# Maximal Distortion of Geodesic Diameters in Polygonal Domains 

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#### Abstract

For a polygon $P$ with holes in the plane, we denote by $\varrho(P)$ the ratio between the geodesic and the Euclidean diameters of $P$. It is shown that over all convex polygons with $h$ convex holes, the supremum of $\varrho(P)$ is between $\Omega\left(h^{1 / 3}\right)$ and $O\left(h^{1 / 2}\right)$. The upper bound improves to $O\left(1+\min \left\{h^{3 / 4} \Delta, h^{1 / 2} \Delta^{1 / 2}\right\}\right)$ if every hole has diameter at most $\Delta \cdot \operatorname{diam}_{2}(P)$; and to $O(1)$ if every hole is a fat convex polygon. Furthermore, we show that the function $g(h)=\sup _{P} \varrho(P)$ over convex polygons with $h$ convex holes has the same growth rate as an analogous quantity over geometric triangulations with $h$ vertices when $h \rightarrow \infty$.


## 1 Introduction

Determining the maximum distortion between two metrics on the same ground set is a fundamental problem in metric geometry. In this paper, we study the maximum ratio between the geodesic (i.e., shortest path) diameter and the Euclidean diameter over polygons with holes. A polygon $P$ with $h$ holes (also known as a polygonal domain) is defined as follows. Let $P_{0}$ be a simple polygon, and let $P_{1}, \ldots, P_{h}$ be pairwise disjoint simple polygons in the interior of $P_{0}$. Then $P=P_{0} \backslash\left(\bigcup_{i=1}^{h} P_{i}\right)$.

The Euclidean distance between two points $s, t \in P$ is $|s t|=\|s-t\|_{2}$, and the shortest path distance geod $(s, t)$ is the minimum arclength of a polygonal path between $s$ and $t$ contained in $P$. The triangle inequality implies that $|s t| \leq \operatorname{geod}(s, t)$ for all $s, t \in P$. The geometric dilation (also known as the stretch factor) between the two distances is $\sup _{s, t \in P} \operatorname{geod}(s, t) /|s t|$. The geometric dilation of $P$ can be arbitrarily large, even if $P$ is a (nonconvex) quadrilateral.

The Euclidean diameter of $P$ is $\operatorname{diam}_{2}(P)=\sup _{s, t \in P}|s t|$ and its geodesic diameter is $\operatorname{diam}_{g}(P)=$ $\sup _{s, t \in P} \operatorname{geod}(s, t)$. It is clear that $\operatorname{diam}_{2}(P) \leq \operatorname{diam}_{g}(P)$. We are interested in the distortion

$$
\begin{equation*}
\varrho(P)=\frac{\operatorname{diam}_{g}(P)}{\operatorname{diam}_{2}(P)} . \tag{1}
\end{equation*}
$$

Note that $\varrho(P)$ is unbounded, even for simple polygons. Indeed, if $P$ is a zig-zag polygon with $n$ vertices, contained in a disk of unit diameter, then $\operatorname{diam}_{2}(P) \leq 1$ and $\operatorname{diam}_{g}(P)=\Omega(n)$, hence $\varrho(P) \geq \Omega(n)$. It is not difficult to see that this bound is the best possible, that is, $\varrho(P) \leq O(n)$.

[^0]In this paper, we consider convex polygons with convex holes. Specifically, let $\mathcal{C}(h)$ denote the family of polygonal domains $P=P_{0} \backslash\left(\bigcup_{i=1}^{h} P_{i}\right)$, where $P_{0}, P_{1}, \ldots, P_{h}$ are convex polygons; and let

$$
\begin{equation*}
g(h)=\sup _{P \in \mathcal{C}(h)} \varrho(P) \tag{2}
\end{equation*}
$$

It is clear that if $h=0$, then $\operatorname{geod}(s, t)=|s t|$ for all $s, t \in P$, which implies $g(0)=1$. Or main result is the following.
Theorem 1. For every $h \in \mathbb{N}$, we have $\Omega\left(h^{1 / 3}\right) \leq g(h) \leq O\left(h^{1 / 2}\right)$.
The lower bound construction is a polygonal domain in which all $h$ holes have about the same diameter $\Theta\left(h^{-1 / 3}\right) \cdot \operatorname{diam}_{2}(P)$. We prove a matching upper bound for all polygons $P$ with holes of diameter $\Theta\left(h^{-1 / 3}\right) \cdot \operatorname{diam}_{2}(P)$. In general, if the diameter of every hole is $o(1) \cdot \operatorname{diam}_{2}(P)$, we can improve upon the bound $g(h) \leq O\left(h^{1 / 2}\right)$ in Theorem 1 .
Theorem 2. If $P \in \mathcal{C}(h)$ and the diameter of every hole is at most $\Delta \cdot \operatorname{diam}_{2}(P)$, then $\varrho(P) \leq$ $O\left(1+\min \left\{h^{3 / 4} \Delta, h^{1 / 2} \Delta^{1 / 2}\right\}\right)$. In particular for $\Delta=O\left(h^{-1 / 3}\right)$, we have $\varrho(P) \leq O\left(h^{1 / 3}\right)$.

However, if we further restrict the holes to be fat convex polygons, we can show that $\varrho(P)=O(1)$ for all $h \in \mathbb{N}$. In fact for every $s, t \in P$, the distortion $\operatorname{geod}(s, t) /|s t|$ is also bounded by a constant.

Informally, a convex body is fat if its width is comparable with its diameter. The width of a convex body $C$ is the minimum width of a parallel slab enclosing $C$. For every $\lambda \in(0, \lambda]$, a convex body $C$ is $\lambda$-fat if the ratio of its width to its diameter is at least $\lambda$, that is, width $(C) / \operatorname{diam}_{2}(C) \geq \lambda$; and $C$ is fat if the inequality holds for a constant $\lambda \in(0,1]$. For instance, a disk is 1 -fat, a $3 \times 4$ rectangle is $\frac{3}{5}$-fat and a line segment is 0 -fat. Let $\mathcal{F}_{\lambda}(h)$ be the family of polygonal domain $P=P_{0} \backslash\left(\bigcup_{i=1}^{h} P_{i}\right)^{\prime}$, where $P_{0}$ is a convex polygon and $P_{1}, \ldots, P_{h}$ are $\lambda$-fat convex polygons.
Proposition 1. For $P \in \mathcal{F}_{\lambda}(h)$, where $h \in \mathbb{N}$ and $\lambda \in(0,1]$, we have $\varrho(P) \leq O\left(\lambda^{-1}\right)$.
The special case when all holes are axis-aligned rectangles is also easy.
Proposition 2. Let $P \in \mathcal{C}(h), h \in \mathbb{N}$, such that all holes are axis-aligned rectangles. Then $\varrho(P) \leq O(1)$.

Triangulations. In this paper, we focus on the diameter distortion $\varrho(P)=\operatorname{diam}_{g}(P) / \operatorname{diam}_{2}(P)$ for polygons $P \in \mathcal{C}(h)$ with $h$ holes. Alternatively, we can also compare the geodesic and Euclidean diameters in $n$-vertex triangulations. In a geometric graph $G=(V, E)$, the vertices are distinct points in the plane, and the edges are straight-line segments between pairs of vertices. The Euclidean diameter of $G, \operatorname{diam}_{2}(G)=\max _{u, v \in V}|u v|$ is the maximum distance between two vertices, and the geodesic diameter $\operatorname{diam}_{g}(G)=\max _{u, v \in V} \operatorname{dist}(u, v)$, where $\operatorname{dist}(u, v)$ is the shortest path distance in $G$, i.e., the minimum Euclidean length of a $u v$-path in $G$. With this notation, we define $\varrho(G)=$ $\operatorname{diam}_{g}(G) / \operatorname{diam}_{2}(G)$,

A Euclidean triangulation $T=(V, E)$ of a point set $V$ is a planar straight-line graph where all bounded faces are triangles, and their union is the convex hull $\operatorname{conv}(V)$. Let

$$
\begin{equation*}
f(n)=\sup _{G \in \mathcal{T}(n)} \varrho(G) \tag{3}
\end{equation*}
$$

where the supremum is taken over the set $\mathcal{T}(n)$ all $n$-vertex triangulations. Recall that $g(n)$ is the supremum of diameter distortions over polygons with $n$ convex holes; see (2). We prove that $f(n)$ and $g(n)$ have the same growth rate.

Theorem 3. We have $g(n)=\Theta(f(n))$.
Alternative problem formulation. The following version of the question studied here may be more attractive to the escape community [12, 19]. Given $n$ pairwise disjoint convex obstacles in a convex polygon of unit diameter (e.g., a square), what is the maximum length of a (shortest) escape route from any given point in the polygon to its boundary? According to Theorem 1, it is always $O\left(n^{1 / 2}\right)$ and sometimes $\Omega\left(n^{1 / 3}\right)$.

Related work. The geodesic distance in polygons with or without holes have been studied extensively from the algorithmic perspective; see [22] for a comprehensive survey. In a simple polygon $P$ with $n$ vertices, one can compute the geodesic distance between two given points in $O(n)$ time [20], trade-offs are also available between time and workspace [15]. A shortest-path data structure can report the geodesic distance between any two query points in $O(\log n)$ time after $O(n)$ preprocessing time [14]. In $O(n)$ time, one can also compute the geodesic diameter [16] and radius [1].

For polygons with holes, more involved techniques are needed. Let $P$ be a polygon with $h$ holes, and a total of $n$ vertices. For any $s, t \in P$, one can compute $\operatorname{geod}(s, t)$ in $O(n+h \log h)$ time and $O(n)$ space [26], improving earlier bounds in [17, 18, 21, 27]. A shortest-path data structure can report the geodesic distance between two query points in $O(\log n)$ query time using $O\left(n^{11}\right)$ space; or in $O(h \log n)$ query time with $O\left(n+h^{5}\right)$ space [8]. The geodesic radius can be computed in $O\left(n^{11} \log n\right)$ time [3, 25], and the geodesic diameter in $O\left(n^{7.73}\right)$ or $O\left(n^{7}(\log n+h)\right)$ time [2]. One can find an $(1+\varepsilon)$-approximation in $O\left(\left(n / \varepsilon^{2}+n^{2} / \varepsilon\right) \log n\right)$ time [2, 3]. The geodesic diameter may be attained by a point pair $s, t \in P$, where both $s$ and $t$ lie in the interior or $P$; in which case it is known [2] that there are at least five different geodesic paths between $s$ and $t$.

The diameter of an $n$-vertex triangulation with Euclidean weights can be computed in $\tilde{O}\left(n^{5 / 3}\right)$ time [7, 13]. For unweighted graphs in general, the diameter problem has been intensely studied in the fine-grained complexity community. For a graph with $n$ vertices and $m$ edges, breadthfirst search (BFS) yields a 2-approximation in $O(m)$ time. Under the Strong Exponential Time Hypothesis (SETH), for any integer $k \geq 2$ and $\varepsilon>0$, a ( $2-\frac{1}{k}-\varepsilon$ )-approximation requires $m n^{1+1 /(k-1)-o(1)}$ time [9]; see also [23].

## 2 Convex Polygons with Convex Holes

In this section, we prove Theorem 1. A lower bound construction is presented in Lemma 1, and the upper bound is established in Lemma 22 below.

Lower Bound. The lower bound is based on the following construction.
Lemma 1. For every $h \in \mathbb{N}$, there exists a polygonal domain $P \in \mathcal{C}(h)$ such that $g(P) \geq \Omega\left(h^{1 / 3}\right)$.
Proof. We may assume w.l.o.g. that $h=k^{3}$ for some integer $k \geq 3$. We construct a polygon $P$ with $h$ holes, where the outer polygon $P_{0}$ is a regular $k$-gon of unit diameter, hence $\operatorname{diam}_{2}(P)=$ $\operatorname{diam}_{2}\left(P_{0}\right)=1$. Let $Q_{0}, Q_{1}, \ldots, Q_{k^{2}}$ be a sequence of $k^{2}+1$ regular $k$-gons with a common center such that $Q_{0}=P_{0}$, and for every $i \in\left\{1, \ldots, k^{2}\right\}, Q_{i}$ is inscribed in $Q_{i-1}$ such that the vertices of $Q_{i}$ are the midpoints of the edges of $Q_{i-1}$; see Fig. 1. Enumerate the $k^{3}$ edges of $Q_{1}, \ldots, Q_{k^{2}}$ as $e_{1}, \ldots, e_{k^{3}}$. For every $j=1, \ldots, k^{3}$, we construct a hole as follows: Let $P_{j}$ be an $(|e|-2 \varepsilon) \times \frac{\varepsilon}{2}$
rectangle with symmetry axis $e$ that contains $e$ with the exception of the $\varepsilon$-neighborhoods of its endpoints. Then $P_{1}, \ldots, P_{k^{3}}$ are pairwise disjoint. Finally, let $P=P_{0} \backslash \bigcup_{j=1}^{k^{3}} P_{j}$.


Figure 1: Left: hexagons $Q_{0}, \ldots, Q_{3}$ for $k=6$. Right: The 18 holes corresponding to the edges of $Q_{1}, \ldots, Q_{3}$.

Assume, w.l.o.g., that $e_{i}$ is an edge of $Q_{i}$ for $i \in\left\{0,1, \ldots, k^{2}\right\}$. As $P_{0}=Q_{0}$ is a regular $k$-gon of unit diameter, then $\left|e_{0}\right| \geq \Omega(1 / k)$. Let us compare the edge lengths in two consecutive $k$-gons. Since $Q_{i+1}$ is inscribed in $Q_{i}$, we have

$$
\left|e_{i+1}\right|=\left|e_{i}\right| \cos \frac{\pi}{k} \geq\left|e_{i}\right|\left(1-\frac{\pi^{2}}{2 k^{2}}\right)
$$

using the Taylor estimate $\cos x \geq 1-x^{2} / 2$. Consequently, for every $i \in\left\{0,1, \ldots, k^{2}\right\}$,

$$
\left|e_{i}\right| \geq\left|e_{0}\right| \cdot\left(1-\frac{\pi^{2}}{2 k^{2}}\right)^{k^{2}} \geq\left|e_{0}\right| \cdot \Omega(1) \geq \Omega\left(\frac{1}{k}\right)
$$

It remains to show that $\operatorname{diam}_{g}(P) \geq \Omega(k)$. Let $s$ be the center of $P_{0}$ and $t$ and arbitrary vertex of $P_{0}$. Consider an st-path $\gamma$ in $P$, and for any two points $a, b$ along $\gamma$, let $\gamma(a, b)$ denote the subpath of $\gamma$ between $a$ and $b$. Let $c_{i}$ be the first point where $\gamma$ crosses the boundary of $Q_{i}$ for $i \in\left\{1, \ldots, k^{2}\right\}$. By construction, $c_{i}$ must be in an $\varepsilon$-neighborhood of a vertex of $Q_{i}$. Since the vertices of $Q_{i+1}$ are at the midpoints of the edges of $Q_{i}$, then $\left|\gamma\left(c_{i}, c_{i+1}\right)\right| \geq \frac{1}{2}\left|e_{i}\right|-2 \varepsilon \geq \Omega\left(\left|e_{i}\right|\right) \geq \Omega(1 / k)$. Summation over $i=0, \ldots, k^{2}-1$ yields $|\gamma| \geq \sum_{i=0}^{k^{2}-1}\left|\gamma\left(c_{i}, c_{i+1}\right)\right| \geq k^{2} \cdot \Omega(1 / k) \geq \Omega(k)=\Omega\left(h^{1 / 3}\right)$, as required.

Upper Bound. Let $P \in \mathcal{C}(h)$ for some $h \in \mathbb{N}$ and let $s \in P$. For every hole $P_{i}$, let $\ell_{i}$ and $r_{i}$ be points on the boundary of $P_{i}$ such that $\overrightarrow{s l_{i}}$ and $\overrightarrow{s r_{i}}$ are tangent to $P_{i}$, and $P_{i}$ lies on the left (resp., right) side of the ray ${\overrightarrow{s \ell_{i}}}_{i}$ (resp., $\overrightarrow{s r}_{i}$ ).

We construct a path from $s$ to some point in the outer boundary of $P$ by the following recursive algorithm; refer to Fig. 2 (left). For a unit vector $\vec{u} \in \mathbb{S}^{1}$, we construct path greedy $_{P}(s, \vec{u})$ as


Figure 2: Left: A polygon $P \in \mathcal{C}(7)$ with 7 convex holes, a point $s \in P$, and a path greedy ${ }_{P}(s, \vec{u})$ from $s$ to a point $t$ on the outer boundary of $P$. Right: A boundary arc $\widehat{p q}$, where $|\widehat{p q}| \leq|p r|+|r q|$.
follows. Start from $s$ along a ray emanating from $s$ in direction $\vec{u}$ until reaching the boundary of $P$ at some point $p$. While $p \notin \partial P_{0}$ do: Assume that $p \in \partial P_{i}$ for some $1 \leq i \leq h$. Extend the path along $\partial P_{i}$ to the point $\ell_{i}$ or $r_{i}$ such that the distance from $s$ monotonically increases; and then continue along the ray $\overrightarrow{s \ell_{i}}$ or ${\overrightarrow{s r_{i}}}_{i}$ until reaching the boundary of $P$ again. When $p \in \partial P_{0}$, the path $\operatorname{greedy}_{P}(s, \vec{u})$ terminates at $p$.

Lemma 2. For every $P \in \mathcal{C}(h)$, every $s \in P$ and every $\vec{u} \in \mathbb{S}^{1}$, we have $\left|\operatorname{greedy}_{P}(s, \vec{u})\right| \leq$ $O\left(h^{1 / 2}\right) \cdot \operatorname{diam}_{2}(P)$, and this bound is the best possible.

Proof. Let $P$ be a polygonal domain with a convex outer polygon $P_{0}$ and $h$ convex holes. We may assume w.l.o.g. that $\operatorname{diam}_{2}(P)=1$. For a point $s \in P$ and a unit vector $\vec{u}$, consider the path $\operatorname{greedy}_{P}(s, \vec{u})$. By construction, the distance from $s$ monotonically increases along greedy $P_{P}(s, \vec{u})$, and so the path has no self-intersections. It is composed of radial segments that lie along rays emanating from $s$, and boundary arcs that lie on the boundaries of holes. By monotonicity, the total length of all radial segments is at most $\operatorname{diam}_{2}(P)$. Since every boundary arc ends at a point of tangency $\ell_{i}$ or $r_{i}$, for some $i \in\{1, \ldots, h\}$, the path greedy $_{P}(s, \vec{u})$ contains at most two boundary arcs along each hole, thus the number of boundary arcs is at most $2 h$. Let $\mathcal{A}$ denote the set of all boundary arcs along greedy ${ }_{P}(s, \vec{u})$; then $|\mathcal{A}| \leq 2 h$.

Along each boundary arc $\widehat{p q} \in \mathcal{A}$, from $p$ to $q$, the distance from $s$ increases by $\Delta_{p q}=|s q|-|s p|$. By monotonicity, we have $\sum_{\widehat{p} q \in \mathcal{A}} \Delta_{p q} \leq \operatorname{diam}_{2}(P)$. We now give an upper bound for the length of $\widehat{p q}$. Let $p^{\prime}$ be a point in $s q$ such that $|s p|=\left|s p^{\prime}\right|$, and let $r$ be the intersection of $s q$ with a line orthogonal to $s p$ passing through $p$; see Fig. 2 (right). Note that $|s p|<|s r|$. Since the distance from $s$ monotonically increases along the $\operatorname{arc} \widehat{p q}$, then $q$ is in the closed halfplane bounded by $p r$ that does not contain $s$. Combined with $|s p|<|s r|$, this implies that $r$ lies between $p^{\prime}$ and $q$ on the line $s q$, consequently $\left|p^{\prime} r\right|<\left|p^{\prime} q\right|=\Delta_{p q}$ and $|r q|<\left|p^{\prime} q\right|=\Delta_{p q}$. By the triangle inequality and the

Pythagorean theorem, these estimates give an upper bound

$$
\begin{aligned}
|\widehat{p q}| & \leq|p r|+|r q|=\sqrt{|s r|^{2}-|s p|^{2}}+|r q| \leq \sqrt{\left(\left|s p^{\prime}\right|+\left|p^{\prime} r\right|\right)^{2}-|s p|^{2}}+|r q| \\
& \leq \sqrt{\left(|s p|+\Delta_{p q}\right)^{2}-|s p|^{2}}+\Delta_{p q} \leq O\left(\sqrt{|s p| \Delta_{p q}}+\Delta_{p q}\right) \\
& \leq O\left(\sqrt{\operatorname{diam}_{2}(P) \cdot \Delta_{p q}}+\Delta_{p q}\right) .
\end{aligned}
$$

Summation over all boundary arcs, using Jensen's inequality, yields

$$
\begin{aligned}
\sum_{\widehat{p q} \in \mathcal{A}}|\widehat{p q}| & \leq \sum_{\widehat{p q} \in \mathcal{A}} O\left(\sqrt{\operatorname{diam}_{2}(P) \cdot \Delta_{p q}}+\Delta_{p q}\right) \\
& \leq \sqrt{\operatorname{diam}_{2}(P)} \cdot O\left(\sum_{\hat{p} q \in \mathcal{A}} \sqrt{\Delta_{p q}}\right)+O\left(\sum_{\widehat{p} G \in \mathcal{A}} \Delta_{p q}\right) \\
& \leq \sqrt{\operatorname{diam}_{2}(P)} \cdot O\left(|\mathcal{A}| \cdot \sqrt{\frac{1}{|\mathcal{A}|} \sum_{\widehat{p} q \in \mathcal{A}} \Delta_{p q}}\right)+O\left(\operatorname{diam}_{2}(P)\right) \\
& \leq \sqrt{\operatorname{diam}_{2}(P)} \cdot O\left(\sqrt{|\mathcal{A}| \cdot \operatorname{diam}_{2}(P)}\right)+O\left(\operatorname{diam}_{2}(P)\right) \\
& \leq O(\sqrt{|\mathcal{A}|}) \cdot \operatorname{diam}_{2}(P) \leq O(\sqrt{h}) \cdot \operatorname{diam}_{2}(P),
\end{aligned}
$$

as claimed.
We now show that the bound $\left|\operatorname{greedy}_{P}(s, \vec{u})\right| \leq O\left(h^{1 / 2}\right) \cdot \operatorname{diam}_{2}(P)$ is the best possible. For every $h \in \mathbb{N}$, we construct a polygon $P \in \mathcal{C}(h)$ and a point $s$ such that for every $\vec{u} \in \mathbb{S}^{1}$, we have $\left|\operatorname{greedy}_{P}(s, \vec{u})\right| \geq \Omega\left(h^{1 / 2}\right)$. Without loss of generality, we may assume $\operatorname{diam}_{2}(P)=1$ and $h=3\left(k^{2}+1\right)$ for some $k \in \mathbb{N}$.

We start with the construction described in the proof of Lemma 1 with $k^{3}$ rectangular holes in a regular $k$-gon $P_{0}$, where $s$ is the center of $P_{0}$. We modify the construction in three steps: (1) Let $T$ be a small equilateral triangle centered at $s$, and construct three rectangular holes around the edges of $T$; to obtain a total of $k^{3}+3$ holes. (2) Rotate each hole $P_{j}$ counterclockwise by a small angle, such that when the greedy path reaches $P_{j}$ in an $\varepsilon$-neighborhood of its center, it would always turn left. (3) For any $\vec{u} \in \mathbb{S}^{1}$, the path greedy $P_{P}(s, \vec{u})$ exits the triangle $T$ at a small neighborhood of a corner of $T$. From each corner of $T$, greedy ${ }_{P}(s, \vec{u})$ continues to the outer boundary along a sequence of $k^{2}$ holes. We delete all holes that $\operatorname{greedy}_{P}(s, \vec{u})$ does not touch for any $\vec{u} \in \mathbb{S}^{1}$, thus we retain $h=3 k^{2}+3$ holes. For every $\vec{u} \in \mathbb{S}^{1}$, we have $\left|\operatorname{greedy}_{P}(s, \vec{u})\right| \geq \Omega(k)$ according to the analysis in Lemma 1, hence $\left|\operatorname{greedy}_{P}(s, \vec{u})\right| \geq \Omega\left(h^{1 / 2}\right)$, as required.

Corollary 1. For every $h \in \mathbb{N}$ and every polygon $P \in \mathcal{C}(h)$, we have $\operatorname{diam}_{g}(P) \leq O\left(h^{1 / 2}\right)$. $\operatorname{diam}_{2}(P)$.

Proof. Let $P \in \mathcal{C}(h)$ and $s_{1}, s_{2} \in P$. By Lemma 2, there exist points $t_{1}, t_{2} \in \partial P_{0}$ such that $\operatorname{geod}\left(s_{1}, t_{1}\right) \leq O\left(h^{1 / 2}\right) \cdot \operatorname{diam}_{2}(P)$ and $\operatorname{geod}\left(s_{2}, t_{2}\right) \leq O\left(h^{1 / 2}\right) \cdot \operatorname{diam}_{2}(P)$. There is a path between $t_{1}$ and $t_{2}$ along the perimeter of $P_{0}$. It is well known [24, [28] that $\left|\partial P_{0}\right| \leq \pi \cdot \operatorname{diam}_{2}\left(P_{0}\right)$ for every convex body $P_{0}$, hence geod $\left(t_{1}, t_{2}\right) \leq O\left(\operatorname{diam}_{2}(P)\right)$. The concatenation of these three paths yields a path in $P$ connecting $s_{1}$ and $s_{2}$, of length $\operatorname{geod}\left(s_{1}, s_{2}\right) \leq O\left(h^{1 / 2}\right) \cdot \operatorname{diam}_{2}(P)$.

## 3 Improved Upper Bound for Holes of Bounded Diameter

In this section we prove Theorem 2. Similar to the proof of Theorem 1, it is enough to bound the geodesic distance from an arbitrary point in $P$ to the outer boundary. We give three such bounds in Lemmas 3, 4 and 7.

Lemma 3. Let $P \in \mathcal{C}(h)$ such that $\operatorname{diam}_{2}\left(P_{i}\right) \leq \Delta \cdot \operatorname{diam}_{2}(P)$ for every hole $P_{i}$. If $\Delta \leq O\left(h^{-1}\right)$, then there exists a path of length $O\left(\operatorname{diam}_{2}(P)\right)$ in $P$ from any point $s \in P$ to the outer boundary $\partial P_{0}$.

Proof. Let $s \in P$ and $t \in \partial P_{0}$. Construct an st-path $\gamma$ as follows: Start with the straight line segment st, and whenever st intersects the interior of a hole $P_{i}$, then the segment $s t \cap P_{i}$ is replaced by an arc along $\partial P_{i}$. Since $\left|\partial P_{i}\right| \leq \pi \cdot \operatorname{diam}_{2}\left(P_{i}\right)$ for every convex hole $P_{i}[24, ~ 28]$, then $|\gamma| \leq$ $|s t|+\sum_{i=1}^{h}\left|\partial P_{i}\right| \leq \operatorname{diam}_{2}(P)+\sum_{i=1}^{h} O\left(\operatorname{diam}_{2}\left(P_{i}\right)\right) \leq \mathcal{O}(1+h \Delta) \cdot \operatorname{diam}_{2}(P) \leq O\left(\operatorname{diam}_{2}(P)\right)$, as claimed.

Lemma 4. Let $P \in \mathcal{C}(h)$ such that $\operatorname{diam}_{2}\left(P_{i}\right) \leq \Delta \cdot \operatorname{diam}_{2}(P)$ for every hole $P_{i}$. Then there exists a path of length $O\left(1+h^{3 / 4} \Delta\right) \cdot \operatorname{diam}_{2}(P)$ in $P$ from any point $s \in P$ to the outer boundary $\partial P_{0}$.

Proof. Assume without loss of generality that $\operatorname{diam}_{2}(P)=1$, and $s$ is the origin. Let $\ell \in \mathbb{N}$ be a parameter to be specified later. For $i \in\{-\ell,-\ell+1, \ldots, \ell\}$, let $H_{i}: y=i \cdot \Delta$ be a horizontal line, and $V_{i}: x=i \cdot \Delta$ a vertical line. Since any two consecutive horizontal (resp., vertical) lines are distance $\Delta$ apart, and the diameter of each hole is at most $\Delta$, then the interior of each hole intersects at most one horizontal and at most one vertical line. By the pigeonhole principle, there are integers $a, b, c, d \in\{1, \ldots, \ell\}$ such that $H_{-a}, H_{b}, V_{-c}$, and $V_{d}$ each intersects the interior of at most $h / \ell$ holes; see Fig. 3 .


Figure 3: Illustration for $\ell=5$ (assuming that $P$ is a unit square centered at $s$ ).
Let $B$ be the axis-aligned rectangle bounded by the lines $H_{-a}, H_{b}, V_{-c}$, and $V_{d}$. Due to the spacing of the lines, we have $\operatorname{diam}_{2}(B) \leq 2 \cdot \sqrt{2} \cdot \ell \Delta=O(\ell \Delta)$.

We can construct a path from $s$ to $\partial P_{0}$ as a concatenation of two paths $\gamma=\gamma_{1} \oplus \gamma_{2}$. Let $\gamma_{1}$ be the initial part of greedy $P_{P}(s, \vec{u})$ from $s$ until reaching the boundary of $B \cap P_{0}$ at some point $p$. If $p \in \partial P_{0}$, then $\gamma_{2}=(p)$ is a trivial one-point path. Otherwise $p$ lies on a line $L \in\left\{H_{-a}, H_{b}, V_{-c}, V_{d}\right\}$ that intersects the interior of at most $h / \ell$ holes. Let $\gamma_{2}$ follow $L$ from $p$ to the boundary of $P_{0}$ such that when it encounters a hole $P_{i}$, it makes a detour along $\partial P_{i}$.

It remains to analyze the length of $\gamma$. By Lemma 2, we have $\left|\gamma_{1}\right| \leq O(\sqrt{h}) \cdot \operatorname{diam}_{2}(B) \leq$ $O\left(h^{1 / 2} \ell \Delta\right)$. The path $\gamma_{2}$ has edges along the line $L$ and along the boundaries of holes whose interior intersect $L$. The total length of all edges along $L$ is at $\operatorname{mostam}_{2}(P)=1$. It is well known that $\operatorname{per}(C) \leq \pi \cdot \operatorname{diam}_{2}(C)$ for every convex body [24, 28], and so the length of each detour is $O\left(\operatorname{diam}_{2}\left(P_{i}\right)\right) \leq O(\Delta)$, and the total length of $O(h / \ell)$ detours is $O(h \Delta / \ell)$. Consequently,

$$
\begin{equation*}
|\gamma| \leq O\left(h^{1 / 2} \ell \Delta+h \Delta / \ell+1\right) . \tag{4}
\end{equation*}
$$

Finally, we set $\ell=\left\lceil h^{1 / 4}\right\rceil$ to balance the first two terms in (4), and obtain $|\gamma| \leq O\left(h^{3 / 4} \Delta+1\right)$, as claimed.

When all holes are line segments, we construct a monotone path from $s$ to the outer boundary. A polygonal path $\gamma=\left(p_{0}, p_{1}, \ldots, p_{m}\right)$ is $\vec{u}$-monotone for a unit vector $\vec{u} \in \mathbb{S}^{1}$ if $\vec{u} \cdot \overrightarrow{v_{i-1} v_{i}} \geq 0$ for all $i \in\{1, \ldots, m\}$; and $\gamma$ is monotone if it is $\vec{u}$-monotone for some $\vec{u} \in \mathbb{S}^{1}$.

Lemma 5. Let $P \in \mathcal{C}(h)$ such that every hole is a line segment of length at most $\Delta \cdot \operatorname{diam}_{2}(P)$. If $\Delta \geq h^{-1}$, then there exists a monotone path of length $O\left(h^{1 / 2} \Delta^{1 / 2}\right) \cdot \operatorname{diam}_{2}(P)$ in $P$ from any point $s \in P$ to the outer boundary $\partial P_{0}$.

Proof. We may assume w.l.o.g. that $\operatorname{diam}_{2}(P)=1$. Denote the line segments by $a_{i} b_{i}$, for $i=$ $1, \ldots, h$, such that $x\left(a_{i}\right) \leq x\left(b_{i}\right)$. Let $\ell=\left\lceil h^{1 / 2} \Delta^{1 / 2}\right\rceil$, and note that $\ell=\Theta\left(h^{1 / 2} \Delta^{1 / 2}\right)$ when $\Delta \geq h^{-1}$. Partition the right halfplane (i.e., right of the $y$-axis) into $\ell$ wedges with aperture $\pi / \ell$ and apex at the origin, denoted $W_{1}, \ldots, W_{\ell}$. For each wedge $W_{i}$, let $\vec{w}_{i} \in \mathbb{S}$ be the direction vector of its axis of symmetry.

Partition the $h$ segments as follows: For $j=1, \ldots, \ell$, let $\mathcal{H}_{j}$ be the set of segments $a_{i} b_{i}$ such that $\overrightarrow{a_{i} b_{i}}$ is in $W_{j}$. Finally, let $\mathcal{H}_{j^{*}}$ be a set with minimal cardinality, that is, $\left|\mathcal{H}_{j^{*}}\right| \leq h / \ell=O\left(h^{1 / 2} / \Delta^{1 / 2}\right)$. Let $\vec{v}=\vec{w}_{j^{*}}^{\perp}$. We construct a $\vec{v}$-monotone path $\gamma$ from $s$ to the outer boundary $\partial P_{0}$ as follows. Start in direction $\vec{v}$ until reaching a hole $a_{i} b_{i}$ at some point $p$. While $p \notin \partial P_{0}$, continue along $a_{i} b_{i}$ to one of the endpoints: to $a_{i}$ if $\vec{v} \cdot \overrightarrow{a_{i} b_{i}} \geq 0$, and to $b_{i}$ otherwise; then continue in direction $\vec{v}$. By monotonicity, $\gamma$ visits every edge at most once.

It remains to analyze the length of $\gamma$. We distinguish between two types of edges: let $E_{1}$ be the set of edges of $\gamma$ contained in $\mathcal{H}_{j^{*}}$, and $E_{2}$ be the set of all other edges of $\gamma$. The total length of edges in $E_{1}$ is at most the total length of all segments in $\mathcal{H}_{j^{*}}$, that is,

$$
\sum_{e \in E_{1}}|e| \leq\left|\mathcal{H}_{j^{*}}\right| \cdot \Delta \leq O\left(h^{1 / 2} / \Delta^{1 / 2}\right) \cdot \Delta=O\left(h^{1 / 2} \Delta^{1 / 2}\right) .
$$

Every edge $e \in E_{2}$ makes an angle at least $\pi /(2 \ell)$ with vector $\vec{v}$. Let proj(e) denote the orthogonal projection of $e$ to a line of direction $\vec{v}$. Then $|\operatorname{proj}(e)| \geq|e| \sin (\pi /(2 \ell))$. By monotonicity, the projections of distinct edges have disjoint interiors. Consequently, $\sum_{e \in E_{2}}|\operatorname{proj}(e)| \leq \operatorname{diam}_{2}(P)=1$. This yields

$$
\begin{aligned}
\sum_{e \in E_{2}}|e| & \leq \sum_{e \in E_{2}} \frac{|\operatorname{proj}(e)|}{\sin (\pi /(2 \ell))}=\frac{1}{\sin (\pi /(2 \ell))} \sum_{e \in E_{2}}|\operatorname{proj}(e)| \\
& =O(\ell)=O\left(h^{1 / 2} \Delta^{1 / 2}\right)
\end{aligned}
$$

Overall, $|\gamma|=\sum_{e \in E_{1}}|e|+\sum_{e \in E_{2}}|e|=O\left(h^{1 / 2} \Delta^{1 / 2}\right)$, as claimed.


Figure 4: Line $L$ traverses a convex polygon $P$, but does not cross the segment $a b$.

For extending Lemma 5 to arbitrary convex holes, we need the following technical lemma; refer to Fig. 4 .

Lemma 6. Let $P$ be a convex polygon with a diametral pair $a, b \in \partial P$, where $|a b|=\operatorname{diam}_{2}(P)$. Suppose that a line $L$ intersects the interior of $P$, but does not cross the line segment ab. Let $p, q \in \partial P$ such that $p q=L \cap P$, and points $a, p, q$, and $b$ appear in this counterclockwise order in $\partial P$; and let $\widehat{p q}$ be the counterclockwise $p q$-arc of $\partial P$. Then $|\widehat{p q}| \leq \frac{4 \pi \sqrt{3}}{9}|p q|<2.42|p q|$.

Proof. Since $|a b|=\operatorname{diam}_{2}(P)$, then $P$ lies in the intersection of two disks of radius $|a b|$ centered at $a$ and $b$, respectively. It follows that for any point $c \in P$, we have $\angle a c b \geq \pi / 3$. For any point $c \in \widehat{p q}$, we have $\angle p c q \geq \angle a c b$ by convexity, whence $\angle p c q \geq \angle a c b \geq \pi / 3$. The locus of points $c$ where $(p, c, q)$ is a counterclockwise triple and $\angle p c q \geq \pi / 3$ is the convex compact set $C$ bounded by $p q$ and a circular arc of radius $\frac{\sqrt{3}}{3}|p q|$. The length of the circular arc is $\frac{2 \pi / 3}{2 \pi}=\frac{2}{3}$ times the circumference of the circle, or $\frac{2}{3} \cdot 2 \pi \cdot \frac{\sqrt{3}}{3}|p q|=\frac{4 \pi \sqrt{3}}{9}|p q|<2.42|p q|$.

Let $D$ be the convex compact set bounded by $p q$ and $\widehat{p q}$. It is well known that for any two convex compact sets $C, D \subset \mathbb{R}^{2}, C \subset D$ implies $\operatorname{per}(C) \leq \operatorname{per}(D)$ (e.g., by Crofton's formula). Consequently, $|\widehat{p q}|<2.42|p q|$.

Lemma 7. Let $P \in \mathcal{C}(h)$ such that $\operatorname{diam}_{2}\left(P_{i}\right) \leq \Delta \cdot \operatorname{diam}_{2}(P)$ for every hole $P_{i}$. If $\Delta \geq h^{-1}$, then there exists a path of length $O\left(h^{1 / 2} \Delta^{1 / 2}\right) \cdot \operatorname{diam}_{2}(P)$ in $P$ from any point $s \in P$ to the outer boundary $\partial P_{0}$.

Proof. We may assume w.l.o.g. that $\operatorname{diam}_{2}(P)=1$. For each hole $P_{i}, i \in\{1, \ldots, h\}$, choose a diametral point pair $a_{i}, b_{i} \in \partial P_{i}$ with $\operatorname{diam}_{2}\left(P_{i}\right)=\left|a_{i} b_{i}\right|$, and $x\left(a_{i}\right) \leq x\left(b_{i}\right)$. By Lemma 5, there exists a monotone path $\gamma$ of length $|\gamma| \leq O\left(h^{1 / 2} \Delta^{1 / 2}\right)$ from $s$ to the outer boundary $\partial \vec{P}_{0}$, that lies in $P_{0}$, and does not cross any of the segments $a_{i} b_{i}$. The edges of $\gamma$ alternate between edges along segments $a_{i} b_{i}$, which lie in distinct holes, and edges between two distinct segments that are all parallel to a common direction vector $\vec{w}$. However, $\gamma$ may intersect some of the holes, so it does not necessarily lie in $P$.

We modify $\gamma$ as follows. For every maximal subpath $\gamma(p, q)$ whose interior lies in the interior of a hole $P_{i}$, we replace $\gamma(p, q)$ with a path $\widehat{p q}$ along $\partial P_{i}$. Note that the segments $p q$ and $a_{i} b_{i}$ do not cross: Indeed, suppose they cross. Then $\gamma(p, q)$ crosses $a_{i} b_{i}$ in the interior of $P_{i}$. However, when $\gamma$ reaches $a_{i} b_{i}$, it follows it to one of its endpoint, so $q \in\left\{a_{i}, b_{i}\right\}$, which is a contradiction.

By Lemma 6, $|\widehat{p q}| \leq 2.42|\gamma(p, q)|$. Consequently, all detours jointly increase the length of $\gamma$ by a factor of at most 2.42, and the resulting path has length $O\left(h^{1 / 2} \Delta^{1 / 2}\right)$.

## 4 Polygons with Fat or Axis-Aligned Convex Holes

In this section, we show that in a polygonal domain $P$ with fat convex holes, the distortion $\operatorname{geod}(s, t) /|s t|$ is bounded by a constant for all $s, t \in P$, and prove Proposition 1.

Let $C$ be a convex body in the plane, and let $P=\mathbb{R}^{2} \backslash C$ be its complement. For any two points $s, t \in \partial C$, we compare the Euclidean distance $|s, t|$ with the geodesic distance geod $(s, t)$, which is the shortest st-path along the boundary of $C$. The geometric dilation of $C$ is $\delta(C)=\sup _{s, t \in \partial C} \frac{\operatorname{geod}(s, t)}{|s t|}$.

Lemma 8. Let $C$ be a $\lambda$-fat convex body, $\lambda \in(0,1]$. Then $\delta(C) \leq \min \left\{\pi \lambda^{-1}, 2\left(\lambda^{-1}+1\right)\right\}=O\left(\lambda^{-1}\right)$.
Proof. It is known [11, Lemma 11] that $\delta(C)=\frac{|\partial C|}{2 h}$, where $h=h(C)$ is the minimum halving distance of $C$ (i.e., the minimum distance between two points on $C$ that divide the length of $C$ in two equal parts). It is also known [10, Thm. 8] that $h \geq \operatorname{width}(C) / 2$. Putting these together one deduces that $\delta(C) \leq \frac{|C|}{\text { width }(C)}$. The isoperimetric inequality $|\partial C| \leq \operatorname{diam}_{2}(C) \pi$ and the obvious inequality $|\partial C| \leq 2 \operatorname{diam}_{2}(C)+2$ width $(C)$ lead to the following dilation bounds: $\delta(C) \leq \pi \frac{\operatorname{diam}_{2}(C)}{\text { width }(C)}$ and $\delta(C) \leq 2\left(\frac{\operatorname{diam}_{2}(C)}{\text { width }(C)}+1\right)$; see also [10, 24]. Since $C$ is $\lambda$-fat, direct substitution yields the two bounds in the lemma; the latter bound is better for small $\lambda$.

Corollary 2. Let $P=P_{0} \backslash\left(\bigcup_{i=1}^{h} P_{i}\right)$ be a polygonal domain, where $P_{0}$ is a convex polygon and $P_{1}, \ldots, P_{h}$ are $\lambda$-fat convex polygons soe some $\lambda \in(0,1]$. Then for any $s, t \in P$, we have $\operatorname{geod}(s, t) \leq O\left(\lambda^{-1}|s t|\right)$.

Proof. If the line segment st is contained in $P$, then $\operatorname{geod}(s, t)=|s t|$, and the proof is complete. Otherwise, segment st is the concatenation of line segments contained in $P$ and line segments $p_{i} q_{i} \subset P_{i}$ with $p_{i}, q_{i} \in \partial P_{i}$, for some indices $i \in\{1, \ldots, h\}$. By replacing each segment $p_{i} q_{i}$ with the shortest path on the boundary of the hole $P_{i}$, we obtain an $s t$-path $\gamma$ in $P$. Since each hole is $\lambda$-fat, we replaced each line segment $p_{i} q_{i}$ with a path of length $O\left(\left|p_{i} q_{i}\right| / \lambda\right)$ by Lemma 8. Overall, we have $|\gamma| \leq O(|s t| / \lambda)$, as required.

Proof. Given a triangulation $T \in \mathcal{T}(n)$, we construct a polygonal domain $P$ as follows. Let the outer polygon $P_{0}$ be the convex hull of $T$, and in the interior of each triangle $t_{i}$, create a triangular hole $P_{i}$ such that $\partial P_{i}$ is in the $\varepsilon$-neighborhood of $t_{i}$ for some small $\varepsilon>0$. A triangulation with $n$ vertices has at most $2 n-5=O(n)$ triangular faces, and so $P \in \mathcal{C}(h)$ with $h=O(n)$ holes.

We claim that $\varrho(P)=\Theta(\varrho(T))$. Note that $\operatorname{diam}_{2}(P)=\operatorname{diam}_{2}(T)$ by construction, and so it is enough to prove that $\operatorname{diam}_{g}(P)=\Theta\left(\operatorname{diam}_{g}(T)\right)$. Let $s, t \in P$ be a diametral pair, where $\operatorname{geod}_{P}(s, t)=\operatorname{diam}_{g}(P)$. By construction, $s$ and $t$ lie in the $\varepsilon$-neighborhood of some edges $e_{s}, e_{t} \in E(T)$, respectively. Let $\gamma$ be a shortest st-path in $P$, and suppose it passes through the $\varepsilon$-neighborhoods of the vertices $\left(v_{1}, \ldots, v_{k}\right)$. Note that all edges and vertices of $T$
are contained in $P$. In particular, the shortest $v_{1} v_{k}$-path in $T$ is also contained in $P$, and so $\operatorname{geod}_{P}\left(v_{1}, v_{k}\right) \leq \operatorname{geod}_{T}\left(v_{1}, v_{k}\right)$. The geodesic distance between $s$ and the $\varepsilon$-neighborhood of $v_{1}$ (resp., $t$ and the $\varepsilon$-neighborhood of $v_{k}$ ) is at $\operatorname{most}^{\operatorname{diam}} 2(T) \leq \operatorname{diam}_{g}(T)$. Overall, $\operatorname{geod}_{P}(s, t) \leq$ $3 \operatorname{diam}_{g}(T)+2 \varepsilon \leq O\left(\operatorname{diam}_{g}(T)\right)$ if $\varepsilon>0$ is sufficiently small, and so $\operatorname{diam}_{g}(P) \leq O\left(\operatorname{diam}_{g}(T)\right)$.

Conversely, let $u, v \in V(T)$ be a diametral pair of vertices, i.e., $\operatorname{geod}_{T}(u, v)=\operatorname{diam}_{g}(T)$. Let $\gamma$ be a shortest $u v$-path in $P$. Clearly, we have $|\gamma| \leq \operatorname{diam}_{g}(P)$. Suppose that $\gamma$ intersects the $\varepsilon$-neighborhoods of vertices $\left(v_{1}, \ldots, v_{k}\right)$, where $s=v_{1}$ and $t=v_{k}$. For $i=1, \ldots, k-1$, denote by $\gamma_{i}$ the subpath of $\gamma$ between the $\varepsilon$-neighborhoods of $v_{i}$ and $v_{i+1}$. Now $\left(v_{1}, \ldots, v_{k}\right)$ is an st-path in $T$ of length

$$
\sum_{i=1}^{k-1}\left|v_{i} v_{i+1}\right| \leq \sum_{i=1}^{k-1}\left(\left|\gamma_{i}\right|+2 \varepsilon\right)<|\gamma|+2 k \varepsilon \leq O\left(\operatorname{diam}_{g}(P)+n \varepsilon\right) \leq O\left(\operatorname{diam}_{g}(P)\right)
$$

if $\varepsilon>0$ is sufficiently small. This implies $\operatorname{diam}_{g}(T)=\operatorname{geod}_{T}(u, v) \leq O\left(\operatorname{diam}_{g}(P)\right)$, and finally that $\varrho(P)=\Theta(\varrho(T))$, as required.

Corollary 3. If $P=P_{0} \backslash\left(\bigcup_{i=1}^{h} P_{i}\right)$ be a polygonal domain, where $P_{0}$ is a convex polygon and $P_{1}, \ldots, P_{h}$ are $\lambda$-fat convex polygons for some $\lambda \in(0,1]$, then $\operatorname{diam}_{g}(P) \leq O\left(\lambda^{-1} \operatorname{diam}_{2}(P)\right)$, hence $\varrho(P) \leq O\left(\lambda^{-1}\right)$.

Proposition 3. Let $P \in \mathcal{C}(h), h \in \mathbb{N}$, such that every hole is an axis-aligned rectangle. Then from any point $s \in P$, there exists a path of length at most $\operatorname{diam}_{2}(P)$ in $P$ to the outer boundary $\partial P_{0}$.

Proof. Let $B=[0, a] \times[0, b]$ be a minimal axis-parallel bounding box containing $P$. We may assume w.l.o.g. that $x(s) \geq a / 2, y(s) \geq b / 2$, and $b \leq a$. We construct a staircase path $\gamma$ as follows. Start from $s$ in horizontal direction $\vec{d}_{1}=(1,0)$ until reaching the boundary $\partial P$ at some point $p$. While $p \notin \partial P_{0}$, make a $90^{\circ}$ turn from $\overrightarrow{d_{1}}=(1,0)$ to $\overrightarrow{d_{2}}=(0,1)$ or vice versa, and continue. We have $|\gamma| \leq \frac{a+b}{2} \leq a \leq \operatorname{diam}_{2}(P)$, as claimed.

## 5 Polygons with Holes versus Triangulations

The proof of Theorem 3 is the combination of Lemmas 9 and 10 below.
Lemma 9. For every triangulation $T \in \mathcal{T}(n)$, there exists a polygonal domain $P \in \mathcal{C}(h)$ with $h=\Theta(n)$ holes such that $\varrho(P)=\Theta(\varrho(T))$.

Proof. Given a triangulation $T \in \mathcal{T}(n)$, we construct a polygonal domain $P$ as follows. Let the outer polygon $P_{0}$ be the convex hull of $T$, and in the interior of each triangle $t_{i}$, create a triangular hole $P_{i}$ such that $\partial P_{i}$ is in the $\varepsilon$-neighborhood of $t_{i}$ for some small $\varepsilon>0$. A triangulation with $n$ vertices has at most $2 n-5=O(n)$ triangular faces, and so $P \in \mathcal{C}(h)$ with $h=O(n)$ holes.

We claim that $\varrho(P)=\Theta(\varrho(T))$. Note that $\operatorname{diam}_{2}(P)=\operatorname{diam}_{2}(T)$ by construction, and so it is enough to prove that $\operatorname{diam}_{g}(P)=\Theta\left(\operatorname{diam}_{g}(T)\right)$. Let $s, t \in P$ be a diametral pair, where $\operatorname{geod}_{P}(s, t)=\operatorname{diam}_{g}(P)$. By construction, $s$ and $t$ lie in the $\varepsilon$-neighborhood of some edges $e_{s}, e_{t} \in E(T)$, respectively. Let $\gamma$ be a shortest st-path in $P$, and suppose it passes through the $\varepsilon$-neighborhoods of the vertices $\left(v_{1}, \ldots, v_{k}\right)$. Note that all edges and vertices of $T$
are contained in $P$. In particular, the shortest $v_{1} v_{k}$-path in $T$ is also contained in $P$, and so $\operatorname{geod}_{P}\left(v_{1}, v_{k}\right) \leq \operatorname{geod}_{T}\left(v_{1}, v_{k}\right)$. The geodesic distance between $s$ and the $\varepsilon$-neighborhood of $v_{1}$ (resp., $t$ and the $\varepsilon$-neighborhood of $v_{k}$ ) is at $\operatorname{most}^{\operatorname{diam}} 2(T) \leq \operatorname{diam}_{g}(T)$. Overall, $\operatorname{geod}_{P}(s, t) \leq$ $3 \operatorname{diam}_{g}(T)+2 \varepsilon \leq O\left(\operatorname{diam}_{g}(T)\right)$ if $\varepsilon>0$ is sufficiently small, and so $\operatorname{diam}_{g}(P) \leq O\left(\operatorname{diam}_{g}(T)\right)$.

Conversely, let $u, v \in V(T)$ be a diametral pair of vertices, i.e., $\operatorname{geod}_{T}(u, v)=\operatorname{diam}_{g}(T)$. Let $\gamma$ be a shortest $u v$-path in $P$. Clearly, we have $|\gamma| \leq \operatorname{diam}_{g}(P)$. Suppose that $\gamma$ intersects the $\varepsilon$-neighborhoods of vertices $\left(v_{1}, \ldots, v_{k}\right)$, where $s=v_{1}$ and $t=v_{k}$. For $i=1, \ldots, k-1$, denote by $\gamma_{i}$ the subpath of $\gamma$ between the $\varepsilon$-neighborhoods of $v_{i}$ and $v_{i+1}$. Now $\left(v_{1}, \ldots, v_{k}\right)$ is an st-path in $T$ of length

$$
\sum_{i=1}^{k-1}\left|v_{i} v_{i+1}\right| \leq \sum_{i=1}^{k-1}\left(\left|\gamma_{i}\right|+2 \varepsilon\right)<|\gamma|+2 k \varepsilon \leq O\left(\operatorname{diam}_{g}(P)+n \varepsilon\right) \leq O\left(\operatorname{diam}_{g}(P)\right)
$$

if $\varepsilon>0$ is sufficiently small. This implies $\operatorname{diam}_{g}(T)=\operatorname{geod}_{T}(u, v) \leq O\left(\operatorname{diam}_{g}(P)\right)$, and finally that $\varrho(P)=\Theta(\varrho(T))$, as required.

Every planar straight-line graph $G=(V, E)$ can be augmented to a triangulation $T=\left(V, E^{\prime}\right)$, with $E \subseteq E^{\prime}$. One of the notable triangulations is the Constrained Delaunay Triangulation, for short, $\operatorname{CDT}(G)$. Bose and Keil [6] proved that $\operatorname{CDT}(G)$ has bounded stretch for so-called visibility edges. Specifically, if $u, v \in V$ and $u v$ does not cross any edge of $G$, then $\operatorname{CDT}(G)$ contains a $u v$-path of length $O(|u v|)$. The constant factor was later improved by Bose et al. [4, 5].
Lemma 10. For every polygonal domain $P \in \mathcal{C}(h)$, there exists a triangulation $T \in \mathcal{T}(n)$ with $n=\Theta(h)$ vertices such that $\varrho(T)=\Theta(\varrho(P))$.
Proof. Assume that $P=P_{0} \backslash \bigcup_{i=1}^{h} P_{i}$. For all $j=1, \ldots, h$, let $a_{i}, b_{i} \in \partial P_{i}$ be a diametral pair, that is, $\left|a_{i} b_{i}\right|=\operatorname{diam}_{2}\left(P_{i}\right)$. The line segments $\left\{a_{i} b_{i}: i=1, \ldots, h\right\}$, together with the four vertices of a minimum axis-aligned bounding box of $P$, form a planar straight-line graph $G$ with $2 h+4$ vertices. Let $T=\operatorname{CDT}(G)$ be the constrained Delaunay triangulation of $G$.

We claim that $\varrho(T)=\Theta(\varrho(P))$. We prove this claim in two steps. For an intermediate step, we define a polygon with $h$ line segment holes: $P^{\prime}=P_{0} \backslash \bigcup_{i=1}^{h}\left\{a_{i} b_{i}\right\}$. For any point pair $s, t \in P$, denote by $\operatorname{dist}(s, t)$ and $\operatorname{dist}^{\prime}(s, t)$, resp., the shortest distance in $P$ and $P^{\prime}$. Since $P \subseteq P^{\prime}$, we have $\operatorname{dist}^{\prime}(s, t) \leq \operatorname{dist}(s, t)$. By Lemma 6, dist $(s, t)<2.42 \cdot \operatorname{dist}^{\prime}(s, t)$. Thus $\operatorname{dist}^{\prime}(s, t)=\Theta(\operatorname{dist}(s, t))$ for all $s, t \in P$.

Every point $s \in P$ lies in one or more triangles in $T$; let $s^{\prime}$ denote a closest vertex of a triangle in $T$ that contains $s$. For $s, t \in P$, let dist ${ }^{\prime \prime}(s, t)$ be the length of the $s t$-path $\gamma$ composed of the segment $s s^{\prime}$, a shortest $s^{\prime} t^{\prime}$-path in the triangulation $T$, and the segment $t^{\prime} t$.

Since $\gamma$ does not cross any of the line segments $a_{j} b_{j}$, we have $\operatorname{dist}^{\prime}(s, t) \leq \operatorname{dist}^{\prime \prime}(s, t)$ for any pair of points $s, t \in P$. Conversely, every vertex in the shortest $s^{\prime} t^{\prime}$-path in $P^{\prime}$ is an endpoint of an obstacle $a_{j} b_{j}$. Consequently, every edge is either an obstacle segment $a_{j} b_{j}$, or a visibility edge between the endpoints of two distinct obstacles. By the result of Bose and Keil [6], for every such edge $p q, T$ contains a $p q$-path $\tau_{p q}$ of length $\left|\tau_{p q}\right| \leq O(|p q|)$. The concatenation of these paths is an $s^{\prime} t^{\prime}$-path $\tau$ of length $|\tau| \leq O\left(\right.$ dist $\left.^{\prime}\left(s^{\prime}, t^{\prime}\right)\right)$. Finally, note that the diameter of each triangle in $T$ is at $\operatorname{most}_{\operatorname{diam}_{2}}\left(P^{\prime}\right)$. Consequently, if $s, t \in P$ maximizes $\operatorname{dist}(s, t)$, then

$$
\operatorname{dist}^{\prime \prime}(s, t)=\left|s s^{\prime}\right|+|\gamma|+\left|t^{\prime} t\right| \leq 2 \cdot \operatorname{diam}_{2}(P)+|\tau| \leq O\left(\operatorname{dist}^{\prime}\left(s^{\prime} t^{\prime}\right)\right)
$$

as required. Consequently, $\operatorname{diam}_{g}(T)=\Theta\left(\operatorname{diam}_{g}(P)\right)$, which in turn implies that $\varrho(T)=\Theta(\varrho(P))$.

## 6 Conclusion

We have shown that in a convex polygonal domain $P$ with $h$ convex holes, the distortion of the geodesic distance, $\varrho(P)=\frac{\operatorname{diam}_{g}(P)}{\operatorname{diam}_{2}(P)}$, is always $O\left(h^{1 / 2}\right)$ and sometimes $\Omega\left(h^{1 / 3}\right)$. Closing the gap between the upper and lower bounds remains an open problem. Generalizations to $d$-dimensional Euclidean spaces for $d \geq 3$ are left for future research. Improving the running times of algorithms for computing the geodesic diameter or radius of a polygon with $h$ holes when all holes as well as the outer polygon are convex remains as another interesting problem.

Acknowledgments. Research on this paper was partially supported by the NSF awards DMS 1800734 and DMS 2154347.

## References

[1] Hee-Kap Ahn, Luis Barba, Prosenjit Bose, Jean-Lou De Carufel, Matias Korman, and Eunjin Oh. A linear-time algorithm for the geodesic center of a simple polygon. Discrete $\mathfrak{E}$ Computational Geometry, 56:836-859, 2016. doi:10.1007/s00454-016-9796-0.
[2] Sang Won Bae, Matias Korman, and Yoshio Okamoto. The geodesic diameter of polygonal domains. Discrete 8 Computational Geometry, 50(2):306-329, 2013. doi:10.1007/ s00454-013-9527-8.
[3] Sang Won Bae, Matias Korman, and Yoshio Okamoto. Computing the geodesic centers of a polygonal domain. Comput. Geom., 77:3-9, 2019. doi:10.1016/j.comgeo.2015.10.009.
[4] Prosenjit Bose, Jean-Lou De Carufel, and André van Renssen. Constrained generalized Delaunay graphs are plane spanners. Comput. Geom., 74:50-65, 2018. doi:10.1016/j.comgeo. 2018.06.006.
[5] Prosenjit Bose, Rolf Fagerberg, André van Renssen, and Sander Verdonschot. On plane constrained bounded-degree spanners. Algorithmica, 81(4):1392-1415, 2019. doi:10.1007/ s00453-018-0476-8.
[6] Prosenjit Bose and J. Mark Keil. On the stretch factor of the constrained Delaunay triangulation. In Proc. 3rd IEEE Symposium on Voronoi Diagrams in Science and Engineering (ISVD), pages 25-31, 2006. doi:10.1109/ISVD.2006.28.
[7] Sergio Cabello. Subquadratic algorithms for the diameter and the sum of pairwise distances in planar graphs. ACM Trans. Algorithms, 15(2):21:1-21:38, 2019. doi:10.1145/3218821.
[8] Yi-Jen Chiang and Joseph S. B. Mitchell. Two-point Euclidean shortest path queries in the plane. In Proc. 10th ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 215-224, 1999. URL: https://dl.acm.org/doi/10.5555/314500.314560.
[9] Mina Dalirrooyfard, Ray Li, and Virginia Vassilevska Williams. Hardness of approximate diameter: Now for undirected graphs. In Proc. 62nd IEEE Symposium on Foundations of Computer Science (FOCS), pages 1021-1032. IEEE, 2021. doi:10.1109/FOCS52979.2021. 00102.
[10] Adrian Dumitrescu, Annette Ebbers-Baumann, Ansgar Grüne, Rolf Klein, and Günter Rote. On the geometric dilation of closed curves, graphs, and point sets. Comput. Geom., 36(1):1638, 2007. doi:10.1016/j.comgeo.2005.07.004.
[11] Annette Ebbers-Baumann, Ansgar Grüne, and Rolf Klein. Geometric dilation of closed planar curves: New lower bounds. Comput. Geom., 37(3):188-208, 2007. doi:10.1016/j.comgeo. 2004.12.009.
[12] Steven R. Finch and John E. Wetzel. Lost in a forest. The American Mathematical Monthly, 111(8):645-654, 2004. doi:10.2307/4145038.
[13] Pawel Gawrychowski, Haim Kaplan, Shay Mozes, Micha Sharir, and Oren Weimann. Voronoi diagrams on planar graphs, and computing the diameter in deterministic $\tilde{O}\left(n^{5 / 3}\right)$ time. SIAM J. Comput., 50(2):509-554, 2021. doi:10.1137/18M1193402.
[14] Leonidas J. Guibas and John Hershberger. Optimal shortest path queries in a simple polygon. J. Comput. Syst. Sci., 39:126-152, 1989. doi:10.1016/0022-0000(89)90041-X.
[15] Sariel Har-Peled. Shortest path in a polygon using sublinear space. J. Comput. Geom., 7:19-45, 2015. doi:10.20382/jocg.v7i2a3.
[16] John Hershberger and Subhash Suri. Matrix searching with the shortest-path metric. SIAM J. Computing, 26(6):1612-1634, 1997. doi:10.1137/S0097539793253577.
[17] John Hershberger and Subhash Suri. An optimal algorithm for Euclidean shortest paths in the plane. SIAM J. Computing, 28(6):2215-2256, 1999. doi:10.1137/S0097539795289604.
[18] Sunjiv Kapoor, Shachindra N. Maheshwari, and Joeseph S. B. Mitchell. An efficient algorithm for Euclidean shortest paths among polygonal obstacles in the plane. Discrete $\&$ Computational Geometry, 18:377-383, 1997. doi:10.1007/PL00009323.
[19] David Kübel and Elmar Langetepe. On the approximation of shortest escape paths. Comput. Geom., 93:101709, 2021. doi:10.1016/j.comgeo.2020.101709.
[20] Der-Tsai Lee and Franco P. Preparata. Euclidean shortest paths in the presence of rectilinear barriers. Networks, 14:393-410, 1984. doi:10.1002/net.3230140304.
[21] Joseph S. B. Mitchell. Shortest paths among obstacles in the plane. Int. J. Comput. Geom. Appl., 6(3):309-332, 1996. doi:10.1142/S0218195996000216.
[22] Joseph S.B. Mitchell. Shortest paths and networks. In Handbook of Discrete and Computational Geometry, chapter 31. CRC Press, Boca Raton, FL, 3 edition, 2017. doi: 10.1201/9781315119601.
[23] Liam Roditty and Virginia Vassilevska Williams. Fast approximation algorithms for the diameter and radius of sparse graphs. In Proc. 45 th Symposium on Theory of Computing Conference (STOC), pages 515-524. ACM, 2013. doi:10.1145/2488608.2488673.
[24] Paul R. Scott and Poh Wah Awyong. Inequalities for convex sets. Journal of Inequalities in Pure and Applied Mathematics, 1:article 6, 2000.
[25] Haitao Wang. On the geodesic centers of polygonal domains. J. Comput. Geom., 9(1):131-190, 2018. doi:10.20382/jocg.v9i1a5.
[26] Haitao Wang. A new algorithm for Euclidean shortest paths in the plane. In Proc. 53rd ACM Symposium on Theory of Computing (STOC), pages 975-988, 2021. doi:10.1145/3406325. 3451037.
[27] Haitao Wang. Shortest paths among obstacles in the plane revisited. In Proc. 32nd ACMSIAM Symposium on Discrete Algorithms (SODA), pages 810-821, 2021. doi:10.1137/1. 9781611976465.51.
[28] Isaak M. Yaglom and Vladimir G. Boltyanskii. Convex Figures. 1951. Translated by P.J. Kelly and L.F. Walton, Holt, Rinehart and Winston, New York, 1961.


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