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right) & =\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2,
\end{aligned}
$$

$$
\begin{aligned}
\beta^{\star}\left(\mathrm{a}^{3} \mathrm{~b}\right) & =\mathrm{b}^{5}(\mathrm{ab})^{k-6} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{a}^{2} \mathrm{~b}\right) & =\mathrm{aaba}^{2 k-8}, \\
\beta^{\star}(\mathrm{ab}) & =\mathrm{bb}^{k-2} \$ \mathrm{aba}^{2 k-6}, \\
\beta^{\star}(\mathrm{b} \$) & =\mathrm{a}, \\
\beta^{\star}(\mathrm{ba}) & =\mathrm{a}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{abb}^{k-2} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right) & =\mathrm{ab}^{2 k-2 j-1} \mathrm{a} \text { for all } 2 \leq j \leq k-1 \text { and } \\
\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{a} .
\end{aligned}
$$

Hence, $B W T\left(w_{k} \mathrm{~b} \$\right)=\beta^{\star}(\$) \cdot\left(\prod_{i=2}^{k-1} \beta^{\star}\left(\mathrm{a}^{k-i} \mathrm{~b}\right)\right) \cdot \beta^{\star}(\mathrm{b} \$) \cdot\left(\prod_{i=1}^{k} \beta^{\star}\left(\mathrm{b}^{i} \mathrm{a}\right)\right)$. Moreover, it holds that $r\left(w_{k} \mathbf{b} \$\right)=6 k-13$.

Proof. The first rotation of $\operatorname{BWT}\left(w_{k} \mathrm{~b} \$\right)$ is $\$ w_{k} \mathrm{~b}$. Hence, $\beta^{\star}(\$)=\mathrm{b}$. There is also a rotation $\mathrm{b} \$ w_{k}$, which ends with an a because $w_{k}$ ends with an a . Hence, $\beta^{\star}(\mathrm{b} \$)=\mathrm{a}$. It lefts to compare the remaining ranges $\beta^{\star}(v)$ with respect to $\beta(v)$.

It is easy to see from Lemma 20, Lemma 21, and Lemma 22 that $\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right)=\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ for all $3 \leq i \leq k-2$.

The rotation starting with $\mathrm{a} s_{2}$ in $w_{k}$ does not exist anymore when $\mathrm{b} \$$ is appended to $w_{k}$. By Lemma 23 the remaining rotations keep their last symbols and relative order. Therefore, $\beta^{\star}(\mathrm{aab})$ is the same as $\beta(\mathrm{aab})$ but with the first character removed, i.e., $\beta^{\star}(\mathrm{aab})=\mathrm{aaba}^{2 k-8}$.

For the rotations starting with ab , it happens that the rotation that originally started with $s_{2}$ in $w_{k}$, now ends with a $\$$ when $\mathrm{b} \$$ is appended. Also, there is a new rotation starting with $\mathrm{ab} \$$ that ends with b , and is clearly the smallest of the range. By Lemma 24, the remaining rotations do not change their last symbol. Also, all the rotations that come from $w_{k}$ keep their relative order. Hence, $\beta^{\star}(\mathrm{ab})=\mathrm{bb}^{k-2} \$ \mathrm{aba}^{2 k-6}$.

In the case of the rotations starting with ba, the rotation that originally started with bas $s_{2}$ now starts with bab $\$ s_{2}$ and can be found just before the rotation starting with $\mathrm{baba}^{k-2}$. From Lemma 25 the remaining rotations keep their last symbols and relative order. Hence, $\beta^{\star}(\mathrm{ba})=\mathrm{a}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{abb}^{k-2} \mathrm{a}$.

For the rotations starting with $\mathrm{b}^{j} \mathrm{a}$ for $2 \leq j \leq k-1$, one can notice that after appending $\mathrm{b} \$$ to $w_{k}$, the rotation that previously started with $\mathrm{b}^{j} \mathrm{a} s_{2}$ and ended with ab , now starts with $\mathrm{b}^{j} \mathrm{ab} \$ s_{2}$ and still ends with a b . Moreover, this rotation is still strictly in between the rotations starting with $\mathrm{b}^{j} \mathrm{aae}_{j}$ and $\mathrm{b}^{j} \mathrm{aba}^{j-2} s_{j+1}\left(q_{k}\right.$ instead of $s_{j+1}$ if $j=k-1$ ). From Lemma 26 and Lemma 27, we can see that the latter two rotations are still the smallest and greatest of the range, and both end with an a. Also, all the other rotations keep their last symbols. Hence, $\beta^{\star}\left(\mathbf{b}^{j} \mathbf{a}\right)=\beta\left(\mathbf{b}^{j} \mathbf{a}\right)$ for all $2 \leq j \leq k-1$.

Finally, it is clear that $\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right)=\mathrm{a}$, as there is only one maximal run of $k$ symbol b's, and it is not preceded by $\$$.

Lemma 38 (BWT of $w_{k} \mathrm{bb} \$$ ). Given an integer $k>5$, for $w_{k} \mathrm{bb} \$$ it holds that

$$
\beta^{\star}(\$)=\mathrm{b}
$$

$$
\begin{aligned}
\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right) & =\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2, \\
\beta^{\star}\left(\mathrm{a}^{3} \mathrm{~b}\right) & =\mathrm{b}^{5}(\mathrm{ab})^{k-6} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{a}^{2} \mathrm{~b}\right) & =\mathrm{aaba}^{2 k-8}, \\
\beta^{\star}(\mathrm{ab}) & =\mathrm{bb}^{k-2} \$ \mathrm{aba}^{2 k-6} \\
\beta^{\star}(\mathrm{b} \$) & =\mathrm{b} \\
\beta^{\star}(\mathrm{ba}) & =\mathrm{a}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{abb}^{k-2} \mathrm{a}, \\
\beta^{\star}(\mathrm{bb} \$) & =\mathrm{a} \\
\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right) & =\mathrm{ab}^{2 k-2 j-2} \mathrm{ab} \text { for all } 2 \leq j \leq k-1 \text { and } \\
\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{a} .
\end{aligned}
$$

Hence, $B W T\left(w_{k} \mathrm{bb} \$\right)=\beta^{\star}(\$) \cdot\left(\prod_{i=2}^{k-1} \beta^{\star}\left(\mathrm{a}^{k-i} \mathrm{~b}\right)\right) \cdot \beta^{\star}(\mathrm{b} \$) \cdot \beta^{\star}(\mathrm{ba}) \cdot \beta^{\star}(\mathrm{bb} \$) \cdot$ $\left(\prod_{i=2}^{k} \beta^{\star}\left(\mathrm{b}^{i} \mathrm{a}\right)\right)$. Moreover, it holds that $r\left(w_{k} \mathrm{~b} \$\right)=8 k-17$.

Proof. The first rotation of $\operatorname{BWT}\left(w_{k} \mathrm{bb} \$\right)$ is $\$ w_{k} \mathrm{bb}$. Hence, $\beta^{\star}(\$)=\mathrm{b}$. There is another new rotation $\mathrm{b} \$ w_{k} \mathrm{~b}$. Hence, $\beta^{\star}(\mathrm{b} \$)=\mathrm{b}$. There is also a rotation $\mathrm{bb} \$ w_{k}$ that ends with an a because $w_{k}$ ends with an a. Hence, $\beta^{\star}(\mathrm{bb} \$)=\mathrm{a}$. It lefts to compare the remaining ranges $\beta^{\star}(v)$ with respect to $\beta(v)$.

It is easy to see from Lemma 20, Lemma 21, and Lemma 22 that $\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right)=\beta\left(\mathrm{a}^{i} \mathrm{~b}\right)$ for all $3 \leq i \leq k-2$.

The rotation starting with $\mathrm{a} s_{2}$ in $w_{k}$ does not exist anymore when $\mathrm{bb} \$$ is appended to $w_{k}$. By Lemma 23 the remaining rotations keep their last symbols and relative order. Therefore, $\beta^{\star}(\mathrm{aab})$ is the same as $\beta(\mathrm{aab})$ but with the first character removed, i.e., $\beta^{\star}(\mathrm{aab})=\mathrm{aaba}^{2 k-8}$.

For the rotations starting with ab , it happens that the rotation that originally started with $s_{2}$ in $w_{k}$, now ends with a $\$$ when $\mathrm{bb} \$$ is appended. Also, there is a new rotation starting with $\mathrm{abb} \$$ that ends with b , and can be found just before the rotation starting with $s_{2}$. By Lemma 24 , the remaining rotations do not change their last symbol. Also, all the rotations that come from $w_{k}$ keep their relative order. Hence, $\beta^{\star}(\mathrm{ab})=\mathrm{b}^{k-2} \mathrm{~b} \$ \mathrm{aba}^{2 k-6}$.

In the case of the rotations starting with ba, the rotation that originally started with bas $s_{2}$ now starts with babb $\$ s_{2}$ and can be found just before the rotation starting with $\mathrm{b} s_{3}$ (the greatest on the range). From Lemma 25 we can see that the remaining rotations keep their last symbols and relative order. Hence, $\beta^{\star}(\mathrm{ba})=$ $\mathrm{a}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{ab}^{k-2} \mathrm{ba}$.

For the rotations starting with $\mathrm{b}^{j} \mathrm{a}$ for $2 \leq j \leq k-1$, one can notice that after appending $\mathrm{bb} \$$ to $w_{k}$, the rotation that previously started with $\mathrm{b}^{j} \mathrm{a} s_{2}$ and ended with a b , now starts with $\mathrm{b}^{j} \mathrm{abb} \$ s_{2}$ and still ends with ab. Moreover, this rotation is greater than the rotation starting with $\mathrm{b}^{j} \mathrm{aba}^{j-2} s_{j+1}\left(q_{k}\right.$ instead of $s_{j+1}$ if $\left.j=k-1\right)$. From Lemma 26 and Lemma 27 we can see that all the other rotations keep their relative order an last symbols. The rotation starting with $\mathrm{b}^{j} \mathrm{aba}^{j-2} s_{j+1}\left(q_{k}\right.$ instead of $s_{j+1}$ if $j=k-1$ ) still ends with an a, but now is the second greatest of its range. Hence, $\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right)=\mathrm{ab}^{2 k-2 j-2} \mathrm{ab}$ for all $2 \leq j \leq k-1$.

Finally, it is clear that $\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right)=\mathrm{a}$, as there is only one maximal run of $k$ symbol b 's, and it is not preceded by $\$$.

Lemma 39 (BWT of $\left.w_{k} \mathrm{a} \$\right)$. Given an integer $k>5$, for $w_{k} \mathrm{a} \$$ it holds that

$$
\begin{aligned}
\beta^{\star}(\$) & =\mathrm{a} \\
\beta^{\star}(\mathrm{a} \$) & =\mathrm{a} \\
\beta^{\star}(\mathrm{aa} \$) & =\mathrm{b} \\
\beta^{\star}\left(\mathrm{a}^{i} \mathrm{~b}\right) & =\mathrm{ba}^{k-i-2} \text { for all } 4 \leq i \leq k-2, \\
\beta^{\star}\left(\mathrm{a}^{3} \mathrm{~b}\right) & =\mathrm{b}^{5}(\mathrm{ab})^{k-6} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{a}^{2} \mathrm{~b}\right) & =\mathrm{aaba}^{2 k-8}, \\
\beta^{\star}(\mathrm{ab}) & =\mathrm{b}^{k-2} \$ \mathrm{aba}^{2 k-6}, \\
\beta^{\star}(\mathrm{ba}) & =\mathrm{ba}^{k-5} \mathrm{bbbab}^{k-5} \mathrm{ab}^{k-2} \mathrm{a}, \\
\beta^{\star}\left(\mathrm{b}^{j} \mathrm{a}\right) & =\mathrm{bab}^{2 k-2 j-2} \mathrm{a} \text { for all } 2 \leq j \leq k-1 \text { and } \\
\beta^{\star}\left(\mathrm{b}^{k} \mathrm{a}\right) & =\mathrm{a} .
\end{aligned}
$$

Hence, $B W T\left(w_{k} \mathrm{a} \$\right)=\beta^{\star}(\$) \cdot\left(\prod_{i=2}^{k-1} \beta^{\star}\left(\mathrm{a}^{k-i} \mathrm{~b}\right)\right) \cdot \beta^{\star}(\mathrm{b} \$) \cdot\left(\prod_{i=1}^{k} \beta^{\star}\left(\mathrm{b}^{i} \mathrm{a}\right)\right)$. Moreover, it holds that $r\left(w_{k} \mathrm{a} \$\right)=8 k-16$.

Proof. We obtain $\operatorname{BWT}\left(w_{k} \mathbf{a} \$\right)=\operatorname{aBWT}\left(w_{k} \$\right)$ by applying Proposition 34 to the words $w_{k} \mathrm{a} \$$ and $w_{k} \$$, and we already know the structure of $\operatorname{BWT}\left(w_{k} \$\right)$ by Lemma 36.

Proposition 40. There exists an infinite family of words such that: (i) $r_{\Phi}(w \mathrm{~b})-$ $r_{\$}(w)=\Theta(\sqrt{n}) ;(i i) r_{\$}(\widehat{w})-r_{\$}(w)=\Theta(\sqrt{n})$; (iii) $r_{\$}(\widehat{w} \mathrm{a})-r_{\$}(w)=\Theta(\sqrt{n})$.

Proof. Such a family is composed of the words $w_{k} \mathrm{~b}$ with $k>5$. The proof follows from Lemma 36, Lemma 37, Lemma 38, Lemma 39, and Observation 17.

### 5.4 The relationship between $r$ and $\boldsymbol{r}_{\$}$

Now we address the differences between the measures $r$ and $r_{\$}$. In fact, not only are the measures $r$ and $r_{\$}$ not equal over the same input, but they may differ by a $\Theta(\log n)$ multiplicative factor, or by a $\Theta(\sqrt{n})$ additive factor.

Proposition 41. There exists an infinite family of words $v$ such that $r_{\$}(v) / r(v)=$ $\Theta(\log n)$, where $n=|v|$.
Proof. The family consists of the reverse of the Fibonacci words of odd order. Let $v=\operatorname{rev}(s)$, with $s$ a Fibonacci word of odd order $2 k+1$. Since $s$ is a standard word, $r(s)=2$. Moreover, its reverse $v$ is a conjugate and thus $\operatorname{BWT}(v)=\operatorname{BWT}(s)$, implying that also $r(v)=2$. Let $v^{\prime}=v \$$. Since $\$<$ a, by Proposition 2 it follows that $r\left(v^{\prime}\right) \in\{2 k+2,2 k+3\}$. Altogether, $r_{\$}(v) / r(v) \leq \frac{2 k+3}{2}=\Theta(k)=\Theta(\log n)$.

Proposition 42. There exists an infinite family of words $w$ such that $r_{\$}(w)-r(w)=$ $\Theta(\sqrt{n})$, where $n=|w|$.

Proof. The family consists of the words $w_{k}$ for all $k>5$, defined in Section 4. From Proposition 28 and Lemma 36, it holds $r_{\$}\left(w_{k}\right)-r\left(w_{k}\right)=2 k-4$. By Observation 17, it holds $r_{\$}\left(w_{k}\right)-r\left(w_{k}\right)=\Theta(\sqrt{n})$.

## 6 Conclusion

In this paper, we studied how a single edit operation on a word (insertion, deletion or substitution of a character) can affect the number of runs $r$ of the BWT of the word. Our contribution is threefold. First, we prove that $\Omega(\log n)$ is a lower bound for all three edit operations, by exhibiting infinite families of words for which each edit operation can increase the number of runs by a multiplicative $\Theta(\log n)$ factor. Since for all of these families, $r=\mathcal{O}(1)$, this also proves that the upper bound $\mathcal{O}(\log n \log r)$ given in [1] is tight in the case of $r=\mathcal{O}(1)$, for each of the three edit operations. Secondly, we improved the best known lower bound of $\Omega(\log n)$ for the additive sensitivity of $r$ [1, 15], by giving an infinite family of words on which insertion, deletion, and substitution of a character can increase $r$ by a $\Theta(\sqrt{n})$ additive factor. Finally, we put in relation the two common variants of the number of runs of the BWT, which we denote as $r$ resp. $r_{\$}$. The latter, $r_{\$}$, is the variant used in articles on string data structures and compression, which assumes that each word is terminated by an end-of-string symbol; for the variant $r$ commonly used in the literature on combinatorics on words, no such assumption is made.

Our work opens several roads of investigation. First, we ask whether there exist families of words with $r=\omega(1)$ for which edit operations can cause a multiplicative increase of $\Omega(\log n)$. In other words, is the bit catastrophe effect restricted to words on which the compression power of $r$ is maximal?

Another interesting question is whether the upper bound $\mathcal{O}(r \log r \log n)$ from [1] for the additive sensitivity of $r$ is tight. A weaker question, an answer to which would make a step in this direction, is whether there exists an infinite family with $r=\omega(1)$ on which one edit operation can cause an additive increase of $\omega(r)$ in the number of runs of the BWT.

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## Authors contributions

All authors contributed equally to the paper.

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